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DEDEKIND CUTS IN $C(X)$

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Abstract. The aim of this paper is to show that every Hausdorff continuous interval-valued function on a completely regular topological space X corresponds to a Dedekind cut in $C(X)$ and conversely.

1. Introduction. The set $\mathbb{R}$ of real numbers can be constructed starting from the rational set $\mathbb{Q}$ by the method developed by R. Dedekind in 1858 and which today is called "the order completion by Dedekind cuts" (see [9] or [16], pp. 17–21). In 1937, using the model of Dedekind cuts, H. M. MacNeille showed how the order completion of any partially ordered set can be obtained [15] (see also [14] or [19]). This completion is called Dedekind–MacNeille completion, Dedekind order completion or Dedekind completion for short. The monographs [14] and [19] contain not only the construction of the Dedekind completion of a partially ordered set, but also the construction of the Dedekind completion of an Archimedean vector lattice.

It is well known that the set $C(X)$ of all real-valued continuous functions on a topological space X is an ordered set which is not Dedekind complete ([14], p. 125). R. P. Dilworth was the first who tried to obtain the Dedekind completion of $C(X)$ by using MacNeille's construction. In 1950 ([10], Theorem 4.1) he proved that the Dedekind completion of $C_b(X)$, the set of all real-valued bounded continuous functions on a completely regular topological space X, is order isomorphic with the lattice of all normal upper semicontinuous functions on X. (See Section 3 for the definition of normal semicontinuous functions.) To obtain this result Dilworth showed first that every normal subset of $C_b(X)$ corresponds to a normal upper semicontinuous function and conversely. (See Section 2 for the definition of normal sets.)

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Not long after, in 1953, A. Horn, in a beautiful paper but unfortunately almost unknown [12], proved that the Dedekind completion of $C(X)$ is order isomorphic with the lattice consisting of all normal lower semicontinuous functions which are bounded above and below by continuous functions. The fact that the functions are normal lower semicontinuous and not normal upper semicontinuous like in Dilworth's result is not important because the two lattices are order isomorphic. The proof of Horn does not use Dedekind cuts or normal subsets of $C(X)$. Horn's idea was to describe directly the Dedekind completion of $C(X)$ like a subset of the Dedekind complete lattice of all the functions defined on X.

In 2004 R. Anguelov [1] constructed the Dedekind completion of $C(X)$ and $C_b(X)$ using Hausdorff continuous interval-valued functions on X. (See Section 3 for the definition and some properties of these functions.) Note that the definition of a Hausdorff continuous interval-valued function used in this paper is that which appears in the book of Sendov [18] and in the paper of Anguelov [1] and differs from that used in set valued analysis. This type of functions, that originally appeared in numerical analysis, are used today to determine solutions of partial differential equations [4, 5, 21, 22].

The aim of this paper is to show that every Hausdorff continuous interval-valued function on a completely regular topological space X corresponds to a Dedekind cut in $C(X)$ and conversely. So we show that the Dedekind completion of $C(X)$ can be constructed in the same manner in which we obtain $\mathbb{R}$ from $\mathbb{Q}$.

2. Cuts in an ordered set. In this section we establish the notation used throughout the paper and discuss the notion of cut in a partially ordered set. For the unexplained terminology about ordered sets, and especially about Dedekind completion, see [11, 14, 17].

Let $(P, \leq)$ be a partially ordered set (called ordered set, for short) and let A be a nonempty subset of it. If $p \in P$ is an upper (lower) bound of A we write this symbolically $A \leq p$ ($p \leq A$). If A is bounded above, then A^u denotes the set of all upper bounds of A, and if A is bounded below, then A^l denotes the set of all lower bounds of A. If $A = \varnothing$ then every $p \in P$ is a lower and at the same time an upper bound of A. Therefore, $\varnothing^u = P$ and $\varnothing^l = P$. In this situation we have: (a) $\sup \varnothing$ exists if and only if P has *a least element* denoted by $\min P$ and hence $\sup \varnothing = \min P$; (b) $\inf \varnothing$ exists if and only if P has *a greatest element* denoted by $\max P$, and hence $\inf \varnothing = \max P$. We will say that the ordered set $(P, \leq)$ has *endpoints* if P has a least element and a greatest element.

Let $(P, \leq)$ be an ordered set without endpoints. The sets A^u have the properties:

(U1) $A^u \neq \varnothing \Leftrightarrow A$ is bounded above.
(U2) $A^u = P \Leftrightarrow A = \varnothing$ (here it is used that P has no least element).
(U3) A^u is an upper-set (that is, if $p \in A^u$ and $q \geq p$, then $q \in A^u$).

Similarly, the sets B^l have the properties:

(L1) $B^l \neq \varnothing \Leftrightarrow B$ is bounded below.
(L2) $B^l = P \Leftrightarrow B = \varnothing$ (here it is used that P has no greatest element).
(L3) B^l is a down-set (that is, if $p \in B^l$ and $q \leq p$, then $q \in B^l$).

A subset A of P is called proper if $A \neq \varnothing$ and $A \neq P$. If A is a proper subset of P which is bounded above, then A^u is a proper subset of P. The set A^u is bounded below by every element of A. Hence the set A^{ul} is also a proper subset of P.

PROPOSITION 2.1. *The sets A^u and B^l have the following properties*:

(i) $A \subset B \Rightarrow B^u \subset A^u$ *and* $B^l \subset A^l$.
(ii) $A \subset A^{ul}$ *and* $A \subset A^{lu}$.
(iii) $A^u = A^{ulu}$ *and* $A^l = A^{lul}$.
(iv) $B \subset A^u \Leftrightarrow A \subset B^l$.
(v) $A^{ul} = \Big(\bigcup_{a\in A} \{a\}^l\Big)^{ul}$ *and* $A^{lu} = \Big(\bigcup_{a\in A} \{a\}^u\Big)^{lu}$.

If $A = A^{ul}$ then A is called a *lower normal* subset of P. If $A = A^{lu}$ then A is called an *upper normal* subset of P.

DEFINITION 2.2. Let $(P, \leq)$ be an ordered set without endpoints. A pair (A, B) of two nonempty subsets of P is called a *cut* of P if $A = B^l$ and $A^u = B$.

First we remark two simple properties of a cut (A, B): (a) A is bounded above and B is bounded below; (b) A and B are normal subsets of P: A is lower normal, $A = A^{ul}$, and B is upper normal, $B = B^{lu}$. (Because a normal set determines also a cut many authors use the name of cut for a normal set.) Hence a cut has one of the form (A^{ul}, A^u) or (B^l, B^{lu}), where A is bounded above and B is bounded below, respectively.

In the next proposition we enumerate the properties of a cut (A, B).

PROPOSITION 2.3. *Any cut (A, B) has the following properties*:

(i) *A and B are proper subsets of P.*
(ii) *A is a down-set and B is an upper-set.*
(iii) *If $p \in A$ and $q \in B$, then $p \leq q$.*
(iv) *$A \cap B$ has at most a common element.*
(v) *$A \cap B = \{p\} \Leftrightarrow p$ is the greatest element of A and the least element of B. In this case $A = \{p\}^l$ and $B = \{p\}^u$.*
(vi) *$A \cap B = \varnothing \Leftrightarrow A$ has no a greatest element or B has no a least element.*

A cut (A, B) with $A \cap B = \varnothing$ is called a *gap*. A cut (A, B) is called a *Dedekind cut* if A does not have a greatest element. Therefore we have two types of Dedekind cuts: some for which B has a least element and then $B = \{p\}^u$ for some $p \in P$, and others for which B has no least element, that is, the gaps.

The cuts are used to construct the Dedekind–MacNeille completion of an ordered set. Usually this construction is made using normal sets [11, 14, 17]. For the sake of completeness we sketch here the construction with cuts following the ideas from [13].

Let $(P, \leq)$ be an ordered set without endpoints. We denote by P^δ the collection of all the cuts of P. On P^δ we define the order relation

$$(A_1, B_1) \preceq (A_2, B_2) \Leftrightarrow A_1 \subseteq A_2.$$

Let us observe that $A_1 \subseteq A_2 \Leftrightarrow B_1 \supseteq B_2$.

The following theorem shows that P^δ is the Dedekind completion of P.

Theorem 2.4. *Let $(P,\leq)$ be an ordered set without endpoints. Then*

(i) *$(P^\delta,\preceq)$ is a Dedekind complete lattice.*

(ii) *The mapping $\Phi : P \longrightarrow P^\delta$, $\Phi(p) = (\{p\}^l,\{p\}^u)$, is an order embedding of P into P^δ which preserves all suprema and infima that exist in P.*

(iii) *For every cut $(A,B)\in P^\delta$ we have the equalities*

$$(A,B) = \bigvee_{P^\delta}\Phi(A) = \bigwedge_{P^\delta}\Phi(B). \tag{1}$$

For a proof of this theorem using normal sets see, for example, [17]. In the following we put in evidence some useful formulas which are obtained in the proof, make some comments and discuss some consequences.

If $\mathcal{S} = \{(A_\gamma,B_\gamma)\}_{\gamma\in\Gamma}$ is a nonempty bounded subset of P^δ, then there exist the supremum and the infimum of $\mathcal{S}$ in P^δ and they are given by the formulas

$$\bigvee_{P^\delta}(A_\gamma,B_\gamma) = \Big(\Big(\bigcup_{\gamma\in\Gamma}A_\gamma\Big)^{ul},\bigcap_{\gamma\in\Gamma}B_\gamma\Big), \tag{2}$$

$$\bigwedge_{P^\delta}(A_\gamma,B_\gamma) = \Big(\bigcap_{\gamma\in\Gamma}A_\gamma,\Big(\bigcup_{\gamma\in\Gamma}B_\gamma\Big)^{lu}\Big). \tag{3}$$

The pair $(\{p\}^l,\{p\}^u)$ is a cut of P since we have $\{p\}^{lu} = \{p\}^u$ and $\{p\}^{ul} = \{p\}^l$. The fact that Φ is an order embedding results from the obvious equivalences: $p_1\leq p_2 \Leftrightarrow \{p_1\}^l\subseteq\{p_2\}^l \Leftrightarrow \Phi(p_1)\preceq\Phi(p_2)$.

If (A,B) is a cut in P^δ, by using Proposition 2.1 and formulas (2) and (3), we have

$$(A,B) = (A^{ul},A^u) = \Big(\Big(\bigcup_{a\in A}\{a\}^l\Big)^{ul},\bigcap_{a\in A}\{a\}^u\Big) = \bigvee_{P^\delta}\Phi(A),$$

$$(A,B) = (B^l,B^{lu}) = \Big(\bigcap_{b\in B}\{b\}^l,\Big(\bigcup_{b\in B}\{b\}^u\Big)^{lu}\Big) = \bigwedge_{P^\delta}\Phi(B).$$

Based on the above theorem we can identify an element $p\in P$ with the cut $(\{p\}^l,\{p\}^u)$ generated by it and hence identify P with its image $\Phi(P)$ in P^δ. With this identification it is clear that the order relation $\preceq$ on P^δ, when is restricted on P, coincides with $\leq$.

Let (A,B) be a cut of P and let $a\in A$ and $b\in B$. Then $a\leq b$ and, since A is a down set and B is an upper set, $\{a\}^l\subseteq A$ and $B\subseteq\{b\}^u$. These inclusions show that we have the inequalities

$$(\{a\}^l,\{a\}^u)\preceq(A,B)\preceq(\{b\}^l,\{b\}^u).$$

Using the above identification between the elements of P and the cuts generated by them we can write

$$a\preceq(A,B)\preceq b, \tag{4}$$

for all $a\in A$ and all $b\in B$. Let us observe that the following converse implication holds: if a and b are two elements of P such that the inequalities (4) hold then $a\in A$ and $b\in B$.

Using the identification between P and $\Phi(P)$ the equalities (1) can be written in the following form:

$$(A,B) = \bigvee_{P^\delta}\{a\in P\mid a\preceq(A,B)\} = \bigwedge_{P^\delta}\{b\in P\mid b\succeq(A,B)\}.$$

3. *H*-continuous interval-valued functions. In this section we present in brief the notion of H-continuous interval-valued function. To keep the presentation as short as possible we adopt like definition an equivalent characterization. The initial definition and other properties of these functions can be found in [1, 3, 18].

Let X be a topological space. If $f : X \longrightarrow \mathbb{R}$ is a locally bounded function on X we denote by $I(f)$ the lower limit function of f and by $S(f)$ the upper limit function of f, that is,

$$I(f) : X \longrightarrow \mathbb{R}, \quad I(f)(x) = \sup_{V \in \mathcal{V}_x} \inf_{y \in V} f(y), \quad x \in X, \tag{5}$$

$$S(f) : X \longrightarrow \mathbb{R}, \quad S(f)(x) = \inf_{V \in \mathcal{V}_x} \sup_{y \in V} f(y), \quad x \in X. \tag{6}$$

where $\mathcal{V}_x$ denotes the set of all neighborhoods of the point x in the space X. The real-valued functions $I(f)$ and $S(f)$ are lower and upper semicontinuous, respectively, and $I(f) \leq f \leq S(f)$.

Let $\mathcal{B}_{\mathrm{loc}}(X)$ be the set of all locally bounded real-valued functions on X. $\mathcal{B}_{\mathrm{loc}}(X)$ is a Dedekind complete lattice and $C(X) \subset \mathcal{B}_{\mathrm{loc}}(X)$. If $f \in \mathcal{B}_{\mathrm{loc}}(X)$ then $I(f), S(f) \in \mathcal{B}_{\mathrm{loc}}(X)$. Thus we have two operators $I, S : \mathcal{B}_{\mathrm{loc}}(X) \longrightarrow \mathcal{B}_{\mathrm{loc}}(X)$. I is called the *lower Baire operator* and S is called the *upper Baire operator* in honor of R. Baire who used these limit functions in his book [7].

Baire operators $I, S : \mathcal{B}_{\mathrm{loc}}(X) \longrightarrow \mathcal{B}_{\mathrm{loc}}(X)$ have the following properties:

(a) $I(f) \leq f \leq S(f)$, for all $f \in \mathcal{B}_{\mathrm{loc}}(X)$;
(b) I and S are idempotent, that is, $I \circ I = I$ and $S \circ S = S$;
(c) I and S are order preserving, that is, $f \leq g \Rightarrow I(f) \leq I(g)$ and $S(f) \leq S(g)$, for all $f, g \in \mathcal{B}_{\mathrm{loc}}(X)$.

As a consequence of these properties the composed operators $I \circ S$ and $S \circ I$ are also idempotent.

Let $f \in \mathcal{B}_{\mathrm{loc}}(X)$. Then: (a) f is lower semicontinuous $\Leftrightarrow I(f) = f$; (b) f is upper semicontinuous $\Leftrightarrow S(f) = f$. A lower (upper) semicontinuous function f is called *normal lower* (*upper*) *semicontinuous* if $I(S(f)) = f$ $(S(I(f)) = f)$. The concept of normal semicontinuous function was introduced by Dilworth [10].

Let $\mathbb{IR} = \{[\underline{a}, \overline{a}] : \underline{a}, \overline{a} \in \mathbb{R}, \underline{a} \leq \overline{a}\}$ be the set of all real closed intervals (intervals, for short). An interval-valued function is a function $f : X \longrightarrow \mathbb{IR}$. Since for a point $x \in X$ the value $f(x)$ is an interval $[\underline{f}(x), \overline{f}(x)]$, we denote an interval-valued function by $\underline{\overline{f}} = [\underline{f}, \overline{f}]$. So we have distinct notation between real-valued and interval-valued functions. An interval-valued function $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ is called locally bounded if its components are locally bounded.

DEFINITION 3.1. A locally bounded interval-valued function $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ is said to be *Hausdorff continuous* (in the sense of Sendov [18] and Anguelov [1]), or *H-continuous* for short, if $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ has the following properties:

(i) $\underline{f}$ is lower semicontinuous and $\overline{f}$ is upper semicontinuous.
(ii) $I(\overline{f}) = \underline{f}$ and $S(\underline{f}) = \overline{f}$.

The set of all H-continuous functions is denoted by $\mathbb{H}_{\mathrm{ft}}(X)$. (The index ft shows that the functions have *finite* interval values.) Endowed with the order relation

$$\overline{\underline{f}} \leq \overline{\underline{g}} \Leftrightarrow \underline{f} \leq \underline{g} \text{ and } \overline{f} \leq \overline{g}, \tag{7}$$

the set $\mathbb{H}_{\mathrm{ft}}(X)$ is a Dedekind order complete lattice [1].

H-continuous functions do not differ too much from the usual real-valued continuous functions because they assume interval values only on a set of first Baire category. More precisely, it is shown in [1] that for every $\overline{\underline{f}} = [\underline{f}, \overline{f}] \in \mathbb{H}_{\mathrm{ft}}(X)$ the set $W_{\overline{\underline{f}}} = \{x \in X \mid \overline{f}(x) - \underline{f}(x) > 0\}$ is of first Baire category. The function $\overline{\underline{f}}$ has point values on the set $D_{\overline{\underline{f}}} = X \setminus W_{\overline{\underline{f}}} = \{x \in X \mid \overline{f}(x) = \underline{f}(x)\}$, that is, $\overline{f}(x) = f(x)$, for all $x \in D_{\overline{\underline{f}}}$, and $\overline{\underline{f}} = f$ is a real-valued continuous function on $D_{\overline{\underline{f}}}$. If X is a Baire space, $D_{\overline{\underline{f}}}$ is a dense subset of X. Therefore a H-continuous function on a Baire space X has the form

$$\overline{\underline{f}}(x) = \begin{cases} f(x), & \text{if } x \in D_{\overline{\underline{f}}}, \\ [\underline{f}(x), \overline{f}(x)], & \text{if } x \in W_{\overline{\underline{f}}}. \end{cases}$$

4. Dedekind cuts in $C(X)$. Let X be a topological space. A function $f : X \longrightarrow \mathbb{R}$ is called *C-bounded on X*, or simply *C-bounded*, if there exist two continuous functions $g_1, g_2 \in C(X)$ such that $g_1 \leq f \leq g_2$. Note that f is C-bounded if and only if $I(f)$ and $S(f)$ are C-bounded. The set of all C-bounded functions is denoted by $\mathcal{B}_c(X)$. $\mathcal{B}_c(X)$ is a Dedekind complete lattice and $\mathcal{B}(X) \subset \mathcal{B}_c(X) \subset \mathcal{B}_{\mathrm{loc}}(X)$. ($\mathcal{B}(X)$ denotes the set of all bounded functions on X.)

If f is a C-bounded function, we denote by L_f the set of all continuous functions on X which are below f, that is, $L_f = \{g \mid g \in C(X),\ g \leq f\}$, and by U_f the set of all continuous functions on X which are above f, that is, $U_f = \{g \mid g \in C(X),\ g \geq f\}$. Obviously, these sets are nonempty since f is C-bounded. Then, for every C-bounded function f, we can define two new real-valued functions by setting

$$L(f)(x) = \sup\{g(x) \mid g \in L_f\}, \quad x \in X,$$
$$U(f)(x) = \inf\{g(x) \mid g \in U_f\}, \quad x \in X.$$

$L(f)$ is a lower semicontinuous function and $U(f)$ is an upper semicontinuous function. So we have two operators $L, U : \mathcal{B}_c(X) \longrightarrow \mathcal{B}_c(X)$, and, obviously,

$$L(f) \leq I(f) \leq S(f) \leq U(f), \quad f \in \mathcal{B}_c(X).$$

The next proposition was proved by Dilworth for real bounded functions ([10], Lemma 4.1) and by Horn for real extended valued functions ([12], Theorem 8). Our proof for C-bounded functions follows the ideas of Dilworth's proof.

Proposition 4.1. *If X is a completely regular topological space, then for every C-bounded function f we have the equalities*

$$S(f) = U(f), \quad I(f) = L(f). \tag{8}$$

Proof. Let f be an arbitrary function in $\mathcal{B}_c(X)$ and let x be an arbitrary point of X. For every $\varepsilon > 0$ there exists an open neighborhood V_x of x such that

$$M_x(f) < S(f)(x) + \varepsilon, \tag{9}$$

where $M_x(f) = \sup\{f(y) \mid y \in V_x\}$. Due to the complete regularity of X there exists a function $h \in C_b(X)$ with the properties: $h(x) = 1$, $h(z) = 0$ for all $z \notin V_x$, and $0 \le h \le 1$. Since f is C-bounded there exist two continuous functions on X, g_1 and g_2, such that $g_1 \le f \le g_2$. Consider the function $g : X \longrightarrow \mathbb{R}$ defined by

$$g(z) = g_2(z) - [g_2(z) - M_x(f) \wedge g_2(z)]\, h(z), \quad z \in X.$$

Because $g_2, h \in C(X)$, the function $g \in C(X)$. Since $h(x) = 1$ we have

$$g(x) = M_x(f) \wedge g_2(x) \le M_x(f). \tag{10}$$

If $z \in V_x$, because $0 \le h(z) \le 1$, we have

$$g(z) \ge M_x(f) \wedge g_2(z) \ge f(z) \wedge g_2(z) = f(z).$$

If $z \notin V_x$ then $h(z) = 0$ and so $g(z) = g_2(z) \ge f(z)$. Therefore $g \ge f$. By using the relations (9) and (10) and the definition of $U(f)(x)$ we obtain

$$S(f)(x) > M_x(f) - \varepsilon \ge g(x) - \varepsilon \ge U(f)(x) - \varepsilon.$$

Since ε is arbitrary, $S(f)(x) \ge U(f)(x)$ for all $x \in X$. Hence $S(f) \ge U(f)$. Because the inverse inequality is always true we have $S(f) = U(f)$.

A similar proof can be given for the equality $I(f) = L(f)$ by using the function $g(z) = g_1(z) + [m_x(f) \vee g_1(z) - g_1(z)]\, h(z)$, $z \in X$, where $m_x(f) = \inf\{f(y) \mid y \in V_x\}$ and V_x is an open neighborhood of x such that $I(f)(x) - \varepsilon < m_x(f)$. ∎

We define by $\mathcal{NL}^{\mathrm{cb}}_{\mathrm{sc}}(X)$ the set of all C-bounded normal lower semicontinuous functions on X and by $\mathcal{NU}^{\mathrm{cb}}_{\mathrm{sc}}(X)$ the set of all C-bounded normal upper semicontinuous functions on X.

The following proposition shows that if f is not only C-bounded but also a normal semicontinuous function then the sets U_f and L_f are normal subsets of $C(X)$. The case when f is a bounded function was proved by Dilworth ([10], Lemma 4.2).

Proposition 4.2. *Let X be a completely regular topological space.*

(i) *If $f \in \mathcal{NU}^{\mathrm{cb}}_{\mathrm{sc}}(X)$ then U_f is an upper normal subset of $C(X)$, that is,*

$$(U_f)^{lu} = U_f.$$

(ii) *If $f \in \mathcal{NL}^{\mathrm{cb}}_{\mathrm{sc}}(X)$ then L_f is a lower normal subset of $C(X)$, that is,*

$$(L_f)^{ul} = L_f.$$

Proof. (i) The inclusion $U_f \subset (U_f)^{lu}$ is always true. We must prove that $(U_f)^{lu} \subset U_f$. Let $h \in (U_f)^{lu}$. This means that $h \in C(X)$ and $h \ge (U_f)^l$. We must show that $h \in U_f$, that is $h \ge f$. Let $x \in X$ and let $\varepsilon > 0$. Because h is continuous at x, there exists a neighborhood V of x such that

$$|h(y) - h(x)| < \varepsilon/2, \quad y \in V. \tag{11}$$

Since f is normal upper semicontinuous, by Theorem 3.1 of [10], there exists a nonempty open set $A \subset V$ such that

$$m_A(f) \ge f(x) - \varepsilon/2, \tag{12}$$

where $m_A(f) = \inf\{f(y) \mid y \in A\}$.

Let y_0 be a point of A. Since the space X is completely regular there exists a continuous function ϕ such that $\phi(y_0) = 1$, $\phi(y) = 0$ for all $y \notin A$, and $0 \leq \phi \leq 1$. Since f is C-bounded there exist $g_1, g_2 \in C(X)$ such that $g_1 \leq f \leq g_2$. We define a function $g_0 : X \longrightarrow \mathbb{R}$ by setting

$$g_0 = g_1 + [m_A(f) \vee g_1 - g_1]\phi.$$

Obviously, $g_0 \in C(X)$. We claim that $g_0 \geq f$. Indeed, if $y \in A$, by using that $0 \leq \phi(y) \leq 1$ we have

$$g_0(y) \leq m_A(f) \vee g_1(y) \leq f(y) \vee g_1(y) \leq f(y).$$

If $y \notin A$, then $\phi(y) = 0$, and so we have $g_0(y) = g_1(y) \leq f(y)$. Hence $g_0 \leq f$ which means that $g_0 \in (U_f)^l$. Since $h \geq (U_f)^l$ results that $h \geq g_0$. By using (11), (12) and $\phi(y_0) = 1$ we have

$$\begin{aligned} h(x) = h(y_0) + (h(x) - h(y_0)) \geq g_0(y_0) - \varepsilon/2 = \\ = m_A(f) \vee g_1(y_0) - \varepsilon/2 \geq m_A(f) - \varepsilon/2 \geq f(x) - \varepsilon. \end{aligned}$$

Since ε is arbitrary, $h(x) \geq f(x)$ for all $x \in X$. This means that $h \in U_f$ and the proof is complete. ∎

An useful observation is given by the following lemma. It shows that if f is a C-bounded function, then f and $I(f)$ have the same set of continuous lower bounds and f and $S(f)$ have the same set of continuous upper bounds.

LEMMA 4.3. *For a C-bounded function f we have the equalities*:

(i) $L_f = L_{I(f)}$ *and* $U_f = U_{S(f)}$.
(ii) $(U_f)^l = L_{S(f)}$ *and* $(L_f)^u = U_{I(f)}$.

Proof. (i) Let $g \in L_f$. Then $g \in C(X)$ and $g \leq f$. Since the operator I is order preserving, $g = I(g) \leq I(f)$. Therefore, $g \in L_{I(f)}$. Conversely, if $g \in L_{I(f)}$ we have $g \leq I(f) \leq f$. Hence $g \in L_f$.

(ii) By using Proposition 4.1 the following equivalences hold

$$g \in (U_f)^l \Leftrightarrow g \leq U_f \Leftrightarrow g \leq \inf U_f = U(f) = S(f) \Leftrightarrow g \in L_{S(f)},$$

which prove the first part of (ii). ∎

Let X be a completely regular topological space and let (A, B) be a cut of $C(X)$. Since A is bounded above and B is bounded below there exist the functions $\underline{f} = \sup A$ and $\overline{f} = \inf B$, where $\sup A$ and $\inf B$ are computed point-wisely. Note that $\underline{f}$ is lower semicontinuous, $\overline{f}$ is upper semicontinuous, $\underline{f} \leq \overline{f}$, and they are C-bounded.

The next proposition gives a converse result of Proposition 4.2.

PROPOSITION 4.4. *Let X be a completely regular topological space, let (A, B) be a cut of $C(X)$ and $\underline{f} = \sup A$ and $\overline{f} = \inf B$. Then the following statements hold*:

(i) $A^u = U_{\underline{f}}$, $(U_{\underline{f}})^l = L_{S(\underline{f})}$, $A = L_{S(\underline{f})}$, *and $\underline{f}$ is a normal lower semicontinuous function.*
(ii) $B^l = L_{\overline{f}}$, $(L_{\overline{f}})^u = U_{I(\overline{f})}$, $B = U_{I(\overline{f})}$, *and $\overline{f}$ is a normal upper semicontinuous function.*

Proof. (i) $A^u = U_{\underline{f}}$, by definitions. The equality $(U_{\underline{f}})^l = L_{S(\underline{f})}$ results from the following equivalences, in which Proposition 4.1 is used.

$$g \in (U_{\underline{f}})^l \Leftrightarrow g \le U_{\underline{f}} \Leftrightarrow g \le \inf U_{\underline{f}} = U(\underline{f}) = S(\underline{f}) \Leftrightarrow g \in L_{S(\underline{f})}.$$

Since (A, B) is a cut the set A is lower normal, that is, $A = A^{ul}$. Then we have

$$A = (A^u)^l = (U_{\underline{f}})^l = L_{S(\underline{f})}.$$

Because $\underline{f}$ is C-bounded, $S(\underline{f})$ is also C-bounded. By using Proposition 4.1 we have

$$\underline{f} = \sup L_{S(\underline{f})} = L(S(\underline{f})) = I(S(\underline{f})),$$

which means that $\underline{f}$ is a normal lower semicontinuous function. ∎

Corollary 4.5. *If X is a completely regular topological space, then every cut (A, B) of $C(X)$ has the form*

$$(L_{S(\underline{f})}, U_{I(\overline{f})}),$$

where $\underline{f} = \sup A$ and $\overline{f} = \inf B$.

The next theorem is the main result of the paper. It shows that every cut of $C(X)$ corresponds to a C-bounded H-continuous interval-valued function and conversely.

Theorem 4.6. *Let X be a completely regular topological space. Then to each cut of $C(X)$ there corresponds a C-bounded H-continuous interval-valued function, and conversely, to each C-bounded H-continuous interval-valued function there corresponds a cut of $C(X)$.*

More precisely, if (A, B) is a cut of $C(X)$ then the interval-valued function $\underline{\overline{f}} = [\underline{f}, \overline{f}]$, where $\underline{f} = \sup A$ and $\overline{f} = \inf B$, is C-bounded and H-continuous.

Conversely, if $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ is a C-bounded and H-continuous interval-valued function then (A, B), where $A = L_{\overline{f}}$ and $B = U_{\underline{f}}$, is a cut of $C(X)$.

Proof. Let (A, B) be a cut of $C(X)$. Then A and B are nonempty subsets of $C(X)$ such that $A^u = B$ and $B^l = A$. Let $\underline{f} = \sup A$ and $\overline{f} = \inf B$. $\underline{f}$ is lower semicontinuous, $\overline{f}$ is upper semicontinuous, $\underline{f} \le \overline{f}$, and they are C-bounded. To prove that $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ is a H-continuous interval-valued function we must show that $I(\overline{f}) = \underline{f}$ and $S(\underline{f}) = \overline{f}$. By using Proposition 4.1 and Proposition 4.4 we have

$$I(\overline{f}) = L(\overline{f}) = \sup L_{\overline{f}} = \sup B^l = \sup A = \underline{f}$$
$$S(\underline{f}) = U(\underline{f}) = \inf U_{\underline{f}} = \inf A^u = \inf B = \overline{f}.$$

Conversely, let $\underline{\overline{f}} = [\underline{f}, \overline{f}]$ be a C-bounded H-continuous interval-valued function. Then $\underline{f}$ is lower semicontinuous, $\overline{f}$ is upper semicontinuous, $I(\overline{f}) = \underline{f}$, and $S(\underline{f}) = \overline{f}$. Define $A = L_{\overline{f}}$ and $B = U_{\underline{f}}$. These sets are nonempty since $[\underline{f}, \overline{f}]$ is C-bounded. To prove that (A, B) is a cut of $C(X)$ we must show that $A^u = B$ and $B^l = A$. By using Lemma 4.3 we have

$$A^u = (L_{\overline{f}})^u = U_{I(\overline{f})} = U_{\underline{f}} = B$$
$$B^l = (U_{\underline{f}})^l = L_{S(\underline{f})} = L_{\overline{f}} = A.$$

The proof is complete. ∎

Let $\mathbb{H}_{\rm cb}(X)$ be the set of all C-bounded H-continuous interval-valued functions on X. By Theorem 2.4 and Theorem 4.6 we obtain another demonstration of Anguelov's result that the Dedekind completion of $C(X)$ is the Dedekind complete lattice $\mathbb{H}_{\rm cb}(X)$. The difference is that now it is known that a C-bounded H-continuous interval-valued functions on X represents a cut in $C(X)$.

THEOREM 4.7. *Let X be a completely regular topological space. Then the Dedekind completion of $C(X)$ is $\mathbb{H}_{\rm cb}(X)$, that is,*

$$C(X)^{\delta} = \mathbb{H}_{\rm cb}(X).$$

For the introduction of a linear structure on $\mathbb{H}_{\rm cb}(X)$ see [3, 6, 20]. The consideration of a norm on $\mathbb{H}_{\rm cb}(X)$ is discussed in [2]. For the relation between H-continuous interval-valued functions and the real-valued quasicontinuous functions see [8]. If we do not prefer to work with interval-valued functions, then we can use equivalence classes of quasicontinuous functions.

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