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LINEAR DIRECT CONNECTIONS

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Abstract. In this paper we study the geometry of direct connections in smooth vector bundles (see N. Teleman [Tn.3]); we show that the infinitesimal part, ∇^τ, of a direct connection τ is a linear connection. We determine the curvature tensor of the associated linear connection ∇^τ.

As an application of these results, we present a direct proof of N. Teleman's Theorem 6.2 [Tn.3], which shows that it is possible to represent the Chern character of smooth vector bundles as the periodic cyclic homology class of a specific periodic cyclic cycle Φ^τ_*, manufactured from a direct connection τ, rather than from a smooth linear connection as the Chern-Weil construction does. In addition, we show that the image of the cyclic cycle Φ^τ_* into the de Rham cohomology (through the A. Connes' isomorphism) coincides with the cycle provided by the Chern-Weil construction applied to the underlying linear connection ∇^τ.

For more details about these constructions, the reader is referred to [M], N. Teleman [Tn.1], [Tn.2], [Tn.3], C. Teleman [Tc], A. Connes [C.1], [C.2] and A. Connes and H. Moscovici [C.M].

1. Introduction. In this paper we address two problems: (i) to better understand the geometry of direct connections and (ii) to provide a direct proof of N. Teleman's theorem [Tn.3], which shows how to modify the Chern-Weil theory from the case of linear connections to the case of direct connections to extract the Chern character of smooth vector bundles.

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Before going into more details, we recall that a direct connection (see N. Teleman [Tn.2], [Tn.3]) in a vector bundle is a law which provides a direct isomorphic transport of fibres, from point to point, rather than along paths. Direct connections were originally called quasi-connections, see [Tn.2].

Concerning the first problem, we show that any direct connection τ has an underlying linear connection ∇^{τ}, which represents its infinitesimal part. We determine the connection coefficients $\Gamma^{\cdot}_{\cdot\cdot}$ of the underlying linear connection and its curvature tensor.

With regard to the second problem, we mention that N. Teleman [Tn.3, Th. 6.2] had shown that the Chern character $Ch_*(\xi)$, of an arbitrary smooth vector bundle $\xi : E \to M$ may be obtained as the cyclic homology, (resp. periodic cyclic homology), *classes*, (resp. *class*) of cyclic cycles, (resp. a periodic cyclic cycle), of the algebra of smooth functions on M, whose chain components are the functions

$$\Phi_k : U_{k+1} \to \mathbb{R},$$

$$\Phi_k\left(x_0, x_1, \ldots, x_k\right) := \operatorname{Tr} \tau\left(x_0, x_1\right) \circ \tau\left(x_1, x_2\right) \circ \cdots \circ \tau\left(x_{k-1}, x_k\right) \circ \tau\left(x_k, x_0\right),$$

where τ is an arbitrary direct connection in ξ and U_{k+1} is a neighborhood of the diagonal in M^{k+1}.

Recall that the periodic cyclic homology of the algebra of smooth functions is isomorphic to the bi-graded de Rham cohomology of the manifold M. This result, due to A. Connes, constitutes the bridge between the classical differential geometry and the noncommutative geometry. The Connes' isomorphism associates with any periodic cyclic cycle f an even/odd non-homogeneous closed differential form $\Omega(f)$ on M.

The proof of Theorem 6.2. [Tn.3] uses the homotopy invariance of the cyclic homology, as well as the non commutative definition of the Chern character derived from the Levi-Civita connection in a vector bundle, results due to A. Connes [C.1], [C.2]; it states the result at the level of homology *classes*.

In this paper we present a *direct* proof of this theorem, showing in addition that the image of the periodic cyclic cocycle $\{\Phi_k\}_k$ through the Connes' isomorphism

$$\Omega_{2k}(\Phi_{2k}) =$$

$$\frac{1}{(2k)!} \sum_{i_1, i_2, .., i_{2k}} \frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \cdots \frac{\partial}{\partial x_{2k}^{i_{2k}}} \Phi_k\left(x_0, x_1, \ldots, x_{2k}\right)_{x_0 = x_1 = \cdots = x_{2k} = x} dx^{i_1} \wedge \cdots \wedge dx^{i_{2k}}$$

coincides with the differential *form* provided by the classical Chern-Weil theory applied upon the underlying linear connection ∇^{τ}.

We recall that within the theory of linear connections the closeness of the form TrR^k in the de Rham complex is a consequence of the Bianchi identity. It is relevant to mention that the same result follows trivially in the context of direct connections as a consequence of the symmetry of mixed partial derivatives (Schwarz lemma)—see Remark 3.

For other applications of direct connections, we refer to A. Mishchenko and N. Teleman [M.T]. For related topics, providing geometric interpretations of the Chern character, we refer to N. Teleman [Tn.1], [Tn.2], [Tn.3], and C. Teleman [Tc].

In the sequel we freely use the Einstein summation convention on repeated indices.

2. Direct connections vs. linear connections. Let E be a real or complex smooth vector bundle over the manifold M.

Definition 1 (see N. Teleman [Tn.2], [Tn.3]). By a *linear direct connection* in a vector bundle E we mean a smooth mapping

$$\tau : U \to GL\,(E)$$

where $U \subset M \times M$ is an open neighborhood of the diagonal $\Delta = \{(x,x)\,;\ x \in M\}$, such that

$$\tau\,(x,y) : E_{|y} \to E_{|x}$$

and

$$\tau\,(x,x) = \mathrm{id} : E_{|x} \to E_{|x}.$$

Direct connections, used also in the paper by A. Mishchenko and N. Teleman [M.T], were called quasi connections.

We intend to extract from a direct connection its infinitesimal part along the diagonal.

Definition 2. Let X be a smooth tangent field over M and ϕ a smooth section in E. Let x_0 be an arbitrary point in M and let $\gamma : (-\varepsilon, \varepsilon) \to M$ be an integral path of the field X with the initial condition $\gamma(0) = x_0$, ($\dot\gamma(0) = X(x_0)$).

We define

$$\nabla^{\tau}_{X(x_0)}(\phi) = \frac{d}{dt}\,\{\tau(\gamma(0), \gamma(t))\,(\phi\,(\gamma\,(t)))\}_{|t=0} \in E_{|x_0}.$$

Remark 1. As the parameter t varies, the function under the derivative sign describes a smooth path in the fibre $E_{|x_0}$ and hence $\nabla^{\tau}_{X(x_0)}(\phi)$ is well defined; it depends only on X, ϕ and x_0. Notice that the condition $\tau\,(x,x) = \mathrm{id}_{E_x}$ is necessary in order to insure that the outcome of the derivation is a vector of the fibre over the point x_0.

We intend to describe $\nabla^{\tau}_{X(x_0)}(\phi)$ locally. For, let $(x^1, x^2, \ldots, x^m)$ $(dimM = m)$ be a local coordinate system on an open neighborhood $\mathcal{V}$ of a point x_0. Using the same local coordinate system on both factors of the direct product $M \times M$, any point $(x,y) \in \mathcal{V} \times \mathcal{V}$ will be given by local coordinates $(x^1, x^2, .., x^m | y^1, y^2, \ldots, y^m)$.

Let n be the $\mathbb{K}$-rank ($\mathbb{K} = \mathbb{R}$, or $\mathbb{C}$) of the bundle E and let $\{e_1, e_2, \ldots, e_n\}$ be a local frame in the bundle E over V.

The direct connection τ is given locally by a matrix

$$\tau(x|y) = \|\tau_i^j(x|y)\| \in M_{n,n}(\mathbb{K}),$$
$$\tau(x|y)\,(e_i(y)) = \sum_j \tau_i^j(x|y) \cdot e_j(x),$$
$$\tau_i^j(x|x) = \delta_i^j.$$

The field ϕ may be represented over $\mathcal{V}$ by $\phi(x) = \sum_i \phi^i(x) e_i(x)$. Then

$$\begin{aligned}\nabla^{\tau}_{X(x_0)}(\phi) &= \frac{d}{dt}\{\tau(\gamma(0),\gamma(t))\,(\phi(\gamma(t)))\}_{|t=0}\\ &= \frac{d}{dt}\Big\{\tau(\gamma(0),\gamma(t))\Big(\sum_i \phi^i(\gamma(t))e_i(\gamma(t))\Big)\Big\}_{|t=0}\\ &= \frac{d}{dt}\Big\{\sum_{i,j}\tau_i^j(x_0|\gamma(t))\cdot\phi^i(\gamma(t))e_j(x_0)\Big\}_{|t=0}\\ &= \sum_{i,j}\frac{d}{dt}\{\tau_i^j(x_0|\gamma(t))\cdot\phi^i(\gamma(t))\}_{|t=0}\cdot e_j(x_0).\end{aligned}$$

Describing the path γ by $\gamma(t)=(y^1(t),y^2(t),\dots,y^m(t))$, we have further

$$\begin{aligned}\nabla^{\tau}_{X(x_0)}(\phi) &= \sum_{i,j}\frac{d}{dt}\{\tau_i^j(x_0|y^1(t),y^2(t),\dots,y^m(t))\cdot\phi^i(y^1(t),y^2(t),\dots,y^m(t))\}_{|t=0}e_j(x_0)\\ &= \sum_{\alpha=1}^{\alpha=m}\sum_{i,j}\Big\{\frac{\partial}{\partial y^\alpha}\tau_i^j(x_0|y^1,y^2,\dots,y^m))_{|y=x_0}\cdot\dot y^\alpha(0)\cdot\phi^i(x_0)\\ &\qquad + \tau_i^j(x_0|x_0)\frac{\partial}{\partial y^\alpha}\phi^i(y^1,y^2,\dots,y^m)_{|y=x_0}\cdot\dot y^\alpha(0)\Big\}e_j(x_0).\end{aligned}$$

Defining

$$\Gamma^j_{i,\alpha}(x_0)=\frac{\partial}{\partial y^\alpha}\tau_i^j(x_0|y^1,y^2,\dots,y^m))_{|y=x_0},$$

we get

$$\begin{aligned}&\nabla^{\tau}_{X(x_0)}(\phi)\\ &= \sum_{\alpha=1}^{\alpha=m}\sum_{i,j}\{\Gamma^j_{i,\alpha}(x_0)\cdot\dot y^\alpha(0)\cdot\phi^i(x_0)\ +\delta_i^j\frac{\partial}{\partial y^\alpha}\phi^i(y^1,y^2,\dots,y^m)_{|y=x_0}\cdot\dot y^\alpha(0)\}e_j(x_0)\\ &= \sum_{\alpha=1}^{\alpha=m}\Big\{\sum_{i,j}\Gamma^j_{i,\alpha}(x_0)\cdot\dot y^\alpha(0)\cdot\phi^i(x_0)e_j(x_0)\\ &\qquad +\sum_i\frac{\partial}{\partial y^\alpha}\phi^i(y^1,y^2,\dots,y^m)_{|y=x_0}\cdot\dot y^\alpha(0)e_i(x_0)\Big\}.\end{aligned}$$

Representing the tangent field X locally

$$X(x)=\sum_\alpha X^\alpha(x)\cdot\frac{\partial}{\partial x^\alpha},$$

one has $\dot y^\alpha(0)=X^\alpha(x_0)$; the above computation shows that

$$\begin{aligned}&\nabla^{\tau}_{X(x_0)}\Big(\sum_i\phi^i e_i\Big)\\ &= \sum_{\alpha=1}^{\alpha=m}\Big\{\sum_{i,j}\Gamma^j_{i,\alpha}(x_0)\cdot X^\alpha(x_0)\cdot\phi^i(x_0)e_j(x_0)\\ &\qquad +\sum_i\frac{\partial}{\partial y^\alpha}\phi^i(y^1,y^2,\dots,y^m)_{|y=x_0}\cdot X^\alpha(x_0)e_i(x_0)\Big\}\\ &= \sum_{\alpha=1}^{\alpha=m}\Big\{\sum_{i,j}\Gamma^j_{i,\alpha}(x_0)\cdot X^\alpha(x_0)\cdot\phi^i(x_0)e_j(x_0)\Big\}+\sum_i(d\phi^i)(X)(x_0)\ e_i(x_0).\end{aligned}$$

Both this formula and Remark 1 above show that ∇^τ is a linear connection in the vector bundle E. The linear connection ∇^τ will be called *associated, or underlying, linear connection to the direct connection* τ. To summarize, we have proved

THEOREM 3. *(i) For any direct connection* τ, *let*

$$\nabla^\tau_{X(x)}(\phi) = \frac{d}{dt}\tau(\gamma(0),\gamma(t))_{|t=0}\left(\phi\left(\gamma\left(t\right)\right)\right) \in E_{|x}, \qquad \gamma\left(0\right) = x, \quad \dot{\gamma}(t) = X(\gamma\left(t\right))$$

be its infinitesimal part along the diagonal. Then ∇^τ *is a linear connection.*

(ii) Let $(x^1, x^2, \ldots, x^m | y^1, y^2, \ldots, y^m)$ *be a local system of coordinates on a neighborhood* $\mathcal{V} \times \mathcal{V}$ *of a point* $(x, x) \in M \times M$ *and let* $\{e_1, e_2, \ldots, e_n\}$ *be a local frame in* E *over* $\mathcal{V}$. *Let* $\tau(x|y)$ *be the matrix describing locally the direct connection* τ*:*

$$\tau(x|y) = \|\tau_i^j(x|y)\| \in M_{n,n}(\mathbb{K}),$$
$$\tau(x,y)\left(e_i(y)\right) = \sum_j \tau_i^j(x|y)\cdot e_j(x), \qquad \tau_i^j(x|x) = \delta_i^j.$$

Then the coefficients $\Gamma^j_{i,\alpha}$ *of the connection* ∇^τ *are given locally by*

$$\nabla^\tau_{\frac{\partial}{\partial x^\alpha}} e_i = \sum_j \Gamma^j_{i,\alpha} e_j,$$

where

$$\Gamma^j_{i,\alpha}(x) = \frac{\partial}{\partial y^\alpha}\tau_i^j(x^1, x^2, \ldots, x^m | y^1, y^2, \ldots, y^m)_{y=x}.$$

In the above formulas it is assumed that on each of the two factors $\mathcal{V}$ *of* $\mathcal{V}\times\mathcal{V}$ *the same coordinate functions are considered.*

Let $R = (\nabla^\tau)^2$ be the curvature tensor of the connection ∇^τ. The components of the curvature R are

$$\begin{aligned} R^j_{i\alpha\beta}(x) &= \frac{\partial}{\partial x^\alpha}\Gamma^j_{i\beta}(x) - \frac{\partial}{\partial x^\beta}\Gamma^j_{i\alpha}(x) + \Gamma^j_{k\alpha}(x)\cdot\Gamma^k_{i\beta}(x) - \Gamma^j_{k\beta}(x)\cdot\Gamma^k_{i\alpha}(x) \\ &= \frac{\partial^2}{\partial x^\alpha \partial y^\beta}\tau_i^j(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha}\tau_i^j(x|y)_{y=x} \\ &\quad + \frac{\partial}{\partial y^\alpha}\tau_k^j(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\beta}\tau_i^k(x|y)_{y=x} - \frac{\partial}{\partial y^\beta}\tau_k^j(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\alpha}\tau_i^k(x|y)_{y=x}. \end{aligned}$$

COROLLARY 4. *The curvature form* R *of the underlying linear connection* ∇^τ, *associated to the direct connection* τ, *is given by*

$$\begin{aligned} R = \Bigg(&\frac{\partial^2}{\partial x^\alpha \partial y^\beta}\tau_i^j(x|y)_{y=x} - \frac{\partial^2}{\partial x^\beta \partial y^\alpha}\tau_i^j(x|y)_{y=x} \\ &+ \frac{\partial}{\partial y^\alpha}\tau_k^j(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\beta}\tau_i^k(x|y)_{y=x} - \frac{\partial}{\partial y^\beta}\tau_k^j(x|y)_{y=x}\cdot\frac{\partial}{\partial y^\alpha}\tau_i^k(x|y)_{y=x}\Bigg) dx^\alpha \wedge dx^\beta. \end{aligned}$$

For further computations it is convenient to introduce the matrices Γ_α whose components are given by

$$\|\Gamma_\alpha\|_i^j := \Gamma^j_{i,\alpha}$$

and matrices $R_{\alpha\beta}$ whose entries are the components of the curvature tensor

$$\|R_{\alpha\beta}\|_i^j := R^j_{i\alpha\beta}.$$

Remark 2. It is clear that if the base manifold M is endowed with an affine connection, then the parallel transport along small geodesics—defined for a given linear connection in the vector bundle E over M—produces a direct connection τ in E. Such examples of direct connections have already been used in various papers, as computational tools, see e.g. C. Teleman [Tc], A. Connes and H. Moscovici [C.M]. We might ask whether any direct connection can be obtained by this procedure. The answer to this question is negative. Indeed, notice that any direct connection τ produced by parallel transport along geodesics has to satisfy the condition $\tau(x,y)\circ\tau(y,x)=Id$, a property which might not be satisfied by an arbitrary direct connection. Moreover, if a direct connection τ derives from the parallel transport along geodesics, then the affine connection on the base manifold might not be uniquely defined by it. For example, if the linear connection in E is flat, then the corresponding direct connection τ satisfies the condition $\tau(z,y)\circ\tau(y,x)=\tau(z,x)$, locally, and it does not depend on the affine connection on the base manifold.

It is interesting to point out that the fulfillment of the condition $\tau(x,y)\circ\tau(y,x)=Id$ is not sufficient to insure that the direct connection τ derives from the parallel transport along geodesics, either. For, we provide the following

Example 5. Let $M=\mathbb{R}$ and let E be the product bundle $M\times\mathbb{R}$ of rank 1. Let $\mathbf{e}_1$ be the constant frame $\mathbf{e}_1(x)=(x,1)$ in E. With respect to this frame, consider the direct connection τ defined by the matrix

$$\tau(y|x)=e^{y-x+(y-x)^3}\in M_{1,1}(\mathbb{R}).$$

As the exponent is an odd function, τ satisfies $\tau(x,y)\circ\tau(y,x)=Id$. On the other hand, the corresponding linear connection ∇^τ is given by

$$\nabla^\tau_{\frac{d}{dx}}(\mathbf{e}_1(x))=\frac{d}{dy}(e^{y-x+(y-x)^3}\mathbf{e}_1(x))_{y=x}=\mathbf{e}_1(x),$$

and the parallel transport of the vector ξ_0 along the line $\mathbb{R}$ from the point x to the point y is the solution of the differential equation

$$\frac{d\xi(y)}{dy}=\xi(y),\qquad \xi(x)=\xi_0,$$

or,

$$\xi(y)=e^{y-x}\xi_0,$$

which differs from $e^{y-x+(y-x)^3}\xi_0$.

Although $\tau(x,y)=(\tau(y,x))^{-1}$ is not true in general, it is true, however, that it holds infinitesimally. In fact, we have

Proposition 6. *For any direct connection τ, its matrix components satisfy the identities*

(i)

$$\frac{\partial}{\partial x^\alpha}\tau_i^j(x|y)_{y=x}+\frac{\partial}{\partial y^\alpha}\tau_i^j(x|y)_{y=x}=0.$$

(ii)

$$\frac{\partial}{\partial x^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x}=0=\frac{\partial}{\partial y^\alpha}\{\tau(x|y)\circ\tau(y|x)\}_{y=x}.$$

Proof. As $\tau(x|x) = Id$, we get that the directional derivative $\left(\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial y^\alpha}\right)$ of τ along the diagonal vanishes. This proves (i). The second identity is a consequence of the first.

3. Direct connections and Chern character

3.1. *Recall of periodic cyclic homology.* For the benefit of the reader and for setting notation, we recall in this section some basic notions and results, due to A. Connes [C.1], [C.2], which lay at the foundations of noncommutative geometry.

Given a locally convex associative algebra $\mathcal{A}$, the space of k-chains $C_k(\mathcal{A})$ over the algebra $\mathcal{A}$, $C_*(\mathcal{A})$ is, by definition, a topological completion (usually, projective completion) of the algebraic tensor product $\otimes^{k+1}\mathcal{A}$. Two boundary operators, b' and b are introduced by the formulas

$$b'(f_0 \otimes f_1 \otimes \dots f_{k-1} \otimes f_k) = \sum_{r=0}^{r=k-1} (-1)^r f_0 \otimes f_1 \otimes \dots \otimes (f_r \cdot f_{r+1}) \otimes \dots \otimes f_{k-1} \otimes f_k$$

and

$$b(f_0 \otimes f_1 \otimes \dots f_{k-1} \otimes f_k) = \sum_{r=0}^{r=k-1} (-1)^r f_0 \otimes f_1 \otimes \dots \otimes (f_r \cdot f_{r+1}) \otimes \dots \otimes f_{k-1} \otimes f_k + (-1)^k f_0.f_1 \otimes \dots \otimes f_{k-1}.$$

The boundary operator b' defines the bar complex; if the algebra $\mathcal{A}$ is unitary then the bar complex is acyclic.

The complex based on the boundary operator b is the Hochschild complex of the algebra $\mathcal{A}$; its homology is the Hochschild homology of the algebra.

The graded cyclic permutation $T : C_k(\mathcal{A}) \to C_k(\mathcal{A})$ is defined on generators by

$$T(f_0 \otimes f_1 \otimes \dots \otimes f_{k-1} \otimes f_k) = (-1)^k f_1 \otimes \dots \otimes f_{k-1} \otimes f_k \otimes f_0.$$

The operator $N : C_k(\mathcal{A}) \to C_k(\mathcal{A})$ is given by

$$N = 1 + T + T^2 + \dots + T^k.$$

The periodic cyclic homology is defined as the homology of the total complex associated to a first and second quadrant direct product bicomplex $\{C_{p,q}\}_{p\in Z, q\geq 0}$ defined by:

(i) $C_{p,q} = C_q(\mathcal{A})$; the boundary operators are considered of degree -1,

(ii) the columns consist of alternating bar and Hochschild complexes: the Hochschild complex on each even order column and the bar complex (with b' replaced by $-b'$) on each odd order column,

(iii) the boundary homomorphisms of the horizontal complexes are given by the alternating homomorphisms N and $1 - T$

$$\dots \xleftarrow{1-T} C_{-1,q} \xleftarrow{N} C_{0,q} \xleftarrow{1-T} C_{1,q} \xleftarrow{N} \dots.$$

This complex is called the periodic bicomplex of the algebra $\mathcal{A}$.

Given the periodicity of the periodic cyclic bicomplex, there are essentially only two periodic cyclic homologies: $H^{\lambda,per}_{even}(\mathcal{A})$, and $H^{\lambda,per}_{odd}(\mathcal{A})$.

THEOREM 7 (A. Connes [C.1], [C.2]).

(i)

$$H^{\lambda,per}_{even}(C^\infty(M)) = \bigoplus^{k=even} H^k_{dR}(M),$$

(ii)

$$H^{\lambda,per}_{odd}(C^\infty(M)) = \bigoplus^{k=odd} H^k_{dR}(M),$$

(iii) Given the periodic cyclic cycle $f = \prod_{p=0}^{p=\infty} f_{\epsilon-p,p} \in \prod_{p=0}^{p=\infty} C_{\epsilon-p,p}(C^\infty(M))$ *(*$\epsilon = 0, 1$*), its (*$\epsilon + 2p$*)-degree component is the de Rham cohomology class of the (closed) differential form*

$$\frac{1}{(\epsilon+2p)!}\frac{\partial^{\epsilon+2p}}{\partial x_1^{i_1}\dots\partial x_{\epsilon+2p}^{i_{\epsilon+2p}}} f_{\epsilon+2p}(x_0, x_1, \dots, x_{\epsilon+2p})_{|\Delta} dx_1^{i_1}\wedge\dots\wedge dx_{\epsilon+2p}^{i_{\epsilon+2p}},$$

where Δ *is the diagonal.*

DEFINITION 8. For any cyclic cycle $f = \prod_{p=0}^{p=\infty} f_{\epsilon-p,p}$, let

$$\Omega_{\epsilon+2p}(f) := \frac{1}{(\epsilon+2p)!}\frac{\partial^{\epsilon+2p}}{\partial x_1^{i_1}\dots\partial x_{\epsilon+2p}^{i_{\epsilon+2p}}} f_{\epsilon+2p}(x_0, x_1, \dots, x_{\epsilon+2p})_{|\Delta} dx_1^{i_1}\wedge\dots\wedge dx_{\epsilon+2p}^{i_{\epsilon+2p}}.$$

3.2. *Direct connections and Chern character forms.* Let τ be a direct connection in the complex vector bundle $\xi : E \to M$ and let ∇^τ be its associated linear connection.

Consider the function $\Phi_k : M^{k+1} \supset U_{k+1} \to \mathbb{C}$ (where $U_{k+1} \subset M^{k+1}$ is a neighborhood of the diagonal in M^{k+1}) defined by the formula

$$\Phi_k(x_0, x_1, \dots, x_k) := Tr\,\tau(x_0, x_1)\tau(x_1, x_2)\dots\tau(x_{k-1}, x_k)\tau(x_k, x_0).$$

THEOREM 9 (N. Teleman, [Tn.2] Theorem 6.2). *Let* ξ *be a complex vector bundle over the paracompact manifold* M *and let* τ *be a smooth linear direct connection in* ξ*. Then*

(i) the infinite chain Φ^τ *with components*

$$f_{-2p,2p} := (-1)^p \frac{(2p)!}{p!}\Phi_{2p} \in C_{-2p,2p}(C^\infty(M))$$

$$f_{-(2p-1),2p-1} := (-1)^{p-1} \frac{(2p)!}{p!}\Phi_{2p-1} \in C_{-(2p-1),2p-1}(C^\infty(M))$$

is an even periodic cyclic cycle over the algebra $C^\infty(M)$*;*

(ii) its homology class is (up to a multiplicative constants) the total Chern character of ξ*.*

The reader should notice that, in view of Connes' result, Theorem 7, a modification of the direct connection away from a small neighborhood of the diagonal does not change the periodic cyclic homology class of the chain Φ^τ as its periodic cyclic homology class depends only on its 1-jet along the diagonal. For more information about the Hochschild homology of the algebra of smooth functions, the reader might refer to N. Teleman [Tn.4]

As explained in the section 1, we intend to present here a direct proof of this theorem providing, in addition, an explicite link between the differential forms $\Omega_{2k}(\Phi^\tau)$ and the classical Chern-Weil forms, at the level of *differential forms* rather than *cohomology classes.*

More precisely, we prove

THEOREM 10. *Let τ be a direct connection and let ∇^τ be its underlying linear connection. Then*

$$\Omega_{2k}(\Phi_{2k}^\tau) = \frac{1}{(2k)!} \cdot \frac{1}{2^k} \cdot Tr\, R^k,$$

where $R = (\nabla^\tau)^2$ is the curvature of the underlying linear connection ∇^τ.

Proof. We have to evaluate

$$\Omega_{2k}(\Phi_{2k}^\tau) :=$$

$$\sum_{i_1,\dots,i_{2k}} \frac{1}{(2k)!} \frac{\partial^{2k}}{\partial x_1^{i_1} \dots \partial x_{2k}^{i_{2k}}} Tr\, \tau(x_0, x_1) \dots \tau(x_{2k-1}, x_{2k}) \tau(x_{2k}, x_0)_{|\Delta} dx_0^{i_1} \wedge \dots \wedge dx_0^{i_{2k}}.$$

We intend to evaluate the differential form $\Omega_{2k}(\Phi_{2k}^\tau)$ at the point $(x_0, x_0, \dots, x_0)$.

LEMMA 11.

$$\sum_{i_1,i_2} \frac{\partial^2}{\partial x_1^{i_1} \partial x_2^{i_2}} (\tau(x_0,x_1)\tau(x_1,x_2)\tau(x_2,x_3))_{|(x_0=x_1=x_2)} dx_0^{i_1} \wedge dx_0^{i_2}$$
$$= \frac{1}{2} \sum_{i_1,i_2} R_{i_1 i_2}(x_0) \tau(x_0|x_3) dx_0^{i_1} \wedge dx_0^{i_2}.$$

Proof. We work in local coordinates. Let us define

$$A(x_0,x_3) := \sum_{i_1,i_2} \frac{\partial^2}{\partial x_1^{i_1} \partial x_2^{i_2}} (\tau(x_0,x_1)\tau(x_1,x_2)\tau(x_2,x_3))_{|(x_0=x_1=x_2)} dx_0^{i_1} \wedge dx_0^{i_2}.$$

An elementary computation gives

$$\begin{aligned}
&A(x_0,x_3)\\
&= \Bigg\{ \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_0|x_1) \right) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_1|x_2) \right) \cdot \tau(x_2|x_3) \\
&\quad + \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_0|x_1) \right) \cdot \tau(x_1|x_2) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_2|x_3) \right) \\
&\quad + \tau(x_0|x_1) \cdot \left(\frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \tau(x_1|x_2) \right) \cdot \tau(x_2|x_3) \\
&\quad + \tau(x_0|x_1) \cdot \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_1|x_2) \right) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_2|x_3) \right) \Bigg\}_{|(x_0=x_1=x_2)} dx_0^{i_1} \wedge dx_0^{i_2} \\
&= \Bigg\{ \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_0|x_1) \right) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_1|x_2) \right) \cdot \tau(x_0|x_3) + \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_0|x_1) \right) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_2|x_3) \right) \\
&\quad + \left(\frac{\partial}{\partial x_1^{i_1}} \frac{\partial}{\partial x_2^{i_2}} \tau(x_1|x_2) \right) \cdot \tau(x_0|x_3) \\
&\quad + \left(\frac{\partial}{\partial x_1^{i_1}} \tau(x_1|x_0) \right) \cdot \left(\frac{\partial}{\partial x_2^{i_2}} \tau(x_2|x_3) \right) \Bigg\}_{|(x_0=x_1=x_2)} dx_0^{i_1} \wedge dx_0^{i_2}.
\end{aligned}$$

Using Proposition 6 and the notation for the matrices Γ_α, $R_{\alpha\beta}$, we have further

$$\begin{aligned} A(x_0,x_3) = \Big\{ &\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0)\cdot\tau(x_0|x_3) + \left(\frac{\partial}{\partial x_1^{i_1}}\tau(x_0|x_1)\right)\cdot\left(\frac{\partial}{\partial x_2^{i_2}}\tau(x_2|x_3)\right) \\ &+ \left(\frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3) \\ &+ \left(\frac{\partial}{\partial x_1^{i_1}}\tau(x_1|x_0)\right)\cdot\left(\frac{\partial}{\partial x_2^{i_2}}\tau(x_2|x_3)\right)\Big\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2} \\ = \Big\{ &\left(\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0) + \frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3) \\ &+ \left(\frac{\partial}{\partial x_1^{i_1}}\tau(x_0|x_1) + \frac{\partial}{\partial x_1^{i_1}}\tau(x_1|x_0)\right)\cdot\frac{\partial}{\partial x_2^{i_2}}\tau(x_2|x_3))\Big\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2}. \end{aligned}$$

From Corollary 4 and Proposition 6, we get

$$\begin{aligned} &A(x_0,x_3) \\ &= \sum_{i_1,i_2}\left\{\left(\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0) + \frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2))\cdot\tau(x_0|x_3\right)\right\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2} \\ &= \frac{1}{2}\sum_{i_1,i_2}\left\{\left(\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0) + \frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3)\right\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2} \\ &\quad - \frac{1}{2}\sum_{i_1,i_2}\left\{\left(\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0) + \frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3)\right\}_{|(x_0=x_1=x_2)} dx_0^{i_2}\wedge dx_0^{i_1} \\ &= \frac{1}{2}\sum_{i_1,i_2}\left\{\left(\Gamma_{i_1}(x_0)\cdot\Gamma_{i_2}(x_0) + \frac{\partial}{\partial x_1^{i_1}}\frac{\partial}{\partial x_2^{i_2}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3)\right\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2} \\ &\quad - \frac{1}{2}\sum_{i_1,i_2}\left\{\left(\Gamma_{i_2}(x_0)\cdot\Gamma_{i_1}(x_0) + \frac{\partial}{\partial x_1^{i_2}}\frac{\partial}{\partial x_2^{i_1}}\tau(x_1|x_2)\right)\cdot\tau(x_0|x_3)\right\}_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2} \\ &= \frac{1}{2}\sum_{i_1,i_2} R_{i_1 i_2}\cdot\tau(x_0|x_3)_{|(x_0=x_1=x_2)} dx_0^{i_1}\wedge dx_0^{i_2}. \end{aligned}$$

This completes the proof of the Lemma.

Lemma 12.

$$\begin{aligned} Tr\,\frac{\partial^{2k}}{\partial x_1^{i_1}\dots\partial x_{2k}^{i_{2k}}}\tau(x_0,x_1)\dots\tau(x_{2k-1},x_{2k})\tau(x_{2k},x_0)_{|\Delta}\cdot(dx_0^{i_1}\wedge dx_0^{i_2})\otimes\cdots\otimes(dx_0^{i_{2k-1}}\wedge dx_0^{i_{2k}}) \\ = \frac{1}{2^k}\sum_{i_1,i_2,\dots,i_{2k-1},i_{2k}} Tr\,R_{i_1 i_2}(x_0)R_{i_3 i_4}(x_0)\dots R_{i_{2k-1}i_{2k}}(x_0) \\ \cdot(dx_0^{i_1}\wedge dx_0^{i_2})\otimes\cdots\otimes(dx_0^{i_{2k-1}}\wedge dx_0^{i_{2k}}). \end{aligned}$$

Proof. We apply Lemma 11, in succession, to each of the pairs of arguments and corresponding partial derivations, beginning with the first pair (x_1,x_2). This procedure, followed by applying the trace operator, leads to the desired relation.

We return to the proof of the theorem. Denoting by Alt the skew-symmetrization of tensor products

$$Alt(v_1 \otimes v_2 \otimes \cdots \otimes v_r) := \frac{1}{r!} \sum_{\iota \in S_r} Sign(\iota) \cdot v_{\iota_1} \otimes v_{\iota_2} \otimes \cdots \otimes v_{\iota_r} := v_1 \wedge v_2 \wedge \cdots \wedge v_r,$$

we have

$$\begin{aligned}
&(2k)! \cdot \Omega_{2k}(\Phi^{\tau}_{2k}) \\
&:= \frac{\partial^{2k}}{\partial x_1^{i_1} \dots \partial x_{2k}^{i_{2k}}} Tr\, \tau(x_0, x_1) \dots \tau(x_{2k-1}, x_{2k}) \tau(x_{2k}, x_0)_{|\Delta} dx_0^{i_1} \wedge \cdots \wedge dx_0^{i_{2k}} \\
&= \frac{\partial^{2k}}{\partial x_1^{i_1} \dots \partial x_{2k}^{i_{2k}}} Tr\, \tau(x_0, x_1) \dots \tau(x_{2k-1}, x_{2k}) \tau(x_{2k}, x_0)_{|\Delta} Alt(dx_0^{i_1} \otimes \cdots \otimes dx_0^{i_{2k}}) \\
&= \frac{\partial^{2k}}{\partial x_1^{i_1} \dots \partial x_{2k}^{i_{2k}}} Tr\, \tau(x_0, x_1) \dots \tau(x_{2k}, x_0)_{|\Delta} Alt[(dx_0^{i_1} \wedge dx_0^{i_2}) \otimes \cdots \otimes (dx_0^{i_{2k-1}} \wedge dx_0^{i_{2k}}))] \\
&= Alt\left[\frac{\partial^{2k}}{\partial x_1^{i_1} \dots \partial x_{2k}^{i_{2k}}} Tr\, \tau(x_0, x_1) \dots \tau(x_{2k}, x_0)_{|\Delta} (dx_0^{i_1} \wedge dx_0^{i_2}) \otimes \cdots \otimes (dx_0^{i_{2k-1}} \wedge dx_0^{i_{2k}}))\right] \\
&= \frac{1}{2^k} \cdot Alt[Tr\, R_{i_1 i_2}(x_0) R_{i_3 i_4}(x_0) \dots R_{i_{2k-1} i_{2k}}(x_0) \cdot (dx_0^{i_1} \wedge dx_0^{i_2}) \otimes \cdots \otimes (dx_0^{i_{2k-1}} \wedge dx_0^{i_{2k}})] \\
&= \frac{1}{2^k} \cdot Tr\, R_{i_1 i_2}(x_0) R_{i_3 i_4}(x_0) \dots R_{i_{2k-1} i_{2k}}(x_0) \cdot Alt[(dx_0^{i_1} \wedge dx_0^{i_2}) \otimes \cdots \otimes (dx_0^{i_{2k-1}} \wedge dx_0^{i_{2k}})] \\
&= \frac{1}{2^k} \cdot Tr\, R_{i_1 i_2}(x_0) R_{i_3 i_4}(x_0) \dots R_{i_{2k-1} i_{2k}}(x_0) \cdot Alt[(dx_0^{i_1} \otimes dx_0^{i_2}) \otimes \cdots \otimes (dx_0^{i_{2k-1}} \otimes dx_0^{i_{2k}})] \\
&= \frac{1}{2^k} \cdot Tr\, R_{i_1 i_2}(x_0) R_{i_3 i_4}(x_0) \dots R_{i_{2k-1} i_{2k}}(x_0) \cdot dx_0^{i_1} \wedge dx_0^{i_2} \wedge \cdots \wedge dx_0^{i_{2k-1}} \wedge dx_0^{i_{2k}} \\
&= \frac{1}{2^k} \cdot Tr\, R^k.
\end{aligned}$$

This completes the proof of Theorem 10.

REMARK 3. We recall that, within the theory of linear connections, the closedness of the form $Tr\, R^k$ in the de Rham complex is a consequence of the Bianchi identity. It is interesting to mention that the same result follows trivially in the context of direct connections as a consequence of three facts: (i) the expression of the curvature of a direct connection depends polynomially only on the functions $\tau(x, y)$ differentiated once or two times, (ii) to each differentiation $\frac{\partial}{\partial x^i}$ there corresponds an exterior derivative factor dx^i, and (iii) the symmetry of mixed partial derivatives (Schwarz lemma).

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