SEMILINEAR PERTURBATIONS OF
HILLE-YOSIDA OPERATORS

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Abstract. The semilinear Cauchy problem

(1) $$u'(t) = Au(t) + G(u(t)), \qquad u(0) = x \in \overline{D(A)},$$

with a Hille-Yosida operator A and a nonlinear operator $G : D(A) \to X$ is considered under the assumption that

$$\|G(x) - G(y)\| \le \|B(x-y)\| \quad \forall x, y \in D(A)$$

with some linear $B : D(A) \to X$,

$$B(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} V(s)x\,ds,$$

where V is of suitable small strong variation on some interval $[0, \varepsilon)$. We will prove the existence of a semiflow on $[0, \infty) \times \overline{D(A)}$ that provides Friedrichs solutions in L_1 for (1). If X is a Banach lattice, we replace the condition above by

$$|G(x) - G(y)| \le Bv \quad \text{whenever } x, y, v \in D(A), |x - y| \le v,$$

with B being positive. We illustrate our results by applications to age-structured population models.

1. Introduction. Let A be a Hille-Yosida operator in a Banach space X, i.e., after equivalent renormalization and an identity shift, an m-dissipative linear operator. Equivalently it is the generator of a locally Lipschitz continuous integrated semigroup S. The part A_0 of A in $\overline{D(A)}$ generates a C_0-semigroup $\dot{S}$ which is the strong derivative of S.

We will consider perturbations $A + G$ with nonlinear operators G that are Lipschitz continuous with respect to A in various ways which we will make precise later, and show that the solutions of the associated Cauchy problem induce a continuous semiflow

(nonlinear semigroup) on $\overline{D(A)}$. This will generalize results that have been obtained for the following three scenarios:

- G satisfies a Lipschitz condition on $\overline{D(A)}$ [30, 11, 21],
- A is densely defined and G a linear operator of Miyadera-Voigt type [22, 35],
- X is an abstract L space and G a positive linear operator on $D(A)$ [36, 32].

Our approach does not cover the case that A is m-dissipative, $D(A)$ is dense, G is continuous, $D(G) = X$, and G continuous and dissipative [38].

For the sake of exposition let us assume that A is the generator of a C_0 semigroup T_0. The nonlinear nature of the perturbation does not allow the use of Dyson-Phillips series like in [35] in constructing the semiflow Θ,

$$\Theta(t, x) = T_0(t)x + \int_0^t T_0(s)G(\Theta(t - s, x))ds.$$

Banach's fixed point theorem will be the basic tool of this paper, but it will not be applied to the previous integral equation, but to

$$(2) \qquad v(t) = G\left(T_0(t)x + \int_0^t T_0(s)v(t - s)ds\right),$$

and Θ is obtained via

$$\Theta(t, x) = T_0(t)x + \int_0^t T_0(s)v(t - s)ds.$$

Actually we do not solve (2), but something more complicated. For $x \in \overline{D(A)}$, $u(t) = \Theta(t, x)$ is the solution of the Cauchy problem

$$(3) \qquad u' = Au + G \circ u, \quad t > 0, \qquad u(0) = x$$

in a generalized integral sense which we will explain later (Section 2) and in an L_1-*Friedrichs* sense:

$(\mathbf{L_1}\text{-}\mathbf{F})$ $u : [0, \infty) \to \overline{D(A)}$ is continuous, $u(0) = x$, and there exists a sequence of continuously differentiable functions (u_j) with values in $D(A)$ such that Au_j is continuous and

$$(4) \qquad \left.\begin{array}{c} u_j \to u, \qquad j \to \infty, \text{ uniformly on } [0, T] \\ (u'_j - Au_j) \text{ and } (G \circ u_j) \text{ converge in } L_1([0, T], X) \\ \lim_{j \to \infty}(u'_j - Au_j) = \lim_{j \to \infty} G \circ u_j \end{array}\right\} \qquad \forall T > 0.$$

We are going to carry out this program in detail in two situations. In either scenario, we will consider nonlinearities that are Lipschitz continuous relatively to A in a global sense. While this can be an unpleasant restriction in some applications, it makes the semiflow satisfy an exponential Lipschitz condition in the state variable:

$$(5) \qquad \|\Theta(t, x) - \Theta(t, \tilde{x})\| \le Ne^{\theta t}\|x - \tilde{x}\| \qquad \forall t \ge 0, x, \tilde{x} \in \overline{D(A)},$$

with constants $\theta \in \mathbf{R}$, $N \ge 1$.

First, in the general case that X is a Banach space (Section 5), we consider a nonlinear operator $G : D(A) \to X$ such that

$$\|G(x) - G(y)\| \le \|B(x - y)\| \qquad \forall x, y \in D(A)$$

for some linear operator $B : D(A) \to Y$ into a Banach space Y such that

$$B \int_0^t T_0(s)x ds = V(t)x, \qquad x \in X,$$

where V is an operator family of suitable small strong variation on some interval $[0, \varepsilon]$.

As a special case we obtain the following nonlinear Miyadera-Voigt type result (see Section 6 for a more general and detailed result).

THEOREM 1. *Let A be the generator of a C_0-semigroup T_0 on a Banach space X. Consider a Banach space Y, a dense subspace D of X and maps $G_0 : D \to X$, $B_0 : D \to Y$, B_0 linear, such that*

$$\|G_0(x) - G_0(y)\| \leq \|B_0(x - y)\| \qquad \forall x, y \in D,$$

and the following conditions are satisfied:

1. *$D \subseteq D(A)$, $T_0(t)D \subseteq D$, and the function $B_0 T_0(\cdot)x$ is continuous for all $x \in D$.*
2. *There are $\alpha, \lambda > 0$ and $\gamma \in (0, 1)$ such that*

$$\int_0^\alpha e^{-\lambda t} \|B_0 T_0(t)x\| dt \leq \gamma \|x\| \qquad \forall x \in D.$$

Then there is a unique A-continuous extension $G : D(A) \to X$ of G_0 and a unique continuous semiflow Θ on X which satisfies the exponential Lipschitz condition (5) such that, for each $x \in X$, $u(t) = \Theta(t, x)$ uniquely solves the Cauchy-problem (3) in the Friedrichs sense $(L_1\text{-}F)$.

In a second scenario, we concentrate on the Banach lattice case and a resolvent positive Hille-Yosida operator A. As illustration we present a result for the case that X is an abstract L space. For a general Banach lattice our assumptions become more involved (Section 7).

THEOREM 2. *Let A be a resolvent positive Hille-Yosida operator in the abstract L-space X. Consider a nonlinear $G : D(A) \to X$ and a linear positive $B : D(A) \to X$ such that*

$$|G(x) - G(y)| \leq Bv \quad \text{whenever} \ \ x, y, v \in D(A), |x - y| \leq v,$$

and the spectral radius of $B(\lambda - A)^{-1}$ is smaller than 1 for some $\lambda > s(A)$.

Then there is a unique continuous semiflow Θ on $\overline{D(A)}$ which satisfies the exponential Lipschitz condition (5) such that, for each $x \in \overline{D(A)}$, $u(t) = \Theta(t, x)$ uniquely solves the Cauchy-problem (3) in the Friedrichs sense $(L_1\text{-}F)$.

Moreover, if G is positive, so is Θ; more precisely: If G maps $D(A) \cap X_+$ into X_+, then $\Theta(t, \cdot)$ maps $\overline{D(A)} \cap X_+$ into itself.

The exponential estimate (5) is reminiscent of the *Crandall-Liggett* theorem [9] and its extensions (see [6] and the references there) such that one might wonder whether Theorem 1 and 2 could be obtained therefrom. However, even for linear G, it has not been possible to show directly that $A + G$ is a Hille-Yosida operator, and the respective results were obtained by constructing the (integrated) semigroup [35, 32].

We will not be able to write the nonlinear integral equation (2) in this strict form for the simple reason that, in general, the expression in $(\cdot)$ is not an element of $D(A)$. We

will rather work with fixed points of the following operator.

$$\Psi_x(v)(t) = \lim_{h \to 0} G\left(\frac{1}{h}[S(t+h)x - S(t)x] + \frac{1}{h}[(S * v)(t+h) - (S * v)(t)]\right),$$

(6)

$$(S * v)(t) = \int_0^t S(r)v(t-r)dr,$$

where the convergence is required to hold in the L_1-sense and S is the integrated semi-group generated by A. In Section 2, we give abstract conditions under which the operator Ψ_x is well defined and has fixed points which we relate to Friedrichs solutions and integral solutions of the semilinear Cauchy problem (3). They will be connected to the results presented above by Stieltjes convolutions (Section 3) and cumulative outputs (Section 4).

Once it is known that the generalized solutions of the Cauchy problem induce a semiflow, the powerful theory of dynamical system is available to study their qualitative behavior ([15, 27], e.g.).

In Section 8, we apply our results to age-structured population models where the births rate does not continuously depend on the population age density.

While, for the ease of exposition, we have restricted this paper to time-autonomous perturbations $A+G$, generalizations to non-autonomous semilinear perturbations $A+G(t)$ seem to be straightforward. Of course, the semiflow in (5) will be non-autonomous as well, a nonlinear evolutionary system, and a suitable generalization of the exponential Lipschitz condition will only hold for all times if the Lipschitz conditions for $G(t)$ will be uniform for all $t \geq 0$.

2. Nonlinear perturbations in Banach spaces. Let X be a Banach space and A a Hille-Yosida operator in X, i.e., A is a linear closed operator such that the resolvent set $\rho(A)$ of A contains an infinite interval (ω, ∞) and there exists some $M \geq 1$ such that

$$\|(\lambda - A)^{-n}\| \leq M(\lambda - \omega)^{-n} \qquad \forall \lambda > \omega, n \in \mathbf{N}.$$

Equivalently [2, 17, 1], A generates an integrated semigroup $S(t), t \geq 0$, such that

$$\|S(t) - S(r)\| \leq M \int_r^t e^{\omega s} ds.$$

In general, an integrated semigroup is a strongly continuous family S of bounded linear operators $S(t), t \geq 0$, that satisfies $S(0) = 0$ and the functional equation

$$S(t)S(r) = \int_0^{r+t} S(s)ds - \int_0^t S(s)ds - \int_0^r S(s)ds, \qquad t, r \geq 0,$$

strongly. The fact that A generates S can be either expressed as

(7) $x \in D(A), y = Ax \iff S(t)x - tx = \int_0^t S(r)ydr \ \forall t \geq 0,$

[29] or, if S is exponentially bounded, equivalently in terms of Laplace transforms as

(8) $(\lambda - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t)dt, \qquad \lambda > \omega,$

[2, 17, 23, 1]. If A is a Hille-Yosida operator, then $S(t)x$ is strongly differentiable in $t \geq 0$ if and only if $x \in \overline{D(A)}$. In this case, $\dot{S}(t)x := \frac{d}{dt}S(t)x$ is a C_0-semigroup on $\overline{D(A)}$ which

is generated by the part of A in $\overline{D(A)}$. The following result in [17] is one of the linear foundations for our nonlinear considerations.

THEOREM 3. *Let $f \in L_1([0,T], X)$. Define*

$$(S * f)(t) = \int_0^t S(r)f(t-r)dr, \qquad t \geq 0.$$

Then $S * f$ is continuously differentiable, takes its values in $D(A)$ with $A(S * f)(t)$ being a continuous function in $t \geq 0$, and $u = (S * f)'$ is the unique solution of

$$(9) \qquad u(t) = A \int_0^t u(s)ds + \int_0^t f(s)ds, \qquad t \in [0,T].$$

Finally

$$\|(S * f)'(t)\| \leq M \int_0^t e^{\omega r}\|f(t-r)\|dr \qquad \forall t \in [0,T].$$

The result that (9) has a unique solution which satisfies the estimate in Theorem 3 has already been obtained in [10].

In the following we will write

$$S \diamond f := (S * f)'.$$

After this linear prelude, let Z be a closed subspace of X and

$$G : D(A) \to Z$$

be A-continuous, i.e., if X_A denotes $D(A)$ equipped with the graph norm, $\|x\|_A = \|x\| + \|Ax\|$, then $G : X_A \to Z$ is continuous. We consider the semilinear Cauchy problem

$$(10) \qquad u'(t) = (A + G)u(t), \qquad u(0) = x \in \overline{D(A)}.$$

We will use two closely connected concepts of solving (10) in a generalized sense. Up to slight modifications, they can be found in [10] and [5].

DEFINITION 1. Let $T > 0$. A continuous function $u : [0,T] \to X$ with $u(0) = x$ is called a *Friedrichs solution* or an *F-solution in L_1* of (10) on $[0,T]$ if there are functions $u_j \in C^1([0,T], X) \cap C([0,T], X_A)$ such that $(Gu_j)_{j \in \mathbf{N}}$ converges in $L_1([0,T], X)$ and

$$u_j \to u, \qquad j \to \infty, \text{ uniformly on } [0,T],$$
$$(u_j' - Au_j) \text{ and } (G \circ u_j) \text{ converge in } L_1([0,T], X),$$
$$\lim_{j \to \infty} (u_j' - Au_j) = \lim_{j \to \infty} G \circ u_j.$$

The second notion is a modification of the concept of integral solution.

DEFINITION 2. A continuous function $u : [0,T] \to X$ is called an *I-solution* of (10) if u can be continuously extended to some interval $[0, T + \epsilon]$, $\epsilon > 0$, such that $\int_0^t u(s)ds \in D(A)$ for all $t \in [0, T + \epsilon]$, $G(\frac{1}{h} \int_t^{t+h} u(s)ds)$ converges in $L_1([0,T]; X)$ as $h \to 0$, and

$$(11) \qquad u(t) = x + A \int_0^t u(s)ds + \lim_{h \to 0} \int_0^t G\left(\frac{1}{h} \int_s^{s+h} u(r)dr\right)ds \qquad \forall t \in [0,T].$$

We establish a first relation between the two solution concepts.

LEMMA 4. *If $0 < T_1 < T_2$, then any I-solution on $[0, T_2]$ is an F-solution in L_1 on $[0, T_1]$.*

Proof. Let u be an I-solution on $[0, T_2]$. Set

$$v(t) = \lim_{h \to \infty} G\left(\frac{1}{h} \int_t^{t+h} u(s)ds\right), \qquad t \in [0, T_2],$$

with convergence holding in $L_1([0, T_2]; Z)$. Then

$$u(t) = x + A \int_0^t u(s)ds + \int_0^t v(s)ds \qquad \forall t \in [0, T_2].$$

Define

$$u_j(t) = j \int_t^{t+\frac{1}{j}} u(s)ds, \qquad v_j(t) = j \int_t^{t+\frac{1}{j}} v(s)ds, \qquad t \in [0, T_1].$$

Then $u_j \to u$ in $C([0, T_1]; X)$ and $v_j \to v$ in $L_1([0, T_1]; Z)$. Further $v = \lim_{j \to \infty} G \circ u_j$ in $L_1([0, T_1]; X)$. Finally

$$u_j'(t) = j\left(u\left(t + \frac{1}{j}\right) - u(t)\right) = Au_j(t) + v_j(t),$$

so

$$u_j'(t) - Au_j(t) = v_j(t) \to v(t) \leftarrow G(u_j(t))$$

in $L_1([0, T_1]; Z)$. ∎

Theorem 3 suggests looking for I-solutions of (10) as solutions of the integral equation

$$u(t) = \dot{S}(t)x + \lim_{h \to 0} \frac{d}{dt} \int_0^t S(s)G\left(\frac{1}{h} \int_{t-s}^{t-s+h} u(r)dr\right)ds,$$

where S is the integrated semigroup generated by A. Therefore the following mapping Ψ from $\overline{D(A)} \times L_1([0, T], Z)$ to $L_1([0, T], Z)$ will play an important role whenever it is well-defined,

(12) $$\Psi(x, v)(t) = \lim_{h \to 0} G\left(\frac{1}{h} \int_t^{t+h} \dot{S}(s)xds + \frac{1}{h} \int_t^{t+h} (S \diamond v)(s)ds\right).$$

The convergence is required to be in $L_1([0, T], Z)$ and to be independent of how v is extended outside of $[0, T]$. Notice that this is the same map as in (6). We have the following fundamental result.

PROPOSITION 5. *Assume the operator Ψ in (12) is well-defined and let $T > 0$, $x \in \overline{D(A)}$, $\Psi_x = \Psi(x, \cdot)$. Then there is a one-to-one correspondence between fixed points of Ψ_x and I-solutions. More precisely, the following two relations hold:*

(1) *If v_x is a fixed point of Ψ_x in $L_1([0, T], Z)$ and is extended by 0 outside of $[0, T]$, then $u = \dot{S}(\cdot)x + S \diamond v_x$ is an I-solution on $[0, T]$.*

(2) *If u is an I-solution of $[0, T]$, then*

$$v(t) = L_1\text{-} \lim_{h \to 0} G\left(\frac{1}{h} \int_t^{t+h} u(s)ds\right)$$

is a fixed point of Ψ_x in $L_1([0, T], Z)$ and $u = \dot{S}(\cdot)x + S \diamond v$.

Proof. (1) Let v_x be a fixed point of Ψ_x and set $u = \dot{S}(\cdot)x + S \diamond v_x$. Extend v_x by 0 outside of $[0, T]$. By definition of Ψ, $v_x(t) = \lim_{h \to 0} G(\frac{1}{h} \int_t^{t+h} u(r)dr)$. u is an I-solution by Theorem 3.

(2) Let u be an I-solution of (10), i.e., satisfy (11) or rather all of Definition 2. Define v as stated in the theorem, then $u(t) = \dot{S}(t)x + (S \diamond v)(t)$ for $t \in [0, T]$ by Theorem 3. Since, by assumption, Ψ is well-defined and independent of how v is extended, v is a fixed point of Ψ_x. ∎

Proposition 5 suggests looking for conditions which make the operator Ψ given by (12) well defined. The following subspace E of $\overline{D(A)} \times L_1([0, T]; Z)$ will play an important role.

By definition, E consists of those pairs (x, v) such that

(13)
$$\begin{cases} x \in \overline{D(A)}, \quad v \in L_1([0, T], Z), \\ \dot{S}(t)x + (S \diamond v)(t) \in D(A) \quad \forall t \in [0, T], \\ A\left(\dot{S}(t)x + (S \diamond v)(t)\right) \text{ is a continuous function of } t \in [0, T]. \end{cases}$$

Since $G : D(A) \to Z$ is A-continuous,

(14)
$$\Psi_0(x, v)(t) = G\left(\dot{S}(t)x + (S \diamond v)(t)\right)$$

defines a map Ψ_0 from E to $C([0, T]; Z) \subseteq L_1([0, T]; Z)$. Notice that the argument of G is in $D(A)$ and is a continuous function of t in the graph norm of A.

PROPOSITION 6. *Let $G : D(A) \to Z$ be A-continuous and E be the subspace defined in (13). Assume that Ψ_0 can be extended to a continuous map Ψ from $\overline{D(A)} \times L_1([0, T]; Z)$ to $L_1([0, T]; Z)$. Then Ψ satisfies (12).*

Proof. Set
$$w_h(t) = \frac{1}{h}(S(t+h) - S(t))x + \frac{1}{h} \int_t^{t+h} (S \diamond v)(s)ds.$$

Here we have extended v beyond T in an arbitrary way such that $v \in L_1([0, T + \epsilon], X)$ for some $\epsilon > 0$. By Theorem 3, $w_h(t) \in D(A)$, and
$$Aw_h(t) = \frac{1}{h}(\dot{S}(t+h) - \dot{S}(t))x + \frac{1}{h}\left((S \diamond v)(t+h) - (S \diamond v)(t)\right) - \frac{1}{h} \int_t^{t+h} v(s)ds$$

is a continuous function of $t \in [0, T]$. Moreover
$$\begin{aligned} w_h(t) &= \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}((S * v)(t+h) - (S * v)(t)) \\ &= \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}\int_0^t S(s)(v(t+h-s) - v(t-s))ds \\ &\quad + \frac{1}{h}\int_t^{t+h} S(s)v(t+h-s)ds \\ &= \frac{1}{h}\dot{S}(t)S(h)x + \frac{1}{h}\int_0^t S(s)\frac{d}{dt}\left(\int_{t-s}^{t+h-s} v(r)dr\right)ds \\ &\quad + \frac{1}{h}\int_0^h S(t+s)v(h-s)ds \end{aligned}$$

$$= \frac{1}{h}\dot{S}(t)S(h)x + \frac{d}{dt}\int_0^t S(s)\left(\frac{1}{h}\int_{t-s}^{t+h-s} v(r)dr\right)ds$$

$$+\frac{1}{h}\int_0^h (S(t+s) - S(t))v(h-s)ds$$

$$= \dot{S}(t)x_h + (S \diamond v_h)(t)$$

with

$$x_h = \frac{1}{h}S(h)x + \frac{1}{h}\int_0^h S(s)v(h-s)ds, \quad v_h(t) = \frac{1}{h}\int_t^{t+h} v(r)dr.$$

Since $w_h(t) \in D(A)$ and $Aw_h(t)$ is continuous in t, by (13),

$$(x_h, v_h) \in E \quad \text{and} \quad G(w_h(t)) = \Psi_0\,(x_h, v_h)\,(t).$$

As $x_h \to x$ and $v_h \to v$ in $L_1([0,T], Z)$ as $h \to 0$ and Ψ is the continuous extension of Ψ_0 by assumption, we have

$$\Psi(x,v) = \lim_{h \to 0} G \circ w_h \quad \text{in } L_1([0,T]; Z). \quad \blacksquare$$

PROPOSITION 7. *Suppose that $T > 0$ and the assumptions of Proposition 6 hold. If u is an F-solution of (9) on $[0,T]$, then $u = \dot{S}(\cdot)x + S \diamond v$ with a fixed point v of Ψ_x and u is an I-solution of (9).*

Proof. Let u be an F-solution in L_1. Then we find $u_j \in C^1([0,T], X) \cap C([0,T], X_A)$, $w_j \in L_1([0,T], X)$ such that $(Gu_j)_{j \in \mathbf{N}}$ converges in $L_1([0,T], X)$ and

$$\lim_{j \to \infty} \|u_j - u\| = 0,$$

$$\lim_{j \to \infty} \|w_j\|_1 = 0$$

and

$$u'_j = Au_j + G \circ u_j + w_j.$$

We set $v_j := G \circ u_j$. By Theorem 3,

(15) $$u_j = \dot{S}(\cdot)u_j(0) + S \diamond (G \circ u_j + w_j) = \dot{S}(\cdot)u_j(0) + S \diamond v_j + S \diamond w_j.$$

Since $u_j \in C([0,T], X_A)$, $(u_j(0), v_j + w_j)$ is an element of E by (13), and we have the following representation of v_j,

$$v_j = G \circ (\dot{S}(\cdot)u_j(0) + S \diamond v_j + S \diamond w_j) = \Psi_0(u_j(0), v_j + w_j).$$

Since Ψ is continuous and $u_j(0) \to u(0)$, $(v_j + w_j) \to v$ in L_1, we have

$$\Psi(u(0), v) = \lim_{j \to \infty} \Psi_0(u_j(0), v_j + w_j) = \lim_{j \to \infty} v_j = v$$

in $L_1([0,T], X)$ and v is a fixed point of Ψ_x, $x = u(0)$. Furthermore, by (15),

(16) $$u(t) = \lim_{j \to \infty} u_j(t) = \lim_{j \to \infty} \dot{S}(\cdot)u_j(0) + (S \diamond v_j)(t) + (S \diamond w_j)(t)$$

$$= \dot{S}(\cdot)u(0) + S \diamond v.$$

Then u is an I-solution by Proposition 5, Part 1. $\blacksquare$

Once we have found the unique fixed point v_x of Ψ_x in $L_1([0,T],Z)$, motivated by Propositions 6 and 7, we define

$$(17) \qquad \Theta(t,x) = \dot{S}(t)x + (S \diamond v_x)(t).$$

THEOREM 8. *Let $T > 0$ and assume that $L_1([0,T],Z)$ can equivalently be renormalized such that*

$$\|\Psi_0(x,v) - \Psi_0(y,w)\| \le K_1\|x-y\| + K_2\|v-w\| \qquad \forall(x,v),(y,w) \in E,$$

where $K_1 > 0$ and $K_2 \in (0,1)$.

Then there exists a continuous semiflow Θ on $[0,\infty) \times \overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and $u(t) = \Theta(t,x)$ are the unique I-solutions and the unique F-solutions of (10) on every finite interval $[0,\tau]$.

Proof. Since E contains the dense subspace $D(A_0) \times C_0^\infty((0,T);Z)$ of the Banach space $\overline{D(A)} \times L_1([0,T];Z)$, Ψ_0 has a Lipschitz extension Ψ to the latter space with the same Lipschitz constants.

Ψ is a uniform contraction on $\overline{D(A)} \times L_1([0,T];Z)$ and, by the parameter version of Banach's fixed point theorem, there are unique fixed points v_x of the mappings Ψ_x for all $x \in \overline{D(A)}$ and the dependence of the v_x on x is Lipschitz continuous on $[0,T]$ (see e.g. the proof of [8], Th. 2.2),

$$\|v_x - v_y\|_1 \le C\|x-y\| \qquad \forall x,y \in \overline{D(A)},$$

with a suitable constant $C > 0$. We define Θ as in (17). By Theorem 3, for $0 < t \le T$,

$$\|\Theta(t,x) - \Theta(t,y)\| \le \sup_{s\in[0,T]} \|\dot{S}(s)\|\|x-y\| + \|S \diamond (v_x - v_y)\|$$
$$\le Me^{\omega_+ T}(\|x-y\| + \|v_x - v_y\|_1) \le K\|x-y\|,$$

where $K = Me^{\omega_+ T}(1+C)$ and $\omega_+ = \max\{\omega,0\}$ is the positive part of ω. From Propositions 5 and 6 we know that $\Theta(\cdot,x)$ is the unique I-solution of (10).

The semiflow property follows from the fact that $u = \Theta(\cdot,x)$ is also the unique F-solution of (10). See Lemma 4 and Lemma 7. Notice that $\tilde{u}(t) = \Theta(t+s,x)$ is the F solution of (10) with initial value $\tilde{u}(0) = \Theta(s,x)$, so $\Theta(t+s,x) = \Theta(t,\Theta(s,x))$.

The semiflow Θ can be extended to $[0,2T]$ in the standard way: Let $t = s+r \in (T,2T]$ with $r,s \in [0,T]$. Set $\Theta(t,x) = \Theta(r,\Theta(s,x))$. The semiflow property on $[0,T]$ guarantees that this definition is independent of the choice of r and s and readily extends to $[0,2T]$. It also implies that $\Theta(\cdot,x)$ is an I-solution on $[0,2T]$,

$$\Theta(T+t,x) - x = \Theta(t,\Theta(T,x)) - x$$
$$= \Theta(T,x) - x + A\int_0^t \Theta(s,\Theta(T,x))ds$$
$$+ \lim_{h\searrow 0} \int_0^t G\left(\frac{1}{h}\int_s^{s+h} \Theta(r,\Theta(T,x))dr\right)ds$$

$$= A \int_0^T \Theta(s,x)ds + \lim_{h \searrow 0} \int_0^T G\Big(\frac{1}{h}\int_s^{s+h} \Theta(r,x)dr\Big)ds$$

$$+ A \int_0^t \Theta(s+T,x)ds + \lim_{h \searrow 0} \int_0^t G\Big(\frac{1}{h}\int_s^{s+h} \Theta(r+T,x)dr\Big)ds$$

$$= A \int_0^{T+t} \Theta(s,x)ds + \lim_{h \searrow 0} \int_0^{T+t} G\Big(\frac{1}{h}\int_s^{s+h} \Theta(r,x)dr\Big)ds,$$

and the limit exists in L_1. The Lipschitz continuity of $\Theta(t,x)$ in x easily extends to $[T, 2T]$ with the Lipschitz constant K^2. Continuing this way we can extend the semiflow to $[0, \infty)$ with

$$\|\Theta(t,x) - \Theta(t,y)\| \leq K^n \|x - y\| \qquad \forall t \in [(n-1)T, nT), \; n \in \mathbf{N}.$$

The exponential Lipschitz condition (5) is satisfied with $N = K$, $\theta = \frac{\ln K}{T}$. ∎

COROLLARY 9. *Let $T > 0$ and assume that $L_1([0,T], Z)$ can equivalently be renormalized such that*

$$(18) \qquad \|\Psi_0(x,v) - \Psi_0(y,w)\| \leq K_1\|x - y\| + K_2\|v - w\| \qquad \forall (x,v), (y,w) \in E,$$

where $K_1, K_2 > 0$. Then Ψ_0 has an extension Ψ satisfying the analogous inequality on $\overline{D(A)} \times L_1([0,T], X)$.

Further assume that there exist $n \in \mathbf{N}$ and $\varepsilon \in (0,1)$ such that for $\Psi_x = \Psi(x, \cdot)$,

$$\|\Psi_x^n(v) - \Psi_x^n(w)\| \leq \varepsilon \|v - w\| \qquad \forall v, w \in L_1([0,T], Z), x \in \overline{D(A)}.$$

Then there is a unique continuous semiflow Θ on $[0, \infty) \times \overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and $u(t) = \Theta(t, x)$ are the unique I-solutions and the unique F-solutions of (10) on every finite interval $[0, \tau]$.

Proof. By assumption, with $\Psi_x := \Psi(x, \cdot)$, Ψ_x^n is a strict contraction on the Banach space $L_1([0,T], Z)$ and so has a unique fixed point v_x. Since $\Psi_x(v_x)$ is also a fixed point of Ψ_x^n, $\Psi_x(v_x) = v_x$. Since every fixed point of Ψ_x is also a fixed point of Ψ_x^n, v_x is the unique fixed point of Ψ_x. For all $x, y \in \overline{D(A)}$,

$$\|v_x - v_y\| = \|\Psi_x^n(v_x) - \Psi_y^n(v_y)\|$$
$$\leq \|\Psi_x^n(v_x) - \Psi_x^n(v_y)\| + \|\Psi_x^n(v_y) - \Psi_y^n(v_y)\|$$
$$\leq \|v_x - v_y\| \, \varepsilon + K_n\|x - y\|,$$

for some constant $K_n > 0$. Reorganizing this inequality,

$$\|v_x - v_y\| \leq (1 - \varepsilon)^{-1} K_n \|x - y\|.$$

The remaining part of the corollary follows as in Theorem 8. ∎

For cases in which the estimate in Theorem 8 is difficult to check directly, we offer the following version which will be used in the Banach lattice case. Let $\phi : \mathbf{R} \to [0, \infty)$ be a continuous non-negative function with compact support and $\int_{\mathbf{R}} \phi(s)ds = 1$. Set

$$\phi_j(t) = j\phi(jt),$$

and for $v \in L_1([0, T]; X)$ define

$$v_j^\phi(t) = \int_0^\infty \phi_j(s - t)v(s)ds,$$

where v has been extended outside of $[0,T]$ in an arbitrary measurable essentially bounded fashion. One readily checks that $v_j^\phi \to v$, $j \to \infty$, uniformly on compact subsets of $(0,T)$, if v is continuous on $(0,T)$, and $v_j^\phi \to v$ in $L_1([0,T];X)$. If $v : [0,T] \to D(A)$ and Av is continuous on $(0,T)$ then $v_j^\phi(t) \in D(A)$ and $Av_j^\phi \to Av$, $j \to \infty$, uniformly on compact subsets of $(0,T)$. ϕ with the properties above is called a *mollifier* because v^ϕ is typically smoother than v.

COROLLARY 10. *Let $G : D(A) \to Z$ be A-continuous and E be the subspace defined in (13). Let $K_1, K_2 \geq 0$ and ϕ a mollifier as above such that, for all $(x,v),(y,w) \in E$,*

$$\liminf_{j \to \infty} \int_0^T \|G((\dot{S}(\cdot)x + S \diamond v)_j^\phi(t)) - G((\dot{S}(\cdot)y + S \diamond w)_j^\phi(t))\| dt$$
$$\leq K_1 \|x - y\| + K_2 \int_0^T \|v(t) - w(t)\| dt.$$

Then the map Ψ_0 in (14) satisfies the inequalities (18) in Corollary 9, and its extension Ψ satisfies the analogous inequalities on $\overline{D(A)} \times L_1([0,T],X)$ and equation (12).

Proof. Since G is A-continuous, we have for all $t \in (0,T)$ and $(x,v) \in E$

$$\lim_{j \to \infty} G((\dot{S}(\cdot)x + S \diamond v)_j^\phi(t)) = G(\dot{S}(t)x + S \diamond v(t)).$$

By Fatou's lemma, the inequalities (18) in Corollary 9 holds, and Ψ_0 has an extension Ψ on $\overline{D(A)} \times L_1([0,T],X)$ which satisfies the analogous inequalities and, by Proposition 6, equation (12). ∎

3. A Stieltjes type convolution of operator families with vector-valued L_1-functions.
The convolution we introduce in this section turns out to be useful in the proof of our perturbation theorems. The results we present are established in [34].

Let $I \subseteq \mathbf{R}$ be an interval which is bounded from below and

$$l(I) := \inf I \qquad \text{and} \qquad r(I) := \sup(I)$$

denote its left and right end-point. Unless explicitly stated otherwise, we always assume that $l(I) \in I$ and $r(I) \notin I$. Recall that, for every $f \in L_1(I,X)$, $\int_{l(I)}^{\cdot} f(t)dt$ is of bounded variation and its variation is given by

$$\mathbf{v}\left(\int_{l(I)}^{\cdot} f(t)dt; I \right) = \|f\|_1 = \int_{l(I)}^{r(I)} \|f(s)\| ds.$$

This motivates our next definition.

DEFINITION 3. A function $f \in L_1(I,X)$ is of *bounded pseudo-variation* if

$$\mathbf{pv}(f;I) := \overline{\lim_{h \searrow 0}} \frac{1}{h} \int_{l(I)}^{r(I)-h} \|f(s+h) - f(s)\| ds < \infty.$$

If $r(I) = \infty$, $r(I) - h$ is to be interpreted as ∞. Recall that $f : I \to X$ is *of bounded variation* if

$$\mathbf{v}(f;I) := \sup \left\{ \sum_{j=1}^k \|f(t_j) - f(t_{j-1})\| \right\} < \infty,$$

where the supremum is taken over all partitions $t_0 < \cdots < t_k$ of I where $k \in \mathbf{N}$ and $t_j \in I$. $f : I \to X$ is *Lipschitz continuous* if

$$\mathbf{pv}_\infty(f; I) := \sup \left\{ \frac{\|f(t) - f(s)\|}{|t - s|}; t, s \in I, t \neq s \right\} < \infty.$$

Every function of bounded pseudo-variation admits a version that is of bounded variation:

LEMMA 11. *Let $f : I \to X$ be of bounded pseudo-variation. Then*

$$(19) \qquad f^\oplus(t) := \lim_{h \searrow 0} \frac{1}{h} \int_t^{t+h} f(s) ds$$

exists for all $t \in I$, $f = f^\oplus$ a.e., and $f^\oplus$ is of bounded variation on I with $\mathbf{v}(f^\oplus, I) = \mathbf{pv}(f, I)$. Further $(f^\oplus)^\oplus = f^\oplus$ and $f^\oplus$ is right continuous.

We set

$$(20) \qquad \mathbf{pv}^\oplus(f; I) := \mathbf{pv}(f; I) + \|f^\oplus(l(I))\|.$$

PROPOSITION 12. *The vector space*

$$\mathcal{PV}(I; X) := \{f^\oplus; \ f : I \to X \text{ is of bounded pseudo-variation}\}$$

equipped with the norm $\mathbf{pv}^\oplus(f; I)$ is a Banach space.

The analogous result for the space of Lipschitz continuous functions is well known. We extend the previous concepts to operator families.

DEFINITION 4. A family of bounded linear operators $(U(t))_{t \in I}$ from a Banach space X into a Banach space Y is called to be of *bounded strong pseudo-variation* if

$$\mathbf{pv}(U(\cdot)x; I) < \infty \qquad \forall x \in X.$$

It is easy to check that $x \mapsto U(\cdot)x$ is a closed linear operator from X into the Banach space $\mathcal{PV}(I; Y)$ and so a bounded linear operator; this implies that

$$(21) \quad \begin{aligned} \mathbf{PV}(U; I, X) &:= \sup_{\|x\| \leq 1, \ x \in X} \mathbf{pv}(U(\cdot)x; I) < \infty, \\ \mathbf{PV}^\oplus(U; I, X) &:= \sup_{\|x\| \leq 1, \ x \in X} \mathbf{pv}^\oplus(U(\cdot)x; I) < \infty, \end{aligned}$$

and

$$(22) \quad \begin{aligned} \mathbf{pv}(U(\cdot)x; I) &\leq \mathbf{PV}(U; I, X) \|x\|, \\ \mathbf{pv}^\oplus(U(\cdot)x; I) &\leq \mathbf{PV}^\oplus(U; I, X) \|x\|, \end{aligned} \qquad \forall x \in X.$$

We call $\mathbf{PV}(U; I, X)$ the *strong pseudo-variation* of U on $I \times X$.

It is well known and follows in the same way that $U(\cdot)$ is Lipschitz continuous in operator norm if it is strongly Lipschitz continuous. We set

$$\mathbf{PV}_\infty(U; I, X) = \sup \left\{ \frac{\|U(t) - U(t)\|}{|t - s|}; t, s \in I, t \neq s \right\}.$$

We introduce

$$U^\oplus(t)x = \lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} U(s)x ds$$

which exists according to Lemma 11. Notice that $U(t)x$ and $U^{\oplus}(t)x$ have the same pseudo-variation.

Before we define a Stieltjes type convolution of U with $f \in L_1(I, X)$, with $l(I) = 0$, we recall that the standard convolution

$$(U * f)(t) := \int_0^t U(s)f(t-s)ds, \qquad t \in I,$$

defines a function in $L_1(I, Y)$, if $f \in L_1(I, X)$ and $U(t)$ is strongly integrable on I. If U is of bounded strong pseudo-variation, we actually have the following result.

PROPOSITION 13. *Let I be a bounded interval, U be of bounded strong pseudo-variation on I and $f \in L_1([0,T], X)$. Then $U * f \in L_\infty(I, Y)$ and*

$$\|U * f\|_\infty \leq \sup_{\|x\| \leq 1} \|U(\cdot)x\|_\infty \|f\|_1.$$

In fact, much more holds.

THEOREM 14. *Let U be of bounded strong pseudo-variation on I and $f \in L_1(I, X)$. Then $U * f$ is continuous on I. Moreover the limit*

$$U \star f = \lim_{h \searrow 0} \frac{1}{h} \int_0^{\cdot} (U(s+h) - U(s))f(\cdot - s)ds,$$

exists in $L_1(I, Y)$,

$$\|U \star f\|_1 \leq \mathbf{PV}(U; I, X) \|f\|_1,$$

and

$$(U * f)(t) = \int_0^t (U \star f)(s)ds + \int_0^t U^{\oplus}(0)f(s)ds.$$

Finally

$$U^{\oplus} \star f = U \star f.$$

We define

(23) $$(U \diamond f)(t) = (U \star f)(t) + U^{\oplus}(0)f(t).$$

Our Stieltjes type convolution relates to the standard convolution as follows.

COROLLARY 15. *Let $I = [0, T)$ and $(U(t))_{t \in I}$ be a family of bounded linear operators from X to Y that is of bounded strong pseudo-variation.*

(a) *Then, for all $f \in L_1(I, Y)$,*

$$(U * f)(t) = \int_0^t (U \diamond f)(s)ds \quad \forall t \in I, \qquad \|U \diamond f\|_1 \leq \mathbf{PV}^{\oplus}(U; [0, T), X) \|f\|_1.$$

(b) *If $f : [0, T) \to X$ is of bounded pseudo-variation, then $U \diamond f$ is of bounded pseudo-variation on $[0, T)$,*

$$\mathbf{pv}^{\oplus}(U \diamond f; I) \leq \mathbf{PV}^{\oplus}(U; I, X)\mathbf{pv}^{\oplus}(f; I).$$

Further $U^{\oplus} \diamond f^{\oplus} = U \diamond f^{\oplus} = U^{\oplus} \diamond f = U \diamond f$ a.e. on I,

$$(U \diamond f)^{\oplus}(0) = U^{\oplus}(0)f^{\oplus}(0).$$

(c) *If U is Lipschitz continuous and $f \in L_1(I, X)$, then $U \star f$ is continuous with*

$$\|U \star f\|_\infty \leq \mathbf{PV}_\infty(U; I, X) \, \|f\|_1.$$

If $U(t)$ and $V(t)$, $t \in I$, are families of bounded linear operators which are of bounded strong pseudo-variation, then, for each x, $U \diamond (V(\cdot)x)$ is defined and a function of bounded pseudo-variation, but difficult to turn into a family of bounded linear operators $W(t)x$, because it is uniquely determined up to a set of Lebesgue measure 0 which depends on x. This situation can be remedied by setting

$$(U \diamond V)(t)x = (U \diamond [V(\cdot)x])^\oplus.$$

By Lemma 11 and Corollary 15 (b), $U \diamond V$ is a family of bounded linear operators that is of locally bounded strong pseudo-variation and strongly right-continuous.

PROPOSITION 16. *Let $f : [0, T) \to X$ be of bounded pseudo-variation and $U(t)$, $V(t)$, $t \in [0, T)$, form families of bounded linear operators on X that are of bounded strong pseudo-variation. Then*

$$(U \diamond V) \diamond f = U \diamond (V \diamond f) \quad a.e.$$

We define recursively

$$U^{\diamond 1} = U^\oplus, \qquad U^{\diamond(n+1)} = U \diamond U^{\diamond n}.$$

COROLLARY 17. *Let $(U(t))_{t \in I}$ be a family of bounded linear operators that is of locally bounded strong variation. Then*

$$\mathbf{PV}^\oplus(U^{\diamond n}; I, X) \leq [\mathbf{PV}^\oplus(U; I, X)]^n \qquad \forall n \in \mathbf{N}.$$

If $U(t)$ maps X into a closed subspace Z, then

$$\mathbf{PV}^\oplus(U^{\diamond(n+1)}; I, X) \leq [\mathbf{PV}^\oplus(U; I, Z)]^n \, \mathbf{PV}^\oplus(U; I, X) \qquad \forall n \in \mathbf{N}.$$

Application to integrated semigroups. If we apply our results to integrated semigroups we obtain the following generalization of Theorem 3. Notice that $S \diamond f = S \star f$ because $S^\oplus(0) = 0$.

LEMMA 18. *Let S be an integrated semigroup which is of locally bounded strong pseudo-variation. Then $(S \diamond f)(t) \in \overline{D(A)}$ and $(S \ast f)(t) \in D(A)$ for all $f \in L_1(I, X)$ and almost all $t \geq 0$. Furthermore we have*

$$A(S \ast f)(t) = (S \diamond f)(t) - \int_0^t f(s)ds \quad for \ a.a. \ t \geq 0$$

and

$$(S \ast f)(t) = \int_0^t (S \diamond f)(s)ds \quad \forall t \in I.$$

4. Cumulative outputs of bounded strong variation.

Recall that X, Y are Banach spaces and A is a Hille-Yosida operator on X. The locally Lipschitz continuous integrated semigroup generated by A is denoted by $(S(t))_{t \geq 0}$ and the C_0-semigroup on $\overline{D(A)}$ generated by A_0 is denoted by $(\dot{S}(t))_{t \geq 0}$.

ASSUMPTION H4. Let B be an A-bounded operator with values in Y, $T \in [0, \infty]$, such that for every $x \in X$,

$$B \int_0^t S(s)x\,ds = \int_0^t V(s)x\,ds, \qquad t \in [0, T),$$

with a family $(V(t))_{t \geq 0}$ of bounded linear operators from X to Y that is of bounded strong pseudo-variation on $I = [0, T)$.

REMARK 1. Replacing V by $V^{\oplus}$ we see that $B \int_0^t S(s)ds$ is right differentiable with derivative $V^{\oplus}(t)x$. So without restriction, we assume this from the very beginning, with $V(t)$ being strongly right continuous.

LEMMA 19. *Let Assumption H4 be satisfied. Then the following holds:*

1. *For all $x \in \overline{D(A)}$ and all $t \in I$, $BS(t)x = V(t)x$ and $V(\cdot)x$ is a.e. differentiable on $[0, T)$ with $\frac{d}{dt}V(\cdot)x \in L_1([0, T); X)$ and $V(t)x = \int_0^t \frac{d}{ds}V(s)x\,ds$.*
   For all $x \in D(A)$ and all $t \in I$, $V(t)x$ is right differentiable and

   $$Bx + V(t)Ax = V'(t)x.$$

   For all $x \in D(A_0)$ and all $t \in I$, $V(t)x$ is continuously differentiable and $B\dot{S}(t)x = V'(t)x$.

2. *For all $f \in L_1(I, X)$ and $t \in I$,*

   $$(V * f)(t) = B(S * f)(t).$$

3. *If $(x, f) \in E$, E as in (13), then $V(\cdot)x + V * f$ is continuously differentiable on $[0, T]$ and*

   $$B(\dot{S}(t)x + (S \diamond f)(t)) = \frac{d}{dt}(V(t)x + (V * f)(t)).$$

   In particular $V(\cdot)x \in W_{1,loc}^1[0, T]$ and

   $$B(\dot{S}(t)x + (S \diamond f)(t)) = V'(t)x + (V * f)'(t).$$

Proof. The first statement in 1. is obvious because $A \int_0^t S(s)x\,ds$ is differentiable for $x \in \overline{D(A)}$ with derivative $AS(t)x$. Similarly, for $x \in D(A_0)$, $AS(t)x$ is differentiable with derivative $A\dot{S}(t)x = \dot{S}(t)A_0x$ and so is $BS(t)x = V(t)x$ with derivative $B\dot{S}(t)x$. For $x \in D(A)$, by part 1 and (7),

$$V(t)x = Btx + B \int_0^t S(s)Ax\,ds = tBx + \int_0^t V(s)Ax\,ds.$$

We see that $V(t)x$ is right differentiable and

$$V'(t)x = Bx + V(t)Ax, \quad x \in D(A).$$

Returning to the case that $x \in D(A_0)$, we have that

$$\int_0^T |V'(t)x|\,dt = \mathbf{pV}(V(\cdot)x; [0, T)) \leq \mathbf{PV}\left(V; [0, T), \overline{D(A)}\right)\|x\|.$$

If $x \in \overline{D(A)} = \overline{D(A_0)}$, we find a sequence (x_n) in $D(A_0)$ such that $x_n \to x$ as $n \to \infty$. By the previous estimate, the functions $V'(\cdot)x_n$ form a Cauchy sequence in the Banach

space $L_1([0, T); X)$ and have a limit v therein. Then

$$\int_0^t v(s)ds = \lim_{n\to\infty} \int_0^t V'(s)x_n ds = \lim_{n\to\infty} V(t)x_n = V(t)x.$$

As for 2. we first recall that $A(S * f)(t)$ is continuous in t (Theorem 3). Since B is A-continuous, $B(S * f)(t)$ is continuous. Changing the order of integration shows that

$$\int_0^t (S * f)(r)dr = \int_0^t \left(\int_0^{t-s} S(r)f(s)dr \right) ds.$$

So

$$\int_0^t B(S * f)(r)dr = B \int_0^t (S * f)(r)dr = \int_0^t B \left(\int_0^{t-s} S(r)f(s)dr \right) ds$$

$$= \int_0^t \left(\int_0^{t-s} V(r)f(s)dr \right) ds = \int_0^t (V * f)(r)dr.$$

Since $V * f$ is continuous, the assertion follows.

3. As B is A-bounded and $A(\dot{S}(t)x + (S \diamond f)(t))$ is continuous,

$$B(\dot{S}(t)x + (S \diamond f)(t))$$

$$= \lim_{h\to 0} \frac{1}{h} B(S(t+h)x - S(t)x + (S * f)(t+h) - (S * f)(t))$$

$$= \lim_{h\to 0} \frac{1}{h} (V(t+h)x - V(t)x + (V * f)(t+h) - (V * f)(t))$$

uniformly for t in bounded intervals. So $V(t)x + (V * f)(t)$ is continuously differentiable in t with derivative $B(\dot{S}(t)x + (S \diamond f)(t))$. ∎

It follows from Lemma 19 (1) and the fact that $S(t)$ maps X into $\overline{D(A)}$ that V has the properties of a cumulative output defined below [12, 24, 32]. In some circumstances it is convenient to start from the operator family $V(t)$ rather than the operator B.

DEFINITION 5. A family $(V(t))_{t\geq 0}$ of bounded operators from X to Y is a *cumulative output* for $(S(t))_{t\geq 0}$ if the following holds: $V(t)$ is strongly right continuous at any $t > 0$, the right limit $V(0+)$ exists strongly, and

$$(24) \qquad V(t)S(r) = \int_0^{t+r} V(s)ds - \int_0^t V(s)ds - \int_0^r V(s)ds \qquad \forall t, r \geq 0.$$

It follows from the uniform boundedness theorem and Lemma 4.14 in [32] that $V(t)$ is exponentially bounded.

REMARK 2. Since $S(\cdot)x$ is continuously differentiable for all $x \in \overline{D(A)}$ we obtain

$$V(t)S'(r)x = V(t+r)x - V(r)x, \qquad t, r > 0.$$

If $w(V)$ denotes the exponentially bound for $(V(t))_{t\geq 0}$, we define

$$F(\lambda)x = \lambda \int_0^\infty e^{-\lambda t} V(t)x dt, \qquad \lambda > w(V),\ x \in X.$$

Since V is a cumulative output for S, $F(\lambda)$ is a resolvent output of A, i.e.,

$$F(\lambda)(\mu - A)^{-1} = \frac{1}{\mu - \lambda}(F(\lambda) - F(\mu)), \qquad \lambda, \mu > w(V), \lambda \neq \mu.$$

So we can define $\hat{B} = F(\lambda)(\lambda - A)$ and show that this definition is independent of $\lambda > w(V)$. A straightforward Laplace transform argument shows that Assumption H4 is satisfied with $\hat{B}$ replacing B.

5. Application to the semilinear Cauchy problem. We state the additional assumptions on G and A.

ASSUMPTION H5. There exists a linear operator $B : D(A) \to Y$ into a Banach space Y, $T > 0$ such that

$$\|G(x) - G(y)\| \le \|B(x - y)\| \qquad \forall x, y \in D(A),$$

and

$$B \int_0^t S(r)x\,dr = \int_0^t V(r)x\,dr, \qquad t \in [0, T), x \in X,$$

with an operator family $(V(t))_{t \in [0,T)}$ from X to Y which is of bounded strong pseudo-variation and $\mathbf{PV}^{\oplus}(V; [0, T), Z) < 1$.

THEOREM 20. *Under Assumption H5, there is a unique continuous semiflow Θ on $\overline{D(A)}$ such that the exponential Lipschitz condition (5) is satisfied and the orbits of Θ uniquely solve the Cauchy problem (3) in the L_1-Friedrichs sense (L_1-F).*

Proof. Let $T > 0$. We show that, under Assumption H5, the assumptions of Theorem 8 are fulfilled with

$$K_1 = \mathbf{PV}(V; [0, T), X), \qquad K_2 = \mathbf{PV}^{\oplus}(V; [0, T), Z).$$

Let $(x, v), (y, w) \in E$, E as in (13). By Lemma 19 and H5,

$$\|G(\dot{S}(t)x + (S \diamond v)(t)) - G(\dot{S}(t)y + (S \diamond w)(t))\|$$
$$\le \|B(\dot{S}(t)x + (S \diamond v)(t)) - B(\dot{S}(t)y + (S \diamond w)(t))\|$$
$$= \|V'(t)(x - y) + (V \diamond (v - w))(t)\|$$

for a.a. $t \ge 0$. By Corollary 15 (a),

$$\int_0^T \|G(\dot{S}(t)x + (S \diamond v)(t)) - G(\dot{S}(t)y + (S \diamond w)(t))\|\,dt$$
$$\le \mathbf{PV}(V; [0, T), X)\,\|x - y\| + \mathbf{PV}^{\oplus}(V; [0, T), Z)\,\|v - w\|. \quad \blacksquare$$

This theorem will work well if we have enough information about the integrated semigroup S generated by A. Aesthetically it may be more pleasing if all assumptions are in terms of A (or its resolvent) and G.

COROLLARY 21. *There exists a linear operator $B : D(A) \to Y$ into a Banach space Y such that*

$$\|G(x) - G(y)\| \le \|B(x - y)\| \qquad \forall x, y \in D(A),$$

and there exist $\mu > 0$ and $\gamma \in (0, 1)$ and $c > 0$ such that

$$\int_0^\infty \|B(\lambda + \mu - A)^{-(k+1)}x\|k\lambda^{k-1}\,d\lambda + \liminf_{\lambda \to \infty} \|B(\lambda - A)^{-1}x\| \le \gamma\|x\|$$
$$\forall x \in Z, k \in \mathbf{N},$$

$$\int_0^\infty \|B(\lambda + \mu - A)^{-(k+1)}x\|k\lambda^{k-1}d\lambda \le c\|x\| \qquad \forall x \in \overline{D(A)}, k \in \mathbf{N}.$$

Then there is a unique continuous semiflow Θ on $\overline{D(A)}$ such that the orbits of Θ uniquely solve the Cauchy problem (3) in the Friedrichs sense (L_1-F) for all $x \in \overline{D(A)}$.

Proof. By Theorem 6.8 and the subsequent remark in [34], there is a family of bounded operators $(V(t))_{t \ge 0}$ and some $w > 0$ such that

$$B(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t}V(t)x\,dt \qquad \forall \lambda > w, x \in X,$$

and V is of bounded strong (pseudo)-variation on every finite interval, $V(0) = 0$. Moreover the strong right limit $V(0+) = V^\oplus(0)$ exists and

$$V^\oplus(0)x = \lim_{\lambda \to \infty} B(\lambda - A)^{-1}x.$$

For sufficiently large $\lambda > 0$,

$$\|G(x) - G(y)\| \le \|B(\lambda - A)^{-1}\|\,\|(\lambda - A)(x - y)\| \qquad \forall x, y \in D(A)$$

and therefore G is A-continuous. A straightforward Laplace transform argument shows that Assumption H4 is satisfied on every finite interval provided we find some $T > 0$ with $\mathbf{PV}^\oplus(V; [0, T), Z) < 1$. To this end, let

$$V^\mu(t) = e^{-\mu t}V(t) + \mu \int_0^t e^{-\mu s}V(s)ds.$$

By Proposition 3.1 in [34],

$$B(\lambda + \mu - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t}V^\mu(t)dt.$$

V^μ is of bounded pseudo-variation on $[0, \infty)$, and $\mathbf{PV}^\oplus(V^\mu; [0, \infty), Z) \le \gamma$ as follows from [39] or [4] (see also [34], Theorem 4.2). It follows from Proposition 3.1 (c) in [34] that, for any $T > 0$,

$$\limsup_{h \to 0} \frac{1}{h} \int_0^{T-h} e^{-\mu t}\|V(t+h)x - V(h)x\|dt + \|V^\oplus(0)x\|$$
$$= \mathbf{pv}^\oplus(V^\mu(\cdot)x), [0, T)) \le \gamma\|x\| \qquad \forall x \in Z.$$

So

$$\mathbf{PV}^\oplus(V; [0, T); Z) < \gamma e^{\mu T}.$$

By choosing $T > 0$ small enough we achieve that $\gamma e^{\mu T} < 1$ and our assertion follows from Theorem 20. ∎

6. Semilinear Miyadera-Voigt theorems. For densely defined A, $X = Z$ and $G = B$ we obtain a new proof of the well known Miyadera-Voigt perturbation theorem for C_0-semigroups [35] that makes no use of the Dyson-Phillips series and works for semi-linear perturbations of Miyadera-Voigt type. If the perturbation is of Kato type, the density of $D(A)$ can be dropped.

THEOREM 22. *Let A be the generator of a C_0-semigroup $\dot{S}$ on a Banach space X, $Z \subseteq X$ be a closed subspace of X and Y be another Banach space. Let D be a dense subspace of X such that $D \cap Z$ is dense in Z. Let $G_0 : D \to Z$, $B_0 : D \to Y$, B_0 linear, such that*

$$\|G_0(x) - G_0(y)\| \le \|B_0(x - y)\| \qquad \forall x, y \in D,$$

and the following conditions are satisfied:

1. *$D \subseteq D(A)$, $\dot{S}(t)D \subseteq D$, and $B_0\dot{S}(\cdot)x$ continuous for all $x \in D$.*
2. *There are α, λ, $K > 0$ and $\gamma \in (0,1)$ such that*
   (a) *$\int_0^\alpha e^{-\lambda t}\|B_0\dot{S}(t)x\|dt \le K\|x\|$ for all $x \in D$.*
   (b) *$\int_0^\alpha e^{-\lambda t}\|B_0\dot{S}(t)x\|dt \le \gamma \|x\|$ for all $x \in D \cap Z$.*

Then there is a unique A-continuous extension $G : D(A) \to Z$ of G_0 and a unique continuous semiflow Θ on X such that the exponential Lipschitz condition (5) and the following are satisfied:

1. *For all $x \in X$, $\Theta(\cdot, x)$ uniquely solves the Cauchy problem (3) in the Friedrichs sense (L_1-F).*
2. *For all $x \in X$ and $t \ge 0$ we have*

$$(25) \qquad \Theta(t,x) = \dot{S}(t)x + \lim_{h \to 0} \int_0^t \dot{S}(t-s)G(\frac{1}{h}\int_s^{s+h}\Theta(r,x)dr)ds.$$

REMARK 3. In the proof we will show that B_0 is $A|_D$ bounded and can be uniquely extended to an A-bounded linear operator $B : D(A) \to Y$, answering a question asked in [35], Remark 5.

This particular result even holds if condition 2. is replaced by the following weaker one: There are $\alpha, K > 0$ such that

$$\|\int_0^t B_0\dot{S}(r)xdr\| \le K\|x\| \qquad \forall x \in D, t \in [0, \alpha].$$

Proof. Define

$$V(t)x = \int_0^t B_0\dot{S}(s)xds, \qquad t \ge 0, x \in D.$$

Since $\dot{S}$ is a semigroup that leaves D invariant,

$$V(t)\dot{S}(r)x = V(t+r)x - V(r)x, \qquad t, r \ge 0, x \in D.$$

By our assumptions,

$$\|V(t)x\| \le e^{\lambda\alpha}K\|x\|, \qquad x \in D, t \in [0, \alpha].$$

$V(t)$ can be uniquely extended to a bounded linear operator from X to Y for $t \in [0, \alpha]$, satisfying the same estimate. The functional relationship between V and $\dot{S}$ shows inductively that $V(t)$ can be extended to a bounded linear operator from X to Y for $t \in [0, n\alpha]$, $n \in \mathbf{N}$,

$$(26) \qquad V(t)\dot{S}(r) = V(t+r) - V(r), \qquad t, r \ge 0.$$

Cf. Remark 2. This functional relationship implies that V is exponentially bounded,

$$\|V(t)\| \le \tilde{M}e^{\tilde{\omega}t}, \qquad t \ge 0,$$

with appropriate constants $\tilde{M}, \tilde{\omega}$. See the proof of Lemma 4.14 in [32]. It also follows that $V(t)x \to 0$, $t \to 0$, first for $x \in D$, but then also for $x \in X$. By (26), $V(t)$ is strongly right continuous. Set

$$F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} V(t) dt, \qquad \lambda > \tilde{\omega}.$$

Taking Laplace transforms of the functional relation (26) between V and $\dot{S}$ shows that $F(\cdot)$ is a resolvent output for A,

$$F(\lambda)(\mu - A)^{-1} = \frac{1}{\mu - \lambda}(F(\lambda) - F(\mu)), \qquad \mu, \lambda > \tilde{\omega}, \lambda \neq \mu.$$

Thus the definition $B := F(\lambda)(\lambda - A)$ is independent of $\lambda > \tilde{\omega}$ and provides an A-bounded operator B. Set

$$S(t) = \int_0^t \dot{S}(r) dr.$$

Then

$$BS(t)x = F(\lambda)(\lambda - A)S(t)x = F(\lambda)[\lambda S(t)x - \dot{S}(t)x + x]$$

forms a strongly continuous exponentially bounded family of bounded linear operators. Taking Laplace transforms,

$$\lambda \int_0^\infty e^{-\lambda t} BS(t) dt = F(\lambda) = \lambda \int_0^\infty e^{-\lambda t} V(t) dt.$$

Since $BS(t)$ is strongly continuous and $V(t)$ strongly right continuous, the uniqueness properties of the Laplace transform imply that $V(t) = BS(t)$ for all $t \geq 0$ and the second formula in Assumption H5 follows. Further, for $x \in D$, both $V(t)x$ and $BS(t)x$ can continuously be differentiated yielding $B_0 \dot{S}(t)x = B\dot{S}(t)x$ for all $t \geq 0$. Specializing to $t = 0$, $B_0 x = Bx$, i.e., B is an A-bounded extension of B_0.

Actually it follows from our assumptions that D is a core for A, i.e., D is dense in $D(A)$ with respect to the graph norm of A. See [13], Proposition 1.7, and [35], p. 168, where additional references can be found.

This allows to extend G_0 to a map G from $D(A)$ to Z as follows. Let $x \in D(A)$. Then there exists a sequence (x_n) in D such that $x_n \to x$ and $Ax_n \to Ax$ as $n \to \infty$. Since B is an A-bounded extension of B_0, $B_0 x_n \to Bx$ as $n \to \infty$. The relation between G_0 and B_0 implies that $(G_0(x_n))$ is a Cauchy sequence, and we can define

$$G(x) = \lim_{n \to \infty} G_0(x_n).$$

This definition is independent of the choice of the sequence (x_n) and

$$\|G(x) - G(y)\| \leq \|B(x - y)\| \qquad \forall x, y \in D(A).$$

Since B is A-bounded, G is an A-continuous extension of G_0, and as such uniquely determined by G_0 because D is dense in $D(A)$ with respect to the graph norm of A.

By Lemma 19 (1), there exists a bounded linear map $\dot{V} : X \to L_1([0, \alpha], Y)$ such that $(\dot{V}x)(s) = B_0 \dot{S}(s)x$ for $x \in D$ and

$$V(t)x = \int_0^t (\dot{V}x)(s) ds$$

and

$$\int_0^\alpha e^{-\lambda t} \|(\dot{V}x)(t)\| dt \le K\|x\| \qquad \forall x \in X,$$

$$\int_0^\alpha e^{-\lambda t} \|(\dot{V}x)(t)\| dt \le \gamma \|x\| \qquad \forall x \in Z.$$

Let $\tau \in (0, \alpha]$. Then

$$\mathbf{pv}^\oplus(V(\cdot)x; [0, \tau)) = \int_0^\tau \|(\dot{V}x)(t)\| dt \le K e^{\lambda\tau}\|x\| \qquad \forall x \in X,$$

$$\mathbf{pv}^\oplus(V(\cdot)x; [0, \tau)) \le \gamma e^{\lambda\tau}\|x\| \qquad \forall x \in Z.$$

Choosing $\tau > 0$ small enough we can achieve that $\gamma e^{\lambda\tau} < 1$. The existence and the properties of the semiflow Θ now follow from Theorem 20. ∎

THEOREM 23. *Let A be a Hille-Yosida operator in a Banach space X, A_0 the part of A in $\overline{D(A)}$ and $\dot{S}(\cdot)$ the C_0-semigroup generated by A_0.*

Let D be a dense subspace of $\overline{D(A)}$ and Y another Banach space. Let $G_0 : D \to X$, $B_0 : D \to Y$, B_0 linear, such that

$$\|G_0(x) - G_0(y)\| \le \|B_0(x - y)\| \qquad \forall x, y \in D,$$

and the following conditions are satisfied:

1. *$D \subseteq D(A_0)$, $\dot{S}(t)D \subseteq D$, and $B_0\dot{S}(\cdot)x$ continuous for all $x \in D$.*
2. *For every $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\int_0^\delta \|B_0\dot{S}(t)x\| dt \le \epsilon \|x\| \text{ for all } x \in D.$$

Then there is a unique A-continuous extension $G : D(A) \to X$ of G_0 such that $G(x) = \lim_{\lambda\to\infty} G\left(\lambda(\lambda - A)^{-1}x\right)$ for all $x \in D(A)$ and a unique continuous semiflow Θ on X such that the exponential Lipschitz condition (5) and the following are satisfied:

1. *For all $x \in X$, $\Theta(\cdot, x)$ uniquely solves the Cauchy problem (3) in the Friedrichs sense (L_1-F).*
2. *For all $x \in X$ and $t \ge 0$ we have*

$$(27) \qquad \Theta(t, x) = \dot{S}(t)x + \lim_{h\to 0} \int_0^t \dot{S}(t - s)G\left(\frac{1}{h}\int_s^{s+h} \Theta(r, x)dr\right)ds.$$

Perturbations $(A + B_0)_{|D}$ of this kind are said to be of *Kato* type [37]. We mention that Hille-Yosida operators are preserved under Kato type perturbations ([34], Theorem 6.11).

Proof. The same proof as for Theorem 22 provides a strongly continuous family of bounded linear operators $V_0(t) : \overline{D(A)} \to Y$, $V_0(0) = 0$, with

$$V_0(t)\dot{S}(r)x = V_0(t + r)x - V_0(r)x, \qquad t, r \ge 0, x \in \overline{D(A)},$$

and

$$\mathbf{PV}^\oplus(V_0; [0, \delta), \overline{D(A)}) \le \epsilon,$$

where δ has been chosen in dependence of ϵ as in 2. of the assumptions of this theorem. In particular $V_0(t) \to 0$, $t \to 0$, in operator norm. The functional relation with $\dot{S}$ implies

that $V_0(t)$ is right continuous in operator norm. Integrating over r,

$$(28) \qquad V_0(t)S(r)x = \int_0^{t+r} V_0(s)xds - \int_0^t V_0(s)xds - \int_0^r V_0(s)xds,$$
$$t, r \geq 0, x \in \overline{D(A)}.$$

Then V_0 can be extended to a cumulative output for S on X (i.e., (28) holds for $x \in X$) by

$$V(t) := \lim_{h \searrow 0} V_0(t)\frac{1}{h}S(h)$$

with convergence holding in operator norm (cf. Definition 5). See the proof of Theorem 6.11 in [34]. It is readily checked that V is a cumulative output for S on X. Further

$$\mathbf{PV}^{\oplus}(V; [0, \delta), X) \leq M\mathbf{PV}^{\oplus}(V; [0, \delta), \overline{D(A)}) \leq M\epsilon,$$

with $M = \liminf_{h \searrow 0} \frac{1}{h}\|S(h)\|$. Choosing $\epsilon > 0$ and $\delta > 0$ small enough,

$$\mathbf{PV}^{\oplus}(V; [0, \delta), X) < 1.$$

By Lemma 4.14 in [32], $V(\cdot)$ is exponentially bounded and, as in the proof of Theorem 22, we define

$$F(\lambda) = \lambda \int_0^\infty e^{-\lambda t}V(t)dt.$$

Then $F(\lambda) \to 0$ as $\lambda \to \infty$ in operator norm. By taking Laplace transforms of the cumulative output relation (28), for V rather than V_0, $F(\cdot)$ is a resolvent output for A and the definition $B = F(\lambda)(\lambda - A)$ is independent of λ. The uniqueness properties of the Laplace transform imply that

$$B \int_0^t S(r)dr = \int_0^t V(r)dr.$$

Let $\tilde{B}$ be another extension of B_0 to $D(A)$, $\tilde{B}(\lambda - A)^{-1} \to 0$, $\lambda \to \infty$, in operator norm. Since D is a core for A_0, $\tilde{B}$ and B coincide on $D(A_0)$. Set $\tilde{F}(\lambda) = \tilde{B}(\lambda - A)^{-1}$. Then

$$\tilde{B}x = \tilde{F}(\lambda)(\lambda - A)x = \lim_{\mu \to \infty} \mu\tilde{F}(\mu)x, \qquad x \in D(A).$$

Similarly $Bx = \lim_{\mu \to \infty} \mu F(\mu)x$, $x \in D(A)$. But $F(\lambda)x = B(\lambda - A)^{-1}x = \tilde{B}(\lambda - A)^{-1}x = \tilde{F}(\lambda)x$ for $x \in D(A)$ because $(\lambda - A)^{-1}$ maps $D(A)$ into $D(A_0)$, so $\tilde{B} = B$.

By the definition of B,

$$Bx = \lim_{\lambda \to \infty} \lambda F(\lambda)x = \lim_{\lambda \to \infty} \lambda B(\lambda - A)^{-1}x, \qquad x \in D(A).$$

As in the proof of Theorem 22, G_0 has a unique A_0-continuous extension $\tilde{G} : D(A_0) \to X$ which satisfies

$$\|\tilde{G}(x) - \tilde{G}(y)\| \leq \|B(x - y)\|, \qquad x, y \in D(A_0).$$

It follows that, for $x \in D(A)$, the family $(\tilde{G}(\lambda(\lambda - A)^{-1}x))$ is a Cauchy net as $\lambda \to \infty$. Hence the limit

$$G(x) = \lim_{\lambda \to \infty} \tilde{G}(\lambda(\lambda - A)^{-1}x), \qquad x \in D(A),$$

exists and G satisfies $\|G(x) - G(y)\| \leq \|B(x-y)\|$ for all $x, y \in D(A)$. Since B is A-bounded, G is A-continuous. Further G is uniquely determined by $\tilde{G}$ and the property $G(x) = \lim_{\lambda \to \infty} G(\lambda(\lambda - A)^{-1}x)$ for all $x \in D(A)$.

We have checked all assumptions of Theorem 20 and our assertions follow. ∎

7. Semilinear perturbations in Banach lattices. We consider a Banach lattice X with positive convex cone X_+. Following [3, 1], we call a linear operator A in X resolvent positive, if $(a, \infty) \subseteq \rho(A)$ for some $a \in \mathbf{R}$ and $(\lambda - A)^{-1} \geq 0$ for all $\lambda > a$. Here $\rho(A)$ denotes the resolvent set of A. We refer to [26] for more details on ordered Banach spaces and positive operators.

We say that a linear operator A in X is *absolutely resolvent majorized* by a resolvent positive operator $\bar{A}$ if $(a, \infty) \in \rho(A) \cap \rho(\bar{A})$ for some $a \in \mathbf{R}$ with

$$|(\lambda - A)^{-1}x| \leq (\lambda - \bar{A})^{-1}|x| \qquad \text{for all } x \in X \text{ and } \lambda > a.$$

Given a family of bounded operators $(V(t))_{t \geq 0}$ of locally bounded strong variation we obtain iterative convolutions $V^{\diamond n}$ for $n \in \mathbf{N}$ by

$$V^{\diamond(n+1)} := (V^{\diamond n} \diamond V).$$

$V^{\diamond n}$ is again a family of bounded operators that is of locally bounded strong variation (see Corollary 17). We make the following assumption. Recall that an operator in a Banach lattice is called *positive* if it maps positive elements in its domain into X_+.

ASSUMPTION H7A. Let X be a Banach lattice and A be resolvent majorized by a resolvent positive Hille-Yosida operator $\bar{A}$. Let $G : D(A) \to Z$ map into a closed sublattice Z of X and assume that there is a linear positive operator $B : D(\bar{A}) \to Z$ such that

$$|G(x) - G(y)| \leq Bv \qquad \text{whenever } x, y \in D(A), v \in D(\bar{A}), |x - y| \leq v,$$

and there exists some $w \in \mathbf{R}$ such that

$$B(\lambda - \bar{A})^{-1}x = \lambda \int_0^\infty e^{-\lambda t} V(t)x \, dt \qquad \forall \lambda > w, \ x \in X,$$

where V is a family of bounded strong pseudo-variation on some interval $[0, T + \epsilon)$ with $T, \epsilon > 0$ and

$$\mathbf{PV}^{\oplus}(V^{\diamond n}; [0, T), Z) < 1$$

for some $n \in \mathbf{N}$.

REMARK 4. (a) Since $(\lambda - \bar{A})^{-1}$ and B are positive we have

$$|G(x) - G(y)| \leq B(\lambda - \bar{A})^{-1}|(\lambda - A)(x - y)|$$

and therefore G is A-continuous.

(b) Inductively one realizes that

$$|(\lambda - A)^{-n}x| \leq (\lambda - \bar{A})^{-n}|x| \qquad \forall x \in X, n \in \mathbf{N},$$

which implies that A is a Hille-Yosida operator as well.

We are going to prove a theorem similar to Theorem 20 and proceed as in the previous section by showing first that Ψ_x is well defined and then that it admits a unique fixed point. Let $(\bar{S}(t))_{\geq 0}$ denote the integrated semigroup generated by $\bar{A}$. We start with

LEMMA 24. *Let $T > 0$. Under Assumptions H7a, Ψ_0 satisfies the inequalities (18) in Corollary 9 with*

$$K_1 = \mathbf{PV}(V; [0, T+), X), \qquad K_2 = \mathbf{PV}^{\oplus}(V; [0, T+), Z),$$

where

$$\mathbf{PV}(V; [0, T+), X) := \limsup_{h \downarrow 0} \mathbf{PV}(V; [0, T + h), X).$$

Further the extension Ψ of Ψ_0 satisfies

$$|\Psi(x, v) - \Psi(x, w)| \leq V \diamond |v - w| \qquad \forall v, w \in L_1([0, T]; Z).$$

Proof. We start by showing for $v \in L_1([0, T], X)$ that

(29)
$$|(S \diamond v)(t)| \leq (\bar{S} \diamond |v|)(t).$$

We have for locally Lipschitz continuous integrated semigroups (see [16] p.224)

$$S(t + h)x - S(t)x = \lim_{k \to \infty} \int_t^{t+h} \left(\frac{k}{s}\right) \left(\frac{k}{s} - A\right)^{-(k+1)} x \, ds.$$

Since A is resolvent dominated by $\bar{A}$,

$$|(S(t + h) - S(t))x| \leq \lim_{k \to \infty} \int_t^{t+h} \left(\frac{k}{s}\right) \left|\left(\frac{k}{s} - A\right)^{-(k+1)} x\right| ds$$

$$\leq \lim_{k \to \infty} \int_t^{t+h} \left(\frac{k}{s}\right) \left(\frac{k}{s} - \bar{A}\right)^{-(k+1)} |x| ds = (\bar{S}(t + h) - \bar{S}(t))|x|,$$

and the inequality (29) follows a.e. from Theorem 14 and $S \diamond v = S \star v$. As a by-product, we see that $\{\bar{S}(t)\}$ is an increasing family. Since $S \diamond v$ and $\bar{S} \diamond |v|$ are continuous by Theorem 3, the inequality holds everywhere.

We apply Corollary 10. Let $\phi : \mathbf{R} \to [0, \infty)$ be infinitely often differentiable with support in $(0, 1)$ and $\int_0^1 \phi(s) ds = 1$. Then, for $x, y \in \overline{D(A)}$,

$$|(\dot{S}(\cdot)x)_j^\phi(t) - (\dot{S}(\cdot)y)_j^\phi(t)| = \left| \int_0^\infty \phi_j(s - t)\dot{S}(s)(x - y) ds \right|$$

$$= \lim_{\lambda \to \infty} \left| \int_0^\infty \phi_j(s - t)\dot{S}(s)\lambda(\lambda - A)^{-1}(x - y) ds \right|$$

$$= \lim_{\lambda \to \infty} \lim_{h \searrow 0} \left| \int_0^\infty \phi_j(s - t)\frac{1}{h}[S(s + h) - S(s)]\lambda(\lambda - A)^{-1}(x - y) ds \right|$$

$$\leq \lim_{\lambda \to \infty} \lim_{h \searrow 0} \int_0^\infty \phi_j(s - t)\frac{1}{h}[\bar{S}(s + h) - \bar{S}(s)]\lambda(\lambda - \bar{A})^{-1}|x - y| ds$$

$$= \lim_{\lambda \to \infty} \int_0^\infty \phi_j(s - t)\dot{\bar{S}}(s)\lambda(\lambda - \bar{A})^{-1}|x - y| ds.$$

Since ϕ is zero on $(-\infty, 0]$,

$$(30) \quad |(\dot{S}(\cdot)x)_j^\phi(t) - (\dot{S}(\cdot)y)_j^\phi(t)|$$

$$\leq -\lim_{\lambda \to \infty} \int_0^\infty \phi_j'(s-t)\bar{S}(s)\lambda(\lambda - \bar{A})^{-1}|x-y|ds$$

$$= -\int_0^\infty \phi_j'(s-t)\bar{S}(s)|x-y|ds = \int_0^\infty \phi_j''(s-t)\left(\int_0^s \bar{S}(r)|x-y|dr\right)ds$$

$$\in D(\bar{A}).$$

Let $v, w \in L_1([0,T], Z)$. We set $v(s) = w(s) = 0$ for $s > T$. Let $t \in [0,T]$. Then

$$|(S \star v)_j^\phi(t) - (S \star w)_j^\phi(t)| \leq \int_0^\infty \phi_j(s-t)(\bar{S} \star |v-w|)(s)ds$$

$$= -\int_0^\infty \phi_j'(s-t)(\bar{S} * |v-w|)(s)ds \in D(\bar{A}).$$

By Assumption H7a,

$$|G((\dot{S}(\cdot)x + S \star v)_j^\phi(t)) - G((\dot{S}(\cdot)y + S \star w)_j^\phi(t))|$$

$$\leq B\int_0^\infty \phi_j''(s-t)\left(\int_0^s \bar{S}(r)|x-y|dr\right)ds$$

$$- B\int_0^\infty \phi_j'(s-t)(\bar{S} * |v-w|)(s)ds$$

$$= \int_0^\infty \phi_j''(s-t)\left(\int_0^s V(r)|x-y|dr\right)ds - \int_0^\infty \phi_j'(s-t)(V * |v-w|)(s)ds$$

$$= \int_0^\infty \phi_j(s-t)V^\oplus(ds)|x-y| + \int_0^\infty \phi_j(s-t)(V \diamond |v-w|)(s)ds.$$

Recall that $V^\oplus(t)x = V(t)x$ for a.a. t and that $V^\oplus(\cdot)$ is of bounded strong variation which equals the strong pseudo-variation of $V(\cdot)$. So the Stieltjes integral in the last equation makes sense. Since the support of ϕ_j is contained in $(0, 1/j)$,

$$\int_0^T \|G((\dot{S}(\cdot)x + S \star v)_j^\phi(t)) - G((\dot{S}(\cdot)y + S \star w)_j^\phi(t))\|dt$$

$$\leq \int_0^T \left\|\int_0^\infty \phi_j(s-t)V^\oplus(ds)|x-y|\right\|dt$$

$$+ \int_0^T \left(\int_0^\infty \phi_j(s-t)\|(V \diamond |v-w|)(s)\|ds\right)dt$$

$$\leq \mathbf{PV}\left(V; \left[0, T + \frac{1}{j}\right), X\right)\|x-y\| \sup_{0 \leq s \leq T}\int_0^\infty \phi_j(s-t)dt$$

$$+ \int_0^{T+\frac{1}{j}}\left(\int_0^\infty \phi_j(s-t)dt\right)\|(V \diamond |v-w|)(s)\|ds$$

$$\leq \mathbf{PV}\left(V; \left[0, T + \frac{1}{j}\right), X\right) \|x - y\|$$

$$+ \mathbf{PV}^{\oplus}\left(V; \left[0, T + \frac{1}{j}\right), Z\right) \int_0^{T + \frac{1}{j}} \|v(s) - w(s)\| ds.$$

In the last inequality we have used Corollary 15 (a). Recalling that we have set $v(t) = 0 = w(t)$ for $t > T$, this implies the assumption in Corollary 10. Now

$$\left| \frac{1}{h} \int_t^{t+h} (S \star v)(s) ds - \frac{1}{h} \int_t^{t+h} (S \star w)(s) ds \right|$$

$$\leq \frac{1}{h} \int_t^{t+h} |(S \star (v - w))(s)| ds \leq \frac{1}{h} \int_t^{t+h} (\bar{S} \star |v - w|)(s) ds$$

$$= \frac{1}{h}((\bar{S} * |v - w|)(t + h) - (\bar{S} * |v - w|)(t)) \in D(\bar{A}).$$

By Assumption H7a and Lemma 19 (2.),

$$\left| G\left(\frac{1}{h} \int_t^{t+h} \dot{S}(s) x ds + \frac{1}{h} \int_t^{t+h} (S \star v)(s) ds \right) \right.$$

$$\left. - G\left(\frac{1}{h} \int_t^{t+h} \dot{S}(s) x ds + \frac{1}{h} \int_t^{t+h} (S \star w)(s) ds \right) \right|$$

$$\leq B \frac{1}{h}((\bar{S} * |v - w|)(t + h) - (\bar{S} * |v - w|)(t))$$

$$= \frac{1}{h}((V * |v - w|)(t + h) - (V * |v - w|)(t)).$$

Then Corollaries 10 and 15 imply the second assertion. ∎

We are now ready to prove a result similar to Theorem 20 for Banach lattices.

THEOREM 25. *Under the assumption H7a there is a unique continuous semiflow Θ on $\overline{D(A)}$ satisfying (5) such that the orbits of Θ are the unique F-solutions in $L_1([0, T], X)$ of (10) for all $x \in X$ and $T > 0$.*

Proof. We want to apply Corollary 9. By Lemma 24, the first assumption holds and Ψ_0 has an extension Ψ. Therefore it only remains to show that there exists $n \in \mathbf{N}$ and $\varepsilon \in (0, 1)$ such that

$$\|\Psi_x^n(v) - \Psi_x^n(w)\|_1 \leq \varepsilon \|v - w\|_1 \qquad \forall v, w \in L_1([0, T], Z), x \in \overline{D(A)}.$$

By Lemma 24,

$$|\Psi_x(v) - \Psi_x(w)| \leq V \diamond |v - w|.$$

Thus by induction

$$|\Psi_x^n(v) - \Psi_x^n(w)| \leq V^{\diamond n} \diamond |v - w|.$$

By Corollary 15,

$$\|\Psi_x^n(v) - \Psi_x^n(w)\|_1 \leq \mathbf{PV}^{\oplus}(V^{\diamond n}; [0, T], X) \|v - w\|_1.$$

and we have reached our goal with $\varepsilon = \mathbf{PV}^{\oplus}(V^{\diamond n}; [0, T], X)$. ∎

We call a semiflow Θ on $\overline{D(A)}$ *positive* if $\Theta(t,\cdot)$ maps $\overline{D(A)} \cap X_+$ into itself for all $t \geq 0$, where X_+ denotes the positive cone of the Banach lattice X.

COROLLARY. *Let X be a Banach lattice and A be a resolvent positive Hille-Yosida operator. Let $G : D(A) \to Z$ map into a closed subspace Z of X and be positive, i.e., G maps $D(A) \cap X_+$ into X_+.*

Assume that there is a linear positive operator $B : D(A) \to Z$ such that

$$|G(x) - G(y)| \leq Bv \quad whenever \;\; x, y, v \in D(A), |x - y| \leq v,$$

and there exists some $w \in \mathbf{R}$ such that

$$B(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t}V(t)x dt \qquad \forall \lambda > w, \; x \in X.$$

where V is a family of bounded strong pseudo-variation on some interval $[0, T + \epsilon)$ with $T, \epsilon > 0$ and

$$\mathbf{PV}^\oplus(V^{\diamond n}; [0, T), Z) < 1 \qquad for \; some \; n \in \mathbf{N}.$$

Then there exists a unique continuous positive semiflow Θ on $\overline{D(A)}$ which satisfies (5) and whose orbits uniquely solve (3) in the Friedrichs sense (L_1-F).

Proof. By Theorem 25 we only need to show the positivity of the semiflow. Since G is positive and the integrated semigroup S is increasing, $S \star v$ takes values in X_+ if $v \in L_1([0,T], Z \cap X_+)$ by Theorem 14 and the map

$$\Psi(x, v) = \lim_{h \downarrow 0} G\left(\frac{1}{h}[S(t+h) - S(t)]x + \frac{1}{h} \int_t^{t+h} (S \star v)(s) ds \right)$$

maps $(\overline{D(A)} \cap X_+) \times L_1([0,T], Z \cap X_+)$ into itself. Hence, if $x \in \overline{D(A)} \cap X_+$, the fixed point v_x is in $L_1([0,T], Z \cap X_+)$ and

$$\Theta(t, x) = \dot{S}(t)x + (S \star v_x)(t) \in X_+$$

by Theorem 14. $\blacksquare$

The Banach lattice X is called an *abstract L-space*, if in addition the norm on X is additive on the positive cone X_+ (in particular X may be a L_1-space or a space of measures).

ASSUMPTION H7B. Let X be an abstract L-space, A a linear operator in X, and

$$G : D(A) \to X.$$

Suppose there are a resolvent positive Hille-Yosida operator $\bar{A}$ and a positive $\bar{A}$-bounded operator $B : D(\bar{A}) \to Z$ such that A is absolutely resolvent majorized by $\bar{A}$,

$$\mathrm{spr}\, B(\mu - \bar{A})^{-1} < 1 \qquad for \; some \; \mu \in \mathbf{R} \cap \rho(\bar{A})$$

and

$$|G(x) - G(y)| \leq Bw \quad for \; all \; x, y \in D(A), w \in D(\bar{A}) \; with \; |x - y| \leq w.$$

THEOREM 26. *The statement of Theorem 5.3 also holds if Assumption H7a is replaced by Assumption H7b.*

Proof. Since $\bar{A}$ is resolvent positive on (w, ∞) for some $w > 0$ and B is positive, we have

$$(-1)^n \frac{d^n}{d\lambda^n} B(\lambda - \bar{A})^{-1} = n! B(\lambda - \bar{A})^{-(n+1)} \geq 0, \quad \lambda > w,$$

i.e., $(B(\lambda - \bar{A}))_{\lambda > w}$ is completely monotonic. By the vector valued version of Bernstein's theorem (see [3], section 5, or [1], Section 2.7), we find an exponentially bounded increasing family $(V(t))_{t \geq 0}$, $V(0) = 0$, such that

$$B(\lambda - \bar{A})^{-1} x = \int_0^\infty e^{-\lambda t} dV(t) x = \lambda \int_0^\infty e^{-\lambda t} V(t) x \, dt, \quad \forall x \in X, \; \lambda > w.$$

Since every increasing family of bounded operators is of bounded strong variation on every finite interval, we know from (23), and Theorem 14 that $(V^{\diamond n}(t))_{t \geq 0}$ is an increasing family of bounded operators that is of bounded strong variation on every finite interval. Since X is an abstract L-space and $V^{\diamond n}$ is increasing,

$$\int_0^{\epsilon - h} \|V^{\diamond n}(s+h)x - V^{\diamond n}(s)x\| ds = \int_0^{\epsilon - h} \||V^{\diamond n}(s+h)x - V^{\diamond n}(s)x\|| ds$$

$$= \left\| \int_0^{\epsilon - h} |V^{\diamond n}(s+h)x - V^{\diamond n}(s)x| ds \right\| \leq \left\| \int_0^{\epsilon - h} (V^{\diamond n}(s+h)|x| - V^{\diamond n}(s)|x|) ds \right\|$$

$$= \left\| \int_{\epsilon - h}^{\epsilon} V^{\diamond n}(s)|x| ds - \int_0^h V^{\diamond n}(s)|x| ds \right\| \leq h\|V^{\diamond n}(\epsilon)|x| - V^{\diamond n}(0)|x|\|.$$

By Definition 3 and (20), for $n \geq 2$,

$$\mathbf{pv}^{\oplus}(V^{\diamond n}(\cdot)x; [0, \epsilon]) = \mathbf{pv}(V^{\diamond n}(\cdot)x; [0, \epsilon]) + \|V^{\diamond n}(0)x\| \leq \|V^{\diamond n}(\epsilon)|x|\|$$

$$\leq e^{\lambda \epsilon} \left\| \int_0^{\epsilon} e^{-\lambda t} V^{\diamond n}(dt)|x| \right\| \leq e^{\lambda \epsilon} \|(B(\lambda - \bar{A})^{-1})^n |x|\| \leq e^{\lambda \epsilon} \|(B(\lambda - \bar{A})^{-1})^n\| \, \|x\|.$$

By (21),

$$\mathbf{PV}^{\oplus}(V^{\diamond n}; [0, \epsilon], X) \leq e^{\lambda \epsilon} \|(B(\lambda - \bar{A})^{-1})^n\|.$$

Choosing λ and n large and $\epsilon > 0$ small enough, we have $\mathbf{PV}^{\oplus}(V^{\diamond n}; [0, \epsilon], X) < 1$ and our statement follows from Theorem 25. ∎

8. Applications to age-structured population dynamics. In the sequel, we present applications to several age-structured population models. We treat the boundary condition at age 0 as in [30, 31, 33, 19]. For alternative approaches see [20, 28] and the references cited there. The first model does not involve any additional structure, but has a very general birth law.

8.1. *Age-structured population models without additional structure.* The development of an age-structured population can be described as a partial differential equation for the age-density $u(t, \cdot)$ at time t,

$$(31) \qquad \begin{aligned} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) \frac{u(t, a)}{\mathcal{F}(a)} &= 0; \quad t > 0, 0 < a < c, \\ u(t, 0) &= f(u(t, \cdot)); \quad t > 0, \\ u(0, a) &= u_0(a); \quad 0 \leq a < c. \end{aligned}$$

The number $c \in (0, \infty]$ denotes the maximum age that an individual can reach. $\mathcal{F}(a)$ is the probability to survive till age a and is a decreasing function of a with $\mathcal{F}(0) = 1$, $\mathcal{F}(a) > 0$ for $a \in [0, c)$ and $\mathcal{F}(a) = 0$ for $a > c$ with the last only making sense if $c < \infty$. $f(u(t, \cdot))$ is the population birth rate as a function of the population age density. We make the following assumption:

ASSUMPTION H8A. There exist a non-negative not necessarily finite Borel measure m on $[0, c)$ such that $f : \mathcal{L}_1([0, c); m + \ell) \to [0, \infty)$, $f(0) = 0$, and

$$|f(u) - f(v)| \leq \int_{[0,c)} |u(a) - v(a)| m(da).$$

ℓ represents the Lebesgue measure and $\mathcal{L}_1([0, c); m + \ell)$ the space of real-valued Borel measurable functions x such that

$$\int_{[0,c)} |x(a)| m(da) + \int_{[0,c)} |x(a)| da < \infty.$$

The assumption $f(0) = 0$ means that the birth rate is 0 if the population size is 0. As state space we choose the abstract L space $X = \mathbf{R} \times L_1([0, c); \mathbf{R})$. However, the solutions u of the abstract Cauchy problem in which we will recast (31) will take values in the subspace $\{0\} \times L_1([0, \infty); \mathbf{R})$ which can be identified with $L_1([0, \infty); \mathbf{R})$. We define the operator

$$A(0, x) = \left(x(0), -\mathcal{F}\left(\frac{x}{\mathcal{F}}\right)' \right), \qquad x \in L_1([0, c)),$$

with $D(A)$ being the set of functions $x \in L_1([0, c))$ such that $x/\mathcal{F}$ is absolutely continuous and $\mathcal{F}(x/\mathcal{F})'$ is integrable on $[0, c)$. (Cf. [33], Section 5). We find that

$$(\lambda - A)^{-1}(\xi, x) = (0, v), \qquad \xi \in \mathbf{R}, x \in L_1[0, c),$$

with

$$v(a) = \xi e^{-\lambda a} \mathcal{F}(a) + \int_0^a e^{-\lambda(a-s)} \frac{\mathcal{F}(a)}{\mathcal{F}(s)} x(s) ds, \qquad a \in [0, c).$$

We see that $(0, \infty)$ is contained in $\rho(A)$, $\|(\lambda - A)^{-1}\| \leq 1/\lambda$ for $\lambda > 0$ and that $(\lambda - A)^{-1}$ maps the positive cone into itself, i.e., A is resolvent-poitive. We make the following assumption:

ASSUMPTION H8B. There is some $w > 0$ such that

$$\int_{[s,c)} e^{-w(a-s)} \frac{\mathcal{F}(a)}{\mathcal{F}(s)} m(da) \quad \text{is a bounded function of } s \in [0, c).$$

With this assumption, $D(A)$ is a subspace of $\mathcal{L}_1([0, c), m + \ell)$ and the following operator B is defined on $D(A)$:

$$B(0, x) = \left(\int_{[0,c)} x(a) m(da), 0 \right), \qquad x \in D(A).$$

Obviously B is a positive operator from $D(A)$ to X. Further $G(x) = (f(x), 0)$ is an operator from $D(A)$ to X satisfying the assumptions of Theorem 2, provided we can verify that the spectral radius of $F(\lambda) = B(\lambda - A)^{-1}$ is smaller than 1. But B maps

X into $Z = \mathbf{R} \times \{0\}$, so the spectral radius of $B(\lambda - A)^{-1}$ is smaller than the norm of $B(\lambda - A)^{-1}$ taken on this closed subspace, which is

$$\|B(\lambda - A)^{-1}\|_Z = \int_{[0,c)} e^{-\lambda a} \mathcal{F}(a) m(da).$$

This expression converges to $m(\{0\})$ for $\lambda \to \infty$. So it is sufficient to assume that

$$m(\{0\}) < 1.$$

This is particularly satisfied if $m(\{0\}) = 0$, which means that an individual that has just been born does not reproduce.

THEOREM 27. *Let the Assumptions H8a and H8b be satisfied and $m(\{0\}) < 1$. For each $u_0 \in L_1([0,c), \mathbf{R}) = L_1[0,c)$, there exists a unique Friedrichs solution u which is defined for all times. More precisely $u(t, \cdot)$ is continuous in $t \geq 0$ as a function with values in $L_1[0,c)$, $u(0, \cdot) = u_0$, and there exists a sequence of functions $u_n : [0, \infty) \times [0, c) \to \mathbf{R}$ with the following properties:*

(a) *For all $t \geq 0$, $u_n(t, \cdot) \in L_1[0, c)$ and*

$$\int_0^c |u_n(t, a) - u(t, a)| da \to 0 \qquad \text{locally uniformly in } t \geq 0,$$

(b) *For all $t \geq 0$, $u_n(t, \cdot)/\mathcal{F}$ is absolutely continuous with $\mathcal{F} \frac{\partial}{\partial a} \frac{u_n(t, \cdot)}{\mathcal{F}} \in L_1[0, c)$, $u_n(t, \cdot)$ is differentiable in t as a function with values in $L_1[0, c)$, and*

$$\int_0^T \int_0^c \left| \frac{\partial}{\partial t} u_n(t, a) + \mathcal{F}(a) \frac{\partial}{\partial a} \frac{u_n(t, a)}{\mathcal{F}(a)} \right| da\, dt \to 0, \qquad n \to \infty$$

where $\frac{\partial}{\partial t} u_n(t, a)$ stands for the derivative of $u_n(t, \cdot)$ with respect to t,

(c) *$u_n(t, \cdot) \in L_1([0, c); m + \ell)$ and, for all $T > 0$,*

$$\int_0^T |u_n(t, 0) - f(u_n(t, \cdot))| dt \to 0, \qquad n \to \infty.$$

Moreover $\Theta(t, u_0) = u(t, \cdot)$ defines a semiflow on $L_1[0, c]$. If f maps non-negative functions in $\mathcal{L}([0, c), m + \ell)$ onto non-negative functions, then Θ is a positive semiflow, i.e., if the initial datum u_0 is non-negative, so is the solution u.

8.2. *An age-structured population model with an additional structure.* Let E be a Banach space that represents the distribution of a population with respect to a structure different from age, e.g., induced by space or body size. Let $u(t, a)$ denote the structural distribution (with respect to this structure) of the individuals with age a at time t. We consider the following model:

(32)
$$\begin{cases} (\partial_t + \partial_a)u(t, a) = Au(t, a) + M(a)u(t, a), & t > 0, \quad 0 < a < c, \\ u(t, 0) = \int_0^c [F(u(t, \cdot))](a) da, & t > 0, \\ u(0, a) = u_0(a), & 0 \leq a < c. \end{cases}$$

The number $c \in (0, \infty)$ denotes the maximum age an individual can reach. The closed linear operator A and the bounded linear operators $M(a)$ describe how individuals (with age a) change with respect to the other structure and also to what extent they die.

The boundary condition describes the birth of individuals. The operators $F(f)(a)$ represents the birth rate of individuals with age a if $f : [0, c) \to E$ is the age-density of the population.

We assume that A is the infinitesimal generator of a C_0-semigroup $\{U(t); t \geq 0\}$ on E. Then the set

$$D = \{f \in C^1([0, c]; E); \ f(0) = 0 = f'(0), \ f(t) \in D(A) \quad \forall t \in [0, c], \ Af \text{ continuous}\}$$

is dense in $L_1([0, c), E)$.

For more general assumptions that allow the operator A to depend on a see [25]. We want to apply the semilinear Miyadera-Voigt theory developed in Section 6.

ASSUMPTION 28. Let the operators $F : D \to E$ and $M(a) : E \to E$, $a \in [0, c)$, have the following properties:

(a) There exist a Banach space $\tilde{E}$ and an A-bounded linear operator $C : D(A) \to \tilde{E}$ such that

$$\|[F(f)](a) - [F(g)](a)\| \leq \|C(f(a) - g(a))\| \qquad \forall f, g \in D, a \in [0, c),$$

and, for each $\epsilon > 0$, there exists some $\delta \in (0, c)$ with

$$\int_0^\delta \|CU(t)x\| dt \leq \epsilon \|x\| \qquad \forall x \in D(A).$$

(b) $\{M(a); a \in [0, c)\}$ is a strongly Borel measurable family of bounded linear operators such that, for every $x \in E$, $\|x\| \leq 1$,

$$\operatorname*{ess\,sup}_{0 \leq a < c} \|M(a)x\| < \infty.$$

Following the approach in [31, 19] and [33, Sec. 5], we let let $X_\circ = L_1([0, c), E)$. The evolutionary semigroup on $X_\circ$ associated with U is defined by

$$(T_0(t)f)(a) = \begin{cases} U(t)f(a - t), & a \geq t \\ 0, & a < t \end{cases}, \qquad f \in X_\circ, 0 \leq a < c.$$

Cf. [7, 25], e.g., and the references therein. It is readily checked that $T_0(\cdot)$ is a C_0-semigroup on $X_\circ$ and $\|T_0(t)\| \to 0$ as $t \to \infty$. So there exist $\omega > 0$ and $\hat{M} \geq 1$ such that $\|T_0(t)\| \leq \hat{M} e^{-\omega t}$. If $\mathcal{A}_0$ denotes the infinitesimal generator of T_0, $D \subseteq D(\mathcal{A}_0)$ and

$$[\mathcal{A}_0(f)](a) = Af(a) - f'(a), \qquad f \in D, a \in [0, c).$$

Moreover $T_0(t)D \subseteq D$ for all $f \in D$. Let $X = E \times X_\circ$. It follows from the considerations in [33, Sec. 5], that there exists a Hille-Yosida operator $\mathcal{A}$ such that

$$(\lambda - \mathcal{A})^{-1}(\eta, f) = (0, g),$$
$$g(a) = e^{-\lambda a} U(a)\eta + \int_0^a e^{-\lambda s} U(s) f(a - s) ds.$$

$\mathcal{A}$ generates an integrated semigroup $S(\cdot)$ on X such that $\dot{S}(t)(0, v) = (0, T_0(t)v)$ and

$$S(t) = \lim_{\lambda \to \infty} \int_0^t \lambda \dot{S}(r)(\lambda - \mathcal{A})^{-1} dr,$$

from which we deduce that

$$\|S(t) - S(r)\| \leq \tilde{M} \int_r^t e^{-\omega s} ds$$

and

$$\|(\lambda - \mathcal{A})^{-n}\| \leq \tilde{M}(\lambda + \omega)^{-n} \qquad \forall \lambda > -\omega, n \in \mathbb{N}.$$

In terms of the original semigroup, the integrated semigroup is given by

$$S(t)(\eta, f) = (0, g), \qquad \eta \in E, f \in X_\circ,$$

$$g(a) = H(t - a)U(a)\eta + \int_0^t [T_0(s)f](a)ds.$$

Here H denotes the Heaviside function, $H(t) = 1$ for $t \geq 0$, $H(t) = 0$ for $t < 0$. The generator $\mathcal{A}$ of the integrated semigroup S is related to the infinitesimal generator A of U by

$$\mathcal{A}(0, f) = (-f(0), \ Af(\cdot) - f'),$$

for $f \in C^1([0, c], E)$ with $f(a) \in D(A)$ for all $a \in [0, c)$. Obviously $\overline{D(\mathcal{A})} = \{0\} \times X_\circ$ which can be identified with $X_\circ$. In order to make the connection to Theorem 23, we identify D and $\{0\} \times D$ and define operators $B_0 : D \to Y$, $Y = L_1([0, c), \tilde{E}) \times L_1([0, c), E)$, and $G_0 : D \to X$ by

$$\left. \begin{array}{l} B_0(0, f) = (Cf(\cdot), M(\cdot)f(\cdot)) \\[2mm] G_0(0, f) = \left(\displaystyle\int_0^c [F(f)](a)da, M(\cdot)f(\cdot) \right) \end{array} \right\} f \in D.$$

The assumptions of Theorem 23 are readily checked, and there exists a semiflow on $L_1([0, c), E)$ which solves the age-structured problem 32 in an appropriate generalized sense.

8.3. *Age-structured population models with diffusion.* We make the additional structure in the preceding example explicit and assume that the individuals are distributed over a bounded open habitat $\Omega \subseteq \mathbf{R}^N$. If, at time t, $u(t, a, x)$ is the density of individuals at age a and location x, then the model reads

(33)
$$\left. \begin{array}{ll} \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a} \right) u(t, a, x) = \gamma(x)\Delta_x u(t, a, x) & \\[2mm] \hspace{3cm} -\mu(a, x)u(t, a, x); & x \in \Omega, \\[2mm] u(t, a, x) = 0; & x \in \partial\Omega, \\[2mm] u(t, 0, x) = \displaystyle\int_0^\infty \beta_0(a, x)u(t, a, x)da & \\[2mm] \hspace{1cm} + \displaystyle\sum_{j=1}^n \beta_j(a, x) \left| \dfrac{\partial}{\partial x_j} u(t, a, x) \right| da; & x \in \Omega, \\[2mm] u(0, a, x) = u_0(a, x); & x \in \Omega. \end{array} \right\} \begin{cases} t > 0, \\ 0 < a < c. \end{cases}$$

The number $c \in (0, \infty)$ denotes the maximum age that an individual can reach. $\mu(a, x)$ is the per capita mortality rate at age a at location x. The individuals move randomly with a diffusion rate $\gamma(x)$ which only depends on location. Δ_x denotes the Laplace operator with respect to x. We assume that the boundary of the habitat Ω is deadly expressed

by Dirichlet boundary conditions. The birth rate at location x is assumed to depend not only on the density of individuals at x but also on the magnitude of the flux through x. This dependence is expressed by the rates β_j.

We assume, for simplicity, that the boundary of Ω is of class C^3 and that μ is continuously differentiable on $[0, c) \times \Omega$ with bounded partial derivatives. Further we assume that γ has bounded continuous partial derivatives up to order three. Moreover $\inf\{\gamma(x); x \in \Omega\} > 0$. For sharper assumptions see the literature mentioned below. The functions β_j are assumed to be non-negative, Borel measurable, and essentially bounded on $[0, c) \times \Omega$.

In order to connect the model (33) to the model (32), we let $E = L_2(\Omega)$, $D(A) = H^2(\Omega) \cap H^1_0(\Omega)$, and

$$[A\phi](x) = \gamma(x)\Delta\phi(x), \qquad [M(a)\phi](x) = -\mu(a, x)\phi(x).$$

Here $H^k(\Omega)$ is the Banach space of functions in $L_2(\Omega)$ whose generalized partial derivatives are also contained in $L_2(\Omega)$, up to order k; the norm on $H^k(\Omega)$ combines the L_2-norms of the function and its generalized derivatives up to order k via the Euclidean norm. $H^k_0(\Omega)$ is the closure of $C^\infty_0(\Omega)$ in $H^k(\Omega)$. Further we set

$$([F(\phi)](t))(x) = \beta_0(t, x)|\phi(x)| + \sum_{j=1}^{N} \beta_j(t, x)\left|\frac{\partial}{\partial x_j}\phi(x)\right|, \qquad \phi \in H^1_0(\Omega).$$

Under our assumptions on γ and μ, A generates a C_0-semigroup $\{U(t); t \geq 0\}$ with $U(t)\phi \in D(A)$ for all $t > 0, \phi \in E$, which is given by a Green's function,

$$[U(t)\phi](x) = \int_\Omega \Gamma(t, x, y)\phi(y)dy.$$

See [18, Thm. 16.2, 16.3] and [14, Sec. VI.2.1]. In order to check the Assumptions 28, we set $\tilde{E} = E^{N+1}$, with the norm $\|(\phi_0, \ldots, \phi_n)\| = \sum_{j=0}^{N} \|\phi_j\|$. Then, for $\phi, \psi \in H^1_0(\Omega)$,

$$\|F(t)\phi - F(t)\psi\| = \left\|\beta_0(t, \cdot)|\phi| + \sum_{j=1}^{N} \beta_j(t, \cdot)\left|\frac{\partial}{\partial x_j}\phi\right| - \beta_0(t, \cdot)|\psi| - \sum_{j=1}^{N} \beta_j(t, \cdot)\left|\frac{\partial}{\partial x_j}\psi\right|\right\|$$

$$\leq \left\|\beta_0(t, \cdot)|\phi - \psi| + \sum_{j=1}^{N} \beta_j(t, \cdot)\left|\frac{\partial}{\partial x_j}(\phi - \psi)\right|\right\|.$$

Since the functions β_j are bounded, there exists a constant κ such that

$$\|F(t)\phi - F(t)\psi\| \leq \left\|\kappa|\phi - \psi| + \sum_{j=1}^{N} \kappa\left|\frac{\partial}{\partial x_j}(\phi - \psi)\right|\right\|$$

$$\leq \|\kappa\phi - \kappa\psi\| + \sum_{j=1}^{N}\left\|\kappa\frac{\partial}{\partial x_j}\phi - \kappa\frac{\partial}{\partial x_j}\psi\right\| = \sqrt{N+1}\|C\phi - C\psi\|,$$

where $C : H^1_0(\Omega) \to \tilde{E}$, $\tilde{E} = E^{N+1}$ is the linear operator

$$C\phi = \kappa(\phi, \nabla\phi), \qquad \phi \in H^1_0(\Omega).$$

Let $\phi \in E$, $t > 0$. Then

$$CU(t)\phi = \kappa(\phi_0, \ldots, \phi_n),$$

$$\phi_0(x) = \int_\Omega \Gamma(t, x, y)\phi(y)dy,$$

$$\phi_j(x) = \int_\Omega \frac{\partial}{\partial x_j}\Gamma(t, x, y)\phi(y)dy.$$

The Green's function satisfies the following estimates [18, Thm. 16.3]:

$$0 \leq \Gamma(t, x, y) \leq \xi_0(t, |x - y|),$$

$$\left|\frac{\partial}{\partial x_j}\Gamma(t, x, y)\right| \leq \xi_1(t, |x - y|),$$

$$\xi_i(t, r) = c_1 t^{-\frac{N+i}{2}} \exp\left(-cr^2 t^{-1}\right), \qquad i = 0, 1.$$

With these estimates,

$$\|\phi_j\| \leq \int_{\mathbf{R}^N} \xi_1(t, \|x\|)dx\|\phi\| = c_N \int_0^\infty \xi_1(t, r)r^{N-1}dr\|\phi\|$$

$$= c_N c_1 t^{-\frac{i}{2}} \int_0^\infty e^{-cr^2} dr\|\phi\| = \tilde{c}_N t^{-\frac{i}{2}}\|\phi\|,$$

where $i = 0$ for $j = 0$ and $i = 1$ for $j = 1, \ldots, N$ and $c_N, \tilde{c}_N$ are appropriate constants. If $0 \leq t \leq 1$, with a constant $\hat{c}_N$,

$$\|CU(t)\phi\| \leq \hat{c}_N t^{-\frac{1}{2}}\|\phi\|.$$

If $\delta \in (0, 1]$, for all $\phi \in E$,

$$\int_0^\delta \|CU(t)\phi\|dt \leq \hat{c}_N \int_0^\delta t^{-\frac{1}{2}}\|\phi\|dt \leq 2\hat{c}_N \delta^{1/2}\|\phi\|.$$

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