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A CLASSIFICATION OF PROJECTORS

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Abstract. A positive operator A and a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ are called *compatible* if there exists a projector Q onto $\mathcal{S}$ such that $AQ = Q^*A$. Compatibility is shown to depend on the existence of certain decompositions of $\mathcal{H}$ and the ranges of A and $A^{1/2}$. It also depends on a certain angle between $A(\mathcal{S})$ and the orthogonal of $\mathcal{S}$.

1. Introduction. Consider the set $\mathcal{Q}$ of all (bounded linear) projectors on a Hilbert space $\mathcal{H}$. Sometimes the elements of $\mathcal{Q}$ are named *oblique projectors* in order to emphasize that they are not necessarily orthogonal. Since the early years of matrix and operator theories, projectors have played a relevant role in many studies on spectral theory, approximation, optimization, orthogonal decompositions, least square methods, and so on. Very recently, several applications of oblique projectors to signal processing [10], [13], [36]; sampling [11], [57]; wavelets [3], [56]; information theory [57]; integral equations [51], [52]; statistics [54]; least square approximation [28], [29], [60] and parallel computing [17] have been found. For these multiple manifestations, many results on projectors are rediscovered once and again by different specialists. It seems that a short survey on several old and new results on oblique projectors may be helpful for the interested reader.

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For each closed subspace $\mathcal{S}$ of $\mathcal{H}$ let $\mathcal{Q}^{\mathcal{S}}$ denote the set of all projectors with range $\mathcal{S}$. For each (bounded linear semidefinite) positive operator A on $\mathcal{H}$ consider the set $\mathcal{P}(A,\mathcal{S}) = \{Q \in \mathcal{Q}^{\mathcal{S}}\colon AQ = Q^*A\}$, i.e., all Q with range $\mathcal{S}$ which are Hermitian with respect to the sesquilinear form $\langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle$. Of course, $\mathcal{P}(A,\mathcal{S})$ can be empty (see examples below); we say that $A, \mathcal{S}$ are *compatible* if $\mathcal{P}(A,\mathcal{S})$ is not empty. This condition can be read in terms of different space decompositions, range inclusions and angles between certain closed subspaces of $\mathcal{H}$. It is known [19] that, if A and $\mathcal{S}$ are compatible then a distinguished element $P_{A,\mathcal{S}}$ of $\mathcal{P}(A,\mathcal{S})$ exists which has optimal properties. We show explicit formulas for $P_{A,\mathcal{S}}$ which are computationally useful.

Many results on oblique projectors can be found in the papers by Afriat [1], Davis [22], Ljance [43], Mizel and Rao [45], Halmos [33], Greville [32], Gerisch [30], Pták [49]. Projectors which are Hermitian with respect to a positive matrix have been studied by Mitra and Rao [44] and Baksalary and Kala [9]. More recently, Hassi and Nordstrom [35] studied projectors which are Hermitian with respect to a self-adjoint operator but with emphasis on the case in which $\mathcal{P}(A,\mathcal{S})$ is a singleton. In [47], Pasternak-Winiarski studied the analyticity of the map $A \to P_{A,\mathcal{S}}$, where A runs over the set of positive invertible operators. The map $(A,\mathcal{S}) \to P_{A,\mathcal{S}}$ is studied by Andruchow, Corach and Stojanoff [6], for positive invertible A. For general selfadjoint A, several results on $\mathcal{P}(A,\mathcal{S})$ can be found in [19] and the present paper can be seen as its continuation. Additional results by the authors are contained in [20] and [21]. The latter makes a link between oblique projectors and abstract splines in the sense of Atteia [8]. It is natural that this type of least square approximation results appears in this context, because $P_{A,\mathcal{S}}$ is a kind of orthogonal projector for an appropriate inner product. In particular, oblique projectors, mainly in the finite-dimensional setting, appear frequently under the form of "scaled projectors", i.e., projectors which are Hermitian with respect to a positive diagonal matrix. The reader is referred to the papers by Stewart [53], O'Leary [46], Hanke and Neumann [34], Gonzaga and Lara [31], Wei [60], Forsgren [28], Vavasis [14], among many others, for results on and applications of scaled projectors. A relationship between scaled and A-Hermitian projectors, also in the infinite-dimensional setting, can be found in [7].

The contents of the paper are the following. Section 2 begins with some preliminaries and a short survey of known results on $\mathcal{P}(A,\mathcal{S})$ and $P_{A,\mathcal{S}}$, taken from [19], [20] and [21]. Then, we prove several characterizations of compatibility in terms of decompositions of $\mathcal{H}$ and of the ranges of A and $A^{1/2}$, of certain range inclusions and also of the angle between the closure of $A(\mathcal{S})$ with the orthogonal complement of $\mathcal{S}$. Most of these results are new and the proof of the remainder has been greatly simplified. We collect in Section 3 several formulas for $P_{A,\mathcal{S}}$ using results from Greville [32], Kerzman and Stein [38], [39], Ljance [43], Pták [49] and Buckholtz [16].

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2. Oblique projectors. In what follows $\mathcal{H}$ denotes a Hilbert space with inner product $\langle\ ,\ \rangle$, $L(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$, $GL(\mathcal{H})$ denotes the group of invertible operators on $\mathcal{H}$, $L(\mathcal{H})^+$ the cone of positive operators, $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap$

$L(\mathcal{H})^+$ and $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ the set of oblique projectors. For an operator W its image is denoted by $R(W)$ and its nullspace by $N(W)$. Recall that if $\mathcal{H}, \mathcal{K}$ are two Hilbert spaces and $C \in L(\mathcal{H}, \mathcal{K})$ has closed range, then there exists a unique $C^\dagger \in L(\mathcal{K}, \mathcal{H})$ such that $CC^\dagger C = C$, $C^\dagger CC^\dagger = C^\dagger$ and $CC^\dagger$, $C^\dagger C$ are Hermitian; $C^\dagger$ is called the Moore-Penrose inverse of C (see [23] and [12] for details).

The following result by R. G. Douglas will be frequently used in this paper. Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{G}$ and operators $A \in L(\mathcal{H}, \mathcal{G})$, $B \in L(\mathcal{K}, \mathcal{G})$ then the following conditions are equivalent:

i) the equation $AX = B$ has a solution in $L(\mathcal{K}, \mathcal{H})$;
ii) $R(B) \subseteq R(A)$;
iii) there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$.

In this case, there exists a unique $D \in L(\mathcal{K}, \mathcal{H})$ such that $AD = B$ and $R(D) \subseteq \overline{R(A^*)}$; moreover, $\|D\|^2 = \inf\{\lambda > 0 : \ BB^* \leq \lambda AA^*\}$. We shall call D the *reduced solution* of $AX = B$. The reader is referred to [26] and [27] for the proof of the Douglas theorem and related results. Let us remark that if $R(A)$ is closed then the reduced solution of $AX = B$ is $A^\dagger B$: this follows quite easily from the properties of the Moore-Penrose pseudoinverse. For a fixed closed subspace $\mathcal{S}$ of $\mathcal{H}$, operators in $\mathcal{H}$ are represented as 2×2 matrices according to the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$; more precisely, for each $B \in L(\mathcal{H})$ the identity

$$B = PBP + PB(I-P) + (I-P)BP + (I-P)B(I-P)$$

where P is the orthogonal projector onto $\mathcal{S}$, can be matricially rephrased as $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, where $b_{11} = PBP|_{\mathcal{S}} \in L(\mathcal{S})$, $b_{12} = PB(I-P)|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp, \mathcal{S})$, $b_{21} = (I-P)BP|_{\mathcal{S}} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and $b_{22} = (I-P)B(I-P)|_{\mathcal{S}^\perp} \in L(\mathcal{S}^\perp)$. In particular, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, any projector Q onto $\mathcal{S}$ has the form $Q = \begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix}$ for some $e \in L(\mathcal{S}^\perp, \mathcal{S})$ and any $A \in L(\mathcal{H})^+$ can be expressed as $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, where $a \in L(\mathcal{S})^+$, $b \in L(\mathcal{S}^\perp, \mathcal{S})$, $c \in L(\mathcal{S}^\perp)^+$ and $|\langle b\eta, \xi\rangle|^2 \leq \langle a\xi, \xi\rangle\langle c\eta, \eta\rangle$ for every $\xi \in \mathcal{S}$, $\eta \in \mathcal{S}^\perp$ [50]. As a consequence (see [4]) it follows that the image of the positive square root of a contains the image of $b : R(a^{1/2}) \supseteq R(b)$.

Given a closed subspace $\mathcal{S}$ let $\mathcal{Q}^{\mathcal{S}}$ be the subset of $\mathcal{Q}$ of all projectors with range (i.e. image) $\mathcal{S}$. Of course $\mathcal{Q}$ is the disjoint union of all $\mathcal{Q}^{\mathcal{S}}$. On the other hand, any positive (bounded linear) operator A on $\mathcal{H}$ defines a (Hermitian semidefinite) positive sesquilinear form

$$\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}\ , \quad \langle \xi, \eta\rangle_A = \langle A\xi, \eta\rangle, \quad \xi, \eta \in \mathcal{H}.$$

A (bounded linear) operator T on $\mathcal{H}$ is called *A-Hermitian* if $\langle T\xi, \eta\rangle_A = \langle \xi, T\eta\rangle_A$ for all $\xi, \eta \in \mathcal{H}$, i.e. if $AT = T^*A$. We shall not study the existence of an A-adjoint of an operator (see [41] and [25] for this type of problems). However, the following result shows that this existence is not irrelevant, even in the case of projectors.

Lemma 2.1. *Given $Q \in \mathcal{Q}$ and $A \in L(\mathcal{H})^+$, there exists $W \in L(\mathcal{H})$ such that $AQ = W^*A$ (i.e. Q admits an A-adjoint) if and only if*

$$R(A) = R(A) \cap N(Q^*) \oplus R(A) \cap R(Q^*). \tag{1}$$

Proof. If $\xi \in R(A)$ then $\xi = A\eta$, for some $\eta \in \mathcal{H}$. Since Q^* decomposes $\mathcal{H}$ as the direct sum $R(Q^*) \oplus N(Q^*)$ there exists $w \in \mathcal{H}$ such that $\xi = A\eta = Q^*w + z$, where $z \in N(Q^*)$. But $Q^*\xi = Q^*A\eta = Q^*w \in R(A)$, because $R(Q^*A) = R(AW) \subseteq R(A)$. Then $Q^*\xi = Q^*w \in R(A) \cap R(Q^*)$. Also $z = A\eta - Q^*w \in R(A) \cap N(Q^*)$, because $Q^*w \in R(A)$. This proves decomposition (1).

If formula (1) holds, then $R(Q^*A) = Q^*(R(A) \cap R(Q^*)) = R(A) \cap R(Q^*)$, so that $R(Q^*A) \subseteq R(A)$. By the Douglas theorem there exists a solution W of the equation $AX = Q^*A$. ■

Denote by $\mathcal{Q}_A$ the set of all A-Hermitian projectors on $\mathcal{H}$ and $\mathcal{P}(A, \mathcal{S}) = \mathcal{Q}^{\mathcal{S}} \cap \mathcal{Q}_A$. In [19] it is remarked that every $Q \in \mathcal{Q}$ belongs to some $\mathcal{P}(A, \mathcal{S})$. Thus, $\mathcal{Q} = \cup \mathcal{P}(A, \mathcal{S})$ where $\mathcal{S}$ runs over the class of all closed subspaces on $\mathcal{H}$ and A over a class of positive operators A. The sets $\mathcal{P}(A, \mathcal{S})$ are the object of our study.

We follow the terminology proposed by Ben-Israel and Greville [12]: the operator $Q: \xi \mapsto Q\xi$ which performs the projection is named *projector*, while $Q\xi$ is the *projection* of ξ (under Q).

In what follows $\mathcal{S}$ denotes a closed subspace of $\mathcal{H}$ and A denotes a positive operator on $\mathcal{H}$. Define

$$\mathcal{S}^{\perp_A} := \{\xi \in \mathcal{H} : \langle \xi, \eta \rangle_A = 0 \ \ \forall\, \eta \in \mathcal{S}\}.$$

The identities $\mathcal{S}^{\perp_A} = A^{-1}(\mathcal{S}^{\perp}) = (A\mathcal{S})^{\perp}$ will be used without further mention. Observe that, if A is invertible, then $\langle\ ,\ \rangle_A$ is an inner product which is equivalent to $\langle\ ,\ \rangle$; so that the subspace $\mathcal{S}$ admits a closed A-complement in $(\mathcal{H}, \langle\ ,\ \rangle_A)$, namely $\mathcal{S}^{\perp_A}$; thus, $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp_A}$. However, if A is not invertible, such a complement may not exist. In fact, $\mathcal{S} \cap \mathcal{S}^{\perp_A}$ may be non-trivial and $\mathcal{S} + \mathcal{S}^{\perp_A}$ may be a proper non-closed subspace of $\mathcal{H}$ (see below).

The next theorem collects several well-known facts on projectors which are due to many mathematicians: Afriat [1], Greville [32], Pták [49], Chung [18], Buckholtz [16]. Indeed, the use of projectors is so extended that many results appear once again in papers in functional analysis, statistics, matrix analysis, signal processing, and so on.

Theorem 2.2. *If $\mathcal{S}$ and $\mathcal{N}$ are closed subspaces of a Hilbert space $\mathcal{H}$ then the following properties are equivalent:*

1. $\mathcal{H} = \mathcal{S} \oplus \mathcal{N}$,
2. *there exists $Q \in \mathcal{Q}$ such that $R(Q) = \mathcal{S}$, and $N(Q) = \mathcal{N}$,*
3. $P_{\mathcal{S}} - P_{\mathcal{N}} \in GL(\mathcal{H})$,
4. $\|P_{\mathcal{S}} + P_{\mathcal{N}} - I\| < 1$,
5. *$P_{\mathcal{S}^\perp}|_{\mathcal{N}}$ is injective and $P_{\mathcal{S}^\perp}(\mathcal{N}) = \mathcal{S}^\perp$.*

In that case $P_{\mathcal{S}}P_{\mathcal{N}^\perp}$ has a closed range,

$$\|P_{\mathcal{S}}P_{\mathcal{N}}\| = \|P_{\mathcal{N}}P_{\mathcal{S}}\| < 1 \ , \quad P_{\mathcal{S}} + P_{\mathcal{N}} - P_{\mathcal{N}}P_{\mathcal{S}} \in GL(\mathcal{H}) \ , \quad P_{\mathcal{N}^\perp}P_{\mathcal{S}} - I \in GL(\mathcal{H})$$

and the projector onto $\mathcal{S}$ with nullspace $\mathcal{N}$ is

$$\begin{aligned} P_{\mathcal{S}//\mathcal{N}} &= (P_{\mathcal{S}}P_{\mathcal{N}^\perp})^\dagger = (I - P_{\mathcal{N}^\perp}P_{\mathcal{S}})^{-1}P_{\mathcal{N}} \\ &= (I - P_{\mathcal{S}}P_{\mathcal{N}})^{-1}P_{\mathcal{S}}(I - P_{\mathcal{S}}P_{\mathcal{N}}) \\ &= (I - P_{\mathcal{N}}P_{\mathcal{S}})^{-1}(I - P_{\mathcal{N}}) \\ &= P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{N}} - P_{\mathcal{N}}P_{\mathcal{S}})^{-1}. \end{aligned}$$

In particular, $\|P_{\mathcal{S}//\mathcal{N}}\| = (I - \|P_{\mathcal{N}}P_{\mathcal{S}}\|^2)^{-1/2}$.

REMARK 2.3. There is a formula, due to Kerzman and Stein [38], [39], which expresses, given a projector Q, the unique orthogonal projector P such that $R(P) = R(Q)$. Some of the expressions of $P_{\mathcal{S}//\mathcal{N}}$ given above follow from Kerzman-Stein's formula.

DEFINITION 2.4. Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and let $A \in L(\mathcal{H})^+$. We say that the pair $(A, \mathcal{S})$ is compatible if the set $\mathcal{P}(A, \mathcal{S})$ is not empty.

The following result, due to M. G. Krein [40], will be used, implicitly or explicitly, several times.

LEMMA 2.5 (Krein). *Let Q be a projector with $R(Q) = \mathcal{S}$. Then Q is A-Hermitian if and only if $N(Q) \subseteq A^{-1}(\mathcal{S}^\perp)$. In particular, $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $N(Q) \subseteq A^{-1}(\mathcal{S}^\perp)$, so that $(A, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$.*

Proof. Suppose that $AQ = Q^*A$ and consider ξ such that $\xi \in N(Q)$, then $\langle A\xi, Q\theta\rangle = \langle Q^*A\xi, \theta\rangle = \langle AQ\xi, \theta\rangle = 0$, for all $\theta \in \mathcal{H}$. Therefore $A\xi \in R(Q)^\perp$, or, equivalently, $\xi \in A^{-1}(R(Q)^\perp)$. Conversely, suppose that $N(Q) \subseteq A^{-1}(R(Q)^\perp)$ and consider $\xi, \eta \in \mathcal{H}$. Decompose $\xi = \nu + \rho$ and $\eta = \nu' + \rho'$, where $Q\rho = \rho$, $Q\rho' = \rho'$ and $Q\nu = Q\nu' = 0$. Then $\langle AQ\xi, \eta\rangle = \langle AQ\rho, \nu' + \rho'\rangle = \langle A\rho, \rho'\rangle$ and $\langle Q^*A\xi, \eta\rangle = \langle A\rho, Q(\nu' + \rho')\rangle = \langle A\rho, \rho'\rangle$. Thus $AQ = Q^*A$. ∎

Observe that two projectors Q_1, Q_2 on $\mathcal{H}$ such that $R(Q_1) = R(Q_2)$ and $N(Q_1) \subseteq N(Q_2)$ are equal: every $\xi \in \mathcal{H}$ can be written as $\xi = \rho + \nu$ with $\rho \in R(Q_1)$, $\nu \in N(Q_1)$; then $Q_1\xi = \rho$ and $Q_2\xi = \rho + Q_2\nu = \rho$ because $\nu \in N(Q_1) \subseteq N(Q_2)$. Using this remark, we prove the next result.

COROLLARY 2.6. *The set $\mathcal{P}(A, \mathcal{S})$ is parametrized by the set of all direct complements of $\mathcal{S}$ contained in $A^{-1}(\mathcal{S}^\perp)$.*

REMARK 2.7. If $\mathcal{S} \cap N(A) = \{0\}$ the pair $(A, \mathcal{S})$ is compatible if and only if $\overline{A(\mathcal{S})} \oplus \mathcal{S}^\perp$ is closed. Indeed if $\mathcal{M}, \mathcal{N}$ are closed subspaces, then $\mathcal{M} + \mathcal{N}$ is closed if and only if $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (see theorem 4.8 of [37]); if $(A, \mathcal{S})$ is compatible then $\mathcal{S} \oplus A(\mathcal{S})^\perp = \mathcal{H}$, a fortiori $\mathcal{S} + A(\mathcal{S})^\perp$ is closed. Then $\mathcal{S}^\perp + \overline{A(\mathcal{S})}$ is closed. Moreover $\mathcal{S}^\perp \cap \overline{A(\mathcal{S})} = (\mathcal{S} + A(\mathcal{S})^\perp)^\perp = \{0\}$. Conversely, if $\mathcal{S}^\perp \oplus \overline{A(\mathcal{S})}$ is closed, then $\mathcal{S}^\perp + \overline{A(\mathcal{S})} = \overline{\mathcal{S}^\perp + \overline{A(\mathcal{S})}} = (\mathcal{S} \cap A(\mathcal{S})^\perp)^\perp = (\mathcal{S} \cap N(A))^\perp = \mathcal{H}$. Again, if $\mathcal{H} = \mathcal{S}^\perp + \overline{A(\mathcal{S})}$ then $\mathcal{S} + A(\mathcal{S})^\perp$ is closed and $(\mathcal{S} + A(\mathcal{S})^\perp)^\perp = \mathcal{S}^\perp \cap \overline{A(\mathcal{S})} = \{0\}$.

The following remarks may be helpful to understand the meaning of compatibility. With the 2×2 matrix representation mentioned above, if Q is a projector onto $\mathcal{S}$ then

$Q \in \mathcal{P}(A,\mathcal{S})$ if and only if

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}\begin{pmatrix} 1 & e \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x^* & 0 \end{pmatrix}\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}.$$

It is easy to see that the four equations reduce to a single one, namely, $ax = b$. By Douglas theorem, $ax = b$ has a solution if and only if $R(b) \subseteq R(a)$ and, in this case, there is a unique $d \in L(\mathcal{S}^\perp, \mathcal{S})$ such that $ad = b$ and $R(d) \subseteq R(a)$.

As we saw, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+$ then $R(a^{1/2}) \supseteq R(b)$. In general, $R(a) \subseteq R(a^{1/2}) \subseteq \overline{R(a)}$. Then, there is no much place for a, b to satisfy $R(b) \subseteq R(a^{1/2})$ and not satisfy $R(b) \subseteq R(a)$. In fact, given $\mathcal{S}$, the set $\Upsilon_{\mathcal{S}} = \{B \in L(\mathcal{H})^+ : (B,\mathcal{S}) \text{ is compatible}\}$ is everywhere dense in $L(\mathcal{H})^+$. Moreover, $GL(\mathcal{H})^+$ is dense in $L(\mathcal{H})^+$ and $GL(\mathcal{H})^+ \subseteq \Upsilon_{\mathcal{S}}$. Indeed, from the comments above, if $A \in GL(\mathcal{H})^+$, then $a \in GL(\mathcal{S})^+$, so that the equation $ax = b$ has the unique solution $x = a^{-1}b$. Then $\mathcal{P}(A,\mathcal{S}) = \{P_{A,\mathcal{S}}\}$, where $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}$.

The following result, which contains another parametrization of $\mathcal{P}(A,\mathcal{S})$, in terms of the set of solutions in $L(\mathcal{S}^\perp, \mathcal{S})$ of the equation $ax = b$, follows from the remarks above.

Theorem 2.8. *The pair $(A,\mathcal{S})$ is compatible if and only if $R(b) \subseteq R(a)$. In this case*

$$\mathcal{P}(A,\mathcal{S}) = \{P + PV(I-P) : V \in L(\mathcal{S}^\perp,\mathcal{S}), PAPV = PA|_{\mathcal{S}^\perp}\}$$

$$= \left\{\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : ax = b\right\}.$$

We summarize the conditions which are equivalent to compatibility in the next statement:

Theorem 2.9. *Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and a positive operator A on $\mathcal{H}$, the following conditions are equivalent:*

1. *$\mathcal{P}(A,\mathcal{S})$ is non-empty;*
2. *$\mathcal{S} + \mathcal{S}^{\perp_A} = \mathcal{H}$;*
3. *there exists a closed subspace $\mathcal{W} \subseteq \mathcal{S}^{\perp_A}$ such that $\mathcal{S} \oplus \mathcal{W} = \mathcal{H}$;*
4. *for the representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ of A under the decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$, we have $R(b) \subseteq R(a)$.*

Example 2.10. If $A \in L(\mathcal{H})^+$ has a dense non-closed image in $\mathcal{H}$, then

$$B = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix}$$

belongs to $L(\mathcal{H} \oplus \mathcal{H})^+$ because $B = TT^*$ for $T : H \to \mathcal{H} \oplus \mathcal{H}$ defined by $T\xi = (A^{1/2}\xi, \xi)$. On the other hand, $R(A^{1/2})$ properly contains $R(A)$, so that B and $\mathcal{H} \oplus \{0\}$ are not compatible. In the same order of ideas, let $C = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+$. Then $(C, \mathcal{H} \oplus \{0\})$ is a compatible pair and $R(C) = R(A)$ is non-closed.

Suppose that $(A, \mathcal{S})$ is compatible. Define $P_{A,\mathcal{S}}$ the unique member of $\mathcal{P}(A, \mathcal{S})$ determined by the reduced solution d of $ae = b$: $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{P}(A, \mathcal{S})$ is an affine manifold identified with $\{T \in L(\mathcal{H}) : T|_{\mathcal{S}} = 0,\ T(\mathcal{S}^\perp) \subseteq \mathcal{N}\}$. In particular, $\mathcal{P}(A, \mathcal{S})$ has a unique element if and only if $\mathcal{N} = \{0\}$. If $\mathcal{N} \neq \{0\}$, then $\|P_{A,\mathcal{S}}\| \leq \|Q\|$ for all $Q \in \mathcal{P}(A, \mathcal{S})$. For a proof of these facts, see [19].

THEOREM 2.11. *Let A and $\mathcal{S}$ be compatible. Denote by $\mathcal{N} = (A\mathcal{S})^\perp \cap \mathcal{S} = N(A) \cap \mathcal{S}$. Then $N(P_{A,\mathcal{S}}) = (A\mathcal{S})^\perp \ominus \mathcal{N}$.*

Proof. Both projectors have the same image, namely $\mathcal{S}$. It suffices to show that $N(P_{A,\mathcal{S}}) \subseteq (A\mathcal{S})^\perp \ominus \mathcal{N}$. Recall that $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$ where $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and d is the reduced solution of $ax = b$, i.e., $ad = b$ and $R(d) \subseteq \overline{R(a)}$. If $\xi = \sigma + \sigma^\perp \in N(P_{A,\mathcal{S}})$ then $\sigma + d\sigma^\perp = 0$ and $\xi = -d\sigma^\perp + \sigma^\perp$. We must prove $-ds^\perp + s^\perp \in \mathcal{W} = (A\mathcal{S})^\perp \ominus \mathcal{N}$. First, let us show $-d\sigma^\perp + \sigma^\perp \in (A\mathcal{S})^\perp$, or, equivalently, that $A(-d\sigma^\perp + \sigma^\perp) \in \mathcal{S}^\perp$; but $(-d\sigma^\perp + \sigma^\perp) = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \begin{pmatrix} -d\sigma^\perp \\ \sigma^\perp \end{pmatrix} = \begin{pmatrix} 0 \\ -b^* d\sigma^\perp + c\sigma^\perp \end{pmatrix} \in \mathcal{S}^\perp$. Next, we must show that $-d\sigma^\perp + \sigma^\perp \in (\mathcal{S} \cap N(A))^\perp$. By the definition of d, $-d\sigma^\perp = \lim a\sigma_n$ for some sequence $\{\sigma_n\}$ in $\mathcal{S}$. Given $\sigma \in \mathcal{S} \cap N(A)$, $a\sigma = A\sigma = 0$, so that $\langle -d\sigma^\perp + \sigma^\perp, \sigma \rangle = \langle -d\sigma^\perp, \sigma \rangle = \lim \langle a\sigma_n, \sigma \rangle = \lim \langle \sigma_n, a\sigma \rangle = 0$. This finishes the proof. ∎

REMARK 2.12. Under additional hypotheses on A, other characterizations of compatibility and formulas for $P_{A,\mathcal{S}}$ can be used. We mention a sample of these, taken from [19] and [20]:

1. If $R(PAP)$ is closed (or, equivalently, if $R(PA^{1/2})$ or $A^{1/2}(\mathcal{S})$ are closed), then $(A, \mathcal{S})$ is compatible. Indeed, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, the positivity of A implies that $R(b) \subseteq R(a^{1/2})$ (see, e.g., [4]). If $R(PAP) = R(a)$ is closed, then $R(b) \subseteq R(a^{1/2}) = R(a)$ so that $(A, \mathcal{S})$ is compatible, by Theorem 2.8. In this case,

   (2) $$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & a^\dagger b \\ 0 & 0 \end{pmatrix},$$

   since $a = PAP$ has closed range, and $a^\dagger b$ is the reduced solution of $ax = b$. In particular, if $\mathcal{N} = N(a) = N(A) \cap \mathcal{S} = \{0\}$ (i. e. $R(a) = \mathcal{S}$), one gets

   (3) $$P_{A,\mathcal{S}} = (PAP)^\dagger PA.$$

   Otherwise, $P_{A,\mathcal{S}} = P_{\mathcal{N}} + (PAP)^\dagger PA$.
2. If A has closed range then the following conditions are equivalent:
   (a) The pair $(A, \mathcal{S})$ is compatible.
   (b) $R(PAP)$ is closed.
   (c) $R(AP)$ is closed.
   (d) $\mathcal{S}^\perp + R(A)$ is closed.
3. If P, Q are orthogonal projectors with $R(P) = \mathcal{S}$, then $(Q, \mathcal{S})$ is compatible if and only if $R(QP)$ is closed. Moreover, if $(Q, \mathcal{S})$ is compatible, then $\mathcal{H} = \mathcal{S} + Q^{-1}(\mathcal{S}^\perp) =$

$\mathcal{S} + (R(Q) \cap \mathcal{S}^\perp) + N(Q)$ and, if $\mathcal{N} = N(Q) \cap \mathcal{S}$ and $\mathcal{M} = \mathcal{S} \ominus \mathcal{N}$, then

(a) $P_{Q,\mathcal{S}} = P_{\mathcal{N}} + P_{Q,\ \mathcal{M}}$.
(b) $\mathcal{M} \oplus (N(Q) \perp (R(Q) \cap \mathcal{S}^\perp)) = \mathcal{H}$, and $P_{Q,\ \mathcal{M}}$ is the projector onto $\mathcal{M}$ given by this decomposition.
(c) In the particular case that $N(Q) \cap \mathcal{S} = \{0\} = R(Q) \cap \mathcal{S}^\perp$, then

$$\mathcal{S} \oplus N(Q) = \mathcal{H}$$

and $P_{Q,\mathcal{S}}$ is the projector given by this decomposition, i.e., $N(P_{Q,\mathcal{S}}) = N(Q)$.
(d) $\|P_{Q,\mathcal{S}}\| = \|P_{Q,\ \mathcal{M}}\| = (1 - \|(I-Q)P_{\mathcal{M}}\|^2)^{-1/2}$.

Remark 2.13. Consider the following conditions:

1. The pair $(A, \mathcal{S})$ is compatible;
2. $A(\mathcal{S})$ is closed in $R(A)$;
3. $A^{-1}(\overline{A(\mathcal{S})}) = \mathcal{S} + N(A)$;
4. $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$;
5. $\mathcal{S} + N(A)$ is closed;
6. $P_{\overline{R(A)}}(\mathcal{S})$ is closed, where $P_{\overline{R(A)}}$ is the orthogonal projector onto $\overline{R(A)}$.

A precise description of the relationships among them is provided by the implications: $1 \to 2 \leftrightarrow 3 \to 4 \to 5 \leftrightarrow 6$. Moreover, $(A, \mathcal{S})$ is compatible if and only if $P_{\overline{R(A)}}\ (\mathcal{S})$ is closed and $(A, P_{\overline{R(A)}}\ (\mathcal{S}))$ is compatible.

The next result is a characterization of compatibility in terms of orthogonal decompositions of $R(A)$ and $R(A^{1/2})$.

Proposition 2.14. *Given $A \in L(\mathcal{H})^+$, the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *$R(A) = A(\mathcal{S}) \oplus \mathcal{S}^\perp \cap R(A)$.*
3. *$R(A^{1/2}) = A^{1/2}(\mathcal{S}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$.*
4. *If $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$, then $R(P_{\mathcal{M}} A^{1/2}) \subseteq R(A^{1/2}P)$.*

Proof. $1 \leftrightarrow 2$: If $(A, \mathcal{S})$ is compatible then $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$ so that

$$R(A) = A(\mathcal{S}) + A(A^{-1}(\mathcal{S}^\perp)) = A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A);$$

conversely, if $R(A) = A(\mathcal{S}) \oplus \mathcal{S}^\perp \cap R(A)$, then $\mathcal{H} = A^{-1}(R(A)) = A^{-1}(A(\mathcal{S})) + A^{-1}(\mathcal{S}^\perp \cap R(A))$. But $A^{-1}(\mathcal{S}^\perp \cap R(A)) = A^{-1}(\mathcal{S}^\perp)$ and $A^{-1}(A(\mathcal{S})) = S + N(A)$, so that $\mathcal{H} = S + N(A) + A^{-1}(\mathcal{S}^\perp) = S + A^{-1}(\mathcal{S}^\perp)$, because $N(A) \subseteq A^{-1}(\mathcal{S}^\perp)$.

$1 \leftrightarrow 3$: similar to (1) $\leftrightarrow$ (2).

$3 \leftrightarrow 4$: If $y \in R(A^{1/2})$ then $y = y_1 + y_2$ for unique $y_1 \in A^{1/2}(\mathcal{S})$ and $y_2 \in A^{1/2}(\mathcal{S})^\perp$; but, then, $P_{\mathcal{M}}(y) = y_1 \in A^{1/2}(\mathcal{S}) = R(A^{1/2}P)$. The converse is similar. ∎

As a consequence of Proposition 2.14, it is easy to see that $(A, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ and

$$R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2}).$$

More generally, given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and $\mathcal{W} = A^{-1/2}(\overline{A^{1/2}(\mathcal{S})})$, then $(A, \mathcal{W})$ is compatible if and only if $R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$: in fact, if $(A, \mathcal{W})$ is compatible then, by Proposition 2.14, $R(A^{1/2}) = A^{1/2}(\mathcal{W}) + A^{1/2}(\mathcal{W})^\perp \cap$

$R(A^{1/2})$. On one hand, $A^{1/2}(\mathcal{W}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2})$; on the other hand, since $A^{1/2}(\mathcal{S}) \subseteq A^{1/2}(\mathcal{W}) \subseteq \overline{A^{1/2}(\mathcal{S})}$, we get $A^{1/2}(\mathcal{S})^{\perp} = A^{1/2}(\mathcal{W})^{\perp}$. Thus,

$$R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) + A^{1/2}(\mathcal{S})^{\perp} \cap R(A^{1/2}),$$

and, of course, the sum is direct. The converse is similar.

A notion which is naturally related to oblique projectors is that of angle between subspaces. We consider here two non-equivalent definitions of angles and we show a characterization of the compatibility of $(A, \mathcal{S})$ in terms of these angles. For excellent treatments on angles in Hilbert spaces the reader is referred to the survey by Deutsch [24] or the book by A. Ben-Israel and T. N. E. Greville [12]

Given two subspaces $\mathcal{S}, \mathcal{T}$, the cosine of the *Friedrichs angle* between them is defined by

$$c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta \rangle| : \ \xi \in \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}, \ \ \|\xi\| < 1, \ \eta \in \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^{\perp}, \ \ \|\eta\| < 1\}.$$

It is well known (see Theorem 13 of [24]) that the following conditions are equivalent:

1. $c(\mathcal{S}, \mathcal{T}) < 1$;
2. $\mathcal{S} + \mathcal{T}$ is closed;
3. $\mathcal{S}^{\perp} + \mathcal{T}^{\perp}$ is closed;
4. $c(\mathcal{S}^{\perp}, \mathcal{T}^{\perp}) < 1$.

The formulas $\|P_{\mathcal{S}} P_{\mathcal{T}}\| = c(\mathcal{S}, \mathcal{T})$ [24] and $\|P_{\mathcal{S}//\mathcal{T}}\| = (1 - c(\mathcal{T}, \mathcal{S})^2)^{-1/2}$ [49] relate this notion with oblique projectors.

The *minimal angle* between $\mathcal{S}$ and $\mathcal{T}$ is the angle whose cosine is defined by

$$c_o(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta \rangle| : \ \xi \in \mathcal{S}, \ \|\xi\| < 1, \ \ \eta \in \mathcal{T}, \ \|\eta\| < 1\}.$$

Observe that $c(\mathcal{S}, \mathcal{T}) \leq c_0(\mathcal{S}, \mathcal{T})$ and $c(\mathcal{S}, \mathcal{T}) = c_0(\mathcal{S}, \mathcal{T})$ when $\mathcal{S} \cap \mathcal{T} = \{0\}$.

THEOREM 2.15. *Consider $A \in L(\mathcal{H})^+$. Then $(A, \mathcal{S})$ is compatible if and only if $c_0(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) < 1$.*

Proof. If $(A, \mathcal{S})$ is compatible then $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$, so that $\mathcal{S} + A^{-1}(\mathcal{S}^{\perp})$ is closed. By the remarks above and the identity $A^{-1}(\mathcal{S}^{\perp}) = (A\mathcal{S})^{\perp}$, we get $c(\mathcal{S}, A^{-1}(\mathcal{S}^{\perp})) < 1$ or equivalently, $c(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) < 1$. But $\mathcal{S}^{\perp} \cap \overline{A(\mathcal{S})} = (\mathcal{S} + A^{-1}(\mathcal{S}^{\perp}))^{\perp} = \mathcal{H}^{\perp} = \{0\}$. Therefore, $c_0(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) = c(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) < 1$.

Conversely, if $c_0(\mathcal{S}^{\perp}, \overline{A(\mathcal{S})}) < 1$ then $\mathcal{S}^{\perp} \cap \overline{A(\mathcal{S})} = \{0\}$ and $\mathcal{S}^{\perp} + \overline{A(\mathcal{S})}$ is closed; therefore, $\mathcal{S} + A(\mathcal{S})^{\perp}$ is closed; also $(\mathcal{S} + A(\mathcal{S})^{\perp})^{\perp} = \mathcal{S}^{\perp} \cap \overline{A(\mathcal{S})} = \{0\}$. Then $\mathcal{S} + A(\mathcal{S})^{\perp} = \mathcal{H}$ and $(A, \mathcal{S})$ is compatible. ∎

REMARK 2.16.

1. If A has closed range then, by Remark 2.12, the pair $(A, \mathcal{S})$ is compatible if and only if $R(AP)$ is closed. Note that this is equivalent to the angle condition $c(N(A), \mathcal{S}) < 1$.
2. If P, Q are orthogonal projectors with $R(P) = \mathcal{S}$, define $\mathcal{N} = N(Q) \cap \mathcal{S}$ and $\mathcal{M} = \mathcal{S} \ominus \mathcal{N}$. Then, again by Remark 2.12,

$$\|P_{Q,\mathcal{S}}\| = \|P_{Q,\ \mathcal{M}}\| = (1 - \|(1-Q)P_{\mathcal{M}}\|^2)^{-1/2} = (1 - c(N(Q), \mathcal{S})^2)^{-1/2}.$$

3. Formulas for $P_{A,\mathcal{S}}$. This section is devoted to presenting several explicit formulas for $P_{A,\mathcal{S}}$ in terms of the orthogonal projectors onto $\mathcal{S}$, $\mathcal{W} = A(\mathcal{S})^\perp \ominus (\mathcal{S} \cap N(A))$ and $\mathcal{W}^\perp$. Afriat [1], Greville [32] and Pták [49] have proven this type of formulas, the first two in finite dimensional settings. Some of these formulas seem to have been known by V. E. Ljance [43]. Consider $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$ such that $(A,\mathcal{S})$ is compatible. Denote $\mathcal{N} = \mathcal{S} \cap A(\mathcal{S})^\perp = \mathcal{S} \cap N(A)$ and $\mathcal{W} = A(\mathcal{S})^\perp \ominus \mathcal{N}$. Then, as shown in Theorem 3.5 of [22], $\mathcal{W}$ is the kernel of $P_{A,\mathcal{S}}$ so that $P_{A,\mathcal{S}} = P_{\mathcal{S}//\mathcal{W}}$, the oblique projector onto $\mathcal{S}$, along $\mathcal{W}$. Afriat [1] and Greville [32] exhibited formulas for an oblique projector Q in terms of the orthogonal projectors onto $R(Q)$ and $N(Q)$, by using the Moore-Penrose pseudoinverse. However, in order to use the same method in our infinite dimensional setting we need to know that the operator whose Moore-Penrose pseudoinverse is considered has closed range [23]. This justifies the need of a proof for the first part of the next result. The rest of the proof follows without change Greville's arguments.

Lemma 3.1.

1. *$(A,\mathcal{S})$ is compatible if and only if $P_{\mathcal{W}^\perp}P_{\mathcal{S}}$ has closed range.*
2. *If the pair $(A,\mathcal{S})$ is compatible then*
   (a) $P_{A,\mathcal{S}} = (P_{\mathcal{W}^\perp}P_{\mathcal{S}})^\dagger$.
   (b) $P_{A,\mathcal{S}} = (I - P_{\mathcal{S}}P_{\mathcal{W}})^{-1}P_{\mathcal{S}}(I - P_{\mathcal{S}}P_{\mathcal{W}})$.
   (c) $P_{A,\mathcal{S}} = (I - P_{\mathcal{W}}P_{\mathcal{S}})^{-1}(I - P_{\mathcal{W}}) = P_{\mathcal{S}}(P_{\mathcal{S}} + P_{\mathcal{W}} - P_{\mathcal{W}}P_{\mathcal{S}})^{-1}$.

Proof. If $(A,\mathcal{S})$ is compatible then $\mathcal{H} = \mathcal{S} \oplus \mathcal{W}$ by the remarks above. Observe first that $R(P_{\mathcal{W}^\perp}P_{\mathcal{S}}) = \mathcal{W}^\perp$: for this, it suffices to show the inclusion $\mathcal{W}^\perp \subseteq R(P_{\mathcal{W}^\perp}P_{\mathcal{S}})$, because the converse is evident. If $\xi \in \mathcal{W}^\perp$, then ξ decomposes as $\xi = \sigma + \omega$, $\sigma \in \mathcal{S}$ and $\omega \in \mathcal{W}$, so that $\xi = P_{\mathcal{W}^\perp}x = P_{\mathcal{W}^\perp}\sigma \in P_{\mathcal{W}^\perp}\mathcal{S} = R(P_{\mathcal{W}^\perp}P_{\mathcal{S}})$.

Conversely, if $P_{\mathcal{W}^\perp}P_{\mathcal{S}}$ has closed range then $(P_{\mathcal{W}^\perp}P_{\mathcal{S}})^\dagger$ is a bounded linear operator. Greville's arguments for matrices [32] can be used almost without changes to prove that $(P_{\mathcal{W}^\perp}P_{\mathcal{S}})^\dagger$ is an idempotent with range $\mathcal{S}$ and kernel $\mathcal{W}$. Then $\mathcal{H} = \mathcal{S} \oplus \mathcal{W} = \mathcal{S} + A(\mathcal{S})^\perp$ and $(A,\mathcal{S})$ is compatible. The formulas of part 2 follow from the fact that $P_{A,\mathcal{S}} = P_{\mathcal{S}//\mathcal{W}}$, using Theorem 2.2. ∎

Corollary 3.2. *If the pair $(A,\mathcal{S})$ is compatible and $\mathcal{N} = \{0\}$ then* $P_{A,\mathcal{S}} = (P_{\overline{A(\mathcal{S})}}P_{\mathcal{S}})^\dagger = (I - P_{\mathcal{S}}P_{A(\mathcal{S})^\perp})^{-1}P_{\mathcal{S}}(I - P_{\mathcal{S}}P_{A(\mathcal{S})^\perp}) = (I - P_{A(\mathcal{S})^\perp}P_{\mathcal{S}})^{-1}(I - P_{A(\mathcal{S})^\perp})$.

The shorted operator of A to $\mathcal{S}$ is $A/_{\mathcal{S}} = \sup\{X \in L(\mathcal{H})^\dagger : X \le A \text{ and } R(X) \subseteq \mathcal{S}\}$. In [48], Pekarev proved

$$A/_{\mathcal{S}^\perp} = A^{1/2}P_{\mathcal{M}^\perp}A^{1/2},$$

where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$. Let us show a formula for $P_{A,\mathcal{S}}$ in the spirit of Pekarev's. The relationship between the projectors in $\mathcal{P}(A,\mathcal{S})$ and $A/_{\mathcal{S}^\perp}$ is given by the formula $A/_{\mathcal{S}^\perp} = AE$, which holds for every projector E such that $I - E \in \mathcal{P}(A,\mathcal{S})$ (see [19]). In particular $A/_{\mathcal{S}^\perp} = A(I - P_{A,\mathcal{S}})$ and, if A were invertible, we can compute

$$P_{A,\mathcal{S}} = A^{-1}(A - A/_{\mathcal{S}^\perp}) = A^{-1/2}P_{\mathcal{M}}A^{1/2}.$$

In order to get a generalization of this formula, we consider firstly the injective case:

PROPOSITION 3.3. *Let $A \in L(\mathcal{H})^+$ injective such that $(A, \mathcal{S})$ is compatible. Then*

$$P_{A,\mathcal{S}} = A^{-1/2} P_{\mathcal{M}} A^{1/2}$$

where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.

Proof. Observe that in this case $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$ because $\mathcal{S} \cap N(A) = \{0\}$. Define $Q = A^{-1/2} P_{\mathcal{M}} A^{1/2}$. Then Q is well defined because $A^{-1/2} : R(A^{1/2}) \to \mathcal{H}$ and $R(P_{\mathcal{M}} A^{1/2}) \subseteq R(A^{1/2})$, by Theorem 2.14. It is easy to see that $Q^2 = Q$ and that $N(Q) = A(\mathcal{S})^{\perp}$: in fact, $\xi \in N(Q)$ if and only if $P_{\mathcal{M}} A^{1/2} \xi = 0$, i.e., $A^{1/2}\xi \in A^{1/2}(\mathcal{S})^{\perp}$, or, what is the same, $\xi \in A^{-1/2}(A^{-1/2}(\mathcal{S}^{\perp})) = A^{-1}(\mathcal{S}^{\perp})$. On the other hand, by the definition of Q, $R(Q) \subseteq A^{-1/2}(\mathcal{M}) = A^{-1/2}(\overline{A^{1/2}(\mathcal{S})}) = A^{-1/2}(\overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2})) = A^{-1/2}(A^{1/2}(\mathcal{S})) = \mathcal{S}$ because, by Theorem 2.13, $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$; this proves that $R(Q) \subseteq \mathcal{S}$. Conversely, if $\sigma \in \mathcal{S}$, then $Q\sigma = A^{-1/2} P_{\mathcal{M}} A^{1/2} \sigma = \sigma$. Then $R(Q) = \mathcal{S}$ and $Q = P_{A,\mathcal{S}}$. ■

We generalize this formula to any (not necessarily injective) $A \in L(\mathcal{H})^+$. For $B \in L(\mathcal{H})^+$ denote

$$B^{\sharp} = (B|_{\overline{R(B)}})^{-1} : R(B) \to \overline{R(B)} \subseteq \mathcal{H}.$$

Observe that $B^{\sharp}$ is a linear, not necessarily bounded operator. If $R(B)$ is closed, then $B^{\sharp} P_{R(B)} = B^{\dagger}$.

PROPOSITION 3.4. *Consider $A \in L(\mathcal{H})^+$ such that $(A, \mathcal{S})$ is compatible. Set $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.*

1. *If $\mathcal{S} \subseteq \overline{R(A)}$ then $P_{A,\mathcal{S}} = (A^{1/2})^{\sharp} P_{\mathcal{M}} A^{1/2}$.*
2. *If $\mathcal{S} \cap N(A) = \{0\}$ then $P_{A,\mathcal{S}} = (P_{\overline{R(A)}} P_{\mathcal{S}})^{\dagger} P_{A, P_{\overline{R(A)}}(\mathcal{S})} = (P_{\overline{R(A)}} P_{\mathcal{S}})^{\dagger} (A^{1/2})^{\sharp} P_{\mathcal{M}} A^{1/2}$.*

Proof. Observe that $\mathcal{P}(A, \mathcal{S}) = \{P_{A,\mathcal{S}}\}$ because $\mathcal{S} \cap N(A) = \{0\}$ in both cases.

1. If $\mathcal{S} \subseteq \overline{R(A)}$ and $Q = (A^{1/2})^{\sharp} P_{\mathcal{M}} A^{1/2}$ then Q is well defined because $P_{\mathcal{M}}(R(A^{1/2})) \subseteq R(A^{1/2})$, by Proposition 2.14. On one hand $P_{\mathcal{M}}(R(A^{1/2})) \subseteq \mathcal{M} \cap R(A^{1/2}) = A^{1/2}(\mathcal{S})$, because, by Remark 2.13, $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ thus, $R(Q) \subseteq (A^{1/2})^{\sharp} A^{1/2}(\mathcal{S}) = \mathcal{S}$. On the other hand, $Q\sigma = \sigma$, for all $\sigma \in \mathcal{S}$, because $\mathcal{S} \subseteq \overline{R(A)}$. Then $R(Q) = \mathcal{S}$. It is easy to see that $N(Q) = A^{-1}(\mathcal{S}^{\perp})$; thus, $Q = P_{A,\mathcal{S}}$.

2. If $\mathcal{S} \cap N(A) = \{0\}$ then the subspace $\mathcal{S}' = P_{\overline{R(A)}}(\mathcal{S})$ is closed because $(A, \mathcal{S})$ is compatible, $\mathcal{S}' \subseteq \overline{R(A)}$ and $(A, \mathcal{S}')$ is compatible (see Proposition 2.13). Also $\overline{A^{1/2}(\mathcal{S}')} = \overline{A^{1/2}(\mathcal{S})} = \mathcal{M}$, so that $P_{A,\mathcal{S}'} = (A^{1/2})^{\sharp} P_{\mathcal{M}} A^{1/2}$. Now, $R(P_{\overline{R(A)}} P_{\mathcal{S}}) = \mathcal{S}'$ is closed and $N(P_{\overline{R(A)}} P_{\mathcal{S}}) = \mathcal{S}^{\perp}$: the proof is straightforward. ■

In the general case the set $\mathcal{P}(A, \mathcal{S})$ can be parametrized by means of the set of complements $\mathcal{L}$ of $\mathcal{N} = \mathcal{S} \cap N(A)$ in $\mathcal{S}$. More precisely:

PROPOSITION 3.5. *Let $Q \in \mathcal{Q}$ and consider $A \in L(\mathcal{H})^+$ such that $(A, \mathcal{S})$ is compatible. Let $\mathcal{N} = \mathcal{S} \cap N(A)$. Then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if there exists a (unique) closed subspace $\mathcal{L} \subseteq \mathcal{S}$ such that $\mathcal{S} = \mathcal{N} \oplus \mathcal{L}$, $\mathcal{L} + N(Q)$ is closed, $\mathcal{S} + N(Q) = \mathcal{H}$ and*

$$Q = P_{A,\mathcal{L}} + P_{\mathcal{N}//(\mathcal{L}+N(Q))}.$$

Proof. $\Leftarrow$) Observe that $\mathcal{N} + \mathcal{L} + N(Q) = \mathcal{S} + N(Q) = \mathcal{H}$ and $\mathcal{L} + N(Q)$ is closed so that $Q' = P_{\mathcal{N}//(\mathcal{L}+N(Q))}$ is a well defined (oblique) projector. If $Q = P_{A,\mathcal{L}} + Q'$ then it is easy to see that $Q \in \mathcal{P}(A, \mathcal{S})$.

$\Rightarrow$) Consider $Q \in \mathcal{P}(A, \mathcal{S})$ and let $W = P_{\mathcal{N}} Q$; then $R(W) = \mathcal{N}$. From $QP_{\mathcal{N}} = P_{\mathcal{N}}$ we get that $W^2 = W$. Let $T = Q - W$; T is A-selfadjoint because Q and W are both A-selfadjoint; equality $T^2 = T$ follows from $QW = W = WQ$. Therefore $Q = T + W$, with $T^2 = T$ and $W^2 = W$. Let $\mathcal{L} = \mathcal{S} \cap N(W)$. It follows easily that $T = P_{A,\mathcal{L}}$, $\mathcal{S} = \mathcal{L} + \mathcal{N}$ and $N(W) = \mathcal{L} + N(Q)$. ■

Let $C \in L(\mathcal{H})$ such that $R(C) = \mathcal{S}$ is closed, and $A \in L(\mathcal{H})^+$ with closed range. Formula (3) suggests the natural generalization, which is widely used in the finite dimensional case:

$$P_{A,\mathcal{S}} \overset{?}{=} C(C^*AC)^\dagger C^*A. \tag{4}$$

In general, the formula is false for many reasons. For instance, $(C^*AC)^\dagger$ is unbounded if $R(C^*AC)$ is not closed; or $C(C^*AC)^\dagger C^*A$ may have range strictly contained in $\mathcal{S}$. However, the wide range of applications of the right side of formula (4) makes it desirable to establish its exact relationship with $P_{A,\mathcal{S}}$. In fact, projectors like $C(C^*AC)^\dagger C^*A$ appear explicitly in papers on scaled projections [53], [46], [34], [31], [60], [14], linear least squares problems [28], [29], linear feasibility [28], [29], [17], signal processing [36], [10], [58] and so on.

A first observation is that one needs to verify if $R(C^*AC)$ is closed. An interesting fact, which generalizes item 2 of Remark 2.12, is that $R(C^*AC)$ is closed if and only if $(A, \mathcal{S})$ is compatible. Indeed, note that $R(C^*AC)$ is closed if and only if $R(A^{1/2}CC^*A^{1/2})$ is closed. Since $R(C) = \mathcal{S}$ is closed, there exist $a, b > 0$ such that $aP \leq CC^* \leq bP$, so that

$$aA^{1/2}PA^{1/2} \leq A^{1/2}CC^*A^{1/2} \leq bA^{1/2}PA^{1/2}.$$

This implies, by the Douglas theorem, the identity

$$R((A^{1/2}CC^*A^{1/2})^{1/2}) = R(A^{1/2}P) = A^{1/2}(\mathcal{S}),$$

which is closed if and only if $(A, \mathcal{S})$ is compatible, by Remark 2.12.

Suppose now that $(A, \mathcal{S})$ is compatible. If $\mathcal{N} = N(A) \cap \mathcal{S}$, we shall see that

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + C(C^*AC)^\dagger C^*A, \tag{5}$$

showing that formula (4) holds if and only if $N(A) \cap \mathcal{S} = \{0\}$.

Define $Q = C(C^*AC)^\dagger C^*A$. It is clear that $Q^2 = Q$, $R(Q) \subseteq R(C) = \mathcal{S}$ and $AQ = Q^*A$. Therefore, Q is an A-selfadjoint projector onto a subspace of $\mathcal{S}$. Also, since C and $(C^*AC)^\dagger$ are injective on $R(C^*)$,

$$N(Q) = N(C^*A) = A^{-1}(N(C^*)) = A^{-1}(\mathcal{S}^\perp).$$

The next step is to show that $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$R(C^*A) = C^*(R(A)) = C^*(R(A^{1/2})) = R(C^*A^{1/2}).$$

Hence $R(C^*A) = R((C^*AC)^\dagger C^*A^{1/2})$ and $R(Q) = R(CC^*A^{1/2}) = R(CC^*A)$. But

$$N(ACC^*) = N(CC^*) \perp (R(CC^*) \cap N(A)) = \mathcal{S}^\perp \perp \mathcal{N},$$

so that $R(Q) = N(ACC^*)^\perp = \mathcal{S} \ominus \mathcal{N}$, as claimed. This fact clearly shows that $Q \in \mathcal{P}(A, \mathcal{S} \ominus \mathcal{N}) = \{P_{A,\mathcal{S}\ominus\mathcal{N}}\}$ (since $(\mathcal{S} \ominus \mathcal{N})^{\perp_A} = \mathcal{S}^{\perp_A}$) and also proves formula (5).

It is shown in [6] that for every projector Q onto a closed subspace $\mathcal{S}$, there exists an invertible positive $A \in L(\mathcal{H})$ such that $Q = P_{A,\mathcal{S}}$. This can be rewritten as follows:

PROPOSITION 3.6. *Let $\mathcal{S} \in \mathcal{H}$ be a closed subspace and $C \in L(\mathcal{H})$ with $R(C) = \mathcal{S}$. Let $A \in L(\mathcal{H})^+$ with closed range. Then*

1. *$(A, \mathcal{S})$ is compatible if and only if $R(C^*AC)$ is closed.*
2. *If $N(A) \cap \mathcal{S} = \mathcal{N}$, then*

$$P_{A,\mathcal{S}} = P_{\mathcal{N}} + C(C^*AC)^\dagger C^*A.$$

3. *For every $Q \in L(\mathcal{H})$ such that $Q^2 = Q$ and $R(Q) = \mathcal{S}$, there exists an invertible positive $A \in L(\mathcal{H})$ such that*

$$Q = C(C^*AC)^{-1}C^*A.$$

Final comments and open problems. The structure of the set $\mp_{\mathcal{S}} = \{A \in L(\mathcal{H})^+ : (A, \mathcal{S}) \text{ is compatible}\}$ is not completely known. We have observed that $GL(\mathcal{H})^+$ is contained in $\mp_{\mathcal{S}}$. Of course, if $\mathcal{S}$ is finite-dimensional, then $\mp_{\mathcal{S}} = L(\mathcal{H})^+$.

The extension of compatibility questions to Hermitian operators instead of positive operators is a much more difficult problem. The reader can find in [35], [19] and [44] some results in this direction.

Compatibility is related to some problems arising from wavelet and frame theory. The paper [7] deals with some problems in this area.

A difficult and very useful problem consists in determining conditions which ensure the convergence of sequences like $\{P_{A_n,\mathcal{S}}\}$ and $\{P_{A,\mathcal{S}_n}\}$. A sample of this type of results can be found in [21].

Given $Q \in \mathcal{Q}^{\mathcal{S}}$, it is known that $\chi_Q = \{A \in L(\mathcal{H})^+ : Q \in \mathcal{P}(A, \mathcal{S})\}$ is not empty and the set $\chi_Q \cap GL(\mathcal{H})^+$ is characterized [6]. However, in general, the structure of χ_Q is unknown and it would be interesting to have optimality criteria for choosing $A \in \chi_Q$.

References

[1] S. N. Afriat, *Orthogonal and oblique projectors and the characteristics of pairs of vector spaces*, Proc. Cambridge Philos. Soc. 53 (1957), 800–816.

[2] A. Aldroubi, *Oblique projections in atomic spaces*, Proc. Amer. Math. Soc. 124 (1996), 2051–2060.

[3] A. Aldroubi, *Oblique and hierarchical multiwavelet bases*, Appl. Comput. Harmon. Anal. 4 (1997), 231–263.

[4] W. N. Anderson and G. E. Trapp, *Shorted operators II*, SIAM J. Appl. Math. 28 (1975), 60–71.

[5] T. Ando, *De Branges spaces and analytic operator functions*, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan, (1990).

[6] E. Andruchow, G. Corach and D. Stojanoff, *Geometry of oblique projections*, Studia Math. 137 (1999), 61–79.

[7] J. Antezana, G. Corach, M. A. Ruiz and D. Stojanoff, *Weighted projectors and Riesz frames*, Linear Algebra Appl., in press.

[8] M. Atteia, *Généralisation de la définition et des propriétés des "spline fonctions"*, C. R. Acad. Sci. Paris 260 (1965), 3550–3553.

[9] J. K. Baksalary and R. Kala, *Two relations between oblique and Λ-orthogonal projections*, Linear Algebra Appl. 24 (1979), 99–103.

[10] R. T. Behrens and L. L. Scharf, *Signal processing applications of oblique projection operators*, IEEE Trans. Signal Process. 42 (1994), 1413–1424.

[11] J. Benedetto, *Frames, sampling, and seizure prediction*, in: Advances in Wavelets (Hong Kong, 1997), Springer, Singapore, 1999, 1–25.

[12] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[13] T. Blu and M. Unser, *Quantitative Fourier analysis of approximation techniques. I. Interpolators and projections*, IEEE Transactions on Signal Processing 47 (1999), 2783–2795.

[14] E. Bobrovnikova and S. A. Vavasis, *A norm bound for projectors with complex weights*, Linear Algebra Appl. 307 (2000), 69–75.

[15] L. de Branges and J. Rovnyak, *Square Summable Power Series*, Holt, Rinehart and Winston, New York, 1966.

[16] D. Buckholtz, *Hilbert space idempotents and involutions*, Proc. Amer. Math. Soc. 128 (2000), 1415–1418.

[17] Y. Censor and T. Elfving, *Block-iterative algorithms with diagonally scaled oblique projections for the linear feasibility problem.* SIAM J. Matrix Anal. Appl. 24 (2002), 40–58.

[18] K. Y. Chung, *Subspaces and graphs*, Proc. Amer. Math. Soc. 119 (1993), 141–146.

[19] G. Corach, A. Maestripieri and D. Stojanoff, *Schur complements and oblique projections*, Acta Sci. Math. (Szeged) 67 (2001), 439–459.

[20] G. Corach, A. Maestripieri and D. Stojanoff, *Generalized Schur complements and oblique projections*, Linear Algebra Appl. 341 (2002), 259–272.

[21] G. Corach, A. Maestripieri and D. Stojanoff, *Oblique projections and abstract splines*, Journal of Approximation Theory 117 (2002), 189–206.

[22] C. Davis, *Separation of two linear subspaces*, Acta Sci. Math. (Szeged) 19 (1958), 172–187.

[23] C. A. Desoer and B. H. Whalen, *A note on pseudoinverses*, Journal of Society for Industrial and Applied Mathematics 11 (1963), 442–447.

[24] F. Deutsch, *The angle between subspaces in Hilbert space*, in: pproximation theory, wavelets and applications, S. P. Singh, (ed.), Kluwer, 1995, 107–130.

[25] J. Dieudonné, *Quasi-hermitian operators*, in: Proc. Internat. Symp. Linear Spaces, Jerusalem (1961), 115–122.

[26] R. G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. 17 (1966), 413–416.

[27] P. A. Fillmore and J. P. Williams, *On operator ranges*, Advances in Math. 7 (1971), 254–281.

[28] A. Forsgren, *On linear least-squares problems with diagonally dominant weight matrices.* SIAM J. Matrix Anal. Appl. 17 (1996), 763–788.

[29] A. Forsgren and G. Sporre, *On weighted linear least-squares problems related to interior methods for convex quadratic programming*, SIAM J. Matrix Anal. Appl. 23 (2001), 42–56.

[30] W. Gerisch, *Idempotents, their Hermitian components, and subspaces in position p of a Hilbert space*, Math. Nachr. 115 (1984), 283–303.

[31] C. C. Gonzaga and H. J. Lara, *A note on properties of condition numbers*, Linear Algebra Appl. 261 (1997), 269–273.

[32] T. N. E. Greville, *Solutions of the matrix equations $XAX = X$ and relations between oblique and orthogonal projectors*, SIAM J. Appl. Math. 26, 4 (1974), 828–832.

[33] P. R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. 144 (1969), 381–389.

[34] M. Hanke and M. Neumann, *The geometry of the set of scaled projections*, Linear Algebra Appl. 190 (1993), 137–148.

[35] S. Hassi and K. Nordström, *On projections in a space with an indefinite metric*, Linear Algebra Appl. 208/209 (1994), 401–417.

[36] S. Kayalar and H. L. Weinert, *Oblique projections: formulas, algorithms and error bounds*, Math. Control Signal Systems 2 (1989), 33–45.

[37] T. Kato, *Perturbation Theory for Linear Operators*, Reprint of the 1980 edition, Classics in Mathematics, Springer-Verlag, Berlin, 1995.

[38] N. Kerzman and E. M. Stein, *The Szegő kernel in terms of Cauchy-Fantappiè kernels*, Duke Math. J. 45 (1978), 197–224.

[39] N. Kerzman and E. M. Stein, *The Cauchy kernel, the Szegő kernel, and the Riemann mapping function*, Math. Ann. 236 (1978), 85–93.

[40] M. G. Krein, *The theory of self-adjoint extensions of semibounded Hermitian operators and its applications*, Mat. Sb. (N.S.) 20 (62) (1947), 431–495.

[41] P. D. Lax, *Symmetrizable linear transformations*, Comm. Pure Appl. Math. 7 (1954), 633–647.

[42] C. Lee, M. Eden and M. Unser, *High quality image resizing using oblique projection operators*, IEEE Trans. Image Proces. 7 (1998), 679–692.

[43] V. E. Ljance, *Certain properties of idempotent operators*, Teoret. Prikl. Mat. Vyp. 1 (1958), 16–22 (in Russian).

[44] S. K. Mitra and C. R. Rao, *Projections under seminorms and generalized Moore Penrose inverses*, Linear Algebra Appl. 9 (1974), 155–167.

[45] V. J. Mizel and M. M. Rao, *Nonsymmetric projections in Hilbert space*, Pacific J. Math. 12 (1962), 343–357.

[46] D. P. O'Leary, *On bounds for scaled projections and pseudoinverses*, Linear Algebra Appl. 132 (1990), 115–117.

[47] Z. Pasternak-Winiarski, *On the dependence of the orthogonal projector on deformations of the scalar product*, Studia Math. 128 (1998), 1–17.

[48] E. L. Pekarev, *Shorts of operators and some extremal problems*, Acta Sci. Math. (Szeged) 56 (1992), 147–163.

[49] V. Pták, *Extremal operators and oblique projections*, Časopis pro pestování Matematiky 110 (1985), 343–350.

[50] Ju. L. Smuljan, *An operator Hellinger integral*, Mat. Sb. (N.S.) 49 (91) (1959), 381–430 (in Russian).

[51] J. Steinberg, *A class of integral transforms which are projections*, Integral Equations Operator Theory 34 (1999), 65–90.

[52] J. Steinberg, *Oblique projections in Hilbert spaces*, Integral Equations Operator Theory 38 (2000), 81–119.

[53] G. W. Stewart, *On scaled projections and pseudoinverses*, Linear Algebra Appl. 112 (1989), 189–193.

[54] Y. Takane and H. Yanai, *On oblique projectors*, in: Linear Algebra and Statistics (Istanbul, 1997), Linear Algebra Appl. 289 (1999), 297–310.

[55] W. S. Tang, *Oblique projections, biorthogonal Riesz bases and multiwavelets in Hilbert spaces*, Proc. Amer. Math. Soc. 128 (2000), 463–473.

[56] W. S. Tang, *Oblique multiwavelets in Hilbert spaces*, Proc. Amer. Math. Soc. 128 (2000), 2017–2031.

[57] M. Unser, *Quasi-orthogonality and quasi-projections*, Applied Comp. Harmonic Anal. 3 (1996), 201–214.

[58] P. Vandaele and M. Moonen, *Two deterministic blind channel estimation algorithms based on oblique projectors*, Signal Process. 80 (2000), 481–495.

[59] M. J. Vrhel, C. Lee and M. Unser, *Rapid computation of the continuous wavelet transform by oblique projectors*, IEEE Trans. Signal Process. 45 (1997), 891–900.

[60] M. S. Wei, *Upper bound and stability of scaled pseudoinverses*, Numer. Math. 72 (1995), 285–293.

[61] H. K. Wimmer, *Canonical angles of unitary spaces and perturbations of direct complements*, Linear Algebra Appl. 287 (1999), 373–379.

[62] H. K. Wimmer, *Lipschitz continuity of oblique projectors*, Proc. Amer. Math. Soc. 128 (1999), 873–876.

[63] X. Yu and L. Tong, *Joint channel and symbol estimation by oblique projectors*, IEEE Transactions on Image Processing 8 (1999), 527–536.