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K-THEORY OVER C^*-ALGEBRAS

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Abstract. The contents of the article represents the minicourse which was delivered at the 7th conference "Geometry and Topology of Manifolds. The Mathematical Legacy of Charles Ehresmann", Będlewo (Poland), 8.05.2005 – 15.05.2005. The article includes the description of the so called Hirzebruch formula in different aspects which lead to a basic list of problems related to noncommutative geometry and topology. In conclusion, two new problems are presented: about almost flat bundles and about the Noether decomposition of the Fredholm operators which may not admit the adjoint operator.

Introduction. In the second half of the last century a new research field was developed intensively, which is customarily referred to as "noncommutative geometry". As a matter of fact this notion covers the circle of problems and methods that are primordially based on a fairly simple idea: to reformulate topological properties of spaces and mappings in terms of corresponding algebras of continuous functions. Though this idea is very old (it goes back to the classical Gelfand–Naimark theorem about one-to-one correspondence between the category of locally compact topological spaces and the category of commutative C^*-algebras) and was developed by many authors both in commutative and noncommutative setting, this idea was proclaimed more or less manifestly as a program by A. Connes in his book "Noncommutative geometry" ([Con94]).

In spite of its self-evidence this idea to consider, along with commutative C^*-algebras, also noncommutative algebras as a family of functions on a non-existing "noncommutative" space has become very productive and therefore allowed to join together a variety of different concepts and methods from many areas such as topology, differential geometry, functional analysis, representation theory, asymptotic methods in analysis. They allowed also to mutually enrich all these areas with new theorems and properties.

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Problems of differential topology considered in the sixties were the sources of noncommutative geometry. In essence, the description of topological and homotopy invariants of smooth and piecewise linear manifolds was only one solved problem. The typical invariants for smooth and piecewise linear manifolds were so called characteristic classes, which were known of three types: the Stiefel–Whitney classes, the Chern classes and the Pontryagin classes. They are closely related also to vector bundles.

Hence the problem can be formulated as follows: to what extent one or another characteristic class depends on smooth structure of manifolds by means of which it is determined? As an answer in the fifties it was established that the Stiefel–Whitney classes are homotopy invariants. More or less using the same methods one can establish that rational Pontryagin classes are topological invariants.

As to homotopy invariance of the Pontryagin classes this problem until now is not clear and continues to intrigue many mathematicians. This problem finally was known as the Novikov conjecture and was one of essential sources of the generation of noncommutative geometry.

0.1. *From Poincaré duality to the Hirzebruch formula.* The Pontryagin classes are not homotopy invariant but have close relation to the problem of a description of smooth structures of given homotopy type. Hence the problem of finding all homotopy Pontryagin classes was considered vital. In reality, another problem happened to be more natural. From the point of view of a classification of smooth structures on a manifold the most appropriate equivalence relation between manifolds turned out to be so called internal homology or bordism of manifolds.

In 1945 Pontryagin suggested that internal homology can be described in terms of certain algebraic expressions in characteristic Pontryagin classes—the Pontryagin numbers ([Pon45], 1945), and established that the Pontryagin numbers are invariant with respect to internal homology ([Pon47], theorem 3). Using the surgery theory for smooth manifolds W. Browder and S. Novikov proved that the unique homotopy invariant characteristic Pontryagin number coincides with the signature of an oriented compact manifold with respect to the Poincaré duality. The formula that identifies the signature of a manifold with a certain characteristic Pontryagin number is now known as the Hirzebruch formula ([Hir53]), though its special case was established by V. A. Rokhlin a year before ([Rok52]).

The Poincaré duality and the Hirzebruch formula have a long history, partly connected with coming into being of noncommutative geometry and results of the Moscow topological school of the second half of the 20th century. The study of these problems began with the prominent paper "Analysis situs" by H. Poincaré in 1895 ([Poi95]). There H. Poincaré in particular for the first time formulated the theorem known now as the Poincaré duality for closed oriented manifolds. Although the complete statement and the proof of the Poincaré duality were presented considerably later, H. Poincaré undoubtedly discovered the theorem. Certainly H. Poincaré oversimplified the statement. He meant the coincidence of the Betti numbers that were equidistant from the ends ([Poi95], p. 490). In any case the notion of the Betti numbers requires a more precise definition which H. Poincaré partly gave in his subsequent papers. There is the excellent paper by P. S. Alexandroff containing his speech devoted to the centennial of H. Poincaré's birthday on the special meeting of the International Congress of Mathematicians (Amsterdam, 1954) (see [Ale72], p. 813). We can hardly add anything more to this excellent characterization of the contribution of H. Poincaré to the homology theory of manifolds.

We should especially point out the following new important objects in algebraic topology: the homology groups (E. Noether, 1925), the cohomology groups (J. Alexander, A. N. Kolomogorov, 1934), duality between them (L. S. Pontryagin). The most significant event was the creation of characteristic classes (Stiefel, Whitney, 1935; Pontryagin, 1947; Chern, 1948). Everything became prepared to connect the Poincaré duality and integral invariants of characteristic classes. This connection is now known as the Hirzebruch formula. The Hirzebruch formula gives us an excellent example of the application of categorical methods in algebraic and differential topology. After introduction of the notion of homology group, the Poincaré duality was meant as coincidence of ranks of homology groups. For the Betti numbers it was inessential what kind of homology groups were considered—integer or rational, since the rank of the integer homology groups coincides with the dimension of the rational homology groups. But the notion of homology groups allowed to enrich the Poincaré duality by considering the homology groups with coefficients in finite fields. Therefore, coincidence of Betti numbers can be interpreted as isomorphism of rational homology groups. Taking into account the torsion one should interpret the coincidence of the torsion as an isomorphism of new groups. But these new groups cannot be homology groups because the torsion coincides in the dimensions different from the coincidence of the Betti numbers. This apparent inconsistency was understood after the creation of the cohomology groups and their relations to the homology groups. Finally, the Poincaré duality was interpreted as an isomorphism between the homology groups and the cohomology groups of complementary dimensions:

$$H_k(M;Z) \approx H^{n-k}(M;Z). \tag{1}$$

The crucial understanding here is that this isomorphism is not abstract but is generated by a natural geometric operation. For example, in the case of the middle dimension where $\dim M = n = 2m$ the condition (1) becomes a trivial one since

$$H^m(M;Q) = \mathbf{Hom}(H_m(M;Q), Q) \equiv H_m(M;Q). \tag{2}$$

But in the equality (1) the isomorphism is not accidental. This isomorphism is generated by the intersection operation with the fundamental homology class $[M]$:

$$\cap\, [M] : H^{n-k}(M;Q) \to H_k(M;Q).$$

This means that to the manifold M one can assign the nondegenerate quadratic form which has an additional invariant—the signature of the quadratic form.

The Hirzebruch formula is the expression of the relation between the signature of the manifold M and some characteristic number of the same manifold M.

The Hirzebruch formula says that for a $4k$–dimensional orientable compact closed manifold X the equality

$$\mathbf{sign}\, X = 2^k \langle L(X), [X]\rangle \tag{3}$$

holds. Here $\mathbf{sign}\, X = \mathbf{sign}(H^{2k}(X, \mathbf{C}), \cup)$ is the signature of the nondegenerate quadratic form in the cohomology groups $H^{2k}(X, \mathbf{C})$, defined by the $\cup$-product:

$$([\omega_1], [\omega_2]) \stackrel{def}{=} \langle [\omega_1] \cup [\omega_2], [X] \rangle = \langle [\omega_1 \wedge \omega_2], [X] \rangle = \int_X \omega_1 \wedge \omega_2.$$

The class

$$L(X) = \prod_j \frac{t_j/2}{\operatorname{th}(t_j/2)}$$

is the Hirzebruch characteristic class defined by formal generators t_j such that

$$\sigma_k(t_1, \ldots, t_n) = c_k\,(cTX) \in H^{2k}(X; \mathbf{Z}),$$

where σ_k is an elementary symmetric polynomial.

There are different ways to generalize the Hirzebruch formula—mainly for non-simply connected manifolds which play a crucial role in different problems of noncommutative geometry and topology.

1. Finite dimensional representations

1.1. *Finite dimensional unitary representations.* Let X be a closed orientable non simply connected manifold and let $\pi = \pi_1(X)$,

$$f_X : X \to B\pi$$

be the canonical mapping defined up to homotopy that induces an isomorphism of fundamental groups

$$(f_X)_* : \pi_1(X) \to \pi.$$

Consider a finite dimensional representation

$$\rho : \pi \to U(N).$$

Using the representation ρ one can construct several objects:

1) The flat (complex) vector bundle ξ^ρ over $B\pi$, induced by the representation ρ.

2) The flat (complex) vector bundle ξ^ρ_X over X induced by the same representation ρ, $\xi^\rho_X = f^*_X \xi^\rho$.

3) The cohomology groups $H^{2k}(X, \rho)$ with the local system of coefficients induced by the representation ρ

$$H^{2k}(X, \rho) = H^{2k}(X, \xi^\rho_X).$$

The $\cup$-product induces a nondegenerate Hermitian form in the group $H^{2k}(X, \rho)$:

$$H^{2k}(X, \xi^\rho_X) \times H^{2k}(X, \xi^\rho_X) \xrightarrow{\cup} H^{4k}(X, \xi^\rho_X \otimes \xi^\rho_X) \xrightarrow{\langle\cdot,\cdot\rangle} H^{4k}(X, \mathbf{C}) \approx \mathbf{C}.$$

The signature of this form will be denoted by

$$\mathbf{sign}_\rho\, X = \mathbf{sign}(H^{2k}(X, \rho), \cup).$$

It is easy to check that

$$\mathbf{sign}_\rho\, X = 2^k \langle L(X) \operatorname{ch} \xi^\rho_X, [X] \rangle. \tag{4}$$

Since ξ^ρ is a flat bundle one has $\operatorname{ch} \xi^\rho = \dim \xi^\rho = N$. Hence both the left side and right side of the formula (4) coincide with that of (3) up to an integer factor N.

This means at least that the signature $\mathbf{sign}_\rho X$ depends only on the dimension of the finite dimensional unitary representation ρ. Nevertheless this case might be useful for further generalizations. Namely one can construct at least the right hand side of the formula (4) for more general representations of the fundamental group π.

1.2. *Finite dimensional unitary representations with respect to a pseudo Hermitian structure of the type* (p,q)*.* Consider a representation

$$\rho : \pi \to U(p,q)$$

into the matrix group $U(p,q)$ that preserves an indefinite Hermitian nondegenerate form of the type (p,q). Then again one can construct the operation of the type $\cup$ which generates a nondegenerate Hermitian form into middle cohomologies $H^{2k}(X;\rho)$.

On the other hand the flat vector bundle ξ^ρ_X can be split into the direct sum

$$\xi^\rho_X = \xi^\rho_+ \oplus \xi^\rho_-,$$

such that on each summand the form is (positive and negative) definite. Then the Hirzebruch formula has the following form:

$$\mathbf{sign}_\rho X = 2^k \langle L(X)\,\mathrm{ch}(\xi^\rho_+ - \xi^\rho_-), [X]\rangle.$$

Here the Chern character of the bundles $\xi^\rho_\pm$ may be nontrivial (Lusztig [Lus72, 1972]).

1.3. *Finite dimensional unitary representations with respect to a skew Hermitian structure.* Taking a skew Hermitian form φ on $\mathbf{C}^N$ and the matrix group $\mathbf{Sp}(N)$ which preserves this form one can consider a representation

$$\rho : \pi \to \mathbf{Sp}(N)$$

and a flat vector (complex) bundle ξ^ρ_X. If $\dim X = 4k+2$ then in the middle dimension one has a nondegenerate Hermitian form in the group $H^{2k+1}(X,\rho)$ generated by the $\cup$-product:

$$H^{2k+1}(X,\xi^\rho_X) \times H^{2k+1}(X,\xi^\rho_X) \xrightarrow{\cup} H^{4k+2}(X,\xi^\rho_X \otimes \xi^\rho_X) \xrightarrow{\langle\cdot,\cdot\rangle} H^{4k+2}(X,\mathbf{C}) \approx \mathbf{C}.$$

The flat vector bundle ξ^ρ_X can be split into the direct sum

$$\xi^\rho_X = \xi^\rho_+ \oplus \xi^\rho_-,$$

such that on each summand the Hermitian form $i\cdot\varphi$ is (positive and negative) definite. Then again the Hirzebruch formula has the following form:

$$\mathbf{sign}_\rho X = 2^k \left\langle L(X)\,\mathrm{ch}(\xi^\rho_+ - \xi^\rho_-), [X]\right\rangle$$

(see M. Gromov [Gro95, § $8\frac{1}{2}$]).

1.4. *Construction of splitting of flat vector bundles for calculation of the signature.* For all three cases above a similar construction is available and is based on the construction of the Hodge operator.

Let X be an oriented compact manifold, $\dim X = n$, $\xi \to X$ be a flat (real) vector bundle with a (skew)symmetric form $\langle \bullet, \bullet\rangle_\varphi$ in the fiber $\mathbf{R}^m$, that is

$$\langle u, v\rangle_\varphi = \varepsilon\langle v, u\rangle_\varphi \in \mathbf{R}, \quad u, v \in \mathbf{R}^m, \quad \varepsilon = \pm 1,$$

and constant transition functions preserve this form. This means in particular that the form $\langle\bullet,\bullet\rangle_\varphi$ forms a bilinear (skew)symmetric map

$$\xi\oplus\xi\to\xi\otimes\xi\xrightarrow{\varphi}\overline{\mathbf{1}}.$$

Consider the de Rham complex of differential forms on the manifold X valued in the bundle ξ:

$$0\to\Omega^0(X;\xi)\xrightarrow{d}\Omega^1(X;\xi)\xrightarrow{d}\Omega^2(X;\xi)\xrightarrow{d}\cdots\xrightarrow{d}\Omega^n(X;\xi)\to 0.$$

The corresponding cohomology complex

$$0\to H^0(X;\xi)\xrightarrow{0}H^1(X;\xi)\xrightarrow{0}H^2(X;\xi)\xrightarrow{0}\cdots\xrightarrow{0}H^n(X;\xi)\to 0$$

possesses the Poincaré duality with respect to the pairing

$$\langle\omega_1,\omega_2\rangle\overset{def}{=}\int_X\varphi(\omega_1\wedge\omega_2),\qquad \omega_1,\omega_2\in H^*(X;\xi).$$

This form is nondegenerate and (skew)symmetric. Namely,

$$\langle\omega_2,\omega_1\rangle=\varepsilon(-1)^{\dim\omega_1\cdot\dim\omega_2}\langle\omega_1,\omega_2\rangle.$$

We intend to calculate the signature of the form in terms of characteristic classes.

For that let us consider an analogue of the scalar forms generated by a Riemannian metric $g=(g_{ij}(x))$ on X and a metric tensor $u=(u_{\alpha\beta}(x))$ in the bundle ξ. Fix a local coordinate system $(x^1,\dots,x^n)=(x^i)$ on X and a basis $(e_1,\dots,e_m)=(e_\alpha)$ in the fibers of the bundle ξ. Any differential form $\omega_1\in\Omega^1(X;\xi)$ can be represented as

$$\omega_1=({}_1f_i^\alpha(x)e_\alpha)dx^i.$$

If we have another form

$$\omega_2=({}_2f_i^\alpha(x)e_\alpha)dx^i,$$

the scalar product is defined as

$$(\omega_1,\omega_2)_{g,u}\overset{def}{=}\int_X(({}_1f_i^\alpha)({}_2f_j^\beta)u_{\alpha\beta}g^{ij}(x))\,d\mu,$$

where $d\mu$ is the differential form which determines the measure on X, associated with the Riemannian metric g_{ij},

$$d\mu\overset{def}{=}\sqrt{\det(g_{ij})}dx^1\wedge\cdots\wedge dx^n.$$

The scalar product $(\omega_1,\omega_2)_{g,u}$ is symmetric and nondegenerate.

The form $(\bullet,\bullet)_{g,u}$ can be extended to spaces of forms of other dimensions: let $\omega_1,\omega_2\in\Omega^k(X,\xi)$,

$$\omega_1=\sum_{i_1,i_2,\dots,i_k}\Big(\sum_\alpha{}_1f^\alpha_{i_1,i_2,\dots,i_k}(x)e_\alpha\Big)dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k},$$

$$\omega_2=\sum_{i_1,i_2,\dots,i_k}\Big(\sum_\alpha{}_2f^\alpha_{i_1,i_2,\dots,i_k}(x)e_\alpha\Big)dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_k}.$$

Then

$$(\omega_1,\omega_2)=\int\Big(\sum_{\substack{i_1,i_2,\dots,i_k\\ j_1,j_2,\dots,j_k}}\Big(\sum_{\alpha\beta}{}_1f^\alpha_{i_1,i_2,\dots,i_k}(x){}_2f^\beta_{j_1,j_2,\dots,j_k}(x)u_{\alpha\beta}\Big)g^{i_1j_1}g^{i_2j_2}\cdots g^{i_kj_k}\Big)d\mu.$$

Thus on the space $\Omega^*(X;\xi)=\bigoplus_{k=0}^{n}\Omega^k(X;\xi)$ one has two forms: $\langle\bullet,\bullet\rangle$ and $(\bullet,\bullet)_{g,u}$, both nondegenerate. Hence there is an invertible operator

$$\ast\ :\Omega^*(X;\xi)\to\Omega^*(X;\xi),\qquad \ast_k:\Omega^k(X;\xi)\to\Omega^{n-k}(X;\xi),$$

such that

$$\langle\omega_1,\omega_2\rangle=(\ast\,\omega_1,\omega_2)_{g,u}.$$

The metric tensor u can be chosen such that

(5) $$\ast_{n-k}\cdot\ast_k=\varepsilon(-1)^{(n-k)k}\mathbf{Id}.$$

Really, any metric tensor v in the bundle ξ defines the form

$$(\omega_1,\omega_2)_{g,v}=(A\omega_1,\omega_2)_{g,u}.$$

Putting $A=\sqrt{|\ast^2|}$, for the new operator with respect to the form v,

$$\langle\omega_1,\omega_2\rangle=(\ast_v\omega_1,\omega_2)_{g,v},$$

one has the relation

$$\ast_v=A^{-1}\ast,$$

that is,

$$(\ast_v)^2=|\ast^2|^{-1}\ast^2.$$

The condition (5) can be checked at each point.

Further, if $\omega_1\in\Omega^k$, $\omega_2\in\Omega^{n-k-1}$, then

$$\begin{aligned}(\ast_{k+1}(d_k\omega_1),\omega_2)&=\langle d_k\omega_1,\omega_2\rangle=(-1)^{k+1}\langle\omega_1,d_{n-k-1}\omega_2\rangle\\&=(-1)^{k+1}(\ast_k(\omega_1),d_{n-k-1}\omega_2)=(-1)^{k+1}(d^*_{n-k-1}\ast_k(\omega_1),\omega_2),\end{aligned}$$

that is,

$$\ast_{k+1}d_k=(-1)^{k+1}d^*_{n-k-1}\ast_k$$

or

$$\ast_{k+1}d_k\ast_{n-k}=\varepsilon(-1)^{k+1+(n-k)k}d^*_{n-k-1}$$

or

$$\delta_k=\varepsilon(-1)^{nk+1}\ast_{n-k+1}d_{n-k}\ast_k,$$

where

$$\delta_k=d^*_{k-1}:\Omega^k\to\Omega^{k-1}.$$

In other words

$$(-1)^{nk+1+(n-k+1)(k-1)}\ast_{k-1}\delta_k=d_{n-k}\ast_k,$$

or

(6) $$\ast_{k-1}\delta_k=(-1)^{(n-k)}d_{n-k}\ast_k,$$

and

(7) $$\ast_{k+1}d_k=(-1)^{(k+1)}\delta_{n-k}\ast_k.$$

Put $\alpha_k=(-1)^{\frac{k(k-1)}{2}}\ast_k$. Then

$$-(\alpha_{k+1}d_k+\ast_{k-1}\delta_k)=(d_{n-k}+\delta_{n-k})\alpha_k.$$

This means that on the space $\Omega^* = \bigoplus \Omega^k$ the relation

$$\ast\ (d+\delta) = -\,(d+\delta)\,\ast$$

holds. If $n = 4s$ and $\varepsilon = 1$, then $\alpha^2 = \mathbf{Id}$. If $n = 4s+2$ and $\varepsilon = -1$, then again $\alpha^2 = \mathbf{Id}$.

In both cases the signature of the quadratic form $\langle \bullet, \bullet \rangle_\varphi$ on cohomology, that is, on $\mathbf{Ker}(d+\delta) = \mathbf{Ker}\, d \cap \mathbf{Ker}\, \delta$ coincides with the index of the operator

$$d+\delta : \Omega^+ \to \Omega^-,$$

where Ω^+ and Ω^- are the eigenspaces of the operator α, which correspond to the eigenvalues 1 and -1.

2. Continuous family of finite dimensional representations. Let us consider a continuous family of representations

$$\rho_t : \pi \to U(N), \quad t \in T. \tag{8}$$

This family generates a family of quadratic forms on the family of the homology spaces

$$(H^{2k}(X, \rho_t), \cup).$$

The problem is to describe this family as a continuous family of quadratic forms. For this we need to include the family $\big(H^{2k}(X, \rho_t), \cup\big)$ into a larger space (see [Mis01]).

Given a combinatorial structure on X, let $C_k = C_k(X)$ denote the group of k-dimensional chains of X with coefficients in $\mathbf{C}^N$. Then the representations ρ_t define boundary homomorphisms d_k and the Poincaré duality homomorphisms D_k which are continuous with respect to $t \in T$:

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow{\scriptstyle D_0} & & \uparrow{\scriptstyle D_1} & & & & \uparrow{\scriptstyle D_n} \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array}$$

The properties are

$$d_{k-1} d_k = 0,$$

$$d_k D_k + (-1)^{k+1} D_{k-1} d_{n-k+1}^* = 0, \tag{9}$$

$$D_k = (-1)^{k(n-k)} D_{n-k}^*,$$

D induces an isomorphism in homology groups.

Put

$$F_k = i^{k(k-1)} D_k. \tag{10}$$

Then a similar diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow{\scriptstyle F_0} & & \uparrow{\scriptstyle F_1} & & & & \uparrow{\scriptstyle F_n} \\
C_n^* & \xleftarrow{d_n^*} & C_{n-1}^* & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C_0^*
\end{array} \tag{11}$$

has more natural properties

$$d_k F_k + F_{k-1} d_{n-k+1}^* = 0, \qquad F_k = F_{n-k}^*. \tag{12}$$

Consider the cone of F, which is an acyclic complex with respect to the total graduation and the sum of differentials d and F:

$$0 \longleftarrow A_0 \xleftarrow{H_1} A_1 \xleftarrow{H_2} \cdots \xleftarrow{H_{2l}} A_{2l} \xleftarrow{H_{2l+1}} A_{2l+1} \xleftarrow{H_{2l+2}} \cdots \xleftarrow{H_{4l}} A_{4l} \xleftarrow{H_{4l+1}} A_{4l+1} \longleftarrow 0 \tag{13}$$

where

$$A_k = C_k \oplus C^*_{n-k+1}, \qquad H_k = \begin{pmatrix} d_k & F_{k-1} \\ 0 & d^*_{n-k+2} \end{pmatrix}. \tag{14}$$

Put

$$A = \bigoplus_{k=0}^{n+1} A_k = A_{ev} \oplus A_{odd},$$

where

$$A_{ev} = \bigoplus_{k=0}^{2l} A_{2k}, \quad A_{odd} = \bigoplus_{k=0}^{2l} A_{2k+1}.$$

Then

$$A_{ev} \approx A_{odd} \approx \bigoplus_{k=0}^{n} C_k,$$

and

$$d + d^* + F : A_{ev} \to A_{odd}$$

is an isomorphism.

Taking in account that $d = d_t$, $F = F_t$ one has

THEOREM 1.

$$\mathbf{sign}(A_t) = \mathbf{sign}(X, \rho_t)$$

where $A_t = d_t + d^*_t + F_t$.

2.1. *New notion of signature for a continuous family* $A_t : V_t \to V_t,\ V_t \approx V$. There is a splitting

$$V = V_t^+ \oplus V_t^-,$$

such that A_t is positive on V_t^+ and negative on V_t^-. Then

$$\xi^+ = \coprod V_t^+, \qquad \xi^- = \coprod V_t^-$$

are subbundles. By definition

$$\mathbf{sign}(A_t) = [\xi^+] - [\xi_-] \in K(T).$$

Thus

$$\mathbf{sign}_{\rho_t}(X) = \mathbf{sign}(d_t + d^*_t + F_t) \in K(T).$$

We have a generalization of the Hirzebruch formula:

$$K(T) \ni \mathbf{sign}_{\rho_t} X = 2^k \langle L(X)\, \mathrm{ch}_X(\xi^{\rho_t}_{X\times T}), [X]\rangle \in K(T) \otimes Q, \tag{15}$$

where $\mathrm{ch}_X(\xi^{\rho_t}_{X\times T}) \in H^*(X; K^*(T) \otimes Q)$.

3. Algebraic setting. The most general picture for the Hirzebruch formula for oriented smooth manifolds can be represented as follows (see [Mis70], [Mis95]).

Let $\Omega_*(B\pi)$ denote the bordism group of pairs (M, f_M). Recall that $\Omega_*(B\pi)$ is a module over the ring $\Omega_* = \Omega_*(\{\mathrm{pt}\})$. One can construct a homomorphism

$$\mathbf{sign} : \Omega_*(B\pi) \to L_*(\mathbf{C}\pi) \tag{16}$$

which to every manifold (M, f_M) assigns the element $\mathbf{sign}(M) \in L_*(\mathbf{C}\pi)$, where $L_*(\mathbf{C}\pi)$ is the Wall group for the group ring $\mathbf{C}\pi$.

3.1. *Algebraic construction of symmetric signature* For combinatorial manifolds we have a similar combinatorial diagram

$$\begin{array}{ccccccc}
C_0 & \xleftarrow{d_1} & C_1 & \xleftarrow{d_2} & \cdots & \xleftarrow{d_n} & C_n \\
\uparrow{\scriptstyle D_0} & & \uparrow{\scriptstyle D_1} & & & & \uparrow{\scriptstyle D_n} \\
C^n & \xleftarrow{d_n^*} & C^{n-1} & \xleftarrow{d_{n-1}^*} & \cdots & \xleftarrow{d_1^*} & C^0
\end{array}$$

where

$$C_k \overset{def}{=} C_k(\widetilde{X}), \quad C^k \overset{def}{=} C_0^k(\widetilde{X}) \approx Hom_{\mathbf{C}\pi}(C_k, \mathbf{C}\pi).$$

Here $C_k(\widetilde{X})$ means the chain complex of the universal covering $\widetilde{X}$ with respect to the combinatorial structure of X, $C_0^k(\widetilde{X})$ means the cochain complex with compact supports, $\mathbf{C}\pi$ is the group ring of the fundamental group π with coefficients in the field $\mathbf{C}$ of rational, real or complex numbers.

The homomorphism $\mathbf{sign}$ satisfies the following conditions:

(a) $\mathbf{sign}(M)$ depends only on the homotopy equivalence class of the manifold M.

(b) If N is a simply connected manifold and $\tau(N)$ is its signature then

$$\mathbf{sign}(M \times N) = \mathbf{sign}(M)\tau(N) \in L_*(\mathbf{C}\pi). \tag{17}$$

We shall be interested only in the groups after tensor multiplication with the field of rational numbers $\mathbf{Q}$, in other words in the homomorphism

$$\mathbf{sign} : \Omega_*(B\pi) \otimes \mathbf{Q} \to L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

One has

$$\Omega_*(B\pi) \otimes \mathbf{Q} \approx H_*(B\pi; \mathbf{Q}) \otimes \Omega_* \approx \Omega_*^{frame}(B\pi) \otimes \mathbf{Q} \otimes \Omega_*$$

where $\Omega_*^{frame}(B\pi)$ is the so called "framed" bordisms, that is, bordisms which are represented by manifolds with trivial normal bundles. Therefore the homomorphism $\mathbf{sign}$ can be considered as a product of two homomorphisms

$$\sigma \otimes \tau : H_*(B\pi; \mathbf{Q}) \otimes \Omega_* \approx \Omega_*^{frame}(B\pi) \otimes \mathbf{Q} \otimes \Omega_* \to L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

Here

$$\sigma : H_*(B\pi; \mathbf{Q}) \approx \Omega_*^{frame}(B\pi) \otimes \mathbf{Q} \to L_*(\mathbf{C}\pi) \otimes \mathbf{Q}.$$

is the restriction of σ to $\Omega_*^{frame}(B\pi) \otimes \mathbf{Q} \subset \Omega_*(B\pi) \otimes \mathbf{Q}$. On the other hand the homomorphism $\mathbf{sign}$ represents the cohomology class

$$\overline{\sigma} \in H^*(B\pi; L_*(\mathbf{C}\pi) \otimes \mathbf{Q}) = \mathbf{Hom}\ (H_*(B\pi; \mathbf{Q}), L_*(\mathbf{C}\pi) \otimes \mathbf{Q})$$

such that if M is a framed manifold, $M \in \Omega_*^{frame}(B\pi)$, then

$$\mathbf{sign}(M) = \sigma(M) = \langle f_M^*(\overline{\sigma}), [M] \rangle.$$

The key idea is that for any manifold (M, f_M) the signature can be represented by a version of the general Hirzebruch formula

$$\mathbf{sign}(M, f_M) = \langle L(M) f_M^*(\overline{\sigma}),\, [M] \rangle \in L_*(\mathbf{C}\pi) \otimes \mathbf{Q}. \tag{18}$$

Indeed, let $M = M_1 \times M_2$, $M_1 \in \Omega_*^{frame}(B\pi)$, $M_2 \in \Omega_*$. Then

$$\begin{aligned}\mathbf{sign}(M) = \mathbf{sign}(M_1)\tau(M_2) &= \langle f_M^*(\overline{\sigma}), [M_1] \rangle \langle L(M_2), [M_2] \rangle \\ &= \langle L(M_1 \times M_2) f_M^*(\overline{\sigma}), [M_1 \times M_2] \rangle = \langle L(M) f_M^*(\overline{\sigma}), [M] \rangle.\end{aligned}$$

3.2. *Higher signature.* Let $x \in H^*(B\pi; \mathbf{Q})$ be an arbitrary cohomology class. Then the number

$$\mathbf{sign}_x(M, f_M) = \langle L(M) f_M^*(x), [M] \rangle \in \mathbf{Q}$$

is called the higher signature due to S. P. Novikov. In the case of an additive functional $\alpha : L_*(\mathbf{C}\pi) \otimes \mathbf{Q} \to \mathbf{Q}$ the higher signature $\mathbf{sign}_x(M, f_M)$, where $x = \alpha(\overline{\sigma}) \in H^*(B\pi; \mathbf{Q})$ arises from the Hirzebruch formula (18) above.

This gives a description of the family of all homotopy-invariant higher signatures. This observation is not a direct consequence from theprevious argument but can be obtained from the theory of non-simply connected surgeries by [Wal71]. Indeed, following [Wal71] one can construct an obstruction to the existence of surgery of normal mapping to a homotopy equivalence which is described as a difference of symmetric signatures (16) of two manifolds. Therefore all other higher signatures behind $x = \alpha(\overline{\sigma})$ are not homotopy invariant.

4. Functional version of the Hirzebruch formula. Infinite dimensional representations. Let $C^*[\pi]$ be the C^*-group algebra of the group π. Any unitary representation

$$\rho : \pi \to \mathbf{End}(\mathbf{H})$$

of the group π, where $\mathbf{H}$ is a Hilbert space, can be uniquely extended to a representation

$$\overline{\rho} : C^*[\pi] \to \mathbf{End}(\mathbf{H})$$

of the algebra $C^*[\pi]$. Put $A = \mathbf{Im}\,\overline{\rho} \subset \mathbf{End}(\mathbf{H})$, $\overline{\rho} : C^*[\pi] \to A$.

By ξ^ρ we denote the vector bundle over $B\pi$ with the fiber A, whose transition functions are induced by the action of the group π on the algebra A by the representation ρ. The vector bundle ξ^ρ generates the element of the K-group

$$\xi^\rho \in K_A(B\pi).$$

There is a generalization of the Chern character to vector bundles over the C^*-algebra A: $\mathrm{ch}_A\, \xi \in H^*(X; K_A(\mathrm{pt}) \otimes Q)$.

Hence we can write the right side of the formula (4):

$$? = 2^k \langle L(X)\, \mathrm{ch}_A\, \xi_X^\rho, [X] \rangle \in K_A(\mathrm{pt}) \otimes Q.$$

The left side of the formula can be calculated as a symmetric signature of the manifold X by replacing of rings, induced by the representation ρ, so we obtain the so called generalized Hirzebruch formula for an arbitrary C^*-algebra A:

$$K_A(\mathrm{pt}) \ni \mathbf{sign}_\rho(X) = 2^{2k}\,\langle L(X)\,\mathrm{ch}_A\,\xi^\rho_X, [X]\rangle \in K_A(\mathrm{pt}) \otimes Q.$$

5. Smooth version of the Hirzebruch formula. The left side of the Hirzebruch formula (4) is described in terms of the combinatorial structure of the manifold X. There is a smooth version of this expression as well. Namely, consider the de Rham complex of differential forms on the manifold X with values in the flat vector bundle ξ^ρ:

$$0 \to \Omega_0(X,\xi^\rho) \xrightarrow{d} \Omega_1(X,\xi^\rho) \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{4k}(X,\xi^\rho) \to 0. \tag{19}$$

It is well known that the cohomology groups of the de Rham complex (19) are isomorphic to the cohomology groups $H^*(X,\xi^\rho)$.

The $\cup$-product is induced by exterior product of differential forms, so the Hermitian form which defines the Poincaré duality can be determined by

$$\langle\omega_1,\omega_2\rangle = \int_X \omega_1 \wedge \omega_2. \tag{20}$$

On the other hand using a Riemannian metric on the manifold X, (ω_1,ω_2), the Poincaré duality (20) can be determined with a bounded operator $*$:

$$(\omega_1,\omega_2) = \int_X \omega_1 \wedge *\omega_2,$$

where

$$* : \Omega_k(X) \to \Omega_{n-k}(X).$$

Put

$$\alpha = i^{k(k+1)} * .$$

Then

$$\alpha d\alpha = -d^*;\ \alpha^2 = 1.$$

Let

$$\Omega^+(X) = \mathbf{Ker}\ (\alpha - 1);\ \Omega^-(\alpha + 1).$$

It is evident that

$$(d + d^*)(\Omega^+(X)) \subset \Omega^-(X).$$

Consider the elliptic operator

$$D = (d + d^*) : \Omega^+(X) \to \Omega^-(X).$$

Then we have

$$\mathbf{index}\,D = \mathbf{sign}(X).$$

Using the Atiyah–Singer index formula for elliptic operators we have

$$\mathbf{index}(D \otimes \xi) = 2^k\langle L(X)\,\mathrm{ch}\,\xi, [X]\rangle. \tag{21}$$

for arbitrary vector bundle ξ over the manifold X.

If the bundle ξ is flat, that is, if there is a representation ρ such that $\xi = \xi^\rho$ then

$$\mathbf{index}\,(D \otimes \xi) = \mathbf{sign}_\rho(X)$$

and we again obtain the Hirzebruch formula:

$$\mathbf{sign}_\rho(X) = \mathbf{index}(D \otimes \xi^\rho) = 2^{2k}\langle L(X)\,\mathrm{ch}\,\xi^\rho, [X]\rangle.$$

6. The notion of almost flat vector bundle. Combinatorial local Hirzebruch formula. A naive point of view is that all transition functions $\varphi_{\alpha\beta}(x), x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$ for a vector bundle ξ are almost constant. Then one can construct a so called almost algebraic Poincaré complex of formal dimension n which consists of chains and cochains with values in fibers of the bundle ξ:

$$\begin{tikzcd} C_0(\xi) & C_1(\xi) \arrow[l, "d_1"'] & \cdots \arrow[l, "d_2"'] & C_n(\xi) \arrow[l, "d_n"'] \\ C^n(\xi) \arrow[u, "D"'] & C^{n-1}(\xi) \arrow[l, "d_n^*"'] \arrow[u, "D"'] & \cdots \arrow[l, "d_{n-1}^*"'] & C^0(\xi) \arrow[l, "d_1^*"'] \arrow[u, "D"'] \end{tikzcd}$$

such that

$$\|d^2\| \le \varepsilon, \quad \|Dd^* \pm dD\| \le \varepsilon$$
$$\|D\| \le \mathbf{const}, \quad D^* = \pm D.$$

If ε is sufficiently small and the number of neighbors for each cell is bounded then the operator

$$d + d^* + D : C_*(\xi) \to C_*(\xi)$$

is invertible and a version of the Hirzebruch formula (local combinatorial Hirzebruch formula) holds:

$$\mathbf{sign}\, C_*(X, \xi) = 2^{2k}\left\langle L(X)\,\mathrm{ch}\,\xi, [X]\right\rangle.$$

If moreover the size of all cells is sufficiently large then we come to the notion of almost flat bundle for which the signature $\mathbf{sign}\, C_*(X, \xi)$ is homotopy invariant (see [Gro95], [Mis99]).

7. Almost flat bundles from the point of view of C*-algebras (jointly with N. Teleman). Connes, Gromov and Moscovici [CGM90] showed that for any almost flat bundle α over the manifold M, the index of the signature operator with values in α is a homotopy invariant of M. It follows that a certain integer multiple n of the bundle α comes from the classifying space $B\pi_1(M)$. Geometric arguments show that the bundle α itself, and not necessarily a certain multiple of it, comes from an arbitrarily large compact subspace $Y \subset B\pi_1(M)$ through the classifying mapping.

For this we modify the notion of almost flat structure on bundles over smooth manifolds and extend this notion to bundles over arbitrary CW-spaces using quasi-connections of N. Teleman ([Tel04]).

Using a natural construction by B. Hanke and T. Schick ([HS04]), one can present a simple description of such bundles as a bundle over a C^*-algebra and clarify the homotopy invariance of corresponding higher signatures.

Moreover it is possible to construct a so called classifying space for almost flat bundles (see [MT05a], [MT05b]).

7.1. *Description of almost flat bundles in terms of C^*-algebras due to B. Hanke and T. Schick.* Due to [CGM90] an element $\alpha \in K(M)$ over a smooth manifold M is called an almost flat bundle if for any $\varepsilon > 0$ there are two vector bundles ξ, η with linear connections ∇^{ξ}, ∇^{η} such that:

1. $\alpha = \xi - \eta \in K(M)$,
2. $\|\Theta^{\xi}\| < \varepsilon$, $\|\Theta^{\eta}\| < \varepsilon$, where

$$\|\Theta\| = \sup_{x\in M}\{\|\Theta_x(X\wedge Y)\| : \|X\wedge Y\| \leq 1\}, \tag{22}$$

and $\Theta_x(X\wedge Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature form of the connection ∇.

If α is an almost flat bundle and β is a trivial bundle then $\alpha \oplus \beta$ is also an almost flat bundle. This means that without loss of generality one can consider elements from K which are represented by real vector bundles.

In other words we can consider two sequences of vector bundles $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed connections ∇^1_k and ∇^2_k such that

$$\xi_k = \eta_k \oplus \alpha, \tag{23}$$

$\dim \alpha = d$, $\dim \xi_k = n_k$, $\dim \eta_n = m_k = n_k - d$.

Assume that

$$\lim_{k\to\infty} \|\Theta^i_k\| = 0, \quad i = 1, 2. \tag{24}$$

So instead of a bundle α we shall consider a finer structure namely so called almost flat bundle structure which consists of the following:

1. Sequences of bundles $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed connections ∇^1_k and ∇^2_k such that

$$\lim_{k\to\infty} \|\Theta^i_k\| = 0, \quad i = 1, 2, \tag{25}$$

2. A sequence of isomorphisms

$$f_k : \xi_k \approx \eta_k \oplus \alpha. \tag{26}$$

The same bundle α may admit several almost flat bundle structures. We say that two almost flat bundle structures

$$\mathcal{P} = \{\alpha; \xi = \{\xi_k, \nabla^1_k\}; \eta = \{\eta_k, \nabla^2_k\}, f = \{f_k\}\}$$

and

$$'\mathcal{P} = \{'\alpha;\ '\xi = \{'\xi_k, '\nabla^1_k\};\ '\eta = \{'\eta_k, '\nabla^2_k\};\ 'f = \{'f_k\}\},$$

are equivalent if one structure can be obtained from the other structure by a sequence of the following operations:

1. Passing to a subsequence;
2. Homotopy of linear connections ∇^1_k and ∇^2_k in the class of connections which satisfy the conditions (25), and homotopy of isomorphisms $f = \{f_k\}$;
3. Stabilization of all bundles, that is, adding trivial bundles both to $\xi = \{\xi_k\}$, $\eta = \{\eta_k\}$, and to α with natural extension of the connections ∇^1_k and ∇^2_k and isomorphisms $f = \{f_k\}$ to direct sums.

An equivalence class of almost flat bundle structures is called an almost flat bundle on the manifold M. Among almost flat bundles a trivial bundle is marked out which is represented by the trivial almost flat bundle structure

$$\mathcal{P}^0 = \{\alpha^0; \xi^0 = \{\xi_k^0, \nabla_k^{1,0}\}; \eta^0 = \{\eta_k^0, \nabla_k^{2,0}\}, f^0 = \{f_k^0\}\},$$

where all bundles α^0, ξ_k^0, η_k^0 and connections $\nabla_k^{1,0}$, $\nabla_k^{2,0}$ are trivial and the isomorphisms f_k^0 are identical. The set of all almost flat bundles is endowed with the operation of direct sum $\mathcal{P} \oplus' \mathcal{P}$, the trivial bundle being the neutral element. So the set $\mathcal{V}ect_{af}(M)$ of equivalence classes of almost flat structures forms a semigroup with respect to the direct sum. The Grothendieck group is denoted by $\mathcal{K}_{af}(M)$.

7.2. *Classifying space for almost flat bundles.* The groups $\mathcal{V}ect_{af}(M)$ yield a functor from the category of CW-spaces to the category of abelian groups. Therefore the natural question arises about existing of classifying space for almost flat bundles. This means that there should be a space $\mathbf{BAF}$ with fixed almost flat bundle $\xi^B = \{\xi_k^B\}$, $\eta^B = \{\eta_k^B\}$, $f_k^B : \xi_k^B \approx \eta_k^B \oplus \alpha^B$ such that any almost flat bundle $\xi = \{\xi_k\}$, $\eta = \{\eta_k\}$, $f_k : \xi_k \approx \eta_k \oplus \alpha$ over M can be constructed by a continuous map $\varphi : M \to \mathbf{BAF}$ up to homotopy.

First of all, consider a description of almost flat bundles due to [HS04] using special structural groups.

Put $\mathcal{A} = \prod_{i=1}^{\infty} \mathcal{K}^+$, where $\mathcal{K}^+ = \mathcal{K}^+(H) = \mathbf{C} \oplus \mathcal{K}(H)$ is the algebra of compact operators with an adjoined identical operator in the Hilbert space H. The algebra $\mathcal{A}$ is a unital subalgebra in the algebra of all bounded operators $B(\mathbf{H})$, $\mathbf{H} = \bigoplus_{i=1}^{\infty} H$.

Consider a sequence of bundles $\xi = \{\xi_k\}$, $\dim \xi_k = n_k$, and fix coordinate skew products on the bundles by transition functions

$$\varphi_{ij}^k(x) \in \mathbf{GL}(n_k, \mathbf{C}), \quad x \in U_{ij}$$

for a chart atlas U_i.

Consider the canonical imbedding of $\mathbf{GL}(n_k, \mathbf{C})$ in the group $\mathbf{G}(\mathcal{K}^+)$ of invertible elements, by assigning each matrix $\Phi \in \mathbf{GL}(n_k, \mathbf{C})$ to its diagonal extension in the Hilbert space $H = C^{n_k} \oplus (C^{n_k})^{\perp}$ as

$$\widetilde{\Phi} = \begin{pmatrix} \Phi & 0 \\ 0 & \mathbf{Id} \end{pmatrix} \in \mathcal{K}^+(H). \tag{27}$$

Then the sequence $\varphi_{ij}^k(x)$ defines the function

$$\Phi_{ij}(x) \stackrel{def}{=} \prod_{k=1}^{\infty} \widetilde{\varphi}_{ij}^k(x) = \begin{pmatrix} \widetilde{\varphi}_{ij}^1(x) & 0 & 0 & 0 & \cdots \\ 0 & \widetilde{\varphi}_{ij}^2(x) & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & \cdots \\ 0 & 0 & 0 & \widetilde{\varphi}_{ij}^k(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in \mathbf{G}(\mathcal{A}). \tag{28}$$

Consider the quotient algebra $\mathcal{Q} \stackrel{def}{=} \mathcal{A}/\mathcal{A}_0$, where $\mathcal{A}_0 \subset \mathcal{A}$ consists of elements $x \in \mathcal{A}$, $x = \{x_i \in \mathcal{K}^+\}$, such that

$$\lim_{i\to\infty} \|x_i\| = 0. \tag{29}$$

Let $\overline{\Phi}_{ij}(x) \in \mathcal{Q}$ be the image of the element $\Phi_{ij}(x)$. Then $\{\overline{\Phi}_{ij}(x)\}$ is a family of transition functions for a bundle $\xi_{\mathcal{Q}}$ over the algebra $\mathcal{Q}$.

If the functions $\varphi_{ij}^k(x)$ satisfy the condition of almost flat structure, that is,

$$\lim_{k\to\infty} \sup_{x,y\in U_{ij}} \|\varphi_{ij}^k(x) - \varphi_{ij}^k(y)\| = 0, \tag{30}$$

then the functions $\overline{\Phi}_{ij}(x)$ don't depend on the argument $x \in X$. This means that the bundle $\xi_{\mathcal{Q}}$ is flat.

To an individual bundle α one can assign the bundle $\alpha_{\mathcal{Q}}$ over $\mathcal{Q}$, by defining the sequence $\widetilde{\alpha} = \{\alpha_k\}$, $\alpha_k = \alpha$, and $\alpha_{\mathcal{Q}} \overset{def}{=} \widetilde{\alpha}_{\mathcal{Q}}$.

Now, consider an almost flat bundle, that is, sequences $\xi = \{\xi_k\}$ and $\eta = \{\eta_k\}$ with fixed transition functions

$$\varphi_{ij}^{k,s}(x), \quad x \in U_{ij}, \quad s = 1, 2,$$

for a chart atlas U_i such that

$$\lim_{k\to\infty} \sup_{x,y\in U_{ij}} \|\varphi_{ij}^{k,s}(x) - \varphi_{ij}^{k,s}(y)\| = 0, \quad s = 1, 2, \tag{31}$$

and a sequence of isomorphisms

$$f_k : \xi_k \approx \eta_k \oplus \alpha. \tag{32}$$

Let $\varphi_{ij}^{k,1}(x) \in \mathbf{GL}(n_k, \mathbf{C})$, $\varphi_{ij}^{k,2}(x) \in \mathbf{GL}(m_k, \mathbf{C})$, $n_k = m_k + d$. Let the bundle α be defined by the transition functions $\psi_{ij}(x) \in \mathbf{GL}(d, \mathbf{C})$. Then the direct sum $\zeta_k = \eta_k \oplus \alpha$ has transition functions $\chi_{ij}(x) \in \mathbf{GL}(n_k, \mathbf{C})$ of the following form

$$\chi_{ij}(x) = \begin{pmatrix} \varphi_{ij}^{k,2}(x) & 0 \\ 0 & \psi_{ij}(x) \end{pmatrix}. \tag{33}$$

Turning to the algebra $\mathcal{Q}$ one has three bundles $\xi_{\mathcal{Q}}$, $\eta_{\mathcal{Q}}$ and $\zeta_{\mathcal{Q}}$ and the isomorphism $\overline{F} : \xi_{\mathcal{Q}} \to \zeta_{\mathcal{Q}}$, generated by the sequence (32). On the other hand the bundle $\zeta_{\mathcal{Q}}$ is isomorphic to the direct sum of $\eta_{\mathcal{Q}}$ and $\alpha_{\mathcal{Q}}$,

$$\xi_{\mathcal{Q}} \overset{\overline{F}}{\approx} \zeta_{\mathcal{Q}} \approx \eta_{\mathcal{Q}} \oplus \alpha_{\mathcal{Q}}. \tag{34}$$

The second bundle is defined by the transition functions $\overline{\Psi}_{ij}(x)$,

$$\Psi_{ij}(x) = \prod_{k=1}^{\infty} \widetilde{\psi}_{ij}(x) = \begin{pmatrix} \widetilde{\psi}_{ij}(x) & 0 & 0 & 0 & \cdots \\ 0 & \widetilde{\psi}_{ij}(x) & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & \cdots \\ 0 & 0 & 0 & \widetilde{\psi}_{ij}(x) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{35}$$

7.3. *Simple proof of homotopy invariance of higher signatures of almost flat bundles.* Using the description above one can obtain a simple and elegant proof of the homotopy invariance of the higher signature of an almost flat bundle, which was first established by A. Connes, M. Gromov and H. Moscovici ([CGM90]).

Namely, if α is an almost flat bundle then the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ in (34) are flat bundles over the algebra $\mathcal{Q}$. Since the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ are flat, the following higher signatures are homotopy invariant:

$$\begin{aligned}\mathbf{sign}_x(M) &\in \mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q},\\ \mathbf{sign}_y(M) &\in \mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q},\\ x=\mathrm{ch}_{\mathcal{Q}}(\xi_{\mathcal{Q}}) &\in H^*(M;\mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q}),\\ y=\mathrm{ch}_{\mathcal{Q}}(\eta_{\mathcal{Q}}) &\in H^*(M;\mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q}).\end{aligned}\tag{36}$$

Hence the higher signature

$$\begin{aligned}\mathbf{sign}_z(M) &\in \mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q},\\ z=\mathrm{ch}_{\mathcal{Q}}(\alpha_{\mathcal{Q}}) &\in H^*(M;\mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})\otimes\mathbf{Q}),\end{aligned}\tag{37}$$

is also homotopy invariant.

Consider the homomorphism

$$\theta:\mathbf{K}^*(M)\to\mathbf{K}^*_{\mathcal{Q}}(M),\tag{38}$$

which associates to a finite dimensional bundle α the bundle $\alpha_{\mathcal{Q}}$ over the algebra $\mathcal{Q}$. In the paper ([HS04, Proposition 3.5]) it was shown that

$$\mathbf{K}^*_{\mathcal{A}}(\mathbf{pt})=\prod_{k=1}^{\infty}\mathbf{Z},\qquad \mathbf{K}^*_{\mathcal{Q}}(\mathbf{pt})=\Big(\prod_{k=1}^{\infty}\mathbf{Z}\Big)/\mathbf{J},\tag{39}$$

where $\mathbf{J}=\sum_{k=1}^{\infty}\mathbf{Z}$. Similarly one can show that the homomorphism (38) for $M=\mathbf{pt}$ maps the group $\mathbf{K}^*(\mathbf{pt})=\mathbf{Z}$ by the formula

$$\theta(a)=\Big[\prod_{k=1}^{\infty}a\Big]\in\Big(\prod_{k=1}^{\infty}\mathbf{Z}\Big)/\mathbf{J}.\tag{40}$$

Hence the homomorphism

$$\theta\otimes\mathbf{Q}:\mathbf{K}^*(M)\otimes\mathbf{Q}\to\mathbf{K}^*_{\mathcal{Q}}(M)\otimes\mathbf{Q}\tag{41}$$

is a monomorphism. This completes the proof of the homotopy invariance of higher signatures for almost flat bundles since

$$\mathbf{sign}_z(M)=\theta(\mathbf{sign}_u(M)),\tag{42}$$

where $u=\mathrm{ch}(\alpha)$.

7.4. *Geometric construction of classifying space for almost flat bundles.* Almost flat bundles (without fixed almost flat structure) are unlikely to be constructed using a classifying space since they do not form a homotopy functor. On the other hand, almost flat bundles with fixed almost flat structure can be represented by the inverse image of a continuous map into a classifying space. Actually, it was shown above that with each almost flat structure there is associated a pair of flat bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ over the algebra $\mathcal{Q}$ and an isomorphism of $\mathcal{Q}$-bundles

$$F:\xi_{\mathcal{Q}}\to(\eta_{\mathcal{Q}}\oplus\alpha_{\mathcal{Q}})\,.\tag{43}$$

The presence of flat bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ and the isomorphism F can be interpreted using maps into classifying spaces. Namely, the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ according to the definition have the structural group $\mathbf{G}(\mathcal{Q})$, which consists of all invertible elements of the algebra $\mathcal{Q}$. In other words, we can consider the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ as one-dimensional bundles over the algebra $\mathcal{Q}$, with the fiber $\mathcal{Q}$. Therefore the bundles are classified by continuous maps from the base M into the classifying space $B\mathbf{G}(\mathcal{Q})$. Since the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ are flat, we should take as the classifying space not $BG(\mathcal{Q})$, but $B\widehat{\mathbf{G}(\mathcal{Q})}$, where $\widehat{\mathbf{G}(\mathcal{Q})}$ is the same group $\mathbf{G}(\mathcal{Q})$ with discrete topology. The identity map $\iota : \widehat{\mathbf{G}(\mathcal{Q})} \to G(\mathcal{Q})$ is continuous, which induces the natural continuous map of classifying spaces

$$B_\iota : B\widehat{\mathbf{G}(\mathcal{Q})} \to B\mathbf{G}(\mathcal{Q}). \tag{44}$$

Let

$$\oplus : B\mathbf{G}(\mathcal{Q}) \times B\mathbf{G}(\mathcal{Q}) \to B\mathbf{G}(\mathcal{Q}) \tag{45}$$

be the map that corresponds to direct sum of bundles. Then the bundles $\xi_{\mathcal{Q}}$ and $\eta_{\mathcal{Q}}$ can be represented as inverse images of the canonical flat $\mathcal{Q}$-bundle $\widehat{E_{\mathcal{Q}}}$ over the space $B\widehat{\mathbf{G}(\mathcal{Q})}$ by continuous maps

$$f_\xi, f_\eta : M \to B\widehat{\mathbf{G}(\mathcal{Q})}, \tag{46}$$

$$\xi_{\mathcal{Q}} = f_\xi^*(\widehat{E_{\mathcal{Q}}}), \qquad \eta_{\mathcal{Q}} = f_\eta^*(\widehat{E_{\mathcal{Q}}}). \tag{47}$$

Since the bundle $\widehat{E_{\mathcal{Q}}}$ is the inverse image of the canonical bundle $E_{\mathcal{Q}}$ over $B\mathbf{G}(\mathcal{Q})$ with respect to the map B_ι,

$$\widehat{E_{\mathcal{Q}}} = B_\iota^*(E_{\mathcal{Q}}), \tag{48}$$

the presence of the isomorphism (43) means that the maps

$$M \xrightarrow{f_\xi} B\widehat{\mathbf{G}(\mathcal{Q})} \xrightarrow{B_\iota} B\mathbf{G}(\mathcal{Q}) \tag{49}$$

and

$$M \xrightarrow{f_\eta \times f_\alpha} B\widehat{\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+) \xrightarrow{B_\iota \times \theta} B\mathbf{G}(\mathcal{Q}) \times B\mathbf{G}(\mathcal{Q}) \xrightarrow{\oplus} B\mathbf{G}(\mathcal{Q}) \tag{50}$$

are homotopic.

The classifying space $B\mathbf{G}(\mathcal{Q})$ is determined uniquely up to homotopy equivalence. Therefore it is appropriate to replace the space $B\mathbf{G}(\mathcal{Q})$ by the homotopy equivalent space that is the union of two mapping cylinders

$$\begin{aligned} \widetilde{B\mathbf{G}(\mathcal{Q})} \stackrel{def}{=} &\ (B\mathbf{G}(\mathcal{Q}) \cup_{(B_\iota,0)} ([0,1] \times B\widehat{\mathbf{G}(\mathcal{Q})})) \\ &\cup_{(\oplus\cdot(B_\iota\times\theta),0)}([0,1] \times (B\widehat{\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+))), \end{aligned} \tag{51}$$

and replace the maps B_ι and $\oplus \cdot (B_\iota \times \theta)$ by the imbeddings

$$\begin{aligned} i_0 &= (\mathbf{Id}, 1) : B\widehat{\mathbf{G}(\mathcal{Q})} \hookrightarrow ((0,1] \times B\widehat{\mathbf{G}(\mathcal{Q})}) \subset \widetilde{B\mathbf{G}(\mathcal{Q})}, \\ i_1 &= (\mathbf{Id}, 1) : (B\widehat{\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+)) \hookrightarrow \\ &\qquad ((0,1] \times (B\widehat{\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+))) \subset \widetilde{B\mathbf{G}(\mathcal{Q})}. \end{aligned} \tag{52}$$

Consider the space of continuous paths

$$\mathbf{BAF} \stackrel{def}{=} \Gamma(B\widehat{\mathbf{G}(\mathcal{Q})}; \widetilde{B\mathbf{G}(\mathcal{Q})}; (B\widehat{\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+))), \tag{53}$$

that consists of all continuous paths

$$\gamma : [0,1] \to \widetilde{B\mathbf{G}(\mathcal{Q})}, \tag{54}$$

that start in the subspace $\widehat{B\mathbf{G}(\mathcal{Q})}$, that is,

$$\gamma(0) \in \widehat{B\mathbf{G}(\mathcal{Q})} = \mathbf{Im}\, i_0, \tag{55}$$

and finish in the subspace $(\widehat{B\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+))$, that is,

$$\gamma(1) \in (\widehat{B\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+)) = \mathbf{Im}\, i_1. \tag{56}$$

Each bundle α, endowed with almost flat structure corresponds to a continuous map from M to the space $\mathbf{BAF}$. Indeed, since the maps (49) are (50) homotopic, the compositions $i_0 \cdot f_\xi$ and $i_1 \cdot (f_\eta \times f_\alpha)$ are also homotopic. It follows that there is a continuous map

$$\Phi : M \times [0,1] \to \widetilde{B\mathbf{G}(\mathcal{Q})}, \tag{57}$$

for which one has

$$\Phi(x,0) = i_0 f_\xi(x), \quad \Phi(x,1) = i_1(f_\eta \times f_\alpha)(x). \tag{58}$$

This means that the map Φ induces a map

$$\widetilde{\Phi} : M \to \mathbf{BAF} \tag{59}$$

by the formula

$$\widetilde{\Phi}(x)(t) = \Phi(x,t), \quad x \in M; \quad t \in [0,1]. \tag{60}$$

Conversely, the same formula (60) defines two homotopic maps

$$\begin{aligned} &\Phi(x,0) : M \to \widehat{B\mathbf{G}(\mathcal{Q})}, \\ &\Phi(x,1) : M \to (\widehat{B\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+)), \end{aligned} \tag{61}$$

which in turn define a flat structure on the bundle generated by the mapping of the second component of the map $\Phi(x,1)$.

Moreover, on the space $\mathbf{BAF}$ there is a bundle, the inverse image of which under the mapping (59) coincides with the flat bundle described above.

Now we can describe homotopy properties of the space $\mathbf{BAF}$ using standard Serre fibrations. Let us denote by

$$\begin{aligned} p_0 &: \mathbf{BAF} \to \widehat{B\mathbf{G}(\mathcal{Q})}, \\ p_1 = (p_1' \times p_1'') &: \mathbf{BAF} \to (\widehat{B\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+)) \end{aligned} \tag{62}$$

two standard maps that take each path $\gamma \in \mathbf{BAF}$ to its initial or final points with respect to the conditions (55) and (56). The combined map

$$p = (p_0 \times p_1) : \mathbf{BAF} \to \widehat{B\mathbf{G}(\mathcal{Q})} \times (\widehat{B\mathbf{G}(\mathcal{Q})} \times B\mathbf{G}(\mathcal{K}^+)) \tag{63}$$

is the Serre fibration whose fiber is the loop space $V = \Omega(\widetilde{B\mathbf{G}(\mathcal{Q})}) \approx \Omega(B\mathbf{G}(\mathcal{Q})) \approx \mathbf{G}(\mathcal{Q})$.

On the other hand consider another Serre fibration

$$\widetilde{p} = (p_0 \times p_1') : \mathbf{BAF} \xrightarrow{\Gamma_0} \widehat{B\mathbf{G}(\mathcal{Q})} \times \widehat{B\mathbf{G}(\mathcal{Q})}, \tag{64}$$

whose fiber is the space $\Gamma_0 = \Gamma(x_0; \widetilde{B\mathbf{G}(\mathcal{Q})}; B\mathbf{G}(\mathcal{K}^+))$ of paths in the space $\widetilde{B\mathbf{G}(\mathcal{Q})} \approx B\mathbf{G}(\mathcal{Q})$, that start at a fixed point x_0 and finish in the space $B\mathbf{G}(\mathcal{K}^+)) \subset \widetilde{B\mathbf{G}(\mathcal{Q})}$. The space Γ_0 is foliated by the projection

$$p'_1 : \Gamma_0 \xrightarrow{V} B\mathbf{G}(\mathcal{K}^+), \tag{65}$$

the fiber being the space of loops $V = \Omega(B\mathbf{G}(\mathcal{Q})) \approx \mathbf{G}(\mathcal{Q}) \overset{i_0}{\subset} \Gamma_0$.

LEMMA 1. *The projection (65) is homotopic to the constant mapping.*

In other words the identical mapping of Γ_0 is homotopic to $\varphi : \Gamma_0 \to \mathbf{G}(\mathcal{Q})$, $\varphi \circ i_0 \sim \mathbf{Id}$.

Proof. The statement of the lemma is equivalent to the injectivity of the homomorphism (38)

$$\theta : \mathbf{K}^*(M) \to \mathbf{K}^*_{\mathcal{Q}}(M),$$

which takes a finite dimensional bundle α to the bundle $\alpha_{\mathcal{Q}}$ over the algebra $\mathcal{Q}$. In fact, let (38) be injective. Consider the bundle ξ over the space Γ_0 generated by the mapping (65). Then $\theta(\xi)$ is the bundle over the algebra $\mathcal{Q}$ generated by the mapping

$$i'_1 \cdot p'_1 : \Gamma_0 \to B\mathbf{G}(\mathcal{K}^+) \subset \widetilde{B\mathbf{G}(\mathcal{Q})}. \tag{66}$$

This mapping is homotopic to the constant mapping as the space $\Gamma_0 = \Gamma(x_0; B\mathbf{G}(\mathcal{Q}); B\mathbf{G}(\mathcal{K}^+))$ is the space of paths in the space $\widetilde{B\mathbf{G}(\mathcal{Q})} \approx B\mathbf{G}(\mathcal{Q})$ that start at the fixed point x_0 and finish in the subspace $B\mathbf{G}(\mathcal{K}^+)) \subset \widetilde{B\mathbf{G}(\mathcal{Q})}$. It follows that we can map each path γ to an intermediate point $\gamma(t)$, $0 \le t \le 1$ (in contrast to the end point $\gamma(1)$ in the mapping (66)).

Conversely, assume that the mapping p'_1 is homotopic to a constant mapping. Let ξ be a bundle over M such that $\theta(\xi)$ is trivial. The bundle ξ is generated by the mapping $q : M \to B\mathbf{G}(\mathcal{K}^+))$ and the bundle $\theta(\xi)$ is generated by $i'_1 \cdot q : M \to \widetilde{B\mathbf{G}(\mathcal{Q})}$. Since the bundle $\theta(\xi)$ is trivial, the mapping $i'_1 \cdot q$ is homotopic to a constant mapping. This means that the homotopy $F(x,t)$, $x \in M$, $0 \le t \le 1$, defines the mapping $\widetilde{F} : M \to \Gamma_0$, and the composition $p'_1 \cdot \widetilde{F}$ is equal to q. Consequently, q is homotopic to a constant mapping.

Let ξ be a bundle such that the bundle $\theta(\xi)$ is trivial over the algebra $\mathcal{Q}$, $\{U_i\}$ be a chart atlas and $\{\varphi_{ij}(x) \in G(\mathcal{K}^+)\}$ be transition functions of the bundle ξ. Without loss of generality we can assume that all transition functions are unitary, that is, $\{\varphi_{ij}(x) \in U(\mathcal{K}^+)\}$. The triviality of the bundle $\theta(\xi)$ means that there are functions $h_i(x) \in U(\mathcal{Q})$ such that

$$[\varphi_{ij}(x)] = h_i(x)h_j^{-1}(x), \quad x \in U_{ij}. \tag{67}$$

The element $h_i(x) \in U(\mathcal{Q}) = (\prod_k U(\mathcal{K}^+))/\overline{(\sum_k U(\mathcal{K}^+))}$ can be realized as a continuous section in the group $(\prod_k U(\mathcal{K}^+))$ of the form $h_i^k(x) \in U(\mathcal{K}^+)$. The condition (67) can be written as

$$\lim_{k\to\infty} \|\varphi_{ij}(x)h_j^k(x) - h_i^k(x)\| = 0. \tag{68}$$

One can use the standard technique of the Urysohn lemma for extension of continuous functions. And, for a sufficiently large k, we can deform the functions $h_i(x)$ to functions $\widetilde{h}_i(x)$ such that

$$\varphi_{ij}(x) = \widetilde{h}_i(x)\widetilde{h}_j^{-1}(x), \quad x \in U_{ij}. \ \blacksquare \tag{69}$$

Corollary 1. *Lemma 1 shows that each almost flat bundle over a simply connected CW-space is trivial.*

Proof. An almost flat structure is defined by a continuous mapping $f: M \to \mathbf{BAF}$. Since the composition $p \cdot f: M \to B\widehat{\mathbf{G}(\mathcal{Q})} \times B\widehat{\mathbf{G}(\mathcal{Q})}$ of the projection (64) and f is homotopic to a constant mapping for a simply connected space M, the mapping f is homotopic to a mapping into a fiber Γ_0. The latter is mapped into the space $B\mathbf{G}(\mathcal{K}^+)$ by means of p_1' that is also homotopic to a constant mapping. ∎

8. Fredholm operators for twisted K-theory due to M. Atiyah and G. Segal (jointly with A. Irmatov). In the paper [AS05] M. Atiyah and G. Segal have considered families of Fredholm operators parametrized by points of a compact space K which are continuous in a topology weaker than the uniform topology, i.e. the norm topology in the space of bounded operators $B(H)$ in a Banach space H.

Therefore, it is interesting to establish whether the conditions, characterizing families of Fredholm operators, from the paper [AS05] precisely describe the families of Fredholm operators which form a Fredholm operator over the C^*-algebra $\mathcal{A} = C(K)$ of all continuous functions on K.

It is not supposed by the authors of the paper [AS05] that an operator over $\mathcal{A}$ admits the adjoint operator or in their terminology, continuity of the adjoint family.

Here we aim to clarify the problem of a description of the class of Fredholm operators which in general case do not admit the adjoint operator. For the first time, operators which play the role of Fredholm operators and may not have adjoints were considered in the paper [MF79]. Since the main class of operators considered in the paper [MF79] is the class of pseudodifferential operators for any element of which the adjoint operator automatically is bounded, it follows that existence of the adjoint operator was not the actual question for the main goals of that paper.

However, in their paper [AS05] the authors have considered operators, which may not have adjoints, in the form of families of operators continuous in the compact-open topology, the adjoint families of which, in general case, may not be continuous.

We can show that the class of Fredholm operators over an arbitrary C^*-algebra, which may not admit adjoints, can be extended to a bigger class. This bigger class is defined by the class of compact operators both with and without adjoints (see [IM05]).

In the case where the C^*-algebra is a commutative algebra of continuous functions on a compact space appropriate topologies in the classic spaces of Fredholm and compact operators in the Hilbert space can be constructed. These topologies fully describe the sets of Fredholm and compact operators over the C^*-algebra without the assumption of existence of bounded adjoint operators over the algebra.

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