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EXPLICIT CONSTRUCTION OF A UNITARY DOUBLE PRODUCT INTEGRAL

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Abstract. In analogy with earlier work on the forward-backward case, we consider an explicit construction of the forward-forward double stochastic product integral $\overrightarrow{\overrightarrow{\prod}}(1+dr)$ with generator $dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)$. The method of construction is to approximate the product integral by a discrete double product $\overrightarrow{\overrightarrow{\prod}}_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} \Gamma(R_{m,n}^{(j,k)}) = \Gamma\big(\overrightarrow{\overrightarrow{\prod}}_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} (R_{m,n}^{(j,k)})\big)$ of second quantised rotations $R_{m,n}^{(j,k)}$ in different planes using the embedding of $\mathbb{C}^m \oplus \mathbb{C}^n$ into $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ in which the standard orthonormal bases of $\mathbb{C}^m$ and $\mathbb{C}^n$ are mapped to the orthonormal sets consisting of normalised indicator functions of equipartitions of finite subintervals of $\mathbb{R}$. The limits as $m, n \longrightarrow \infty$ of such double products of rotations are constructed heuristically by a new method, and are shown rigorously to be unitary operators. Finally it is shown that the second quantisations of these unitary operators do indeed satisfy the quantum stochastic differential equations defining the double product integral.

1. Introduction. In earlier work [2] the first author constructed an explicit family of unitary operators $\big({}_a^b\overrightarrow{\overleftarrow{W}}{}_s^t\big)_{0\leq a\leq b<\infty,\ 0\leq s\leq t<\infty}$ on the Hilbert space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ as limits of forward-backward directed discrete double products constructed as follows. For fixed m and n we first form the discrete directed double product $\overrightarrow{\overleftarrow{\prod}}_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_m} (R_{m,n}^{(j,k)})$ where $R_{m,n}^{(j,k)}$ is the $(m+n)\times(m+n)$ matrix got by embedding the 2×2 rotation matrix through the angle $\lambda\sqrt{\frac{(b-a)(t-s)}{mn}}$, where λ is a fixed real parameter, into the intersections of the j-th and $(m+k)$-th rows and columns, and completing the $(m+n)\times(m+n)$ matrix by putting 1 in the remaining diagonal and 0 in the off-diagonal positions. The

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$(m+n)\times(m+n)$ matrix $\overrightarrow{\overleftarrow{\prod}}_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_m}(R^{(j,k)}_{m,n})$ is regarded as an operator on the direct sum $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ by embedding the direct sum $\mathbb{C}^m\oplus\mathbb{C}^n$ into $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ by mapping the standard orthonormal bases in $\mathbb{C}^m$ and $\mathbb{C}^n$ to the orthonormal sets in $L^2(\mathbb{R})$ consisting of the normalised indicator functions of equipartitions into m and n subintervals respectively of the intervals $[a,b[$ and $[s,t[$. ${}^{b}_{a}\overrightarrow{\overleftarrow{W}}{}^{t}_{s}$ is then constructed heuristically as the limit as $m,n\to\infty$ of this operator. An explicit form for ${}^{b}_{a}\overrightarrow{\overleftarrow{W}}{}^{t}_{s}$ is found using a double version of the limiting procedure originally used [5] to construct the so-called time-orthogonal unitary dilation [6].

Using their explicit form the operators ${}^{b}_{a}\overrightarrow{\overleftarrow{W}}{}^{t}_{s}$ were shown [2] rigorously to be unitary. In [3] it was shown that the second quantisations $\Gamma\big({}^{b}_{a}\overrightarrow{\overleftarrow{W}}{}^{t}_{s}\big)$ in the Fock space

$$\Gamma\big(L^2(\mathbb{R})\oplus L^2(\mathbb{R})\big)=\Gamma\big(L^2(\mathbb{R})\big)\otimes\Gamma\big(L^2(\mathbb{R})\big)$$

constitute the family of double stochastic product integrals ${}^{b}_{a}\overrightarrow{\overleftarrow{\prod}}{}^{t}_{s}(1+\lambda(dA^\dagger\otimes dA-dA\otimes dA^\dagger))$ in so far as they satisfy the stochastic differential equations which define these double products. The construction is motivated by the heuristic approximation

$$\begin{aligned}
&{}^{b}_{a}\overrightarrow{\overleftarrow{\prod}}{}^{t}_{s}\,(1+\lambda(dA^\dagger\otimes dA-dA\otimes dA^\dagger))\\
&\simeq \overrightarrow{\overrightarrow{\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}}}\exp\Big\{1+i\lambda\sqrt{\frac{(b-a)(t-s)}{mn}}\big(a_j^\dagger\otimes b_k-a_j\otimes b_k^\dagger\big)\Big\}\\
&\simeq \overrightarrow{\overrightarrow{\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}}}\Gamma(R^{(j,k)}_{m,n})\\
&=\Gamma\Big\{\overrightarrow{\overrightarrow{\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}}}R^{(j,k)}_{m,n}\Big\}
\end{aligned}$$

where the $(a_j,a_j^\dagger)$ and $(b_k,b_k^\dagger)$ are the creation and annihilation pairs formed from the normalised increments of the creation and annihilation processes.

The original motivation for studying double products came from quantum groups, where a purely algebraic "indefinite integral" form of them was used to construct explicit solutions of the quantum Yang–Baxter equation [8]. This motivation prompted the choice of the "forward-backward" form $\overrightarrow{\overleftarrow{\prod}}$ of double product for study. On the other hand "causal" double products living in the single Fock space $\Gamma(L^2(\mathbb{R}))$ of the form $\overrightarrow{\prod}_{0\le x<y\le t}(1+\lambda(dA^\dagger(x)dA(y)-dA(x)\,dA^\dagger(y)))$ have recently [4] become of interest in connection with quantum versions of Lévy area $L(t)$, and in particular of the Lévy area formula [9, 10] for its characteristic function $\mathbb{E}\big[e^{ixL(t)}\big]=\operatorname{sech}(\frac{xt}{2})$ which is that of the Gamma distribution, one of the family of Meixner distributions [11]. Such causal double products are closely associated with double products of rectangular type, for example

through the formula

$$\begin{aligned}
&\overrightarrow{\prod_{a\le x<y\le c}} (1+\lambda(dA^\dagger(x)\,dA(y)-dA(x)\,dA^\dagger(y)))\\
&=\overrightarrow{\prod_{a\le x<y\le b}} (1+\lambda(dA^\dagger(x)\,dA(y)-dA(x)\,dA^\dagger(y)))\\
&\quad {}_a^b\overrightarrow{\overrightarrow{\prod}}{}_b^c(1+\lambda(dA^\dagger\otimes dA-dA\otimes dA^\dagger))\\
&\quad \overrightarrow{\prod_{b\le x<y\le c}} (1+\lambda(dA^\dagger(x)\,dA(y)-dA(x)\,dA^\dagger(y)))
\end{aligned}$$

where in the middle product the single Fock space $\Gamma(L^2(\mathbb{R}))$ is identified with the double Fock space $\Gamma(L^2([-\infty,b[))\otimes\Gamma(L^2([b,\infty[))$ using splitting at b. For this reason it is of interest to study "forward-forward" rectangular double products ${}_a^b\overrightarrow{\overrightarrow{\prod}}{}_s^t(1+\lambda(dA^\dagger\otimes dA-dA\otimes dA^\dagger))$ (henceforth we usually write $\overrightarrow{\prod}$ instead of $\overrightarrow{\overrightarrow{\prod}}$) by methods similar to those of [2, 3]. However the method of the present paper differs from that of [2] in that, instead of embedding 2×2 matrices into $(m+n)\times(m+n)$ matrices by filling in the missing diagonal terms as 1s, we fill in all the missing terms as 0s. This new method allows for a direct evaluation of the limit as $m,n\to\infty$ without going through the iterated double limit procedure of [5]. In addition it will be shown [4] that a modification of the new method allows explicit construction of the corresponding causal product, from which a quantum version of the Lévy area formula will follow.

The paper is organised as follows. In Sections 2 and 3 we recall the definition and properties of quantum stochastic double product integrals. In Section 4 we compute the matrix product $\overrightarrow{\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}}(1+M^{(j,k)})$ where $M^{(j,k)}$ is the $(m+n)\times(m+n)$ matrix formed from a given 2×2 matrix M by embedding the four elements of M at the four intersections of the j-th and $(m+k)$-th rows and columns and taking all remaining elements to be 0. In Section 5 we find the limit as $m,n\to\infty$ of the corresponding operator on $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ in the case when M is the rotation through the angle $\lambda\sqrt{\frac{(b-a)(t-s)}{mn}}$. Unitarity of this operator is established in Section 6. In Section 7 it is shown that its second quantisation is indeed ${}_a^b\overrightarrow{\overrightarrow{\prod}}{}_s^t(1+\lambda(dA^\dagger\otimes dA-dA\otimes dA^\dagger))$ insofar as it satisfies the quantum stochastic differential equations defining the latter.

2. Quantum stochastic double product integrals: definition. Let $\mathcal{I}=\mathbb{C}\langle dA^\dagger, dA, dT\rangle$ denote the Itô algebra spanned by the stochastic differentials of the standard creation and annihilation processes of quantum stochastic calculus [7] and the time process T equipped with the product defined by the quantum Itô product rule

	$dA^\dagger$	dA	dT
$dA^\dagger$	0	0	0
dA	dT	0	0
dT	0	0	0

.

Let dr be an element of the tensor product $\mathcal{I}\otimes\mathcal{I}$.

The double product $\overrightarrow{\prod}(1+dr)$ is a family $\left({}_a^b\overrightarrow{\prod}_s^t(1+dr)\right)_{a\leq b, s\leq t}$ of operators, indexed by pairs of finite half-open subintervals $]a,b],]s,t]\subset\mathbb{R}_+$, each acting in the tensor product $\mathcal{F}(L^2(\mathbb{R}_+))\otimes\mathcal{F}(L^2(\mathbb{R}_+))$ with itself of the Fock space $\mathcal{F}(L^2(\mathbb{R}_+))$ over $L^2(\mathbb{R}_+)$. The operator ${}_a^b\overrightarrow{\prod}_s^t(1+dr)$ is defined in two different but equivalent ways as

$$ {}_a^b\overrightarrow{\prod}{}_s^t(1+dr)=\begin{cases} {}_a^b\overrightarrow{\prod}(1+\widehat{\overrightarrow{\prod}}{}_s^t(1+dr)) & \text{(i)}\\ \overrightarrow{\prod}{}_s^t(1+{}_a^b\widehat{\overrightarrow{\prod}}(1+dr)) & \text{(ii).}\end{cases} $$

Here, in the first definition (i) the inner "decapitated" simple product integral $\widehat{\overrightarrow{\prod}}{}_s^t(1+dr)$ is defined to be the solution $X(t)$ at time t of the quantum stochastic differential equation

$$ dX^{1,2,3}=(X^{1,2}+1^2)\,dr^{1,3},\quad X(s)=0 $$

in which the superscripts $1,2,3$ denote places in the tensor product $\mathcal{I}\otimes\mathcal{P}\otimes\mathcal{I}$, where $\mathcal{P}$ is the algebra of iterated stochastic integral processes living in $\mathcal{F}(L^2(\mathbb{R}_+))$ and the first copy of $\mathcal{I}$ in $\mathcal{I}\otimes\mathcal{I}$ is taken to be the non-unital (left) system algebra, so that $X\in\mathcal{I}\otimes\mathcal{P}$. Specifically, since $\mathcal{I}$ is nilpotent, we have

$$ \widehat{\overrightarrow{\prod}}{}_s^t(1+dr)=X(t)=\int_{s<x<t}dr(\cdot,x)+\int_{s<x_1<x_2<t}dr(\cdot,x_1)\,dr(\cdot,x_2) \tag{1} $$

which is of the form $dA^\dagger\otimes P_-+dA\otimes P_++dT\otimes P_0$ where P_-, P_+ and P_0 are iterated integrals over the interval $]s,t]$. Thus we can define the simple product integral

$$ {}_a^b\overrightarrow{\prod}\left(1+\widehat{\overrightarrow{\prod}}{}_s^t(1+dr)\right)={}_a^b\overrightarrow{\prod}(1+dA^\dagger\otimes P_-+dA\otimes P_++dT\otimes P_0) $$

as the solution $Y(b)$ at b of the quantum stochastic differential equation

$$ dY=Y(dA^\dagger\otimes P_-+dA\otimes P_++dT\otimes P_0),\quad Y(a)=1 \tag{2} $$

in which the unital algebra $\mathcal{P}$ is the (right) system algebra.

Similarly in the second definition (ii) the inner decapitated product integral ${}_a^b\widehat{\overrightarrow{\prod}}(1+dr)$ is the solution $U(b)$ at time b of the stochastic differential equation

$$ dU^{1,2,3}=(U^{1,3}+1^1)\,dr^{2,3},\quad U(a)=0 $$

in which the second copy of $\mathcal{I}$ in $\mathcal{I}\otimes\mathcal{I}$ is taken to be the (right) system algebra, which is of the form $Q_-\otimes dA^\dagger+Q_+\otimes dA+Q_0\otimes dT$ where Q_-, Q_+ and Q_0 are iterated stochastic integrals over the interval $]a,b]$ and can define the simple product integral

$$ \overrightarrow{\prod_s^t}\left(1+{}_a^b\widehat{\overrightarrow{\prod}}(1+dr)\right)=\overrightarrow{\prod_s^t}(1+Q_-\otimes dA^\dagger+Q_+\otimes dA+Q_0\otimes dT) $$

as the solution $V(t)$ of the stochastic differential equation

$$ dV=V(Q_-\otimes dA^\dagger+Q_+\otimes dA+Q_0\otimes dT),\quad V(s)=1. \tag{3} $$

Let us prove that the two definitions (i) and (ii) are equivalent.

THEOREM 2.1. *If Y and V denote the solutions of the stochastic differential equations* (2) *and* (3) *then* $Y(b) = V(t)$.

Proof. We recall the product rule $I_s^t(\alpha)I_s^t(\gamma) = I_s^t(\alpha\beta)$ for iterated stochastic integrals [1]. Here for $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots) \in \bigoplus_{n=0}^{\infty}(\bigotimes^n \mathcal{I})$ an element of the tensor space over $\mathcal{I}$, we define $I_s^t(\alpha)$ by linear extension of the rule that for each $\alpha_0 \in \mathbb{C} = \bigotimes^0 \mathcal{I}$, $I_s^t(\alpha_0) = \alpha_0 1$ and for each n-th rank homogeneous product tensor $dL_1 \otimes dL_2 \otimes \dots \otimes dL_n \in \bigotimes^n \mathcal{I}$ with $dL_1, dL_2, \dots, dL_n \in \mathcal{I}$

$$I_s^t(dL_1 \otimes dL_2 \otimes \dots \otimes dL_n) = \int_{s<x_1<x_2<\dots<x_n<t} dL_1(x_1)\, dL_2(x_2) \dots dL_n(x_n).$$

The product $\alpha\beta$ is defined by

$$(\alpha\beta)_n = \sum_{A\cup B=\mathbb{N}_n} \alpha_{|A|}^A \beta_{|B|}^B\,.$$

Here the sum is over all 3^n ordered pairs (A, B) whose union is $\mathbb{N}_n = \{1, 2, \dots, n\}$, the notation $\alpha_{|A|}^A$ indicates that the homogeneous component $\alpha_{|A|}$ of α of rank $|A|$ is to be placed in the copies of $\mathcal{I}$ within the n-fold tensor product $\bigotimes^n \mathcal{I}$ labelled by the elements of $A \subset \{1, 2, \dots, n\}$ so that, with $\beta_{|B|}^B$ defined analogously, all n copies of $\mathcal{I}$ within $\bigotimes^n \mathcal{I}$ are occupied by the combination $\alpha_{|A|}^A \beta_{|B|}^B$, and finally double occupancies are reduced to single occupancies by using the multiplication map in $\mathcal{I}$.

Using this multiplication rule we can solve (2) iteratively

$${}_a^b\overrightarrow{\prod}\Big(1 + \widehat{\overrightarrow{\prod}}{}_s^t(1 + dr)\Big) = 1 + \big(I_a^b \otimes I_s^t\big)\Big\{\sum_{m,n=1}^{\infty}\Big(\sum_{A_1\cup A_2\cup\dots\cup A_m=\mathbb{N}_n}\prod_{j=1}^{m} dr^{j,m+A_j}\Big)\Big\}. \quad (4)$$

Here the inner summation is over all ordered m-tuples $(A_1, A_2, \dots, A_m)$ of non-empty subsets whose union is $\mathbb{N}_n$ and the notation is as follows. For $A_j = \{a_1, a_2, \dots, a_{|A_j|}\}$ with $a_1 < a_2 < \dots < a_{|A_j|}$ we define $dr^{j,m+A_j}$ to be the element $\prod_{l=1}^{|A_j|} dr^{j,m+a_l}$ of $\mathcal{I}^j \bigotimes \mathcal{I}^{m+a_1,m+a_2,\dots,m+a_{|A_j|}}$ where the superscripts indicate places within the $(m+n)$-fold tensor product $(\bigotimes^m \mathcal{I}) \otimes (\bigotimes^n \mathcal{I})$. (In fact by nilpotence of $\mathcal{I}$ $dr^{j,m+A_j} = 0$ whenever $|A_j| > 2$ so that the inner summation may be restricted to subsets which are either singletons or pairs).

A similar argument shows that, with a similar notation,

$$\overrightarrow{\prod}{}_s^t\Big(1 + {}_a^b\widehat{\overrightarrow{\prod}}(1 + dr)\Big) = 1 + (I_a^b \otimes I_s^t)\Big\{\sum_{m,n=1}^{\infty}\Big(\sum_{B_1\cup B_2\cup\dots\cup B_n=\mathbb{N}_m}\prod_{k=1}^{n} dr^{B_k,m+k}\Big)\Big\}, \quad (5)$$

where now the inner summation is over all ordered n-tuples $(B_1, B_2, \dots, B_n)$ of non-empty subsets whose union is $\mathbb{N}_m$. That the sums (4) and (5) are equal follows from the fact that

$$\begin{aligned}\sum_{A_1\cup A_2\cup\dots\cup A_m=\mathbb{N}_n}\prod_{j=1}^{m} dr^{j,m+A_j} &= \sum_{M\in\mathcal{M}_{m,n}}\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}\big((dr)^{M_{j,k}}\big)^{m+j,k}\\ &= \sum_{B_1\cup B_2\cup\dots\cup B_n=\mathbb{N}_m}\prod_{k=1}^{n} dr^{B_k,m+k}.\end{aligned} \quad (6)$$

Here $\mathcal{M}_{m,n}$ denotes the set of all $m \times n$ matrices $M = [M_{j,k}]_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}$ with each $M_{j,k} \in \{0,1\}$ and such that each row $(M_{j,1}, M_{j,2}, \dots, M_{j,n})$ and each column $(M_{1,k}, M_{2,k}, \dots, M_{m,k})$ contains at least one entry 1. $(dr)^{M_{j,k}}$ is by definition dr if $M_{j,k} = 1$ and void (formally 1 in the product) if $M_{j,k} = 0$ and the superscripts $m+j, k$ refer to places in $(\bigotimes^m \mathcal{I})\otimes(\bigotimes^n \mathcal{I})$. The equality of the three sums (6) is established using the correspondence

$$k \in A_j \Leftrightarrow M_{j,k} = 1 \Leftrightarrow j \in B_k. \ \blacksquare$$

The forward-backward, backward-forward and backward-backward double products $\overset{\rightarrow\leftarrow}{\prod}(1+dr)$, $\overset{\leftarrow\rightarrow}{\prod}(1+dr)$ and $\overset{\leftarrow\leftarrow}{\prod}(1+dr)$ are defined by appropriately replacing stochastic differential equations driven on the right by ones driven on the left. For example

$${}_a^b\overset{\rightarrow\leftarrow}{\prod}{}_s^t(1+dr) = \begin{cases} {}_a^b\overrightarrow{\prod}\Big(1+\overleftarrow{\widehat{\prod}}{}_s^t(1+dr)\Big) \\ \overleftarrow{\prod}{}_s^t\Big(1+{}_a^b\overrightarrow{\widehat{\prod}}(1+dr)\Big), \end{cases}$$

where $\overleftarrow{\widehat{\prod}}{}_s^t(1+dr)$ is the solution $X(t)$ at t of the stochastic differential equation

$$dX^{1,2,3} = dr^{1,3}(X^{1,2} + 1^2), \quad X(s) = 0$$

and $\overleftarrow{\prod}{}_s^t(1+{}_a^b\overrightarrow{\widehat{\prod}}(1+dr))$ is the solution $V(t)$ at t of

$$dV = (Q_- \otimes dA^\dagger + Q_+ \otimes dA + Q_0 \otimes dT)V, \quad V(s) = 1$$

where, as before, ${}_a^b\overrightarrow{\widehat{\prod}}(1+dr)$ is expressed as $Q_- \otimes dA^\dagger + Q_+ \otimes dA + Q_0 \otimes dT$.

3. Quantum stochastic double product integrals: properties.

By nilpotence of $\mathcal{I}$ each element dr of $\mathcal{I}\otimes\mathcal{I}$ has a unique quasi-inverse, that is an element $d\hat{r}$ satisfying

$$dr + d\hat{r} + dr\,d\hat{r} = dr + d\hat{r} + dr\,d\hat{r} = 0;$$

in fact $d\hat{r} = -dr + (dr)^2$.

Theorem 3.1. *${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$ has multiplicative right inverse ${}_a^b\overleftarrow{\prod}{}_s^t(1+d\hat{r})$.*

Proof. Denote ${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$ by Y and ${}_a^b\overleftarrow{\prod}{}_s^t(1+d\hat{r})$ by $\hat{Y}$ so that

$$Y\hat{Y} = {}_a^b\prod\Big(1+\overrightarrow{\widehat{\prod}}{}_s^t(1+dr)\Big)\,{}_a^b\prod\Big(1+\overleftarrow{\widehat{\prod}}{}_s^t(1+d\hat{r})\Big).$$

We compute the differential at time b of the product $Y\hat{Y}$ using the Leibniz-Itô formula as

$$\begin{aligned} d(Y\hat{Y}) &= (dY)\hat{Y} + Y(d\hat{Y}) + (dY)\,(d\hat{Y}) \\ &= Y\Big(\overrightarrow{\widehat{\prod}}{}_s^t(1+dr) + \overleftarrow{\widehat{\prod}}{}_s^t(1+d\hat{r}) + \overrightarrow{\widehat{\prod}}{}_s^t(1+dr)\,\overleftarrow{\widehat{\prod}}{}_s^t(1+d\hat{r})\Big)\hat{Y} \\ &= Y\big(X + \hat{X} + X\hat{X}\big)\hat{Y}, \end{aligned} \tag{7}$$

where

$$X = \widehat{\overrightarrow{\prod}}{}_s^t(1+dr), \quad \hat{X} = \widehat{\overleftarrow{\prod}}{}_s^t(1+d\hat{r}).$$

Let us now compute the differential at time t of $X + \hat{X} + X\hat{X}$; again using the Leibniz-Itô formula it is

$$\begin{aligned} &d\big(X + \hat{X} + X\hat{X}\big) \\ &= dX + d\hat{X} + (dX)\hat{X} + X\,d(\hat{X}) + (dX)(d\hat{X}) \\ &= (X+1)\,dr + d\hat{r}\,(\hat{X}+1) + (X+1)\,dr\hat{X} + X\,d\hat{r}\,(\hat{X}+1) + (X+1)\,dr\,d\hat{r}\,(\hat{X}+1) \\ &= (X+1)(dr + d\hat{r} + dr\,d\hat{r})(\hat{X}+1) = 0, \end{aligned}$$

independently of $t > s$. Since $X(s) + \hat{X}(s) + X(s)\hat{X}(s) = 0$ it follows that $X + \hat{X} + X\hat{X} = 0$. It follows from (7) that $d(Y\hat{Y}) = 0$ independently of $b > a$. Since $Y(a)\hat{Y}(a) = 1$ it follows that $Y\hat{Y} = 1$, that is that ${}_a^b\overleftarrow{\prod}{}_s^t(1+d\hat{r})$ is a right inverse to ${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$. ∎

Elementary adjunction properties of quantum stochastic integrals show that

$$\Big({}_a^b\overrightarrow{\prod}{}_s^t(1+dr)\Big)^\dagger = {}_a^b\overleftarrow{\prod}{}_s^t(1+dr^\dagger)$$

where on the left † denotes the Hilbert space adjoint and on the right the tensor product involution on $\mathcal{I} \otimes \mathcal{I}$ of the natural involution on $\mathcal{I}$ in which $dA^\dagger$ and dA are mutually adjoint and dT is self-adjoint. It follows from Theorem 3.1 that the condition $dr^\dagger = d\hat{r}$ is necessary and sufficient for ${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$ to be coisometric. Isometry, and hence also unitarity, is more difficult to characterise. We shall show by direct construction that, in the case $dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)$, which satisfies the coisometry condition, ${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$ is unitary.

4. Some matrix products. Fix $m, n \in \mathbb{N}$ throughout this section. Let $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be a given 2×2 matrix. For $1 \le j < j' \le m$ and $1 \le k < k' \le n$ we consider the $(m+n) \times (m+n)$ matrix

$$M^{(j,j';k,k')} = \alpha\,|j\rangle\,\langle j'| + \beta\,|j\rangle\,\langle m+k'| + \gamma\,|m+k\rangle\,\langle j'| + \delta\,|m+k\rangle\,\langle m+k'|$$

where for $1 \le l, l' \le m+n$ the dyad $|l\rangle\,\langle l'|$ is the $(m+n) \times (m+n)$ matrix with a single non-zero element 1 at the intersection of the l-th row and l'-th column. We shall also find it convenient to write $M^{(j,j';k,k')}$ as a 2×2 matrix whose elements are themselves matrices, namely

$$M^{(j,j';k,k')} = \begin{bmatrix} \alpha\,|j\rangle\,\langle j'| & \beta\,|j\rangle\,\langle k'| \\ \gamma\,|k\rangle\,\langle j'| & \delta\,|k\rangle\,\langle k'| \end{bmatrix}$$

where now the dyads $|j\rangle\,\langle j'|$, $|j\rangle\,\langle k'|$, $|k\rangle\,\langle j'|$ and $|k\rangle\,\langle k'|$ are respectively $m \times m$, $m \times n$, $n \times m$ and $n \times n$ matrices. Then given a second 2×2 matrix $\tilde{M} = \begin{bmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\gamma} & \tilde{\delta} \end{bmatrix}$ the product $M^{(j,j';k,k')}\tilde{M}^{(\tilde{j},\tilde{j}';\tilde{k},\tilde{k}')}$ is

$$\begin{bmatrix} \alpha\tilde{\alpha}\big|j\big\rangle\big\langle j'\big|\big|\tilde{j}\big\rangle\big\langle\tilde{j}'\big| + \beta\tilde{\gamma}\big|j\big\rangle\big\langle k'\big|\big|\tilde{k}\big\rangle\big\langle\tilde{j}'\big| & \alpha\tilde{\beta}\big|j\big\rangle\big\langle j'\big|\big|\tilde{j}\big\rangle\big\langle\tilde{k}'\big| + \beta\tilde{\delta}\big|j\big\rangle\big\langle k'\big|\big|\tilde{k}\big\rangle\big\langle\tilde{k}'\big| \\ \gamma\tilde{\alpha}\big|k\big\rangle\big\langle j'\big|\big|\tilde{j}\big\rangle\big\langle\tilde{j}'\big| + \delta\tilde{\gamma}\big|k\big\rangle\big\langle k'\big|\big|\tilde{k}\big\rangle\big\langle\tilde{k}'\big| & \gamma\tilde{\beta}\big|k\big\rangle\big\langle j'\big|\big|\tilde{j}\big\rangle\big\langle\tilde{k}'\big| + \delta\tilde{\delta}\big|k\big\rangle\big\langle k'\big|\big|\tilde{k}\big\rangle\big\langle\tilde{k}'\big| \end{bmatrix}.$$

We use the abbreviated notation $M^{(j,j;k,k)} = M^{(j;k)}$. Then it follows from the fact that $|l\rangle\langle l'|\,|l''\rangle\langle l'''| = 0$ unless $l' = l''$ that

$$M^{(j;k)}M^{(j';k')} = 0 \tag{8}$$

unless either $j = j'$ or $k = k'$.

We are interested in the discrete double product $\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}(I + M^{(j;k)})$ where I is the $(m+n)\times(m+n)$ identity matrix. To evaluate it we first introduce some notation. We temporarily denote the matrix $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ by $\begin{bmatrix} \alpha_{00} & \alpha_{01} \\ \alpha_{10} & \alpha_{11} \end{bmatrix}$ and raise it to the ν-th power, so that the elements of the resulting 2×2 matrix M^ν are given by

$$(M^{\nu})_{j,k} = \sum_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}=0}^{1} \alpha_{j\sharp_1}\alpha_{\sharp_1,\sharp_2}\alpha_{\sharp_2,\sharp_3}\cdots\alpha_{\sharp_{\nu-1}k}.$$

Then we define monomials of total degree ν by

$$\begin{aligned}
a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) &= \alpha_{0\sharp_1}\alpha_{\sharp_1,\sharp_2}\alpha_{\sharp_2,\sharp_3}\cdots\alpha_{\sharp_{\nu-1}0},\\
b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) &= \alpha_{0\sharp_1}\alpha_{\sharp_1,\sharp_2}\alpha_{\sharp_2,\sharp_3}\cdots\alpha_{\sharp_{\nu-1}1},\\
c_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) &= \alpha_{1\sharp_1}\alpha_{\sharp_1,\sharp_2}\alpha_{\sharp_2,\sharp_3}\cdots\alpha_{\sharp_{\nu-1}0},\\
d_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) &= \alpha_{1\sharp_1}\alpha_{\sharp_1,\sharp_2}\alpha_{\sharp_2,\sharp_3}\cdots\alpha_{\sharp_{\nu-1}1}.
\end{aligned} \tag{9}$$

so that

$$M^\nu = \sum_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}=0}^{1} \begin{bmatrix} a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) & b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) \\ c_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) & d_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) \end{bmatrix}$$

For $\sharp \in \{0,1\}$ we denote $1-\sharp$ by $\flat$.

Now we can state

Theorem 4.1. *For an arbitrary 2×2 matrix $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$,*

$$\overrightarrow{\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_m}} (I + M^{(j,k)}) = I + \sum_{\nu=1}^{\infty} \sum_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}\in\{0,1\}}$$
$$\begin{bmatrix} a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)A^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}} & b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)B^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}} \\ c_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)C^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}} & d_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)D^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}} \end{bmatrix}$$

where the monomials $a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$, $b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$, $c_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$ and $d_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$ are defined by (9) and the $m\times m$, $m\times n$, $n\times m$ and $n\times n$ matrices $A^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$, $B^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$, $C^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$ and $D^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$ are defined by

$$(A^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}})_{j,j'} = \begin{cases} \left(\frac{(j'-j-1)(j'-j-2)\cdots(j'-j-(\sharp_1+\sharp_2+\dots+\sharp_{\nu-1}))}{(\sharp_1+\sharp_2+\dots+\sharp_{\nu-1})!}\right) \\ \quad\left(\frac{n(n-1)\cdots(n-(\flat_1+\flat_2+\dots+\flat_{\nu-1}))}{(\flat_1+\flat_2+\dots+\flat_{\nu-1}+1)!}\right) & \text{if } j < j' \\ 0 & \text{if } j \geq j', \end{cases} \tag{10}$$

$$(B^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{j,k} = \frac{(m-j-1)(m-j-2)\cdots(m-j-(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}))}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!} \frac{(k-1)(k-2)\cdots(k-(\flat_1+\flat_2+\ldots+\flat_{\nu-1}))}{(\flat_1+\flat_2+\ldots+\flat_{\nu-1})!}, \tag{11}$$

$$(C^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{k,j} = \frac{(j-1)(j-2)\cdots(j-(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}))}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!} \frac{(n-k-1)(n-k-2)\cdots(n-(\flat_1+\flat_2+\ldots+\flat_{\nu-1}))}{(\flat_1+\flat_2+\ldots+\flat_{\nu-1})!}, \tag{12}$$

$$(D^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{k,k'} = \begin{cases} \left(\frac{m(m-1)\cdots(m-(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}))}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}+1)!}\right) \left(\frac{(k'-k-1)(k'-k-2)\cdots(k'-k-(\flat_1+\flat_2+\ldots+\flat_{\nu-1}))}{(\flat_1+\flat_2+\ldots+\flat_{\nu-1})!}\right) & \textit{if } k < k' \\ 0 & \textit{if } k \geq k'. \end{cases} \tag{13}$$

Proof. In view of (8)

$$\prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_m} (I+M^{(j,k)})$$
$$= I + \sum_{\nu=1}^{\infty} \sum_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}=0}^{1} \sum_{1\leq j_1<j_2<\ldots<j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}\leq m} \sum_{1\leq k_1<k_2<\ldots<k_{1+\flat_1+\flat_2+\ldots+\flat_{\nu-1}}\leq n}$$
$$M^{(j_1,k_1)} M^{(j_{1+\sharp_1},k_{1+\flat_1})} \ldots M^{(j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}},k_{1+\flat_1+\flat_2+\ldots+\flat_{-1\nu}})}$$
$$= I + \sum_{\nu=1}^{\infty} \sum_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}=0}^{1} \sum_{1\leq j_1<j_2<\ldots<j_{1+\sharp_1+\ldots+\sharp_{\nu-1}}\leq m} \sum_{1\leq k_1<k_2<\ldots<k_{1+\flat_1+\flat_{2+\ldots+}\flat_{\nu-1}}\leq n}$$
$$\begin{bmatrix} \alpha|j_1\rangle\langle j_1| & \beta|j_1\rangle\langle k_1| \\ \gamma|k_1\rangle\langle j_1| & \delta|k_1\rangle\langle k_1| \end{bmatrix} \begin{bmatrix} \alpha|j_{1+\sharp_1}\rangle\langle j_{1+\sharp_1}| & \beta|j_{1+\sharp_1}\rangle\langle k_{1+\flat_1}| \\ \gamma|k_{1+\flat_1}\rangle\langle j_{1+\sharp_1}| & \delta|k_{1+\flat_1}\rangle\langle k_{1+\flat_1}| \end{bmatrix}$$
$$\cdots \begin{bmatrix} \alpha|j_{1+\sharp_1+\ldots+\sharp_\nu}\rangle\langle j_{1+\sharp+\ldots+\sharp_\nu}| & \beta|j_{1+\sharp_1+\ldots+\sharp_\nu}\rangle\langle k_{1+\flat_1+\ldots+\flat_\nu}| \\ \gamma|k_{1+\flat_1+\ldots+\flat_\nu}\rangle\langle j_{1+\sharp_1+\ldots+\sharp_\nu}| & \delta|k_{1+\flat_1+\ldots+\flat_\nu}\rangle\langle k_{1+\flat_1+\ldots+\flat_\nu}| \end{bmatrix}$$
$$= \sum_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}\in\{0,1\}} \begin{bmatrix} a_{\sharp_1,\ldots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)A^{\sharp_1,\ldots,\sharp_{\nu-1}} & b_{\sharp_1,\ldots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)B^{\sharp_1,\ldots,\sharp_{\nu-1}} \\ c_{\sharp_1,\ldots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)C^{\sharp_1,\ldots,\sharp_{\nu-1}} & d_{\sharp_1,\ldots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta)D^{\sharp_1,\ldots,\sharp_{\nu-1}} \end{bmatrix},$$

where $a_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $b_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $c_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ and $d_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ are defined by (9) and the $m\times m$, $m\times n$, $n\times m$ and $n\times n$ matrices $A^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $B^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $C^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ and $D^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ are given by

$$(A^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{j,j'} = \begin{cases} \sum_{j<j_1<j_2<\ldots<j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}<j'} \sum_{1\leq k_1<k_2<\ldots<k_{1+\flat_1+\flat_{2+\ldots+}\flat_{\nu-1}}\leq n} 1 & \text{if } j<j' \\ 0 & \text{otherwise,} \end{cases}$$

$$(B^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{j,k} = \sum_{j<j_1<j_2<\ldots<j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}<m} \sum_{1\leq k_1<k_2<\ldots<k_{1+\flat_1+\flat_{2+\ldots+}\flat_{\nu-1}}\leq k} 1,$$

$$(C^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{k,j} = \sum_{1\leq j_1<j_2<\ldots<j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}<j} \sum_{k\leq k_1<k_2<\ldots<k_{1+\flat_1+\flat_{2+\ldots+}\flat_{\nu-1}}\leq n} 1,$$

$$(D^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{k,k'} = \begin{cases} \sum_{1\leq j_1<j_2<\ldots<j_{1+\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}\leq m} & \\ \quad \sum_{k<k_1<k_2<\ldots<k_{1+\flat_1+\flat_2+\ldots+\flat_{\nu-1}}<k'} 1 & \text{if } k<k' \\ 0 & \text{otherwise.} \end{cases}$$

Carrying out the counting sums we obtain $A^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $B^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $C^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ and $D^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ in the form claimed in the theorem. ∎

5. Limit of a product of rotation matrices. Now let real numbers λ, a, b, s, t be fixed with $\lambda \neq 0$, $0 \leq a \leq b$ and $0 \leq s \leq t$. For $m, n \in \mathbb{N}$ let

$$\theta_{m,n} = \lambda\sqrt{\frac{(b-a)(t-s)}{mn}}\,.$$

We are interested in the double product of $(m+n)\times(m+n)$ rotation matrices

$$R_{m,n} = \prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}^{\rightarrow} \begin{bmatrix} & & & (j) & & (m+k) & & \\ & 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ (j) & 0 & \cdots & \cos\theta_{m,n} & \cdots & -\sin\theta_{m,n} & \cdots & 0 \\ & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ (m+k) & 0 & \cdots & \sin\theta_{m,n} & \cdots & \cos\theta_{m,n} & \cdots & 0 \\ & \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \cdots \\ & 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

which we can express as in the previous section as

$$R_{m,n} = \prod_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}^{\rightarrow} (I + M^{(j,k)}) \tag{14}$$

where the 2×2 matrix M is given by

$$M = \begin{bmatrix} \cos\theta_{m,n} - 1 & -\sin\theta_{m,n} \\ \sin\theta_{m,n} & \cos\theta_{m,n} - 1 \end{bmatrix} = -2\begin{bmatrix} s(\theta_{m,n})^2 & s(\theta_{m,n})c(\theta_{m,n}) \\ -s(\theta_{m,n})c(\theta_{m,n}) & s(\theta_{m,n})^2 \end{bmatrix}$$

and

$$s(\theta_{m,n}) = \sin\frac{\theta_{m,n}}{2}, \quad c(\theta_{m,n}) = \cos\frac{\theta_{m,n}}{2}\,.$$

We use the matrix (14) to construct an operator $W_{m,n}$ on $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ by embedding the standard orthonormal basis in $\mathbb{C}^{m+n}$ as the orthonormal set $\big((\chi_j, 0), (0, \chi'_k),\ j = 1, 2, \ldots, m,\ k = 1, 2, \ldots, n\big)$ where χ_j and χ'_k are the normalised indicator functions

$$\chi_j(x) = \begin{cases} \sqrt{\frac{b-a}{m}} & \text{if } a + \frac{(j-1)(b-a)}{m} \leq x < a + \frac{j(b-a)}{m} \\ 0 & \text{otherwise,} \end{cases}$$

$$\chi'_k(x) = \begin{cases} \sqrt{\frac{t-s}{n}} & \text{if } s + \frac{(k-1)(t-s)}{n} \leq x < s + \frac{k(t-s)}{n} \\ 0 & \text{otherwise,} \end{cases}$$

and defining $W_{m,n}$ to be the identity I on the orthogonal complement of the latter orthonormal set. Our objective in this section is to show informally that the limit

$W = \lim_{m,n\to\infty} W_{(m,n)}$ exists as an operator on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ of the form

$$W = \begin{bmatrix} I + A & B \\ C & I + D \end{bmatrix}$$

where I is the identity operator on $L^2(\mathbb{R})$ and A, B, C and D are integral operators on $L^2(\mathbb{R})$ whose kernels are to be determined from limiting forms of the $m\times m$, $m\times n$, $n\times m$ and $n \times n$ matrices $A_{m,n}$, $B_{m,n}$, $C_{m,n}$ and $D_{m,n}$ defined by writing

$$R_{m,n} = \begin{bmatrix} I_m + A_{m,n} & B_{m,n} \\ C_{m,n} & I_n + D_{m,n} \end{bmatrix}.$$

We use (14) and Theorem 4.1 (in which the dependence on m, n is suppressed) to write these matrices in the form

$$\begin{bmatrix} A_{m,n} & B_{m,n} \\ C_{m,n} & D_{m,n} \end{bmatrix} = \sum_{\nu=1}^{\infty} \sum_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}\in\{0,1\}}$$
$$\begin{bmatrix} a_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) A_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}} & b_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) B_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}} \\ c_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) C_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}} & d_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) D_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}} \end{bmatrix}$$

where

$$\begin{aligned}
a_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) &= (-2)^\nu \tau_{0\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{\nu-1}0}(\theta_{m,n}),\\
b_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) &= (-2)^\nu \tau_{0\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{\nu-1}1(\theta_{m,n})},\\
c_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}) &= (-2)^\nu \tau_{1\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{\nu-1}0}(\theta_{m,n}),\\
d_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\alpha,\beta,\gamma,\delta) &= (-2)^\nu \tau_{1\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{\nu-1}1}(\theta_{m,n}),
\end{aligned}$$

$\tau_{\sharp,\sharp'}(\theta_{m,n})$ is defined by

$$\tau_{\sharp,\sharp'}(\theta_{m,n}) = \begin{cases} s(\theta_{m,n})^2 & \text{if } \sharp = 0, \sharp' = 0 \\ s(\theta_{m,n})c(\theta_{m,n}) & \text{if } \sharp = 0, \sharp' = 1 \\ -s(\theta_{m,n})c(\theta_{m,n}) & \text{if } \sharp = 1, \sharp' = 0 \\ s(\theta_{m,n})^2 & \text{if } \sharp = 1, \sharp' = 1 \end{cases}$$

and $A_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $B_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$, $C_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ and $D_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}$ are given by (10), (11), (12) and (13).

Let us consider the limiting form of $A_{m,n}$. By (10) for $j < j'$, $\left(A_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}\right)_{j,j'}$ is given by

$$\left(\frac{(j'-j-1)(j'-j-2)\cdots(j'-j-(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}))}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!}\right)$$
$$\left(\frac{n(n-1)\cdots(n-(\flat_1+\flat_2+\ldots+\flat_{\nu-1}))}{(\flat_1+\flat_2+\ldots+\flat_{\nu-1}+1)!}\right).$$

Thus for large m, n

$$(A_{m,n}^{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}})_{j,j'} \cong \frac{(j'-j)^{\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}} n^{\flat_1+\flat_2+\ldots+\flat_{\nu-1}+1}}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!(\flat_1+\flat_2+\ldots+\flat_{\nu-1}+1)!}, \tag{15}$$

which is of order $m^{\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}} n^{\flat_1+\flat_2+\ldots+\flat_{\nu-1}+1}$ for generic $j' > j$. On the other hand, denoting by

$$P_{\sharp,\sharp'}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}) = |l, 1\le l\le \nu : \sharp_{l-1}\ne\sharp_l|$$

the number of changes in parity in the sequence $\sharp=\sharp_0,\sharp_1,\sharp_2,\dots,\sharp_{\nu-1},\sharp_\nu=\sharp'$,

$$\begin{aligned}
&a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\theta_{m,n}) = (-2)^\nu \tau_{0\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{\nu-1}0}(\theta_{m,n})\\
&\cong \pm(-2)^\nu s(\theta_{m,n})^{2(\nu-P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}))}\Big(s(\theta_{m,n})c(\theta_{m,n})\Big)^{P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}\\
&= \pm(-2)^\nu\Big(\sin\frac{\theta_{m,n}}{2}\Big)^{2\nu-P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}\Big(\cos\frac{\theta_{m,n}}{2}\Big)^{P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}\\
&\cong \pm(-2)^\nu\Big(\frac{\theta_{m,n}}{2}\Big)^{2\nu-P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}\\
&= \pm(-2)^\nu\Big(\frac{\lambda^2(b-a)(t-s)}{2^2mn}\Big)^{\nu-\frac12 P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}
\end{aligned}\tag{16}$$

which is of order $(mn)^{-(\nu-\frac12 P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}))}$. Thus, only when

$$\begin{aligned}
\nu-\frac12 P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}) &= \sharp_1+\sharp_2+\ldots+\sharp_{\nu-1} = \flat_1+\flat_2+\ldots+\flat_{\nu-1}+1\\
&= \frac12\left(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}+\flat_1+\flat_2+\ldots+\flat_{\nu-1}+1\right)=\frac{\nu}{2},
\end{aligned}$$

that is $P_{0,0}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})=\nu$, can the product $a_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\theta_{m,n})\ A_{m,n}^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}$ approach a non-zero limiting form as $m,n\to\infty$. This happens only when $\nu=2N$ is even and the sequence $(0,\sharp_1,\sharp_2,\dots,\sharp_{\nu-1},0)=(0,1,0,1,\dots,0,1,0)$ with

$$\sharp_1=\flat_2=\sharp_3=\flat_4=\ldots=\flat_{2N-2}=\sharp_{2N-1}=1,$$

and (15) becomes

$$(A_{m,n}^{1,0,1,\dots,0,1})_{j,j'}\cong\frac{(j'-j)^{N-1}n^N}{(N-1)!N!}\,.$$

Also

$$\begin{aligned}
&a_{1,0,1,\dots,0,1}(\theta_{m,n}) = (-2)^{2N}\big(\tau_{0,1}(\theta_{m,n})\tau_{1,0}(\theta_{m,n})\big)^N\\
&\quad=(-1)^N2^{2N}\Big(\sin\frac{\theta_{m,n}}{2}\cos\frac{\theta_{m,n}}{2}\Big)^{2N}\simeq(-1)^N(\theta_{m,n})^{2N}=\Big(\frac{-\lambda^2(b-a)(t-s)}{mn}\Big)^N.
\end{aligned}$$

Thus for $j<j'$

$$\begin{aligned}
[A_{m,n}]_{j,j'}&\simeq\Big(\frac{-\lambda^2(b-a)(t-s)}{mn}\Big)^N\frac{(j'-j)^{N-1}n^N}{(N-1)!N!}\\
&=\Big(\frac{(b-a)(j'-j)}{m}\Big)^{N-1}\frac{(-\lambda^2(t-s))^N}{(N-1)!N!}\frac{(b-a)}{m}\,.
\end{aligned}$$

The corresponding operator $\sum_{1\le j<j'\le m}[A_{m,n}]_{j,j'}\,|\chi_j\rangle\,\langle\chi_{j'}|$ acts on $f\in L^2(\mathbb{R})$ by

$$\begin{aligned}
&\Big(\sum_{1\le j<j'\le N}[A_{m,n}]_{j,j'}\,|\chi_j\rangle\,\langle\chi_{j'}|\,f\Big)(x)\\
&=\sum_{1\le j<j'\le m}\Big(\frac{(b-a)(j'-j)}{m}\Big)^{N-1}\frac{(-\lambda^2(t-s))^N}{(N-1)!N!}\sqrt{\frac{(b-a)}{m}}\int_{a+\frac{(j'-1)(b-a)}{m}}^{a+\frac{j'(b-a)}{m}}f(y)\,dy\,\chi_j(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le j<j'\le m} \Big(\frac{(b-a)(j'-j)}{m}\Big)^{N-1} \frac{(-\lambda^2(t-s))^N}{(N-1)!N!} \int_{a+\frac{(j'-1)(b-a)}{m}}^{a+\frac{j'(b-a)}{m}} f(y)\,dy\, \chi_{a+\frac{(j-1)(b-a)}{m}}^{a+\frac{j(b-a)}{m}}(x) \\
&\simeq \int <_a^b (x,y) \frac{(y-x)^{N-1}}{(N-1)!} \frac{(-\lambda^2(t-s))^N}{N!} f(y)\,dy
\end{aligned}
$$

where the function $<_a^b$ is defined by

$$
<_a^b (x,y) = \begin{cases} 1 & \text{if } a \le x < y < b \\ 0 & \text{otherwise.} \end{cases}
$$

Summing over all such ν we see from (15) and (16) that in the limit the operator represented by the matrix $A_{m,n}$ approximates the integral operator ${}_a^bA_s^t$ with kernel

$$
\big(\ker\{{}_a^bA_s^t\}\big)(x,y) = \ <_a^b (x,y) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!}\,.
$$

In the same way it can be shown that as $m,n\to\infty$ the matrix $D_{m,n}$ approximates the integral operator ${}_a^bD_s^t$ with kernel

$$
\big(\ker\{{}_a^bD_s^t\}\big)(x,y) = \ <_s^t (x,y) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(b-a))^{N+1}}{N!(N+1)!}\,.
$$

Now consider the limiting form of

$$
B_{m,n} = \sum_{\nu=1}^{\infty} \sum_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}\in\{0,1\}} b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\theta_{m,n}) B_{m,n}^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}},
$$

where, on the one hand

$$
\begin{aligned}
(B_{m,n}^{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}})_{j,k} &= \frac{1}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!(\flat_1+\flat_2+\ldots+\flat_{\nu-1})!} \\
&\qquad (m-j-1)(m-j-2)\cdots(m-j-1-(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})) \\
&\qquad (k-1)(k-2)\cdots(k-1-(\flat_1+\flat_2+\ldots+\flat_{\nu-1})) \\
&\cong \frac{(m-j)^{\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}} k^{\flat_1+\flat_2+\ldots+\flat_{\nu-1}}}{(\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1})!(\flat_1+\flat_2+\ldots+\flat_{\nu-1})!}
\end{aligned} \tag{17}
$$

which is of order $m^{\sharp_1+\sharp_2+\ldots+\sharp_{\nu-1}}n^{\flat_1+\flat_2+\ldots+\flat_{\nu-1}}$ for generic j,k, and on the other hand

$$
\begin{aligned}
b_{\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}}(\theta_{m,n}) &= (-2)^{\nu}\tau_{0\sharp_1}(\theta_{m,n})\tau_{\sharp_1,\sharp_2}(\theta_{m,n})\tau_{\sharp_2,\sharp_3}(\theta_{m,n})\cdots\tau_{\sharp_{N-1}1}(\theta_{m,n}) \\
&\cong \pm(-2)^{\nu}\Big(\frac{\lambda^2(b-a)(t-s)}{2^2mn}\Big)^{\nu-\frac12 P_{0,1}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})}
\end{aligned}
$$

which is of order $(mn)^{-(\nu-\frac12 P_{0,1}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1}))}$. Again for there to be a non-zero limiting form we must have $P_{0,1}(\sharp_1,\sharp_2,\dots,\sharp_{\nu-1})=\nu$, which requires that $\nu=2N+1$ is odd and

$$
(0,\sharp_1,\sharp_2,\dots,\sharp_{\nu-1},1) = (0,1,0,1,\dots,1,0,1)
$$

with

$$
\sharp_1 = \flat_2 = \sharp_3 = \flat_4 = \ldots = \sharp_{2N-1} = \flat_{2N} = 1
$$

and (17) becomes

$$
(B_{m,n}^{1,0,\dots,1,0})_{j,k} \cong \frac{(m-j)^N k^N}{(N!)^2}\,.
$$

Also

$$\begin{aligned} b_{1,0,\ldots,1,0}(\theta_{m,n}) &= (-2)^{2N+1}\tau_{0,1}(\theta_{m,n})(\tau_{1,0}(\theta_{m,n})\tau_{0,1}(\theta_{m,n}))^N \\ &= (-1)^N 2^{2N+1}\Big(\sin\frac{\theta_{m,n}}{2}\cos\frac{\theta_{m,n}}{2}\Big)^{2N+1} \\ &\simeq (-1)^N(\theta_{m,n})^{2N+1} = \lambda\Big(\frac{-\lambda^2(b-a)(t-s)}{mn}\Big)^N\sqrt{\frac{(b-a)(t-s)}{mn}}\,. \end{aligned} \tag{18}$$

Thus

$$[B_{m,n}]_{j,k'} \simeq \lambda\Big(-\lambda^2\frac{(b-a)(m-j)}{m}\frac{(t-s)k}{n}\Big)^N\frac{1}{(N!)^2}\sqrt{\frac{(b-a)(t-s)}{mn}}$$

and the operator represented by the matrix $B_{m,n}$ approximates the integral operator ${}_a^bB_s^t$ with kernel

$$\big(\ker\{{}_a^bB_s^t\}\big)(x,y) = \lambda\chi_a^b(x)\chi_s^t(y)\sum_{N=0}^{\infty}\frac{(-\lambda^2(b-x)(y-s))^N}{(N!)^2}$$

where χ_a^b denotes the indicator function of the interval $[a,b[$. Similarly, as $m,n\to\infty$ the operator represented by $C_{m,n}$ approximates the integral operator ${}_a^bC_s^t$ with kernel

$$\big(\ker\{{}_a^bC_s^t\}\big)(x,y) = -\lambda\chi_a^b(y)\chi_s^t(x)\sum_{N=0}^{\infty}\frac{(-\lambda^2(t-x)(y-a))^N}{(N!)^2}$$

where the change of sign occurs because, instead of (18),

$$c_{0,1,\ldots,0,1}(\theta_{m,n}) = (-2)^{2N+1}\big(\tau_{1,0}(\theta_{m,n})\tau_{0,1}(\theta_{m,n})\big)^N\tau_{1,0}(\theta_{m,n}) = -b_{\sharp_1,\sharp_2,\ldots,\sharp_{\nu-1}}(\theta_{m,n}).$$

We summarise the results of this section in:

THEOREM 5.1. *As $m,n\to\infty$ the operator $W_{(m,n)}$ approaches the matrix of operators*

$${}_a^bW_s^t = \begin{bmatrix} I+{}_a^bA_s^t & {}_a^bB_s^t \\ {}_a^bC_s^t & I+{}_a^bD_s^t \end{bmatrix}$$

on $L^2(\mathbb{R})\oplus L^2(\mathbb{R})$ where the integral operators ${}_a^bA_s^t$, ${}_a^bB_s^t$, ${}_a^bC_s^t$ and ${}_a^bD_s^t$ have kernels

$$\begin{aligned} \big(\ker\{{}_a^bA_s^t\}\big)(x,y) &= <_a^b(x,y)\sum_{N=0}^{\infty}\frac{(y-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!} \\ \big(\ker\{{}_a^bB_s^t\}\big)(x,y) &= \lambda\chi_a^b(x)\chi_s^t(y)\sum_{N=0}^{\infty}\frac{(-\lambda^2(b-x)(y-s))^N}{(N!)^2} \\ \big(\ker\{{}_a^bC_s^t\}\big)(x,y) &= -\lambda\chi_a^b(y)\chi_s^t(x)\sum_{N=0}^{\infty}\frac{(-\lambda^2(t-x)(y-a))^N}{(N!)^2} \\ \big(\ker\{{}_a^bD_s^t\}\big)(x,y) &= <_s^t(x,y)\sum_{N=0}^{\infty}\frac{(y-x)^N(-\lambda^2(b-a))^{N+1}}{N!(N+1)!}\,. \end{aligned}$$

6. Unitarity. We introduce the integral operator Δ_a^b whose kernel is $<_a^b$. Then

$$\ker\big\{(\Delta_a^b)^N\big\} = <_a^b(x,y)\frac{(y-x)^{N-1}}{(N-1)!}\,, \tag{19}$$

and so

$$I + {}^b_aA^t_s = \exp\left\{-\lambda^2(t-s)\Delta^b_a\right\}, \quad I + {}^b_aD^t_s = \exp\left\{-\lambda^2(b-a)\Delta^t_s\right\}. \tag{20}$$

Using the fact that

$$\ker(SB) = (S\otimes I)\ker(B), \quad \ker(BS) = (I\otimes S^*)\ker(B), \tag{21}$$

where S is a bounded operator on $L^2(\mathbb{R}^2)$ and the kernel of an integral operator is regarded as an element of $L^2(\mathbb{R}^2) = L^2(\mathbb{R}^2)\otimes L^2(\mathbb{R}^2)$, and the relations

$$\big((\Delta^b_a)^N\chi^b_a\big)(x) = \frac{(b-x)^N}{N!}\chi^b_a(x), \quad \big(\big((\Delta^t_s)^*\big)^N\chi^t_s\big)(x) = \frac{(x-s)^N}{N!}\chi^t_s(x) \tag{22}$$

the kernels of ${}^b_aA^t_s$, ${}^b_aB^t_s$, ${}^b_aC^t_s$ and ${}^b_aD^t_s$ can be expressed alternatively as

$$\big(\ker\{{}^b_aA^t_s\}\big)(x,y) = \lambda^2 <^b_a (x,y)\left\langle\chi^t_s, \exp\{-\lambda^2(y-x)\Delta^t_s\}\chi^t_s\right\rangle, \tag{23}$$

$$\begin{aligned}\big(\ker\{{}^b_aB^t_s\}\big)(x,y) &= \lambda\chi^t_s(y)\big(\exp\{-\lambda^2(y-s)\Delta^b_a\}\chi^b_a\big)(x)\\ &= \lambda\chi^b_a(x)\big(\exp\{-\lambda^2(b-x)(\Delta^t_s)^*\}\chi^t_s\big)(y)\end{aligned} \tag{24}$$

$$\begin{aligned}\big(\ker\{{}^b_aC^t_s\}\big)(x,y) &= -\lambda\chi^b_a(y)\big(\exp\{-\lambda^2(y-a)\Delta^t_s\}\chi^t_s\big)(x)\\ &= -\lambda\chi^t_s(x)\big(\exp\{-\lambda^2(t-x)(\Delta^b_a)^*\}\chi^b_a\big)(y),\end{aligned} \tag{25}$$

$$\big(\ker\{{}^b_aD^t_s\}\big)(x,y)\lambda^2 = \ <^t_s (x,y)\big\langle\chi^b_a, \exp\{-\lambda^2(y-x)\Delta^b_a\}\chi^b_a\big\rangle, \tag{26}$$

respectively.

Notice that

$${}^b_aD^t_s = {}^t_sA^b_a, \quad {}^b_aC^t_s = -({}^b_aB^t_s)^* \tag{27}$$

THEOREM 6.1. *The operator* ${}^b_aW^t_s$ *is unitary.*

Proof. We lighten notation by writing ${}^b_aW^t_s = W = \begin{bmatrix} I+A & B \\ C & I+D\end{bmatrix}$. Then $W^* = \begin{bmatrix} I+A^* & C^* \\ B^* & I+D^*\end{bmatrix}$ and we have to show that

$$(I+A)(I+A^*) + BB^* = (I+A^*)(I+A) + C^*C = I, \tag{28}$$

$$(I+A)C^* + B(I+D^*) = (I+A^*)B + C^*(I+D) = 0, \tag{29}$$

$$C(I+A^*) + (I+D)B^* = B^*(I+A) + (I+D^*)C = 0, \tag{30}$$

$$CC^* + (I+D)(I+D^*) = B^*B + (I+D^*)(I+D) = I. \tag{31}$$

To prove (28) we observe that, from (20)

$$(I+A)(I+A^*) = \exp\big\{-\lambda^2(t-s)\Delta^b_a\big\}\exp\big\{-\lambda^2(t-s)\left(\Delta^b_a\right)^*\big\}. \tag{32}$$

Regarding $A = {}^b_aA^t_s$ as a function of $t\ge s$ for fixed s, a and b we deduce from (32) that

$$\frac{d}{dt}\{(I+A)(I+A^*)\} = -\lambda^2\exp\big\{-\lambda^2(t-s)\Delta^b_a\big\}\big(\Delta^b_a + \left(\Delta^b_a\right)^*\big)\exp\big\{-\lambda^2(t-s)(\Delta^b_a)^*\big\}.$$

Now $\big(\Delta^b_a + (\Delta^b_a)^*\big)$ is the integral operator with kernel

$$(x,y)\mapsto\ <^b_a (x,y) +\ <^b_a (y,x) = \chi^b_a(x)\chi^b_a(y).$$

Hence by (21), $\frac{d}{dt}\{(I+A)(I+A^*)\}$ is the integral operator with kernel

$$(x,y)\mapsto -\lambda^2\big(\exp\{(t-s)\Delta^b_a\}\chi^b_a\big)(x)\big(\exp\{(t-s)\Delta^b_a\}\chi^b_a\big)(y). \tag{33}$$

Also, from (24)

$$\ker B(x,y) = \lambda \chi_s^t(y)\big(\exp\{-\lambda^2(y-s)\Delta_b^a\}\chi_a^b\big)(x)$$

and so using (21)

$$\begin{aligned}(\ker BB^*)(x,y) &= \int_{\mathbb{R}} \ker B(x,z)\ker B(y,z)\,dz \\ &= \lambda^2 \int_s^t \big(\exp\{-\lambda^2(z-s)\Delta_b^a\}\chi_a^b\big)(x)\big(\exp\{-\lambda^2(z-s)\Delta_b^a\}\chi_a^b\big)(y)\,dz \\ &= \lambda^2\Big(\int_s^t \exp\{-\lambda^2(z-s)\Delta_b^a\}\big(\left|\chi_a^b\right\rangle \left\langle\chi_a^b\right|\big)\exp\{-\lambda^2(z-s)\Delta_b^a\}\,dz\Big)(x,y).\end{aligned}$$

Thus $\frac{d}{dt}(BB^*)$ is the integral operator with kernel

$$(x,y) \mapsto \lambda^2\big(\exp\{-\lambda^2(t-s)\Delta_b^a\}\chi_a^b\big)(x)\big(\exp\{-\lambda^2(t-s)\Delta_b^a\}\chi_a^b\big)(y). \tag{34}$$

Combining (33) and (34) we see that

$$\frac{d}{dt}\big\{(I+A)(I+A^*)+BB^*\big\} = 0$$

for all $t \geq s$. Since $(I+A)(I+A^*)+BB^* = I$ when $t=s$, it follows that $(I+A)(I+A^*)+BB^* = I$ for all values of $t \geq s$. Hence the first part of (28) is proved. The second part is proved similarly using (25). (31) follows from (28) by exchanging the roles of the intervals $[a,b[$ and $[s,t[$.

We prove the first equation of (29), that is $(I+A)C^*+B(I+D^*)=0$. Since

$$(\ker C^*)(x,y) = -\lambda\chi_s^t(y)\big(\exp\big\{-\lambda^2(t-y)(\Delta_a^b)^*\big\}\chi_a^b\big)(x),$$

we have

$$\big(\ker(I+A)C^*\big)(x,y) = \lambda\chi_s^t(y)\big(\exp\big\{-\lambda^2(t-s)\Delta_a^b\big\}\exp\big\{-\lambda^2(t-y)(\Delta_a^b)^*\big\}\chi_a^b\big)(x)$$

and since

$$\begin{aligned}(\ker B)(x,y) &= \lambda\chi_a^b(x)\big(\exp\big\{-\lambda^2(b-x)(\Delta_s^t)^*\big\}\chi_s^t\big)(y), \\ (\ker B(I+D^*))(x,y) &= \lambda\chi_a^b(x)\big(\exp\big\{-\lambda^2(b-a)\Delta_s^t\big\}\exp\big\{-\lambda^2(b-x)(\Delta_s^t)^*\big\}\chi_s^t\big)(y).\end{aligned}$$

Hence we need to show that

$$\begin{aligned}&\chi_s^t(y)\big(\exp\big\{-\lambda^2(t-s)\Delta_a^b\big\}\exp\big\{-\lambda^2(t-y)(\Delta_a^b)^*\big\}\chi_a^b\big)(x) \\ &\qquad = \chi_a^b(x)\big(\exp\big\{-\lambda^2(b-a)\Delta_s^t\big\}\exp\big\{-\lambda^2(b-x)(\Delta_s^t)^*\big\}\chi_s^t\big)(y).\end{aligned} \tag{35}$$

From (19),

$$\begin{aligned}&(\Delta_a^b)^m\big\{(\Delta_a^b)^*\big\}^n\chi_a^b(x) \\ &= \int < {}_a^b(x,z_1) < {}_a^b(z_1,z_2)\cdots < {}_a^b(z_{m-1},z_m)\frac{(z_m-a)^n}{n!}\,\chi_a^b(z_m)\,dz_m\,dz_{m-1}\ldots dz_1 \\ &= \chi_a^b(x)\int_{x<z_1<z_2<\ldots<z_m<b}\frac{(z_m-a)^n}{n!}\,dz_m\,dz_{m-1}\ldots dz_1\end{aligned}$$

$$= \chi_a^b(x) \int_{x<z_1<z_2<\ldots<z_m<b} \sum_{r=0}^{n} (-1)^r \frac{(b-z_m)^r}{r!} \frac{(b-a)^{n-r}}{(n-r)!} \, dz_m \, dz_{m-1} \ldots dz_1$$

$$= \chi_a^b(x) \sum_{r=0}^{n} (-1)^r \frac{(b-x)^{m+r}}{(m+r)!} \frac{(b-a)^{n-r}}{(n-r)!}$$

and so the left hand side of (35) is

$$\chi_a^b(x)\chi_s^t(y) \sum_{m,n=0}^{\infty} \frac{(-\lambda^2(t-s))^m}{m!} \frac{(-\lambda^2(t-y))^n}{n!} \sum_{r=0}^{n} (-1)^r \frac{(b-x)^{m+r}}{(m+r)!} \frac{(b-a)^{n-r}}{(n-r)!} .$$

Similarly the right hand side becomes

$$\chi_a^b(x)\chi_s^t(y) \sum_{m,n=0}^{\infty} \frac{(-\lambda^2(b-a))^m}{m!} \frac{(-\lambda^2(b-x))^n}{n!} \sum_{r=0}^{n} (-1)^r \frac{(t-y)^{m+r}}{(m+r)!} \frac{(t-s)^{n-r}}{(n-r)!} .$$

Writing the first triple sum as

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{m=0}^{\infty} (-1)^r (-\lambda^2)^{m+n} \frac{(t-s)^m}{m!} \frac{(b-x)^{m+r}}{(m+r)!} \frac{(t-y)^n}{n!} \frac{(b-a)^{n-r}}{(n-r)!}$$

and the second as

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \sum_{m=0}^{\infty} (-1)^r (-\lambda^2)^{m+n} \frac{(b-a)^m}{m!} \frac{(t-y)^{m+r}}{(m+r)!} \frac{(b-x)^n}{n!} \frac{(t-s)^{n-r}}{(n-r)!}$$

establishes the equality by the substitutions $m' = n-r$, $n' = m+r$ and $r' = r$ in the former.

The second equation of (29) is established similarly. (30) is deduced from (29) by exchanging $[a,b[$ and $[s,t[$. ■

The proof of the next theorem is closely analogous to that of the corresponding result for the forward-backward case, Theorem 6.1 of [2], and will not be given here.

THEOREM 6.2. *The family of unitary operators* $({}_a^bW_t^s)_{a\le b, s\le t}$ *is a forward evolution in* a *and* b *for fixed* s *and* t, *and also in* s *and* t *for fixed* a *and* b, *that is for* $a \le b \le c$ *and* $s \le t \le u$

$${}_a^bW_t^s \, {}_b^cW_t^s = {}_a^cW_t^s, \quad {}_a^bW_s^t \, {}_a^bW_t^u = {}_a^bW_s^u .$$

7. Construction of the double product integral. When $dr = \lambda(dA^\dagger \otimes dA - dA \otimes dA^\dagger)$ (1) becomes

$$\widehat{\overrightarrow{\prod}}{}_s^t(1+dr) = \lambda\big(dA^\dagger \otimes a(\chi_s^t) - dA \otimes a^\dagger(\chi_s^t)\big) - \lambda^2 \, dT \otimes \int_{s<x<y<t} dA^\dagger(x)\, dA(y)$$

$$= \lambda\big(dA^\dagger \otimes a(\chi_s^t) - dA \otimes a^\dagger(\chi_s^t)\big) - \lambda^2 \, dT \otimes \int_s^t a^\dagger(\chi_s^y)\, dA(y)$$

where we denote by $a^\dagger(u)$ and $a(u)$ the creation and annihilation operators of strength $u \in L^2(\mathbb{R})$. Combining this with the analogous expression for ${}_a^b\widehat{\overrightarrow{\prod}}(1+dr)$ the two alternative definitions of the double product ${}_a^b\overrightarrow{\prod}{}_s^t(1+dr)$ are

(i) The solution at b of the quantum stochastic differential equation

$$dY = Y\Big\{\lambda\big(dA^\dagger \otimes a(\chi_s^t) - dA \otimes a^\dagger(\chi_s^t)\big) - \lambda^2\, dT \otimes \int_s^t a^\dagger(\chi_s^y)\, dA(y)\Big\}, \tag{36}$$

with initial condition $Y(a) = I$.

(ii) The solution at t of the quantum stochastic differential equation

$$dV = V\Big\{\lambda\big(a^\dagger(\chi_a^b) \otimes dA - a(\chi_a^b) \otimes dA^\dagger\big) - \lambda^2 \int_a^b a^\dagger(\chi_a^y)\, dA(y) \otimes dT\Big\}, \tag{37}$$

with initial condition $V(s) = I$.

THEOREM 7.1. *The second quantisation* $\Gamma({}_a^bW_s^t)$*of* ${}_a^bW_s^t$ *solves* (36) *for fixed* a, s *and* t *and* $b \geq a$, *and solves* (37) *for fixed* a, b *and* t *and* $s \geq t$

Proof. Let us establish that $\Gamma({}_a^bW_s^t)$ solves (36). We have to show that, for fixed a, s and t and $b \geq a$

$$\Gamma({}_a^bW_s^t) = I + \int_a^b \Gamma({}_a^xW_s^t)\Big\{\lambda\big(dA^\dagger(x) \otimes a(\chi_s^t) - dA(x) \otimes a^\dagger(\chi_s^t)\big) - \lambda^2\, dT(x) \otimes \int_s^t a^\dagger(\chi_s^y)\, dA(y)\Big\},$$

or, equivalently, by the first fundamental formula of quantum stochastic calculus [12], that for arbitrary f_1, g_1, f_2 and $g_2 \in L^2(\mathbb{R})$,

$$\begin{aligned}
&\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^bW_s^t) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle - \exp\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= \int_a^b \lambda \bar{f}_1(x) \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^xW_s^t)(I \otimes a(\chi_s^t)) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle dx \\
&\quad - \int_a^b \lambda f_2(x) \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^xW_s^t)(I \otimes a^\dagger(\chi_s^t)) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle dx \\
&\quad - \int_a^b \lambda^2 \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^xW_s^t)\Big(I \otimes \int_s^t a^\dagger(\chi_s^y)\, dA(y)\Big) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle dx.
\end{aligned}$$

Since this holds when $b = a$, using the fundamental theorem of calculus it is sufficient to show that

$$\begin{aligned}
&\frac{d}{db}\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^bW_s^t) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= \lambda \bar{f}_1(b) \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^bW_s^b)(I \otimes a(\chi_s^t)) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&\quad - \lambda f_2(b) \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^bW_s^t)(I \otimes a^\dagger(\chi_s^t)) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&\quad - \lambda^2 \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}_a^bW_s^t)(I \otimes \int_s^t a^\dagger(\chi_s^y)\, dA(y)) e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle.
\end{aligned} \tag{38}$$

The left hand side of (38) is

$$\begin{aligned}
&\frac{d}{db}\exp\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, {}^{b}_{a}W^{t}_{s}\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= \left\{\frac{d}{db}\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, {}^{b}_{a}W^{t}_{s}\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle\right\}\exp\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, {}^{b}_{a}W^{t}_{s}\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= \left\{\frac{d}{db}\left(\langle f_1, {}^{b}_{a}A^{s}_{t} f_2\rangle + \langle f_1, {}^{b}_{a}B^{s}_{t} g_2\rangle + \langle g^{b}_{1a}C^{s}_{t} f_2\rangle + \langle g_1, {}^{b}_{a}D^{s}_{t} g_2\rangle\right)\right\}\Theta
\end{aligned} \tag{39}$$

where

$$\Theta = \left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{t}_{s})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle .$$

We express the three terms on the right hand side of (38) similarly as multiples of Θ as follows. The first term is

$$\lambda\bar{f}_1(b)\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{b}_{s})(I\otimes a(\chi^{t}_{s}))e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle = \lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx\,\Theta. \tag{40}$$

For the second we use the commutation relation

$$\Gamma({}^{b}_{a}W^{s}_{t})a^{\dagger}\begin{pmatrix} 0 \\ \chi^{t}_{s} \end{pmatrix} = a^{\dagger}\left({}^{b}_{a}W^{s}_{t}\begin{pmatrix} 0 \\ \chi^{t}_{s} \end{pmatrix}\right)\Gamma({}^{b}_{a}W^{s}_{t}) = a^{\dagger}\begin{pmatrix} {}^{b}_{a}B^{t}_{s}\chi^{t}_{s} \\ (I+{}^{b}_{a}D^{s}_{t})\chi^{t}_{s} \end{pmatrix}\Gamma({}^{b}_{a}W^{s}_{t}) \tag{41}$$

to express it as

$$\begin{aligned}
&-\lambda f_2(b)\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, a^{\dagger}\begin{pmatrix} {}^{b}_{a}B^{t}_{s}\chi^{t}_{s} \\ (I+{}^{b}_{a}D^{s}_{t})\chi^{t}_{s} \end{pmatrix}\Gamma({}^{b}_{a}W^{s}_{t})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= -\lambda f_2(b)\left\langle a\begin{pmatrix} {}^{b}_{a}B^{t}_{s}\chi^{t}_{s} \\ (I+{}^{b}_{a}D^{s}_{t})\chi^{t}_{s} \end{pmatrix}e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{s}_{t})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle \\
&= -\lambda f_2(b)\big(\langle f_1, {}^{b}_{a}B^{t}_{s}\chi^{t}_{s}\rangle + \langle g_1, (I+{}^{b}_{a}D^{s}_{t})\chi^{t}_{s}\rangle\big)\Theta.
\end{aligned} \tag{42}$$

To deal with the third term we use the evolution property, Theorem 6.2, together with (41) to write it as

$$\begin{aligned}
&-\lambda^2\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{t}_{s})\Big(I\otimes\int_s^t a^{\dagger}(\chi^{y}_{s})\Big)e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle g_2(y)\,dy \\
&= -\lambda^2\int_s^t\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{y}_{s})\Gamma({}^{b}_{a}W^{t}_{y})(I\otimes a^{\dagger}(\chi^{y}_{s}))e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle g_2(y)\,dy \\
&= -\lambda^2\int_s^t\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \Gamma({}^{b}_{a}W^{y}_{s})(I\otimes a^{\dagger}(\chi^{y}_{s}))\Gamma({}^{b}_{a}W^{t}_{y})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle g_2(y)\,dy \\
&= -\lambda^2\int_s^t\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, a^{\dagger}\begin{pmatrix} {}^{b}_{a}B^{y}_{s}\chi^{y}_{s} \\ (I+{}^{b}_{a}D^{y}_{t})\chi^{y}_{s} \end{pmatrix}\Gamma({}^{b}_{a}W^{y}_{t})\Gamma({}^{b}_{a}W^{t}_{y})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle g_2(y)\,dy \\
&= -\lambda^2\int_s^t\left\langle e\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, a^{\dagger}\begin{pmatrix} {}^{b}_{a}B^{y}_{s}\chi^{y}_{s} \\ (I+{}^{b}_{a}D^{y}_{t})\chi^{y}_{s} \end{pmatrix}\Gamma({}^{b}_{a}W^{t}_{s})e\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right\rangle g_2(y)\,dy \\
&= -\lambda^2\int_s^t\big(\langle f_1, {}^{b}_{a}B^{y}_{s}\chi^{y}_{s}\rangle + \langle g_1, (I+{}^{b}_{a}D^{y}_{t})\chi^{y}_{s}\rangle\big)\,g_2(y)\,dy\,\Theta.
\end{aligned} \tag{43}$$

Thus we have to prove that

$$\begin{aligned}
&\frac{d}{db}\big(\langle f_1, {}^b_aA^s_t f_2\rangle + \langle f_1, {}^b_aB^s_t g_2\rangle + \langle g_1, {}^b_aC^s_t f_2\rangle + \langle g_1, {}^b_aD^s_t g_2\rangle\big)\\
&= \lambda \bar{f}_1(b)\int_s^t g_2(x)\,dx - \lambda f_2(b)\big(\langle f_1, {}^b_aB^t_s\chi^t_s\rangle + \langle g_1, (I + {}^b_aD^s_t)\chi^t_s\rangle\big)\\
&\qquad - \lambda^2\int_s^t \big(\langle f_1, {}^b_aB^y_s\chi^y_s\rangle + \langle g_1, (I + {}^b_aD^y_t)\chi^y_s\rangle\big) g_2(y)\,dy. \qquad (44)
\end{aligned}$$

To do this we use the kernels of the operators ${}^b_aA^s_t$, ${}^b_aB^s_t$, ${}^b_aC^s_t$ and ${}^b_aD^s_t$ given by Theorem 4.1 to compare the terms on the two sides of (44) which are sesquilinear in (f_1, f_2), (f_1, g_2), (g_1, f_2) and (g_1, g_2), respectively. For (f_1, f_2) we have, on the left,

$$\begin{aligned}
\frac{d}{db}\langle f_1, {}^b_aA^s_t f_2\rangle &= \frac{d}{db}\int\int \bar{f}_1(x) <^b_a (x,y)\sum_{N=0}^{\infty}\frac{(y-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!} f_2(y)\,dy\,dx\\
&= \frac{d}{db}\int_a^b\int_a^y \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(y-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!} f_2(y)\,dx\,dy\\
&= f_2(b)\int_a^b \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(b-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!}\,dx
\end{aligned}$$

using the fundamental theorem of calculus, and on the right

$$\begin{aligned}
-\lambda f_2(b)\langle f_1, {}^b_aB^t_s\chi^t_s\rangle &= -\lambda^2 f_2(b)\int_a^b\int_s^t \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(-\lambda^2(b-x)(y-s))^N}{(N!)^2}\,dy\,dx\\
&= f_2(b)\int_a^b \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(b-x)^N(-\lambda^2(t-s))^{N+1}}{N!(N+1)!}\,dx
\end{aligned}$$

as required. For (f_1, g_2), on the left we have, differentiating under the integral and again using the fundamental theorem of calculus,

$$\begin{aligned}
\frac{d}{db}\langle f_1, {}^b_aB^s_t g_2\rangle &= -\lambda\frac{d}{db}\int_a^b\int_s^t \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(-\lambda^2(b-x))^N(y-s)^N}{(N!)^2} g_2(y)\,dy\,dx\\
&= \lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx - \lambda\int_a^b\int_s^t \bar{f}_1(x)\sum_{N=1}^{\infty}\frac{(b-x)^{N-1}(-\lambda^2(y-s))^N}{(N-1)!N!} g_2(y)\,dy\,dx\\
&= \lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx - \lambda\int_a^b\int_s^t \bar{f}_1(x)\sum_{N=1}^{\infty}\frac{(b-x)^N(-\lambda^2(y-s))^{N+1}}{N!(N+1)!} g_2(y)\,dy\,dx,
\end{aligned}$$

and on the right,

$$\begin{aligned}
&\lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx - \lambda^2\int_s^t \langle f_1, {}^b_aB^y_s\chi^y_s\rangle\, g_2(y)\,dy\\
&= \lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx + \lambda^3\int_a^b\int_s^t\int_s^y \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(-\lambda^2(b-x)(z-s))^N}{(N!)^2} g_2(y)\,dz\,dy\,dx\\
&= \lambda\bar{f}_1(b)\int_s^t g_2(x)\,dx - \lambda\int_a^b\int_s^y \bar{f}_1(x)\sum_{N=0}^{\infty}\frac{(b-x)^N(-\lambda^2(y-s))^{N+1}}{N!(N+1)!} g_2(y)\,dy\,dx.
\end{aligned}$$

For (g_1, f_2), on the left we have

$$\frac{d}{db}\langle g_1, {}^b_a C^s_t f_2\rangle = -\lambda \frac{d}{db}\int_a^b \int_s^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(-\lambda^2(t-x)(y-a))^N}{(N!)^2} f_2(y)\,dx\,dy$$
$$= -\lambda f_2(b) \int_s^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(-\lambda^2(t-x)(b-a))^N}{(N!)^2}\,dx,$$

and, on the right,

$$-\lambda f_2(b)\langle g_1, (I + {}^b_a D^s_t)\chi^t_s\rangle$$
$$= -\lambda f_2(b) \int_s^t \bar{g}_1(x)\,dx - \lambda f_2(b) \int \int_x^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(b-a))^{N+1}}{N!(N+1)!}\,dy\,dx$$
$$= -\lambda f_2(b) \int_s^t \bar{g}_1(x)\,dx - \lambda f_2(b) \int_x^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(t-x)^{N+1}(-\lambda^2(b-a))^{N+1}}{(N+1)!(N+1)!}\,dx$$
$$= -\lambda f_2(b) \int_s^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(-\lambda^2(t-x)(b-a))^N}{(N!)^2}\,dx.$$

Finally for (g_1, g_2), on the left we have

$$\frac{d}{db}\langle g_1, {}^b_a D^s_t g_2\rangle = \frac{d}{db}\int_s^t \int_s^y \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(b-a))^{N+1}}{N!(N+1)!} g_2(y)\,dx\,dy$$
$$= -\lambda^2 \int_s^t \int_x^y \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(b-a))^N}{(N!)^2} g_2(y)\,dx\,dy$$

while on the right we have

$$-\lambda^2 \int_s^t \langle g_1, (I + {}^b_a D^z_t)\chi^z_s\rangle g_2(z)\,dz$$
$$= -\lambda^2 \int_s^z \bar{g}_1(x) g_2(z)\,dz$$
$$\quad - \lambda^2 \int_s^t \int\int \bar{g}_1(x) <^z_s (x,y) \sum_{N=0}^{\infty} \frac{(y-x)^N}{N!} \frac{(-\lambda^2(b-a))^{N+1}}{(N+1)!}\,dx\,dy\,g_2(z)\,dz$$
$$= -\lambda^2 \int_s^z \bar{g}_1(x) g_2(z)\,dz$$
$$\quad - \lambda^2 \int_s^t \int \bar{g}_1(x)\chi^t_s(z) \sum_{N=0}^{\infty} \int_x^z \frac{(y-x)^N}{N!} \frac{(-\lambda^2(b-a))^{N+1}}{(N+1)!}\,dy\,dx\,g_2(z)\,dz$$
$$= -\lambda^2 \int_s^z \bar{g}_1(x) g_2(z)\,dz$$
$$\quad - \lambda^2 \int_s^t \int_s^t \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(z-x)^{N+1}}{(N+1)!} \frac{(-\lambda^2(b-a))^{N+1}}{(N+1)!} g_2(z)\,dx\,dz$$
$$= -\lambda^2 \int_s^t \int_x^y \bar{g}_1(x) \sum_{N=0}^{\infty} \frac{(y-x)^N(-\lambda^2(b-a))^N}{(N!)^2} g_2(y)\,dx\,dy.$$

The proof that $\Gamma({}^b_a W^t_s)$ solves (37) is similar. ∎

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