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ON THE NECESSITY OF REIDEMEISTER MOVE 2 FOR SIMPLIFYING IMMERSED PLANAR CURVES

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Abstract. In 2001, motivated by his results on finite-type knot diagram invariants, Östlund conjectured that Reidemeister moves 1 and 3 are sufficient to describe a homotopy from any generic immersion $S^1 \to \mathbb{R}^2$ to the standard embedding of the circle. We show that this conjecture is false.

1. Introduction. We wish to consider the problem of simplifying immersed planar curves, in a sense which will later be made precise. Intuitively, a generic immersion $S^1 \to \mathbb{R}^2$ can be considered as a knot diagram without the crossing data, and for such immersions we can apply planar versions of the Reidemeister moves for knot diagrams. By applying all three Reidemeister moves to such a diagram, one is able to obtain a standardly embedded circle with no double points. One approach is to add crossing data so as to give a knot diagram of the unknot, then apply the standard three Reidemeister moves to this knot diagram to obtain the standardly embedded circle.

In [Oest1] (part of which appears in [Oest2]), Östlund observed that Reidemeister move 1 is the only move that changes the degree of the Gauss map, and showed that Reidemeister move 3 is the only move that can change the signed number of instances of certain subdiagrams of the Gauss diagram for an embedding. These properties were used to show that any knot K admits a pair of diagrams such that every sequence of Reidemeister moves connecting them contains instances of Reidemeister moves 1 and 3.

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Planar versions of the same arguments give immersions of the circle in which every connecting sequence contains instances of Reidemeister moves 1 and 3.

In order to address the question of the necessity of move 2, Östlund developed a diagrammatic generalization of finite-type knot invariants and analyzed the degree[1] of those invariants used to establish necessity of Reidemeister moves 1 and 3. In particular, he defined a stratification $\{\Sigma_i\}_{i\in\mathbb{N}}$ of the space of projected knot diagrams, based on the number and type of defects from general-position. Paths in Σ_0 are planar isotopies. Paths in $\Sigma_0 \cup \Sigma_1$ which pass through Σ_1 transversely are sequences of planar isotopies and Reidemeister moves. A diagram invariant is a function on knot diagrams which is constant on connected components of Σ_0.

From here, the definition of finite-degree diagram invariants is entirely similar to that of finite-type knot invariants; the definition of a diagram invariant in Σ_0 is extended to each Σ_i by computing the difference between the two neighboring Σ_{i-1} invariants. A diagram invariant is then finite-degree of degree i if it is 0-valued for diagrams in Σ_i.

Östlund showed that any finite-degree diagram invariant which is invariant under Reidemeister moves 1 and 3 is automatically invariant under Reidemeister move 2. This result significantly constrains the methods usable to demonstrate independence of the two moves, and motivates the conjecture that any two equivalent knot diagrams could be connected by a sequence of moves consisting only of planar isotopies and Reidemeister 1 and 3 moves. Östlund chose to formulate the weaker conjecture: two generic plane immersions may be taken one to the other by a sequence consisting of planar isotopies and the first two Reidemeister moves.

The knot diagram version of the conjecture turned out to be false. A counterexample for the case of knots appears in [Mant1] and [Mant2]. Independently, the first author of this paper showed in [Hag] that every knot type admits pairs of diagrams such that every connecting sequence contains every Reidemeister move type. However, both of these arguments are combinatorial arguments based on examples, and after a certain point it is difficult to see how one may extract useful information about the space of knot diagrams from them. Both proofs, however, clearly rely on the three-dimensional structure of examples used in order to establish the result.

In this paper, we show that the (planar) conjecture of Östlund is false. Since every sequence of Reidemeister moves becomes a sequence of planar Reidemeister moves when one forgets the crossing information, the argument also serves as an alternate disproof of the knot theoretic case.

2. Definitions and main results

Definition 2.1. An *immersed curve* is the image of a map $f : S^1 \to F$, where F is some surface, such that any point in the pair $(F, f(S^1))$ has a neighborhood homeomorphic to a neighborhood in the picture right. The pair $(F, f(S^1))$ shall denote the immersed curve.

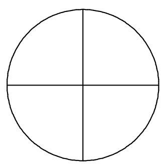


Fig. 1

In this paper, F will usually be $\mathbb{R}^2$ or S^2.

[1] the terms finite-degree and finite-type are here used interchangeably

Definition 2.2. Given an immersed curve, the Reidemeister moves are given, as numbered below (1a, 1b, 2a, 2b, or 3), by identifying a disk in $(F, f(S^1))$ homeomorphic to the disk on the left side of the numbered picture and replacing it with the homeomorphic preimage of the disk on the right.

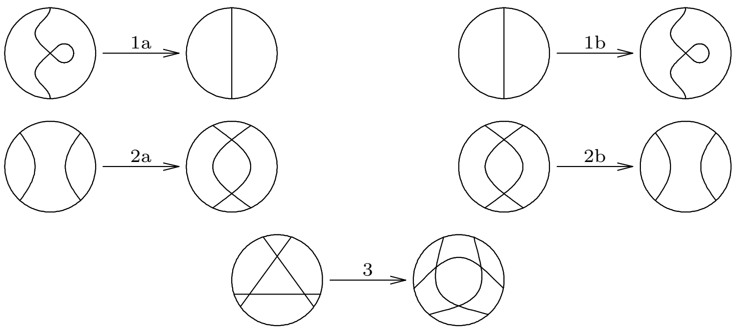


Fig. 2

By convention, planar isotopies are always allowed as moves, even when not explicitly mentioned. Any two homotopic immersed curves are connected by a sequence of Reidemeister moves and planar isotopies.

Definition 2.3. An immersed curve c_0 is $(1,3)$-*simplifiable* if for some N there exists a sequence of immersed curves $\{c_i\}_{i=0}^N$ such that c_{i+1} is obtained from c_i by applying one of Reidemeister moves 1a, 1b, or 3, and $c_N = (F, f(S^1))$, where f is an embedding. The sequence $\{c_i\}_{i=0}^N$ is called a *simplifying sequence* for the curve c_0.

Example 2.4. If F is a surface of genus at least 1 and c_0 is not null-homotopic, then c_0 is not $(1,3)$-simplifiable. This is because the Reidemeister moves applied to curves preserve homotopy type.

Östlund's conjecture, stated in our language, is that every immersed planar curve is $(1,3)$-simplifiable. Since any curve which is $(1,3)$-simplifiable in $\mathbb{R}^2$ is $(1,3)$-simplifiable in its one point compactification S^2, the next theorem suffices to disprove the conjecture:

Main Theorem 2.5. *The following curve is not* $(1,3)$-*simplifiable in* S^2:

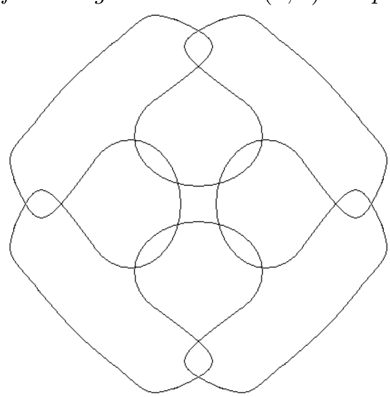


Fig. 3

The proof of this theorem does not rely on heavy machinery. It should be noted that there are immersed curves without an obvious simplifying sequence, which are nonetheless $(1, 3)$-simplifiable. For example, consider the following:

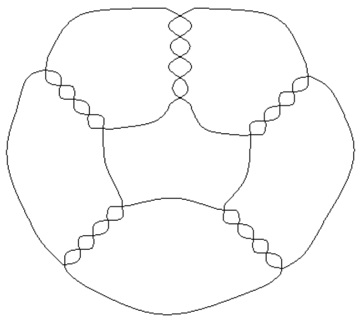
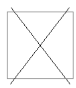
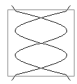


Fig. 4

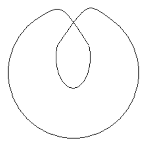
The easiest way to show that this curve is $(1, 3)$-simplifiable is to apply this theorem:

Theorem 2.6. *Let c be a $(1, 3)$-simplifiable curve. Suppose that in c we replace some instances*

of the local picture  *with the local picture* 

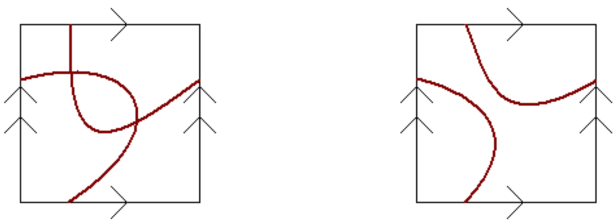
relative boundary (i.e. double bigons replace double points) to obtain curve c'. Since c is $(1, 3)$-simplifiable, c' is $(1, 3)$-simplifiable.

It should be noted that Theorem 2.6 does not say that moves 1 and 3 may be used to replace a double point with a double bigon in an arbitrary diagram. Nonetheless, applying Theorem 2.6 repeatedly to Fig. 4 gives the $(1, 3)$-simplifiable immersed curve on the right.



One could generalize Östlund's conjecture and ask whether two homotopic curves on a surface are related by only the first and third Reidemeister moves. This generalized conjecture is much easier to falsify. It is in fact a generalization because all generic curves on $\mathbb{R}^2$ or S^2 are homotopically trivial.

Theorem 2.7. *The following two curves on T^2 are homotopic, but are not related by a sequence of Reidemeister moves consisting of only the first and third moves.*



3. Proof of Main Theorem. Consider the following shaded regions in the curve from Fig. 3, interpreted as a diagram on S^2:

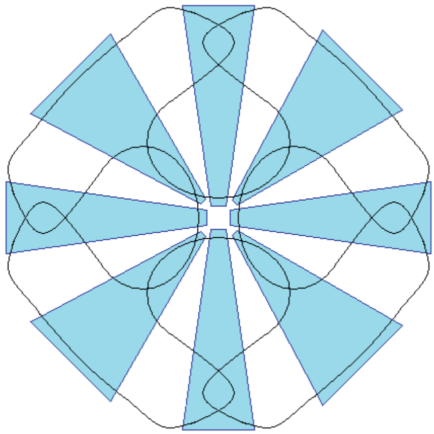


Fig. 5

Reinterpret the diagram as a collection of eight shaded boxes containing immersed tangles, connected by lines with no double points. Each box has a left and right side, as labeled below; the left side of a given box is connected to the right side of its neighbor. Two polygons in the diagram deserve special attention and are marked with a star.

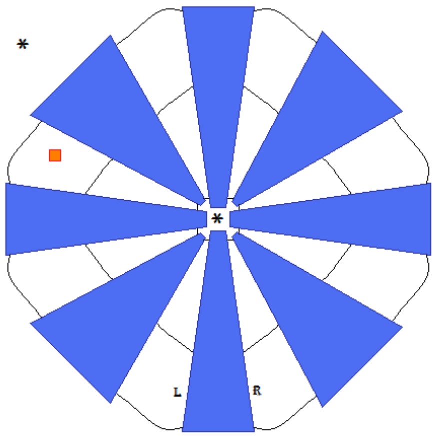


Fig. 6

The diagram satisfies the following properties:

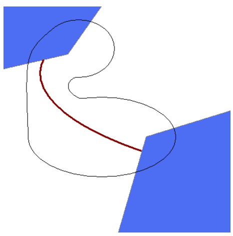
1. Each shaded box contains a tangle with three strands. One of the strands, denoted by strand 1, begins and ends at the left side. Strand 2 begins and ends at the right side. Strand 3 has one endpoint on each side of the box.
2. In each box, the left side of strand 3 connects to strand 2 in the adjacent box to the left. The right side of strand 3 connects to strand 1 in the adjacent box on the right.
3. Strands 1 and 2 intersect in exactly two double points.
4. The polygons marked with a star have at least four sides.

We will show that any application of moves 1a, 1b, or 3 to any copy of Fig. 5 with immersed tangles satisfying the above properties results in a diagram which may be interpreted as a copy of Fig. 5 with immersed tangles still satisfying those properties. Since property 3 implies that any shaded box has at least two double points, every sequence of such moves results in a diagram with at least sixteen double points. This proves that the curve is not $(1, 3)$-simplifiable.

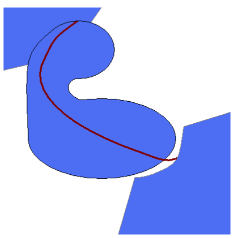
First, note that a move of type 1a, 1b, or 3 occurring entirely within one of the shaded boxes gives a diagram (with the same boxes) satisfying all of the above properties. Such a move fixes the endpoints of the strands, so Properties 1 and 2 remain satisfied. None of these moves change the number of times one strand intersects another within a box, so Property 3 holds after a move.

Property 4 actually follows from the arrangement of the boxes and the other three properties. Fix a starred polygon and consider the portion of its boundary lying within a single shaded box. If the end points of that boundary portion belong to different strands within the box, that box contains at least one vertex for the starred polygon. Otherwise, Property 1 implies that both ends belong to strand 3. Then Property 2 implies that the end points for the portion of the starred polygon's boundary lying in each of the adjacent shaded boxes belong to different strands within that box. Thus each of the adjacent boxes contains a vertex for the starred polygon. Therefore, allowed moves cannot reduce the number of vertices (or edges) for a starred polygon below four.

It remains to show that it suffices to consider only moves lying within a single shaded box. First, consider Reidemeister move 1b. Performing this move requires a disk in our immersed curve that is homeomorphic to the disk on the left side of picture 1b in Fig. 2 in Definition 2.2. Suppose that the segment on the left side of picture 1b in Definition 2.2 is not contained completely inside one of the shaded boxes as specified above. Then one can redefine the shaded box before performing the move so that it occurs entirely within a single shaded box. For example, suppose that the disk for move 1b is the following:



One can then isotop the shaded boxes, while leaving crossings fixed, as follows:



Move 1a, on the other hand, removes a one-sided polygon. This polygon must lie entirely within a single shaded box, for the following reason: Clearly, a one-sided polygon cannot separate the two starred regions. If a closed smooth subcurve of $f(S^1)$ does not lie in a single shaded box, and does not separate the two starred regions, then it must enter and exit one of the shaded boxes on the same side. Such a curve contains a segment of strand type 1 or 2, and by Property 3, any such curve will have at least two crossings.

Finally, move 3 always occurs on a neighborhood of a triangle (which can never be marked with a star). If that triangle lies entirely in one shaded box, that box may be isotoped as above to include the entire disk on which the move occurs. Otherwise, the triangle intersects the white region (an example of such a potential triangle is marked with a red dot in Fig. 6). One can verify that this implies that one of the shaded boxes intersects the triangle only in a single corner, as shown in Fig. 7, for example.

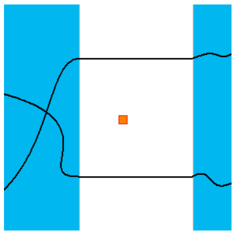


Fig. 7

Assume without loss of generality that the triangle extends to the right of the shaded box containing just the corner, as in Fig. 7. There are two possibilities for the strand ends on the right side of the leftmost box shown in Fig. 7. Either exactly one of the ends belongs to strand 3, or both ends belong to strand 2. In the box to the right, either both of the pictured left ends belong to strand 1, or exactly one belongs to strand 3, respectively. In either case, isotoping the shaded boxes in Fig. 7 to give the shaded boxes in Fig. 8 preserves the required properties and reduces the number of white regions in the triangle. After at most two such box adjustments, all three vertices of the triangle must lie in the same shaded box. Then, since there are no isolated shaded corners, the entire triangle must lie within a single shaded box.

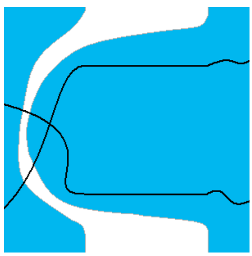


Fig. 8

One could also prove this theorem using Gauss diagrams. We give the main outline, leaving the proof to the reader. The Gauss diagram for the immersed curve in Fig. 3 is as follows:

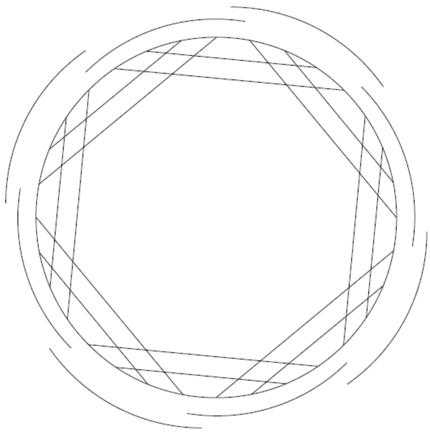


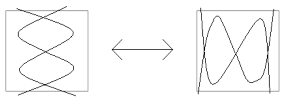
Fig. 9

Move 1b adds a chord to the diagram, which by convention shall be colored gray. The following properties are preserved by Reidemeister moves 1 and 3.

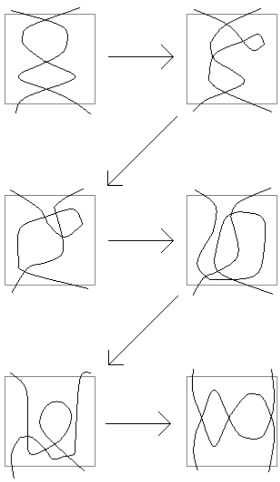
1. If all the gray lines are erased, the resulting diagram is exactly as shown in Fig. 9 above except that some of the adjacent and parallel pairs of black lines may be replaced with crossed pairs.
2. Both endpoints of every gray chord lie in one of the eight regions indicated in Fig. 9.

4. Proof of other theorems

LEMMA 4.1. *The following pictures are connected by a sequence of Reidemeister moves* 1 *and* 3:



Proof. This is the necessary sequence of Reidemeister moves:



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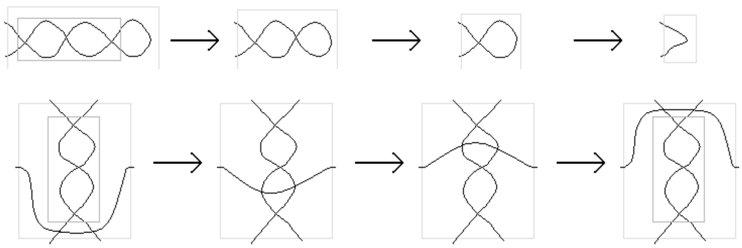
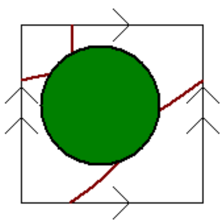
Proof of Theorem 2.6. Consider two immersed curves L and R, equal except inside of a box. The contents of the box for the curves L and R are given respectively by the following pictures on the left and right:

Suppose L is $(1,3)$-simplifiable. Then Reidemeister moves performed on L that are supported away from the box have analogous Reidemeister moves on R. However, the simplifying sequence for L may contain moves 1a and 3 which involve the box. The following sequences of moves on R are analogous to moves 1a and 3 on L which involve the box. In these sequences it may be necessary to first apply Lemma 4.1 to obtain the leftmost picture.



Applying the moves on R analogous to the moves in a simplifying sequence for L gives a simplifying sequence for R. ■

Proof of Theorem 2.7. It will be sufficient to show that by applying R1 and R3 moves to a curve on T^2 of the form



such that the two strands of the tangle inside the disk intersect, one can never obtain an embedded curve (i.e. a curve without double points). Observe that in the picture above, there is exactly one region not contained in the disk, and this region has genus one. Up to isotopy every R1a, R1b and R3 move is supported within the disk. However, as noted in the proof of the main theorem, R1 and R3 moves on tangles do not change the number of intersections between strands. ■

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