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EXISTENCE OF INFINITE-DIMENSIONAL LIE ALGEBRA FOR A UNITARY GROUP ON A HILBERT SPACE AND RELATED ASPECTS

HIROSHI ANDO

University of Copenhagen, Denmark
Research Institute for Mathematical Sciences, Kyoto University
Kyoto, 606-8502, Japan
E-mail: andonuts@kurims.kyoto-u.ac.jp

YASUMICHI MATSUZAWA*

Mathematisches Institut, Universität Leipzig
Johannisgasse 26, 04103 Leipzig, Germany
Department of Mathematics, Hokkaido University
Kita 10, Nishi 8, Kita-ku, Sapporo, 060-0810, Japan
E-mail: matsuzawa@math.sci.hokudai.ac.jp

Abstract. We show that for any strongly closed subgroup of a unitary group of a finite von Neumann algebra, there exists a canonical Lie algebra which is complete with respect to the strong resolvent topology. Our analysis is based on the comparison between measure topology induced by the tracial state and the strong resolvent topology we define on the particular space of closed operators on the Hilbert space. This is an expository article of the paper by both authors in Hokkaido Math. J. 41 (2012), 31–99, with some open problems.

Contents

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1. Introduction and main theorem. In this article we consider the following problem: Let $\mathcal{H}$ be an infinite-dimensional Hilbert space, $U(\mathcal{H})$ be its unitary group equipped with the strong operator topology (SOT for short). Let G be a strongly closed subgroup of $U(\mathcal{H})$. Is there a natural Lie algebra for G? Of course there is no problem when one considers norm topology, but many infinite-dimensional unitary representations are only strongly continuous. Therefore it is natural to consider the strong topology.

In view of the Stone Theorem, the natural candidate would be the set $\mathrm{Lie}(G)$ of all skew-adjoint operators A such that the one parameter unitary group generated by A belongs to G:

$$\mathrm{Lie}(G) := \{A^* = -A \text{ on } \mathcal{H} : e^{tA} \in G \text{ for all } t \in \mathbb{R}\}.$$

However, there remain some problems to be settled. First, due to the domain problem of unbounded operators, $\mathrm{Lie}(G)$ may not be a Lie algebra: the domain $\mathrm{dom}(A+B) = \mathrm{dom}(A) \cap \mathrm{dom}(B)$ or $\mathrm{dom}(AB) = \{\xi \in \mathrm{dom}(B) : B\xi \in \mathrm{dom}(A)\}$ may no longer be dense for $A, B \in \mathrm{Lie}(G)$. Even worse, it can be $\{0\}$. Secondly, as the object in question is infinite-dimensional, we need to choose a suitable topology for $\mathrm{Lie}(G)$. Since we consider a correspondence between Lie groups and Lie algebras, it is natural to require the map $\exp : \mathrm{Lie}(G) \to G$ to be continuous, with G equipped with the strong operator topology. It is well-known that a sequence $\{A_n\}_{n=1}^{\infty}$ of skew-adjoint operators on a Hilbert space converges to a skew-adjoint operator A in the strong resolvent sense if and only if e^{tA_n} converges strongly to e^{tA} for all $t \in \mathbb{R}$. Therefore it seems natural to consider the strong resolvent topology for $\mathrm{Lie}(G)$. However, it is not clear whether the vector space operations and the Lie bracket operations are continuous with respect to the strong resolvent topology of $\mathrm{Lie}(G)$. In general, even if sequences $\{A_n\}_n$, $\{B_n\}_n$ of skew-adjoint operators converge, respectively, to skew-adjoint operators A, B with respect to the strong resolvent topology, the sequences $\{A_n + B_n\}_n$ and $\{A_nB_n - B_nA_n\}_n$ are not guaranteed to converge, respectively, to $A+B$, $AB-BA$. In summary, the group G is in general too large to have a decent topological Lie algebra.

We can solve this difficulty in the case of G being a closed subgroup of the unitary group $U(\mathfrak{M})$ of some finite von Neumann algebra $\mathfrak{M}$ by applying noncommutative integration theory, and prove that the Lie algebraic operations are continuous with respect to the strong resolvent topology and that $\mathrm{Lie}(G)$ is complete as a uniform space. Hence $\mathrm{Lie}(G)$ forms a complete topological Lie algebra.

The starting point in our study is the classical result of Murray–von Neumann stating that the set of all closed operators affiliated with a finite von Neumann algebra $\mathfrak{M}$ has a natural *-algebra structure (see Section 2 for the terminology). By this and the results of

Trotter–Kato and Nelson, it is easily shown that $\mathrm{Lie}(G)$ actually has a well-defined Lie algebra structure. The non-trivial part in our analysis is the topological consideration on the set $\overline{\mathfrak{M}}$ of all closed operators affiliated with $\mathfrak{M}$, which we will explain in the subsequent chapter. In short, we compare two different topologies on $\overline{\mathfrak{M}}$, one of which comes from operator theory and the other from noncommutative integration theory. As a byproduct, we also discuss the following problem: *What kind of unbounded operator algebras can be represented in the form of* $\overline{\mathfrak{M}}$*?* Here, $\mathfrak{M}$ is a finite von Neumann algebra. We give their characterization from the viewpoint of a tensor category. We show that $\mathscr{R}$ can be represented as $\overline{\mathfrak{M}}$ if and only if it is an object of the category **fRng** (cf. Definition 5.3). More precisely, we prove that the category **fRng** is isomorphic to the category **fvN** of finite von Neumann algebras on a Hilbert space as a tensor category.

Notes. After finishing this work, the authors were informed by Professor Daniel Beltiţă that he had recently written a paper whose subject was closely related to ours [3]. He proved, prior to our work, related results in a different approach and motivation. Although having some connections, our focus was on the strong resolvent topology and its connection with other linear topologies on $\overline{\mathfrak{M}}$ and we found that it plays more important roles than the measure topology if $\mathfrak{M}$ is not countably decomposable. Also, the above topological analysis is a crucial part in the characterization of the tensor category of $\overline{\mathfrak{M}}$.

2. Notation and Murray–von Neumann's result. We fix the notation used in the later sections. All the proofs of results without any reference can be found in [1]. For the details about operator theory or operator algebra theory, see for instance, Reed–Simon [11] and Takesaki [17]. Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle \xi, \eta \rangle$, which is linear with respect to η. Let $\mathfrak{M}$ be a von Neumann algebra on $\mathcal{H}$. $\mathfrak{M}' := \{a \in \mathfrak{B}(\mathcal{H}) : ab = ba, \text{ for all } b \in \mathfrak{M}\}$ is the commutant of $\mathfrak{M}$. The group of all unitary operators in $\mathfrak{M}$ is denoted by $U(\mathfrak{M})$. The lattice of all projections in $\mathfrak{M}$ is denoted by $P(\mathfrak{M})$. The orthogonal projection onto the closed subspace $\mathcal{K} \subset \mathcal{H}$ is denoted by $P_{\mathcal{K}}$. The domain of an operator T on $\mathcal{H}$ is written as $\mathrm{dom}(T)$. If T is a closable operator, we write $\overline{T}$ for the closure of T.

Definition 2.1. A densely defined closable operator T on $\mathcal{H}$ is said to be *affiliated* with a von Neumann algebra $\mathfrak{M}$ if for any $u \in U(\mathfrak{M}')$, $uTu^* = T$ holds. In this case we write $T \eta \mathfrak{M}$. If T is affiliated with $\mathfrak{M}$, so is $\overline{T}$. The set of all densely defined closed operators affiliated with $\mathfrak{M}$ is denoted by $\overline{\mathfrak{M}}$.

Note that $T \eta \mathfrak{M}$ if and only if $xT \subset Tx$ for all $x \in \mathfrak{M}'$. A von Neumann algebra with no non-unitary isometry is called *finite*. It is known that a countably decomposable von Neumann algebra is finite if and only if there exists a faithful finite normal trace on it (for the definition and properties of traces, see [17]). A von Neumann algebra is called *atomic* if each nonzero projection dominates a minimal projection. A von Neumann algebra with no nonzero minimal projection is called *diffuse*.

In general, $\overline{\mathfrak{M}}$ is not a *-algebra under these operations. This is the reason for the difficulty of constructing Lie theory in infinite dimensions. However, Murray and von Neumann proved, in the pioneering paper [8], that for a finite von Neumann algebra $\mathfrak{M}$, $\overline{\mathfrak{M}}$ does constitute a *-algebra of unbounded operators:

THEOREM 2.2 (Murray–von Neumann). *The set $\overline{\mathfrak{M}}$ of all densely defined closed operators affiliated with a finite von Neumann algebra $\mathfrak{M}$ on $\mathcal{H}$ constitutes a *-algebra under the sum $\overline{A+B}$, the scalar multiplication $\overline{\lambda A}$ ($\lambda \in \mathbb{C}$), the product $\overline{AB}$ and the involution A^*, where $\overline{X}$ denotes the closure of a closable operator X.*

Through the proof of the above theorem, they introduced an important notion of complete density, which also play significant roles in our study.

DEFINITION 2.3. A subspace $\mathcal{D}$ is said to be *completely dense* for $\mathfrak{M}$ if there exists an increasing net $\{p_\alpha\} \subset P(\mathfrak{M})$ of projections in $\mathfrak{M}$ such that

(1) $p_\alpha \nearrow 1$ (strongly);
(2) $p_\alpha \mathcal{H} \subset \mathcal{D}$ for any α.

It is clear that a completely dense subspace is dense in $\mathcal{H}$. We often omit the phrase "for $\mathfrak{M}$" when the von Neumann algebra in consideration is obvious from the context.

REMARK 2.4.

(1) In [8], Murray and von Neumann used the term "strongly dense". However, this terminology is somewhat confusing. Therefore we tentatively use the term "completely dense".
(2) We do not assume the separability of $\mathcal{H}$, hence we modify the definition of complete density using net instead of sequence. For the subtle difference between them, see [1, 8].

The completely dense subspaces can be understood as noncommutative version of "sets of measure 1 in a probability space" as follows:

PROPOSITION 2.5 (Murray–von Neumann [8]). *Let $\mathfrak{M}$ be a finite von Neumann algebra.*

(1) *If $\{\mathcal{D}_i\}_{i=1}^\infty$ is a sequence of completely dense subspaces for $\mathfrak{M}$, then $\bigcap_{i=1}^\infty \mathcal{D}_i$ is also completely dense.*
(2) *For each $X \in \overline{\mathfrak{M}}$ and a completely dense subspace $\mathcal{D}$ for $\mathfrak{M}$, the subspace $\{\xi \in \mathrm{dom}(X) : X\xi \in \mathcal{D}\}$ is also completely dense. In particular, $\mathrm{dom}(X)$ is completely dense for all $X \in \overline{\mathfrak{M}}$.*

The above proposition, together with the following one, will prove Theorem 2.2.

PROPOSITION 2.6 (Murray–von Neumann [8]). *Let $\mathfrak{M}$ be a finite von Neumann algebra.*

(1) *Every closed symmetric operator in $\overline{\mathfrak{M}}$ is self-adjoint.*
(2) *There are no proper closed extensions of operators in $\overline{\mathfrak{M}}$. Namely, if $X, Y \in \overline{\mathfrak{M}}$ satisfy $X \subset Y$, then $X = Y$.*
(3) *Let $\{X_n\}_{n=1}^\infty$ be a sequence in $\overline{\mathfrak{M}}$. The intersection of domains*

$$\mathcal{D}_{\mathcal{P}} := \bigcap_{p \in \mathcal{P}} \mathrm{dom}(p(X_1, X_1^*, X_2, X_2^*, \dots))$$

of all unbounded operators obtained by substituting $\{X_n\}_{n=1}^\infty$ into the noncommutative polynomial $p(x_1, y_1, \dots)$ is completely dense for $\overline{\mathfrak{M}}$, where $\mathcal{P}$ is the set of all noncommutative polynomials with indefinite elements $\{x_n, y_n\}_{n=1}^\infty$.

Finally, we remark that converse of the Murray–von Neumann's result is also true. Namely,

THEOREM 2.7. *Let $\mathfrak{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Assume that, for all $A, B \in \mathfrak{M}$, the domains* $\mathrm{dom}(A+B)$ *and* $\mathrm{dom}(AB)$ *are dense in $\mathcal{H}$. If the set $\overline{\mathfrak{M}}$ forms a *-algebra with respect to the sum $\overline{A+B}$, the scalar multiplication $\overline{\alpha A}$ ($\alpha \in \mathbb{C}$), the multiplication $\overline{AB}$ and the involution A^*, then $\mathfrak{M}$ is a finite von Neumann algebra.*

3. Topological analysis of $\overline{\mathfrak{M}}$ and existence of the Lie algebra. We keep the notation from Section 2. Let $\mathfrak{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. We investigate topological properties of $\overline{\mathfrak{M}}$. Our goal is to prove that the algebraic operations on $\overline{\mathfrak{M}}$ are continuous with respect to the strong resolvent topology. This can be done by comparing this topology with the τ-measure topology defined below. They seem quite different from each other, but in fact they coincide. To prove this fact, we further introduce another notion of convergence, called the almost everywhere convergence. The main topic of the present section is to compare various convergence relations between them.

3.1. Strong resolvent topology. First of all, we define a topology called the strong resolvent topology on the suitable subset of densely defined closed operators. Let $\mathcal{H}$ be a Hilbert space. We say that a densely defined closed operator A on $\mathcal{H}$ belongs to the *resolvent class* $\mathscr{RC}(\mathcal{H})$ if A satisfies the following two conditions:

(RC.1) there exist self-adjoint operators X and Y on $\mathcal{H}$ such that the intersection $\mathrm{dom}(X) \cap \mathrm{dom}(Y)$ is a core of X and Y,
(RC.2) $A = \overline{X+iY}$, $A^* = \overline{X-iY}$.

Note that (RC.1) implies $\mathrm{dom}(X) \cap \mathrm{dom}(Y)$ is dense, so $X+iY$ and $X-iY$ are closable. Thus $\overline{X+iY}$ and $\overline{X-iY}$ are always defined. Furthermore, we have

$$\frac{1}{2}(A+A^*) = \frac{1}{2}(\overline{X+iY} + \overline{X-iY}) \supset X|_{\mathrm{dom}(X)\cap\mathrm{dom}(Y)}.$$

Hence $A+A^*$ is closable and by (RC.1), we get $\frac{1}{2}(\overline{A+A^*}) \supset X$. As X is self-adjoint, X has no non-trivial symmetric extension, we have $\frac{1}{2}(\overline{A+A^*}) = X$. Therefore, X is uniquely determined. Similarly, Y is also unique and $\frac{1}{2i}(\overline{A-A^*}) = Y$. We define

$$\mathrm{Re}(A) := X = \frac{1}{2}(\overline{A+A^*}), \quad \mathrm{Im}(A) := Y = \frac{1}{2i}(\overline{A-A^*}).$$

Also note that bounded operators and (possibly unbounded) normal operators belong to $\mathscr{RC}(\mathcal{H})$. Now we endow $\mathscr{RC}(\mathcal{H})$ with the *strong resolvent topology* (SRT for short), the weakest topology for which the mappings

$$\mathscr{RC}(\mathcal{H}) \ni A \longmapsto \{\mathrm{Re}(A)-i\}^{-1}, \quad \{\mathrm{Im}(A)-i\}^{-1} \in (\mathfrak{B}(\mathcal{H}), \mathrm{SOT})$$

are continuous. Thus a net $\{A_\alpha\}_\alpha$ in $\mathscr{RC}(\mathcal{H})$ converges to $A \in \mathscr{RC}(\mathcal{H})$ with respect to the strong resolvent topology if and only if

$$\{\mathrm{Re}(A_\alpha)-i\}^{-1}\xi \to \{\mathrm{Re}(A)-i\}^{-1}\xi, \quad \{\mathrm{Im}(A_\alpha)-i\}^{-1}\xi \to \{\mathrm{Im}(A)-i\}^{-1}\xi,$$

for each $\xi \in \mathcal{H}$. This topology is well-studied in the field of unbounded operator theory and suitable for the operator theoretical study.

Let $\mathfrak{M}$ be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$. We shall show that $\overline{\mathfrak{M}}$ is a closed subset of the resolvent class $\mathscr{RC}(\mathcal{H})$. This fact follows from Propositions 2.5, 2.6 and the following lemmata [1].

LEMMA 3.1. *Let $\mathfrak{M}$ be a finite von Neumann algebra on a Hilbert apace $\mathcal{H}$, A be in $\overline{\mathfrak{M}}$. Then there exist unique self-adjoint operators B and C in $\overline{\mathfrak{M}}$ such that*

$$A = \overline{B + iC}.$$

LEMMA 3.2. *Let $\mathfrak{M}$ be a finite von Neumann algebra. Then $\overline{\mathfrak{M}}$ is closed with respect to the strong resolvent topology.*

REMARK 3.3. In general, the strong resolvent topology is not linear. Indeed, there exist sequences $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ of self-adjoint operators and self-adjoint operators A, B such that the following conditions hold:

(1) $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ converge to A and B in the strong resolvent topology, respectively;
(2) $A_n + B_n$ is essentially self-adjoint for each $n \in \mathbb{N}$;
(3) $A + B$ is essentially self-adjoint;
(4) $\left\{\overline{A_n + B_n}\right\}_{n=1}^{\infty}$ converges to some self-adjoint operator C in the strong resolvent topology, but $C \neq \overline{A + B}$.

For the details, see [15]. However, as we see in the sequel, the strong resolvent topology is linear on $\overline{\mathfrak{M}}$.

The next is the main technical theorem.

THEOREM 3.4. *Let $\mathfrak{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then $\overline{\mathfrak{M}}$ is a complete topological *-algebra with respect to the strong resolvent topology.*

To prove this theorem, we next consider τ-measure topology.

3.2. τ-measure topology. We first prove Theorem 3.4 in a countably decomposable case. In this case, we can use the noncommutative integration theory thanks to a faithful normal tracial state. Noncommutative integration theory was initiated by I. E. Segal [14] and has been well studied by many people. We follow the definition of τ-measure topology due to E. Nelson [10]. Let $\mathfrak{M}$ be a countably decomposable finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Fix a faithful normal tracial state τ on $\mathfrak{M}$. The *τ-measure topology* (MT for short) on $\overline{\mathfrak{M}}$ is the linear topology whose fundamental system of neighborhoods at 0 is given by

$$N(\varepsilon, \delta) := \left\{A \in \overline{\mathfrak{M}} : \text{there exists a projection } p \in \mathfrak{M} \text{ such that } \|Ap\| < \varepsilon,\ \tau(p^{\perp}) < \delta\right\},$$

where ε and δ run over all strictly positive real numbers. E. Nelson showed that $\overline{\mathfrak{M}}$ is a complete topological *-algebra with respect to this topology [10]. Note that the τ-measure topology satisfies the first countability axiom.

REMARK 3.5. In this context, the operators in $\overline{\mathfrak{M}}$ are sometimes called *τ-measurable operators*. In [4], various theorems of integration theory are obtained.

Thus there are two topologies on $\overline{\mathfrak{M}}$, the strong resolvent topology and the τ-measure topology. These two topologies look quite different, but they coincide on $\overline{\mathfrak{M}}$, i.e.,

Lemma 3.6. *Let $\mathfrak{M}$ be a countably decomposable finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Then the strong resolvent topology and the τ-measure topology coincide on $\overline{\mathfrak{M}}$. In particular, $\overline{\mathfrak{M}}$ forms a complete topological *-algebra with respect to the strong resolvent topology. Moreover, the τ-measure topology is independent of the choice of a faithful normal tracial state τ.*

To proceed further, we next consider the noncommutative analogue of almost everywhere convergence.

3.3. Almost everywhere convergence. Let $\mathfrak{M}$ be a countably decomposable finite von Neumann algebra on a Hilbert space $\mathcal{H}$.

Definition 3.7. A sequence $\{A_n\}_{n=1}^{\infty} \subset \overline{\mathfrak{M}}$ *converges almost everywhere* (with respect to $\mathfrak{M}$) to $A \in \overline{\mathfrak{M}}$ if there exists a completely dense subspace $\mathcal{D}$ such that

(i) $\mathcal{D} \subset \bigcap_{n=1}^{\infty} \mathrm{dom}(A_n) \cap \mathrm{dom}(A)$,
(ii) $A_n\xi$ converges to $A\xi$ for each $\xi \in \mathcal{D}$.

We remark the analogy between convergence notions in finite von Neumann algebras and those used in probability theory.

non-Abelian	Abelian
"a.e." convergence $X_n \to X$ on $\mathcal{D}$ (completely dense)	a.e. convergence $X_n(\omega) \to X(\omega)$ P-a.e.
convergence in τ-MT: $\forall\varepsilon > 0\,\exists\{e_n\}$ s.t. $\|(X_n - X)e_n\| < \varepsilon,\ \tau(e_n^{\perp}) \to 0$	convergence in probability $\forall\varepsilon > 0\ P(\lvert X_n - X\rvert > \varepsilon) \to 0$
convergence "in law" $\tau(E^{X_n}(\cdot)) \overset{\text{weak}^*}{\to} \tau(E^X(\cdot))$	convergence in law $\mu^{X_n} \overset{\text{weak}^*}{\to} \mu^X$

Here, $E^{X_n}(\cdot)$ and $E^X(\cdot)$ mean the spectral resolution of a self-adjoint operators X_n and X in $\overline{\mathfrak{M}}$, respectively. The relations between the almost everywhere convergence and the other topologies, as expected from the above analogy, turn out to be true [1].

Lemma 3.8. *Let $\{A_n\}_{n=1}^{\infty} \subset \overline{\mathfrak{M}}$ be a sequence, $A \in \overline{\mathfrak{M}}$. Suppose A_n converges to A in the τ-measure topology, then there exists a subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ of $\{A_n\}_{n=1}^{\infty}$ such that A_{n_k} converges almost everywhere to A.*

Lemma 3.9. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence in $\overline{\mathfrak{M}}$ converging almost everywhere to $A \in \overline{\mathfrak{M}}$. Suppose $\{{A_n}^*\}_{n=1}^{\infty}$ also converges almost everywhere to A^*, then $\{A_n\}_{n=1}^{\infty}$ converges to A in the strong resolvent topology.*

From these considerations with an additional technical result we can prove Lemma 3.6.

3.4. Direct sums of algebras of unbounded operators. To finish the proof of Theorem 3.4 in a general case, we consider some results about the direct sums of unbounded operators.

Let $\{\mathcal{H}_\alpha\}_\alpha$ be a family of Hilbert spaces and $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ be the direct sum Hilbert space of $\{\mathcal{H}_\alpha\}_\alpha$, i.e.,

$$\mathcal{H} := \Big\{\xi = \{\xi^{(\alpha)}\}_\alpha : \xi^{(\alpha)} \in \mathcal{H}_\alpha,\ \sum_\alpha \|\xi^\alpha\|^2 < \infty\Big\}.$$

For a subspace $\mathcal{D}_\alpha$ of $\mathcal{H}_\alpha$, we set

$$\widehat{\bigoplus}_\alpha \mathcal{D}_\alpha := \Big\{\xi = \{\xi^{(\alpha)}\}_\alpha \in \mathcal{H} : \xi^{(\alpha)} \in \mathcal{D}_\alpha,\ \xi^{(\alpha)} = 0 \text{ except finitely many } \alpha\Big\}.$$

It is known that $\widehat{\bigoplus}_\alpha \mathcal{D}_\alpha$ is dense in $\mathcal{H}$ whenever each $\mathcal{D}_\alpha$ is dense in $\mathcal{H}_\alpha$.

Next we recall the direct sum of unbounded operators. Let A_α be a (possibly unbounded) linear operator on $\mathcal{H}_\alpha$. We define the linear operator $A = \oplus_\alpha A_\alpha$ on $\mathcal{H}$ as follows:

$$\begin{aligned}
&\mathrm{dom}(A) := \Big\{\xi = \{\xi^{(\alpha)}\}_\alpha \in \mathcal{H} : \xi^{(\alpha)} \in \mathrm{dom}(A_\alpha),\ \sum_\alpha \|A_\alpha \xi^\alpha\|^2 < \infty\Big\},\\
&(A\xi)^{(\alpha)} := A_\alpha \xi^{(\alpha)}, \quad \xi \in \mathrm{dom}(A).
\end{aligned}$$

A is said to be the *direct sum* of $\{A_\alpha\}_\alpha$. It is easy to see that if each A_α is a densely defined closed operator then so is A. In this case,

$$A^* = \bigoplus_\alpha {A_\alpha}^*.$$

The next lemma follows immediately.

LEMMA 3.10. *Let $\mathcal{H}_\alpha$ be a Hilbert space, $\mathcal{H}$ be the direct sum Hilbert space of $\{\mathcal{H}_\alpha\}_\alpha$. For each α, we consider a net $\{A_{\alpha,\lambda}\}_{\lambda\in\Lambda}$ of self-adjoint operators on $\mathcal{H}_\alpha$ and self-adjoint operator A_α on $\mathcal{H}_\alpha$. Set*

$$A_\lambda := \bigoplus_\alpha A_{\alpha,\lambda},$$

and

$$A := \bigoplus_\alpha A_\alpha,$$

on the Hilbert space $\mathcal{H}$. Then A_λ converges to A in the strong resolvent topology if and only if each $\{A_{\alpha,\lambda}\}_{\lambda\in\Lambda}$ converges to A_α in the strong resolvent topology.

The next lemma is the key to prove Theorem 3.4. Here, the symbol $\bigoplus^b$ denotes the ℓ^∞-direct sum of von Neumann algebras.

LEMMA 3.11. *Let $\mathfrak{M}_\alpha$ be a finite von Neumann algebra acting on $\mathcal{H}_\alpha$, and put*

$$\mathfrak{M} := \bigoplus_\alpha^b \mathfrak{M}_\alpha.$$

Then

$$\overline{\mathfrak{M}} = \bigoplus_\alpha \overline{\mathfrak{M}}_\alpha.$$

The sum, the scalar multiplication, the multiplication and the involution are given by

$$\begin{aligned}
\overline{(\bigoplus_\alpha A_\alpha) + (\bigoplus_\alpha B_\alpha)} &= \bigoplus_\alpha (\overline{A_\alpha + B_\alpha}),\\
\overline{\lambda(\bigoplus_\alpha A_\alpha)} &= \bigoplus_\alpha (\overline{\lambda A_\alpha}), \quad \textit{for all } \lambda \in \mathbb{C},\\
\overline{(\bigoplus_\alpha A_\alpha)(\bigoplus_\alpha B_\alpha)} &= \bigoplus_\alpha (\overline{A_\alpha B_\alpha}),\\
(\bigoplus_\alpha A_\alpha)^* &= \bigoplus_\alpha (A_\alpha{}^*).
\end{aligned}$$

*In addition, if each $\overline{\mathfrak{M}}_\alpha$ is countably decomposable, then $\overline{\mathfrak{M}}$ is a complete topological *-algebra with respect to the strong resolvent topology on $\overline{\mathfrak{M}}$.*

We also use

Lemma 3.12. *Let $(\mathfrak{M}, \mathcal{H})$ and $(\mathfrak{N}, \mathcal{K})$ be spatially isomorphic finite von Neumann algebras. If a unitary operator U of $\mathcal{H}$ onto $\mathcal{K}$ induces the spatial isomorphism, then the map*

$$\Phi : \ \overline{\mathfrak{M}} \to \overline{\mathfrak{N}}, \ X \mapsto UXU^*,$$

*is a *-isomorphism. Moreover, Φ is a homeomorphism with respect to the strong resolvent topology.*

Combining all the above results, we can prove Theorem 3.4.

Proof of Theorem 3.4. Since $\mathfrak{M}$ is finite, there exists a family of countably decomposable finite von Neumann algebras $\{\mathfrak{M}_\alpha\}_\alpha$ such that $\mathfrak{M}$ is spatially isomorphic onto $\bigoplus_\alpha^b \mathfrak{M}_\alpha$. From Lemma 3.12, there exists a *-isomorphism of $\overline{\mathfrak{M}}$ onto $\bigoplus_\alpha \overline{\mathfrak{M}}_\alpha$ which is homeomorphic with respect to the strong resolvent topology. By Lemma 3.11, $\bigoplus_\alpha \overline{\mathfrak{M}}_\alpha$ is a complete topological *-algebra and so is $\overline{\mathfrak{M}}$. Hence the proof is complete. ∎

3.5. Local convexity. We also study the local convexity of $\big(\overline{\mathfrak{M}}, \mathrm{SRT}\big)$. This will be important when we consider the extension of a σ-weakly continuous map between finite von Neumann algebras to algebras of affiliated operators. The absence of atomic projections leads to the fact that $\overline{\mathfrak{M}}$ is no longer locally convex, although it is an F-space on which open mapping theorem, uniform boundedness principle and so on hold.

Proposition 3.13. *Let $\mathfrak{M}$ be a finite von Neumann algebra. Then the following are equivalent*:

(1) $(\overline{\mathfrak{M}}, \mathrm{SRT})$ *is locally convex.*
(2) $\mathfrak{M}$ *is atomic.*

We need some lemmata to prove the above proposition.

Above all, we use the following result.

Proposition 3.14. *Let $\mathfrak{M}$ be a finite von Neumann algebra. Then the following are equivalent*:

(1) *There exists no nonzero SRT-continuous linear functional on $\overline{\mathfrak{M}}$.*
(2) $\mathfrak{M}$ *is diffuse.*

In particular, we cannot extend any normal states SRT-continuously to $\overline{\mathfrak{M}}$. For the proof, see [1]. Also see Section 6 for more discussions.

4. Lie group-Lie algebra correspondences. Based on the above study, we show the main result of this paper. Let $\mathfrak{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, G be a strongly closed subgroup of $U(\mathfrak{M})$.

Recall that a densely defined closable operator A is called a *skew-adjoint operator* if $A^* = -A$, and A is called *essentially skew-adjoint* if $\overline{A}$ is skew-adjoint.

DEFINITION 4.1. For a strongly closed subgroup G of $U(\mathfrak{M})$, the set

$$\mathfrak{g} = \mathrm{Lie}(G) := \{A : A^* = -A \text{ on } \mathcal{H},\ e^{tA} \in G,\ \text{for all } t \in \mathbb{R}\}$$

is called the *Lie algebra* of G. The *complexification* $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}_{\mathbb{C}} := \left\{\overline{A + iB} : A, B \in \mathfrak{g}\right\}.$$

If $G = U(\mathfrak{M})$, we sometimes write $\mathfrak{g}$ as $\mathfrak{u}(\mathfrak{M})$.

The next lemma shows we can freely do algebraic operations on $\mathfrak{g}$.

LEMMA 4.2. *Under the above notation,* $\mathfrak{g} \subset \overline{\mathfrak{M}}$ *holds.*

Therefore the sum $\overline{A + B}$ and the Lie bracket $\overline{AB - BA}$ are well-defined operations in $\overline{\mathfrak{M}}$, but it is not clear whether they belong to $\mathfrak{g}$ again. The following lemma helps us answer the question.

LEMMA 4.3 (Trotter-Kato, Nelson [9]). *Let A, B be skew-adjoint operators on a Hilbert space $\mathcal{H}$.*

(1) *If $A + B$ is essentially skew-adjoint on* $\mathrm{dom}(A) \cap \mathrm{dom}(B)$, *then*

$$e^{t(\overline{A+B})} = \operatorname*{s-lim}_{n\to\infty}\left(e^{tA/n}e^{tB/n}\right)^n,$$

for all $t \in \mathbb{R}$.

(2) *If $(AB - BA)$ is essentially skew-adjoint on*

$$\mathrm{dom}(A^2) \cap \mathrm{dom}(AB) \cap \mathrm{dom}(BA) \cap \mathrm{dom}(B^2),$$

then

$$e^{t[A,B]} = \operatorname*{s-lim}_{n\to\infty}\left(e^{-\sqrt{t/n}\,A}e^{-\sqrt{t/n}\,B}e^{\sqrt{t/n}\,A}e^{\sqrt{t/n}\,B}\right)^{n^2},$$

for all $t > 0$, where $[A, B] := \overline{AB - BA}$.

LEMMA 4.4. *Let G be a strongly closed subgroup of $U(\mathfrak{M})$. Then $\mathfrak{g}$ is a real Lie algebra with the Lie bracket $[X, Y] := \overline{XY - YX}$.*

Using the above lemmata and topological results from Section 3, we prove the following main theorem.

THEOREM 4.5. *Let G be a strongly closed subgroup of the unitary group $U(\mathfrak{M})$ of a finite von Neumann algebra $\mathfrak{M}$. Then $\mathfrak{g}$ is a complete topological real Lie algebra with respect to the strong resolvent topology. Moreover, $\mathfrak{g}_{\mathbb{C}}$ is a complete topological Lie *-algebra.*

REMARK 4.6. It is easy to see that for $G = U(\mathfrak{M})$, its Lie algebra $\mathfrak{u}(\mathfrak{M})$ is equal to $\{A \in \overline{\mathfrak{M}}; A^* = -A\}$ and the exponential map

$$\exp : \mathfrak{u}(\mathfrak{M}) \to U(\mathfrak{M})$$

is continuous and surjective by the spectral theorem.

Group homomorphisms induce Lie algebra homomorphisms in a natural way.

PROPOSITION 4.7. *Let $\mathfrak{M}_1, \mathfrak{M}_2$ be finite von Neumann algebras on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, respectively. Let G_i be a strongly closed subgroup of $U(\mathfrak{M}_i)$ $(i = 1, 2)$. For any strongly continuous group homomorphism $\varphi : G_1 \to G_2$, there exists a unique SRT-continuous Lie algebra homomorphism $\Phi : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ such that $\varphi(e^A) = e^{\Phi(A)}$ for all $A \in \mathrm{Lie}(G_1)$. In particular, if G_1 is isomorphic to G_2 as a topological group, then $\mathrm{Lie}(G_1)$ and $\mathrm{Lie}(G_2)$ are isomorphic as topological Lie algebras.*

On the other hand, it also has an infinite-dimensional character.

PROPOSITION 4.8. *Let $\mathfrak{M}$ be a finite von Neumann algebra, then the following are equivalent;*

1. *The exponential map $\exp : \mathfrak{u}(\mathfrak{M}) \ni X \mapsto \exp(X) \in U(\mathfrak{M})$ is locally injective. Namely, the restriction of the map onto some SRT-neighborhood of $0 \in \overline{\mathfrak{M}}$ is injective.*
2. *$\mathfrak{M}$ is finite-dimensional.*

REMARK 4.9. $\mathrm{Lie}(G)$ is not always locally convex, whereas most of infinite-dimensional Lie theories, by contrast, assume local convexity. Indeed, by Proposition 3.13, $\mathfrak{u}(\mathfrak{M})$ is locally convex if and only if $\mathfrak{M}$ is atomic.

Next, we characterize closed *-subalgebras of $\overline{\mathfrak{M}}$.

PROPOSITION 4.10. *Let $\mathfrak{M}$ be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$, $\mathscr{R}$ be a SRT-closed *-subalgebra of $\overline{\mathfrak{M}}$ with $1_{\mathcal{H}}$. Then there exists a unique von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathscr{R} = \overline{\mathfrak{N}}$.*

COROLLARY 4.11. *Let $\mathfrak{M}$ be a finite von Neumann algebra on a Hilbert space $\mathcal{H}$, $\mathfrak{g}$ be a real SRT-closed Lie subalgebra of $\mathfrak{u}(\mathfrak{M})$. Then the following are equivalent:*

1. *there exists a von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$ such that $\mathfrak{g} = \mathfrak{u}(\mathfrak{N})$,*
2. *$1_{\mathcal{H}} \in \mathfrak{g}$ and for all $A, B \in \mathfrak{g}$, $i\left(\overline{AB + BA}\right) \in \mathfrak{g}$.*

In the above case, $\mathfrak{N}$ is unique.

5. Categorical characterization of $\overline{\mathfrak{M}}$

5.1. Motivation. In this section we consider the following problem.

PROBLEM 5.1. Characterize those *-algebras $\mathscr{R}$ of unbounded operators on a Hilbert space $\mathcal{H}$ which are isomorphic to $\overline{\mathfrak{M}}$ for some finite von Neumann algebra $\mathfrak{M}$ on $\mathcal{H}$.

We answer this problem with the aid of tensor category. Let $\mathfrak{M}$ be a finite von Neumann algebra on $\mathcal{H}$. As shown in the previous section, the topological properties of $\overline{\mathfrak{M}}$ are quite different from those of $\mathfrak{M}$. In particular, it may not be locally convex. On the other hand, they have many similar algebraic properties in common. For example, it can be shown that if $X, Y \in \overline{\mathfrak{M}}$ satisfy $\overline{XY} = 1$ then $\overline{YX} = 1$ follows automatically. These similarities are best explained from the categorical viewpoint.

5.2. fvN and fRng as tensor categories. First we note that the usual tensor product $(\mathfrak{M}_1, \mathfrak{M}_2) \mapsto \mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2$ of von Neumann algebras and the tensor product of σ-weakly continuous homomorphisms $(\phi_1, \phi_2) \mapsto \phi_1 \otimes \phi_2$ makes the category of finite von Neumann algebras a tensor category.

Definition 5.2. The category **fvN** is a category whose objects are pairs $(\mathfrak{M}, \mathcal{H})$ of a finite von Neumann algebra $\mathfrak{M}$ acting on a Hilbert space $\mathcal{H}$ and whose morphisms are σ-weakly continuous unital *-homomorphisms. The unit object is $(\mathbb{C}1_{\mathbb{C}}, \mathbb{C})$. The tensor functor is the usual tensor product functor of von Neumann algebras. The definition of left and right unit constraint functors should be obvious.

Next, we study the category of $\overline{\mathfrak{M}}$'s for $\mathfrak{M}$ being finite von Neumann algebras. For this purpose, we have to settle some subtleties due to the fact that we cannot use von Neumann algebraic structure from the outset. This difficulty can be overcome thanks to the notion of resolvent class operators, whose definitions are independent of von Neumann algebras (see Section 3). We define **fRng** as follows.

Definition 5.3. The category **fRng** is a category whose objects $(\mathscr{R}, \mathcal{H})$ consist of a SRT-closed subset $\mathscr{R}$ of the resolvent class $\mathscr{RC}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with the following properties:

(1) $X + Y$ and XY are closable for all $X, Y \in \mathscr{R}$.
(2) $\overline{X+Y}$, $\overline{\alpha X}$, $\overline{XY}$ and X^* again belong to $\mathscr{R}$ for all $X, Y \in \mathscr{R}$ and $\alpha \in \mathbb{C}$.
(3) $\mathscr{R}$ forms a *-algebra with respect to the sum $\overline{X+Y}$, the scalar multiplication $\overline{\alpha X}$, the multiplication $\overline{XY}$ and the involution X^*.
(4) $1_{\mathcal{H}} \in \mathscr{R}$.

The morphism set between $(\mathscr{R}_1, \mathcal{H}_1)$ and $(\mathscr{R}_2, \mathcal{H}_2)$ consists of SRT-continuous unital *-homomorphisms from $\mathscr{R}_1$ to $\mathscr{R}_2$.

From the definition of **fRng**, it is not clear whether, for each objects in **fRng**, its algebraic operations are continuous or not. The next lemma answers this problem.

Lemma 5.4. *Let $(\mathscr{R}, \mathcal{H})$ be an object in* **fRng**. *Then there exists a unique finite von Neumann algebra $\mathfrak{M}$ on $\mathcal{H}$ such that $\mathscr{R} = \overline{\mathfrak{M}}$. Furthermore, $\mathfrak{M} = \mathscr{R} \cap \mathfrak{B}(\mathcal{H})$.*

Note that for each finite von Neumann algebra $\mathfrak{M}$ on a Hilbert space $\mathcal{H}$, $(\overline{\mathfrak{M}}, \mathcal{H})$ is an object in **fRng**. Now we answer Problem 5.1 in the following form.

Theorem 5.5. *The category* **fRng** *is a tensor category. Moreover,* **fRng** *and* **fvN** *are isomorphic as tensor categories.*

To prove this theorem, we need many lemmata. We give an outline of the proof. First, we define the tensor product $\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2$ of objects $\mathscr{R}_i$ $(i = 1, 2)$ in **fRng**. Let A, B be densely defined closed operators on Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, respectively. Let $A \otimes_0 B$ be an operator defined by

$$\operatorname{dom}(A \otimes_0 B) := \operatorname{dom}(A) \otimes_{\mathrm{alg}} \operatorname{dom}(B),$$
$$(A \otimes_0 B)(\xi \otimes \eta) := A\xi \otimes B\eta, \quad \xi \in \operatorname{dom}(A),\ \eta \in \operatorname{dom}(B).$$

It is easy to see that $A \otimes_0 B$ is closable. Denote its closure by $A \otimes B$.

Lemma 5.6. *Let $\mathfrak{M}_1$, $\mathfrak{M}_2$ be finite von Neumann algebras acting on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, respectively. Let $A \in \overline{\mathfrak{M}_1}$ and $B \in \overline{\mathfrak{M}_2}$. Then we have $A \otimes B \in \overline{\mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2}$.*

The next lemma says that the tensor product of algebras of affiliated operators has a natural *-algebraic structure.

Lemma 5.7. *Let $\mathfrak{M}$, $\mathfrak{N}$ be finite von Neumann algebras acting on Hilbert spaces $\mathcal{H}$, $\mathcal{K}$ respectively. Let $A, C \in \overline{\mathfrak{M}}$, $B, D \in \overline{\mathfrak{N}}$. Then we have*

(1) $\overline{(A \otimes B)(C \otimes D)} = \overline{AC} \otimes \overline{BD}$.
(2) $(A \otimes B)^* = A^* \otimes B^*$.
(3) $\overline{A + C} \otimes \overline{B + D} = \overline{A \otimes B + A \otimes D + C \otimes B + C \otimes D}$.
(4) $\overline{\lambda(A \otimes B)} = \overline{\lambda A} \otimes B = A \otimes \overline{\lambda B}$ $(\lambda \in \mathbb{C})$.

Now we define the tensor product $\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2$ of $(\mathscr{R}_1, \mathcal{H}_1)$ and $(\mathscr{R}_2, \mathcal{H}_2)$ in $\mathrm{Obj}(\mathbf{fRng})$. Let $\mathfrak{M}_i$ be finite von Neumann algebras on $\mathcal{H}_i$ such that $\mathscr{R}_i = \overline{\mathfrak{M}_i}$ $(i = 1, 2)$, respectively (cf. Lemma 5.4). From Lemma 5.7, the linear space $\mathscr{R}_1 \otimes_{\mathrm{alg}} \mathscr{R}_2$ spanned by $\{A_1 \otimes A_2 : A_i \in \mathscr{R}_i,\ i = 1, 2\}$ is a *-algebra. Since $\mathscr{R}_1 \otimes_{\mathrm{alg}} \mathscr{R}_2$ is a subset of $\overline{\mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2}$, it belongs to $\mathscr{RC}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Therefore:

Definition 5.8. Under the above notation, we define $\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2$ to be the SRT-closure (for $\mathcal{H}_1 \otimes \mathcal{H}_2$) of $\mathscr{R}_1 \otimes_{\mathrm{alg}} \mathscr{R}_2$.

Lemma 5.9. *Let $\mathscr{R}_i$ $(i = 1, 2)$ be as above. Then $\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2$ is also an object in* **fRng**. *More precisely, if $\mathscr{R}_i = \overline{\mathfrak{M}_i}$, where $\mathfrak{M}_i$ is a finite von Neumann algebra $(i = 1, 2)$, then $\overline{\mathfrak{M}_1} \overline{\otimes} \overline{\mathfrak{M}_2} = \overline{\mathfrak{M}_1 \overline{\otimes} \mathfrak{M}_2}$.*

The above Lemma says that $(\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2, \mathcal{H}_1 \otimes \mathcal{H}_2)$ is again an object in **fRng**.

Next, we discuss the extension of morphisms in **fvN** to ones in **fRng**.

Lemma 5.10. *Let $(\mathfrak{M}_1, \mathcal{H}_1)$, $(\mathfrak{M}_2, \mathcal{H}_2)$ be finite von Neumann algebras. Then the mapping*

$$(\overline{\mathfrak{M}_1}, \mathrm{SRT}) \times (\overline{\mathfrak{M}_2}, \mathrm{SRT}) \longrightarrow (\overline{\mathfrak{M}_1} \overline{\otimes} \overline{\mathfrak{M}_2}, \mathrm{SRT}),$$
$$(A, B) \longmapsto A \otimes B,$$

is continuous.

The next proposition guarantees the existence and the uniqueness of the extension of morphisms in **fvN** to the morphisms in **fRng**. Note that the claim is not trivial, because many σ-weakly continuous linear mappings between finite von Neumann algebras cannot be extended SRT-continuously to the algebra of affiliated operators. Indeed, we cannot extend any σ-weakly continuous state on a finite von Neumann algebra $\mathfrak{M}$ SRT-continuously onto $\overline{\mathfrak{M}}$ if $\mathfrak{M}$ is diffuse.

PROPOSITION 5.11. *Let $\mathfrak{M}_1, \mathfrak{M}_2$ be finite von Neumann algebras on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, respectively.*

(1) *For each SRT-continuous unital *-homomorphism $\Phi : \overline{\mathfrak{M}_1} \to \overline{\mathfrak{M}_2}$, the restriction φ of Φ onto $\mathfrak{M}_1$ is a σ-weakly continuous unital *-homomorphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$.*
(2) *Conversely, for each σ-weakly continuous unital *-homomorphism $\varphi : \mathfrak{M}_1 \to \mathfrak{M}_2$, there exists a unique SRT-continuous unital *-homomorphism $\Phi : \overline{\mathfrak{M}_1} \to \overline{\mathfrak{M}_2}$ such that $\Phi|_{\mathfrak{M}_1} = \varphi$.*

The next lemmata, together with Lemma 5.9, imply that **fRng** is a tensor category.

LEMMA 5.12. *Let $\mathscr{R}_i$, $\mathscr{S}_i$ $(i = 1, 2)$ be objects in* Obj(**fRng**). *If $\Psi_1 : \mathscr{R}_1 \to \mathscr{S}_1$, $\Psi_2 : \mathscr{R}_2 \to \mathscr{S}_2$ are SRT-continuous unital *-homomorphisms, then there exists a unique SRT-continuous unital *-homomorphism $\Psi : \mathscr{R}_1 \overline{\otimes} \mathscr{R}_2 \to \mathscr{S}_1 \overline{\otimes} \mathscr{S}_2$ such that $\Psi(A \otimes B) = \Psi_1(A) \otimes \Psi_2(B)$, for all $A \in \mathscr{R}_1$ and $B \in \mathscr{R}_2$. We define $\Psi_1 \otimes \Psi_2$ to be the map Ψ.*

LEMMA 5.13. *Let $(\mathscr{R}_i, \mathcal{H}_i)$ $(i = 1, 2, 3)$ be objects in* **fRng**. *Then we have a unique *-isomorphism which is homeomorphic with respect to the strong resolvent topology:*

$$(\mathscr{R}_1 \overline{\otimes} \mathscr{R}_2) \overline{\otimes} \mathscr{R}_3 \cong \mathscr{R}_1 \overline{\otimes} (\mathscr{R}_2 \overline{\otimes} \mathscr{R}_3)$$
$$(X_1 \otimes X_2) \otimes X_3 \mapsto X_1 \otimes (X_2 \otimes X_3), \textit{for all } X_i \in \mathscr{R}_i$$

We denote the map by $\alpha_{\mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3}$.

PROPOSITION 5.14. **fRng** *is a tensor category.*

Now we will prove that **fvN** is isomorphic to **fRng** as tensor categories. Define two functors $\mathcal{E}$: **fvN**→ **fRng**, $\mathcal{F}$: **fRng**→**fvN**.

DEFINITION 5.15. Define two correspondences $\mathcal{E}$, $\mathcal{F}$ as follows:

(1) For each object $(\mathfrak{M}, \mathcal{H})$ in **fvN**,
$$\mathcal{E}(\mathfrak{M}, \mathcal{H}) := (\overline{\mathfrak{M}}, \mathcal{H}),$$
which is an object in **fRng**. For each morphism $\varphi : \mathfrak{M}_1 \to \mathfrak{M}_2$ in **fvN**, $\mathcal{E}(\varphi) : \overline{\mathfrak{M}_1} \to \overline{\mathfrak{M}_2}$ is the unique SRT-continuous extension of φ to $\overline{\mathfrak{M}_1}$, so that $\mathcal{E}(\varphi)$ is a morphism in **fRng** by Proposition 5.11.
(2) For each object $(\mathscr{R}, \mathcal{H})$ in **fRng**,
$$\mathcal{F}(\mathscr{R}, \mathcal{H}) := (\mathscr{R} \cap \mathfrak{B}(\mathcal{H}), \mathcal{H}).$$
For each morphism $\Phi : \mathscr{R}_1 \to \mathscr{R}_2$ in **fRng**, $\mathcal{F}(\Phi) := \Phi|_{\mathscr{R}_1 \cap \mathfrak{B}(\mathcal{H})}$, which is a morphism in **fvN** by Proposition 5.11.

LEMMA 5.16. *$\mathcal{E}$ and $\mathcal{F}$ are tensor functors.*

Finally we can show that $\mathcal{E} \circ \mathcal{F} = 1_{\mathbf{fRng}}, \mathcal{F} \circ \mathcal{E} = 1_{\mathbf{fvN}}$. Therefore **fvN** and **fRng** are isomorphic as tensor categories. This completes the proof of Theorem 5.5.

6. Related problems. Here we discuss further research directions.

PROBLEM 6.1. Is it possible to characterize those topological *-algebras $\mathscr{R}$ which are isomorphic to $\overline{\mathfrak{M}}$ for some finite von Neumann algebra $\mathfrak{M}$ without referring to any von Neumann algebra $\mathfrak{M}$ and any Hilbert space $\mathcal{H}$?

Probably the notion of von Neumann-regular ring is helpful for solving the above problem.

PROBLEM 6.2. Let $\mathfrak{M}_1, \mathfrak{M}_2$ be finite von Neumann algebras, $f : \mathfrak{M}_1 \to \mathfrak{M}_2$ be a nonzero positive σ-weakly continuous linear map. As mentioned in the previous chapter, f cannot always be extended to an SRT-continuous linear map $F : \overline{\mathfrak{M}_1} \to \overline{\mathfrak{M}_2}$. Find a necessary and sufficient condition for f to have the unique extension to $\overline{\mathfrak{M}_1}$.

REMARK 6.3. For any diffuse von Neumann subalgebra $\mathfrak{N}_1$ of $\mathfrak{M}_1$, the range $f(\mathfrak{N}_1)$ must not be contained in any atomic von Neumann subalgebra of $\mathfrak{M}_2$.

Proof. Suppose by contradiction $f(\mathfrak{N}_1)$ is contained in an atomic subalgebra $\mathfrak{N}_2$ of $\mathfrak{M}_2$ and f has an SRT-continuous extension $F : \overline{\mathfrak{M}_1} \to \overline{\mathfrak{M}_2}$. Let $\overline{f}$ be a restriction of F to $\overline{\mathfrak{N}_1}$. Since $\mathfrak{N}_2$ is atomic, there exists a orthogonal family of central projections $\{z_i\}_{i\in I}$ of $\mathfrak{N}_2$ with sum equal to 1, such that $\mathfrak{N}_{2,z_i}$ is a finite type I factor for all $i \in I$. Then $\mathfrak{N}_{2,z_i}$ is SRT-closed and the map $\overline{f}_{z_i} : \overline{\mathfrak{N}_1} \ni x \mapsto \overline{f}(x)z_i$ takes values in $\mathfrak{N}_{2,z_i}$. Then for any σ-weakly continuous linear functional $\varphi \in (\mathfrak{N}_2)_*$, $\varphi \circ \overline{f}_{z_i} : \overline{\mathfrak{N}_1} \to \mathbb{C}$ is a SRT-continuous linear functional, hence is equal to 0 by Proposition 3.14. In particular, $\varphi(\overline{f}_{z_i}(1)) = 0$ for any φ, which means $\overline{f}_{z_i}(1) = f(1)z_i = 0$ for all $i \in I$. Therefore we have $f(1) = \sum_{i\in I} f(1)z_i = 0$. Since f is positive, this means $f = 0$, a contradiction. ∎

Therefore the question is whether this is also a sufficient condition. Clearly the complete positivity is not enough, as seen in Proposition 3.14. On the other hand, convex combinations of σ-weakly continuous unital *-homomorphisms can be extended to $\overline{\mathfrak{M}_1}$ by Proposition 5.11.

Let $\mathfrak{M}$ be a von Neumann algebra acting on $\mathcal{H}$. Since every isometry in $\mathfrak{M}$ is a strong limit of unitaries in $\mathfrak{M}$, $U(\mathfrak{M})$ is strongly closed if and only if $\mathfrak{M}$ is finite. Therefore in some sense $U(\mathfrak{M})$ is "maximally large" among those strongly closed subgroups G of $U(\mathcal{H})$ which have natural Lie algebras. Therefore:

PROBLEM 6.4. Let $\mathcal{H}$ be a Hilbert space, G be a strongly closed subgroup of $U(\mathcal{H})$. Suppose Lie(G) is a Lie algebra which is complete with respect to the strong resolvent topology. Is the von Neumann algebra $\mathfrak{M}$ generated by G of finite type?

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References

[1] H. Ando, Y. Matsuzawa, *Lie group-Lie algebra correspondences of unitary groups in finite von Neumann algebras*, Hokkaido Math. J. 41 (2012), 31–99.

[2] A. Arai, *Fock Spaces and Quantum Fields*, Nippon-Hyouronsha, Tokyo, 2000 (in Japanese).

[3] D. Beltiţă, *Lie theoretic significance of the measure topologies associated with a finite trace*, Forum Math. 22 (2010), 241–253.

[4] T. Fack, H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. 123 (1986), 269–300.

[5] C. Kassel, *Quantum Groups*, Grad. Texts in Math. 155, Springer, New York, 1995.

[6] H. Kosaki, *On intersections of domains of unbounded positive operators*, Kyushu J. Math. 60 (2006), 3–25.

[7] S. MacLane, *Categories for the Working Mathematician*, Grad. Texts in Math. 5, Springer, New York, 1998.

[8] F. Murray, J. von Neumann, *On rings of operators*, Ann. of Math. (2) 37 (1936), 116–229.

[9] E. Nelson, *Topics in Dynamics. I. Flows*, Princeton Univ. Press, Princeton, NJ, 1969.

[10] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. 15 (1974), 103–116.

[11] M. Reed, B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis*, Academic Press, New York, 1972.

[12] K. Schmüdgen, *On domains of powers of closed symmetric operators*, J. Operator Theory 9 (1983), 53–75.

[13] K. Schmüdgen, *Unbounded Operator Algebras and Representation Theory*, Oper. Theory Adv. Appl. 37, Birkhäuser, Basel, 1990.

[14] I. E. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. (2) 57 (1953), 401–457.

[15] B. Simon, *Quadratic forms and Klauder's phenomenon: a remark on very singular perturbations*, J. Funct. Anal. 14 (1973), 295–298.

[16] W. Stinespring, *Integration theorems for gages and duality for unimodular groups*, Trans. Amer. Math. Soc. 90 (1959), 15–56.

[17] M. Takesaki, *Theory of Operator Algebras I*, Springer, Berlin, 1979.