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DUAL COMPLEMENTORS IN TOPOLOGICAL ALGEBRAS

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Abstract. We deal with dual complementors on complemented topological (non-normed) algebras and give some characterizations of a dual pair of complementors for some classes of complemented topological algebras. The study of dual complementors shows their deep connection with dual algebras. In particular, we refer to Hausdorff annihilator locally C^*-algebras and to proper Hausdorff orthocomplemented locally convex H^*-algebras. These algebras admit, by their nature, the same type of dual pair of complementors. Dual pairs of complementors are also obtained on their closed 2-sided ideals or even on particular 1-sided ideals. If $(\perp_l, \perp_r)$ denotes a pair of complementors on a complemented algebra, then through the notion of a $\perp_l$ (resp. $\perp_r$)-projection, we get a structure theorem (analysis via minimal 1-sided ideals) for a semisimple annihilator left complemented Q'-algebra. Actually, such an algebra contains a maximal family, say $(x_i)_{i\in\Lambda}$, of mutually orthogonal minimal $\perp_l$-projections and the respective minimal ideals (factors of the analysis) are the Ex_i and x_iE, $i \in \Lambda$. As a consequence, an analysis is given for a certain locally C^*-algebra. In this case, the respective x_i's are, in particular, projections in both (left and right) complementors.

1. Introduction and peliminaries. Dual complementors were introduced by B. J. Tomiuk [16] in the framework of semisimple annihilator Banach complemented algebras.

Here we extend Tomiuk's point of view, by considering dual complementors on complemented (non-normed) topological algebras (Definition 2.1) and seek the interrelations between dual algebras and dual complementors. Moreover, we deal with two classes of involutive topological algebras admitting the same type of dual pair of complementors (Theorems 3.2, 3.8). Dual pairs of complementors are also obtained on their closed 2-sided ideals and on particular 1-sided ideals (Proposition 3.5, Theorem 3.8). Finally, a structural analysis is provided for a semisimple annihilator left complemented Q'-algebra through

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appropriate $\perp_l$-projections (Theorem 4.4). As a consequence, an analogous analysis is given for a certain locally C^*-algebra (Corollary 4.5).

Throughout this paper, all algebras are over the field $\mathbb{C}$ of complexes. A topological algebra E (separately continuous multiplication) is called a Q'_l (resp. Q'_r)-*algebra*, if every maximal regular left (resp. right) ideal is closed. E is a Q'-*algebra*, if it is both a Q'_l and a Q'_r-algebra (see [6: p. 148, Definition 1.1]). If S is a non-empty subset of an algebra E, $\mathcal{A}_l(S)$ (resp. $\mathcal{A}_r(S)$) denotes the left (right) annihilator of S and it is a left (right) ideal, which, in particular, is 2-sided if S is a left (right) ideal. In the case of a topological algebra, the previous ideals are closed. $\mathcal{L}_l(E) \equiv \mathcal{L}_l$ ($\mathcal{L}_r(E) \equiv \mathcal{L}_r$, $\mathcal{L}(E) \equiv \mathcal{L}$) denotes the set of all closed left (right, 2-sided) ideals in a topological algebra E. An algebra E is called *left* (*right*) *preannihilator* if $\mathcal{A}_l(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$; in this case, E is also called *proper*). If $\mathcal{A}_l(E) = \mathcal{A}_r(E) = (0)$, E is called a *preannihilator algebra.* A topological algebra E is said to be an *annihilator algebra* if it is preannihilator and satisfies the conditions:

$$\text{If } \mathcal{A}_l(I) = (0) \text{ with } I \in \mathcal{L}_r, \text{ then } I = E, \tag{1.1}$$

$$\text{if } \mathcal{A}_r(J) = (0) \text{ with } J \in \mathcal{L}_l, \text{ then } J = E. \tag{1.2}$$

A topological algebra E is called a *left* (resp. *right*) *dual algebra* if $I = \mathcal{A}_l(\mathcal{A}_r(I))$ for every $I \in \mathcal{L}_l$ (resp. $J = \mathcal{A}_r(\mathcal{A}_l(J))$ for every $J \in \mathcal{L}_r$). A left and right dual algebra is simply called a *dual algebra.* A topological algebra E such that $I \in \mathcal{L}$ and $I^2 = (0)$ implies $I = (0)$ is called *topologically semiprime.* An *idempotent* (*projection*) of an algebra E is an element $x \in E$ with $x = x^2$. In particular, x is *minimal* if it is non-zero and xEx is a division algebra. $\mathcal{I}d(E)$ denotes the non-zero idempotent elements of E. Two idempotents $x, y \in E$ are *mutually orthogonal* if $xy = yx = 0$. A family $(x_i)_{i \in K}$ of elements in E is called *algebraically orthogonal* if for every $i \neq j$ in K, $x_i x_j = 0$. An algebra E is called *semisimple* if its Jacobson radical $R(E)$ is zero. A topological algebra E is called *left complemented* if there is a mapping

$$\perp_l : \mathcal{L}_l \to \mathcal{L}_l : I \mapsto I^{\perp_l} \tag{1.3}$$

such that

$$\text{if } I \in \mathcal{L}_l, \text{ then } E = I \oplus I^{\perp_l} \text{ (complementarity)} \tag{1.4}$$

($I^{\perp_l}$ is called a (*left*) *complement* of I in E);

$$\text{if } I, J \in \mathcal{L}_l,\ I \subseteq J, \text{ then } J^{\perp_l} \subseteq I^{\perp_l} \text{ (order reversion)}, \tag{1.5}$$

and

$$\text{if } I \in \mathcal{L}_l, \text{ then } (I^{\perp_l})^{\perp_l} = I \text{ (reflexivity)}. \tag{1.6}$$

A mapping $\perp_l$ as above is called a *left complementor on E.*

A *right complemented algebra* is defined analogously, via a *right complementor* $\perp_r$. A left and right complemented algebra is called a *complemented algebra.* $(E, \perp_l)$ (resp. $(E, \perp_r)$) denotes a left (resp. right) complemented algebra, while a complemented algebra is denoted by $(E, \perp_l, \perp_r)$. A topological algebra E satisfying (1.3), (1.5), (1.6), as well as

$$\text{if } I \in \mathcal{L}_l, \text{ then } E = \overline{I \oplus I^{\perp_l}} \tag{1.7}$$

is called a *weakly left complemented algebra* and $I^{\perp_l}$ is a *weak (left) complement of* I. In this case, the map $\perp_l$ is called a *weak left complementor on* E. Analogous notions are given "on the right" (see [3] and [8]). A topological algebra E is called a *left* (*right*) *precomplemented algebra* if for every $I \in \mathcal{L}_l$ (resp. $J \in \mathcal{L}_r$) there exists $I' \in \mathcal{L}_l$ (resp. $J' \in \mathcal{L}_r$) with $E = I \oplus I'$ (resp. $E = J \oplus J'$). E is a *precomplemented algebra* if it is both left and right precomplemented [8]. A *locally* m*-convex algebra* is a topological algebra E whose topology is defined by a family $(p_\alpha)_{\alpha \in A}$ (A a directed index set) of submultiplicative seminorms (see [12] and/or [13]). Such a topological algebra is denoted by $(E, (p_\alpha)_{\alpha \in A})$. A C^**-seminorm* is a seminorm p on an involutive algebra E, satisfying $p(x^*x) = p(x)^2$ for all $x \in E$ (C^*-condition; [15: p. 1, Definition 1]). Such a seminorm is submultiplicative and $*$-preserving [ibid. p. 2, Theorem 2]. A *locally* C^**-algebra* is an involutive complete locally (-m) convex algebra $(E, (p_\alpha)_{\alpha \in A})$ such that each p_α, $\alpha \in A$ is a C^*-seminorm [11: p. 198, Definition 2.2]. A *locally convex* H^**-algebra* is an algebra E equipped with a family $(p_\alpha)_{\alpha \in A}$ of *Ambrose seminorms* in the sense that p_α, $\alpha \in A$ arises from a positive semi-definite (pseudo-)inner product $< , >_\alpha$ such that the induced topology makes E into a locally convex topological algebra. Moreover, the following conditions are satisfied:

For any $x \in E$, there is an $x^* \in E$ such that

$$< xy, z >_\alpha \;=\; < y, x^* z >_\alpha \tag{1.8}$$

$$< yx, z >_\alpha \;=\; < y, z x^* >_\alpha \tag{1.9}$$

for any $y, z \in E$ and $\alpha \in A$. x^* is not necessarily unique. In case E is proper and Hausdorff, x^* is unique and $* : E \to E : x \mapsto x^*$ is an involution (see [4: p. 451, Definition 1.1 and p. 452, Theorem 1.3]). If $(E, (p_\alpha)_{\alpha \in A})$ is a locally convex H^*-algebra, the orthogonal $S^\perp$ of a non-empty subset S of E is

$$S^\perp = \{x \in E :< x, y >_\alpha = 0 \;\text{ for every }\; y \in S, \alpha \in A\}, \tag{1.10}$$

a closed linear subspace of E. If $I \in \mathcal{L}_l$, then $I^\perp$ is a closed left ideal in E. An analogous result holds for any $I \in \mathcal{L}_r$ or $I \in \mathcal{L}$ [ibid. p. 456, Lemma 3.2].

2. Dual pairs of complementors

DEFINITION 2.1. Let $(E, \perp_l)$ be a left complemented algebra. We say that the map

$$\perp_r\colon \mathcal{L}_r \to \mathcal{L}_r \;\text{ with }\; I \mapsto I^{\perp_r} := \mathcal{A}_r[\mathcal{A}_l(I)^{\perp_l}] \tag{2.1}$$

is derived from $\perp_l$. If $\perp_r$ defines on E a right complementor, in the sense that $(E, \perp_r)$ is a right complemented algebra, then we say that $\perp_l$ is a *dual left complementor* on E.

If $(E, \perp_r)$ is a right complemented algebra, we say that the map

$$\perp_l\colon \mathcal{L}_l \to \mathcal{L}_l \;\text{ with }\; J \mapsto J^{\perp_l} := \mathcal{A}_l[\mathcal{A}_r(J)^{\perp_r}] \tag{2.2}$$

is derived from $\perp_r$. If $(E, \perp_l)$ is a left complemented algebra, $\perp_r$ is called a *dual right complementor on* E.

As we see from the previous definition, if the complementor of a left or right complemented algebra is dual, then this topological algebra is a complemented one.

In case of a weakly left (resp. right) complemented algebra, we have the analogous notion of a *dual weak left* (resp. *right*) *complementor.*

DEFINITION 2.2. Let $(E, \perp_l, \perp_r)$ be a complemented (resp. weakly complemented) algebra. Then the pair $(\perp_l, \perp_r)$ is called a *dual pair of complementors* (resp. of *weak complementors*) on E if each one of the $\perp_l, \perp_r$ is derived from the other.

EXAMPLE 2.3. Let X be a completely regular k-space and $\mathcal{C}_c(X)$ the complete locally m-convex algebra of all $\mathbb{C}$-valued continuous functions on X with the point-wise defined operations and the compact open topology. $\mathcal{C}_c(X)$ is, in particular, a locally C^*-algebra under the involution $f^*(x) = \overline{f(x)}$, $f \in \mathcal{C}_c(X)$, $x \in X$. By Theorem 3.1 in [12: p. 337], the points of X correspond to the closed maximal (regular) ideals of $\mathcal{C}_c(X)$, so that $\mathcal{C}_c(X)$ is an annihilator algebra (see [8: p. 3724, Example 2.6]). Hence, by [ibid. p. 3724, Theorem 2.5] and/or [9: p. 198, Theorem 2.4], $\mathcal{C}_c(X)$ is a dual complemented algebra with complementors $\perp_l$ $(=\perp_r) = * \circ \mathcal{A}_r = * \circ \mathcal{A}_l$, which are obviously dual. See also Theorem 3.1 below.

As we shall see in the sequel, dual complementors often appear in dual algebras. A dual algebra is an annihilator one. The converse is not in general true (see the comments at the end of Section 2 in [6: p. 151]). We provide now necessary and sufficient conditions under which a semisimple precomplemented Q_l'-algebra is a dual or an annihilator algebra. This is Theorem 2.5 below, in the proof of which we apply the next result stated in [14: p. 106, Theorem 2.8.29] for semisimple Banach algebras. Its proof also holds in our, more general, form and thus it is omitted.

THEOREM 2.4. *Let E be a preannihilator topological algebra which is equal to the topological sum of a given family $(K_\alpha)_{\alpha\in A}$ of its closed 2-sided ideals* (i.e. $\sum_{\alpha\in A} K_\alpha$ is dense in E). *Then*

(i) E is an annihilator algebra if each K_α is an annihilator algebra.

(ii) E is a dual algebra if each K_α is a dual algebra and

$$x \in \overline{Ex} \cap \overline{xE} \quad \text{for all} \quad x \in E. \tag{2.3}$$

Notice that (2.3) *is*, for instance, *fulfilled for preannihilator precomplemented algebras.* See also [1: p. 34, Lemma 3].

THEOREM 2.5. *Let E be a semisimple precomplemented Q_l'-algebra and $(K_\alpha)_{\alpha\in A}$ the family of its minimal closed 2-sided ideals. Then the following are equivalent:*

1) E is an annihilator (resp. dual) algebra.

2) E is the topological direct sum of the K_α's (i.e. $\bigoplus_{\alpha\in A} K_\alpha$ is dense in E) *and each K_α is an annihilator (resp. dual) algebra.*

Proof. Apply Theorem 2.4 (see also the comment following it), as well as [6: p. 161, Theorem 4.12] and [10: Theorem 3.13]. Notice that the assumption that E is a precomplemented algebra is used only in the "dual case" of 2) $\Rightarrow$ 1). ∎

In the commutative case, *there are semisimple dual (pre)complemented algebras which are not Q'-algebras.* Take, for instance, the algebra $\mathcal{C}_c(X)$ with X completely regular, k-space (non-compact) (see Example 2.3). The spectrum of the algebra $\mathcal{C}_c(X)$, being identical with X (within a homeomorphism of topological spaces; see, for instance [12: p. 223, Theorem 1.2]) is non-compact. Hence, by Proposition 1.3 in [6: p. 149] and by Proposition 1 in [18: p. 296], $\mathcal{C}_c(X)$ is not a Q'-algebra.

LEMMA 2.6. *Let E be a semisimple precomplemented algebra. Suppose that K is a closed 2-sided ideal in E. Then the following are equivalent:*

1) K satisfies the relations:

$$\overline{K\mathcal{A}_l^K(\mathcal{A}_r^K(I))} \subseteq I \quad \textit{for every} \quad I \in \mathcal{L}_l(K) \tag{2.4}$$

(here $\mathcal{A}_l^K$, $\mathcal{A}_r^K$ denote the left and right annihilator on K) *and*

$$\overline{\mathcal{A}_r^K(\mathcal{A}_l^K(J))K} \subseteq J \quad \textit{for every} \quad J \in \mathcal{L}_r(K). \tag{2.5}$$

2) K is a dual algebra.

Proof. 1) $\Rightarrow$ 2): Since $R(K) = R(E) \cap K$, K is a semisimple algebra (see also [2: p. 126, Corollary 20]). Since E is, in particular, preannihilator, K is a precomplemented algebra (see [8: p. 3728, Corollary 3.2; see also the comments preceding it]). Therefore, by (2.3) and (2.4), we get $I \subseteq \mathcal{A}_l^K(\mathcal{A}_r^K(I)) \subseteq \overline{K\mathcal{A}_l^K(\mathcal{A}_r^K(I))} \subseteq I$ for every $I \in \mathcal{L}_l(K)$. Namely, K is a left dual algebra. Likewise, K is a right dual algebra.

2) $\Rightarrow$ 1): It is straightforward. ■

For convenience and based on Lemma 2.6, we set the next

DEFINITION 2.7. A topological algebra E is called a *predual algebra* if it satisfies the conditions: (i) $\overline{E\mathcal{A}_l(\mathcal{A}_r(I))} \subseteq I$, $I \in \mathcal{L}_l$ and (ii) $\overline{\mathcal{A}_r(\mathcal{A}_l(J))E} \subseteq J$, $J \in \mathcal{L}_r$.

Obviously, a *dual algebra is predual.*

THEOREM 2.8. *Let E be a semisimple precomplemented Q_l'-algebra. Suppose that every minimal closed 2-sided ideal in E is a predual algebra. Then E is a dual algebra.*

Proof. We first note that E is the topological orthogonal direct sum of its minimal closed 2-sided ideals, each one of which is a semisimple, topologically simple, precomplemented algebra (see [3: p. 969, Theorem 3.3]). It follows from Lemma 2.6 that the aforementioned ideals are dual algebras and hence E is a dual algebra, as well (see Theorem 2.4 and the comment following it). ■

By the previous proof, we see that each minimal closed 2-sided ideal in E is actually a preannihilator precomplemented algebra. Hence, the "preduality" implies that these ideals are dual algebras. Actually, for preannihilator, precomplemented algebras the notions "predual algebra" and "dual algebra" are the same. Namely, we have the next, more general, result.

PROPOSITION 2.9. *Let E be a preannihilator precomplemented algebra. Then the following are equivalent:*

1) *E is a dual algebra.*
2) *E is a predual algebra.*
3) $\mathcal{A}_l(\mathcal{A}_r(I)) \cap I' = (0)$ *for every* $I \in \mathcal{L}_l$ *and* $\mathcal{A}_r(\mathcal{A}_l(J)) \cap J' = (0)$ *for every* $J \in \mathcal{L}_r$. (I' resp. J' denotes a complement of I resp. J in E).

Proof. We prove the assertion for closed left ideals. Analogously, on the "right".

1) $\Rightarrow$ 2): It is obvious.

2) $\Rightarrow$ 1): Observe that $x \in \overline{Ex} \cap \overline{xE}$ for every $x \in E$ (see the comment after Theorem 2.4). Thus, for any $I \in \mathcal{L}_l$, $\mathcal{A}_l(\mathcal{A}_r(I)) \subseteq \overline{E\mathcal{A}_l(\mathcal{A}_r(I))} \subseteq I$ and hence $\mathcal{A}_l(\mathcal{A}_r(I)) = I$.

1) $\Leftrightarrow$ 3): See [8: p. 3725, Proposition 2.8]. ∎

By Lemma 2.6, Theorem 2.8 and Proposition 2.9, we get the next

Corollary 2.10. *Every semisimple precomplemented Q_l'-algebra is (pre)dual if and only if every minimal closed 2-sided ideal in E is a (pre)dual algebra.*

The following result is stated in [16: p. 816, Proposition 3.1] for semisimple annihilator complemented Banach algebras (hence dual ones; [ibid. p. 816, Remark]), but its proof is still valid in the more general case of dual, complemented algebras. Namely, we have

Proposition 2.11. *For every dual complemented algebra $(E, \perp_l, \perp_r)$ the following are equivalent:*

1) $\perp_r$ is derived from $\perp_l$.

2) $\perp_l$ is derived from $\perp_r$.

The next result concerns characterizations of dual complementors over all preannihilator, complemented algebras.

Theorem 2.12. *Let $(E, \perp_l, \perp_r)$ be a preannihilator complemented algebra. Then the following are equivalent:*

1) The pair $(\perp_l, \perp_r)$ is dual.

2) E is a dual algebra and $[\mathcal{A}_r(I)]^{\perp_r} = \mathcal{A}_r(I^{\perp_l})$ for all $I \in \mathcal{L}_l$.

3) E is a dual algebra and $[\mathcal{A}_l(J)]^{\perp_l} = \mathcal{A}_l(J^{\perp_r})$ for all $J \in \mathcal{L}_r$.

Proof. We only prove that, if the pair $(\perp_l, \perp_r)$ is dual, then E is a dual algebra. For the rest of the assertion see the proof of Theorem 3.2 in [16: p. 817]. So, let J be a closed right ideal in E. Then $J^{\perp_r} = \mathcal{A}_r(\mathcal{A}_l(J)^{\perp_l})$ (see Definitions 2.1 and 2.2). Consider the closed left ideal $\mathcal{A}_l(J)$. Since E is preannihilator and $E = \mathcal{A}_l(J) \oplus \mathcal{A}_l(J)^{\perp_l}$, we get

$$(0) = \mathcal{A}_r(\mathcal{A}_l(J)) \cap \mathcal{A}_r(\mathcal{A}_l(J)^{\perp_l}) = \mathcal{A}_r(\mathcal{A}_l(J)) \cap J^{\perp_r}.$$

Likewise, $\mathcal{A}_l(\mathcal{A}_r(I)) \cap I^{\perp_l} = (0)$ for every $I \in \mathcal{L}_l$. The duality of E follows now from Proposition 2.9. ∎

By Theorem 2.12 and [8: p. 3726, Proposition 2.11], we get the next.

Corollary 2.13. *Let $(E, \perp_l, \perp_r)$ be a preannihilator complemented algebra, such that every $I \in \mathcal{L}_l \cup \mathcal{L}_r$ commutes with its complements in E. Then $(\perp_l, \perp_r)$ is a dual pair of complementors and E is a dual algebra.*

The following result specializes to [16: p. 817, Theorem 3.2].

Corollary 2.14. *Let E be a semisimple complemented Q_l'-algebra in which every minimal closed 2-sided ideal is a (pre)dual algebra. Then the following are equivalent:*

1) $(\perp_l, \perp_r)$ is a dual pair of complementors.

2) $[\mathcal{A}_r(I)]^{\perp_r} = \mathcal{A}_r(I^{\perp_l})$ for all $I \in \mathcal{L}_l$.

3) $[\mathcal{A}_l(J)]^{\perp_l} = \mathcal{A}_l(J^{\perp_r})$ for all $J \in \mathcal{L}_r$.

Proof. By Theorem 2.8, E is a dual algebra.

1) $\Rightarrow$ 3): For any $J \in \mathcal{L}_r$, $J^{\perp_r} = \mathcal{A}_r(\mathcal{A}_l(J)^{\perp_l})$. Hence $\mathcal{A}_l(J^{\perp_r}) = \mathcal{A}_l(\mathcal{A}_r(\mathcal{A}_l(J)^{\perp_l}))$. So, since E is a dual algebra, we finally get $\mathcal{A}_l(J^{\perp_r}) = [\mathcal{A}_l(J)]^{\perp_l}$. See also the proof of Theorem 3.2 in [16: p. 817].

3) $\Rightarrow$ 1): By the assumption and the duality of E, we get that

$$J^{\perp_r} = \mathcal{A}_r(\mathcal{A}_l(J^{\perp_r})) = \mathcal{A}_r(\mathcal{A}_l(J)^{\perp_l}).$$

Thus, $\perp_r$ is derived from $\perp_l$. It follows from Proposition 2.11 that $\perp_l$ is derived from $\perp_r$, as well.

2) $\Leftrightarrow$ 3): This is an immediate consequence of the duality of E and Theorem 2.12. ∎

THEOREM 2.15. *Let $(E, \perp_l)$ be a right preannihilator left complemented algebra. Then the following are equivalent:*

1) $\perp_l$ is a dual complementor and E is a predual algebra.
2) E is a dual algebra such that

$$E = \mathcal{A}_r(I) + \mathcal{A}_r(I^{\perp_l}) \quad \textit{for every} \quad I \in \mathcal{L}_l. \tag{2.6}$$

Proof. 1) $\Rightarrow$ 2): By the comments after Definition 2.1, E is a right complemented algebra with a right complementor $\perp_r$ given by

$$J^{\perp_r} = \mathcal{A}_r[\mathcal{A}_l(J)^{\perp_l}], \; J \in \mathcal{L}_r. \tag{2.7}$$

Then, by using the reasoning in the proof of Theorem 2.12, we get $\mathcal{A}_r(\mathcal{A}_l(J)) = J$ for all $J \in \mathcal{L}_r$. Namely, E is a right dual algebra. Moreover, for any right preannihilator, left complemented algebra, we get $x \in \overline{Ex}$ for every $x \in E$ (see also the comment after Theorem 2.4). Thus, by the proof of 2) $\Rightarrow$ 1) in Proposition 2.9 and the fact that E is a predual algebra, we get that E is a left dual algebra, as well. Consider a closed left ideal I in E and the closed right ideal $K = \mathcal{A}_r(I)$. Then (see also (2.7)),

$$K^{\perp_r} = \mathcal{A}_r([\mathcal{A}_l(\mathcal{A}_r(I))]^{\perp_l}) = \mathcal{A}_r(I^{\perp_l})$$

and hence (2.6) follows. Actually, $E = \mathcal{A}_r(I) \oplus \mathcal{A}_r(I)^{\perp_l}$, since for $x \in \mathcal{A}_r(I) \cap \mathcal{A}_r(I^{\perp_l})$, $Ex \subseteq Ix + I^{\perp_l}x = (0)$, and thus $x = 0$.

2) $\Rightarrow$ 1): Apply a proof analogous to that of Theorem 3.5 in [16: p. 816]. We note that here the duality of E is not needed in the proof that $\perp_r$ reverses the inclusions. ∎

Concerning the assumption that E of Theorem 2.15 is right preannihilator, we give an *example of a right preannihilator algebra which is not a left preannihilator algebra.*

Suppose $x = x^2$ and $0 \neq y = xy$, $yx = 0$ and $y^2 = 0$, while if $\lambda x + \mu y = 0$, $\lambda, \mu \in \mathbb{C}$, then $\lambda = \mu = 0$. Let $E = \{\lambda x + \mu y : \lambda, \mu \in \mathbb{C}\}$. Then E is an algebra with $\mathcal{A}_r(E) = (0)$ and $\mathcal{A}_l(E) \neq (0)$.

PROPOSITION 2.16. *Let E be a topological algebra without divisors of zero. Then the following are equivalent:*

1) E is a dual algebra.
2) E is a weakly complemented algebra with weak complementors $\perp_l = \perp_r = \mathcal{A}_l (= \mathcal{A}_r)$.

Proof. We first note that

$$\mathcal{A}_l(S) = \mathcal{A}_r(S) \quad \text{for any subset } S \text{ of } E. \tag{2.8}$$

Suppose there exists $0 \neq x \in \mathcal{A}_l(S)$ with $Sx \neq \{0\}$. Then $xs = 0$ and $sx \neq 0$ for some $0 \neq s \in S$. So, since $xsx = 0$, $x = 0$, a contradiction. Thus $\mathcal{A}_l(S) \subseteq \mathcal{A}_r(S)$. Likewise, $\mathcal{A}_r(S) \subseteq \mathcal{A}_l(S)$. Moreover, a straightforward computation (see also (2.8)) yields

$$S \cap \mathcal{A}_l(S) = \{0\} = S \cap \mathcal{A}_r(S) \quad \text{for any subset } S \text{ of } E. \tag{2.9}$$

1) $\Rightarrow$ 2) : E, as a dual algebra, is an annihilator one. Thus, by Theorem 2.12 in [8: p. 3726] (see also (2.8)), $E = \overline{I \oplus \mathcal{A}_l(I)}$, $I \in \mathcal{L}_l$. So, since $\mathcal{A}_l$ is reflexive and reverses the inclusions, $(E, \perp_l = \mathcal{A}_l)$ is a weakly left complemented algebra. A similar argument shows the assertion on the "right".

2) $\Rightarrow$ 1) : It is obvious. ∎

It follows from Proposition 2.16 (see also the proof of Corollary 2.13) that *every topological algebra E without divisors of zero admits a dual pair of weak complementors if and only if E is a dual algebra if and only if E satisfies* 2) *of Proposition* 2.16.

3. Dual complementors on topological algebras with an involution. In [8: p. 3724, Theorem 2.5] it was mentioned that a Hausdorff, locally C^*-algebra E is an annihilator algebra if and only if it is a dual algebra, if and only if E is a complemented algebra with complementor

$$\mathcal{A}_l = \mathcal{A}_r \mid_{\mathcal{L}_l \cup \mathcal{L}_r} . \tag{3.1}$$

The proof of (3.1) was based on Corollary 2.3 [ibid. p. 3723]; we do not know if it is true for any locally C^*-algebra. Since Theorem 2.5 in [8: p. 3724] is used in the sequel, we amend (3.1), giving also the relative adapted proof of the theorem concerned. So, we have the next

THEOREM 3.1. *Let $(E, (p_\alpha)_{\alpha \in A})$ be a Hausdorff locally C^*-algebra. Then the following are equivalent:*

1) *E is an annihilator algebra.*
2) *E is a dual algebra.*
3) *E is a complemented algebra with left (resp. right) complementor $\perp_l = * \circ \mathcal{A}_r$ (resp. $\perp_r = * \circ \mathcal{A}_l$).*

Proof. 1) $\Rightarrow$ 2) : We first prove that

$$E = I \oplus \mathcal{A}_r(I)^*, \ I \in \mathcal{L}_l. \tag{3.2}$$

Indeed, if $x \in I \cap \mathcal{A}_l(I^*)$, $xx^* = 0$, $p_\alpha(x) = 0$ for all $\alpha \in A$ and hence $x = 0$. Thus $I \cap \mathcal{A}_r(I)^* = (0)$. Moreover, the ideal $J = I \oplus \mathcal{A}_r(I)^*$ is closed and $\mathcal{A}_r(J) = (0)$: If $x \in \bar{J}$, $x = \lim_\delta x_\delta$ with $(x_\delta)_{\delta \in \Delta}$ a net in J and $x_\delta = y_\delta + z_\delta$, $y_\delta \in I$, $z_\delta \in \mathcal{A}_l(I^*)$. Therefore, $x_\delta y_\delta^* = y_\delta y_\delta^*$ and thus $p_\alpha(y_\delta) \leq p_\alpha(x_\delta)$, $\alpha \in A$. Hence $(y_\delta)_{\delta \in \Delta}$ is a Cauchy net in I and hence $y_\delta \underset{\delta}{\rightarrow} y \in I$. Likewise, $z_\delta \underset{\delta}{\rightarrow} z \in \mathcal{A}_l(I^*)$. Thus, $x = y + z \in J$ and hence $J \in \mathcal{L}_l$. On the other hand, if $x \in \mathcal{A}_r(J)$, $Jx = \{0\}$, from which, $Ix = \{0\}$ and $\mathcal{A}_r(I)^* x = \{0\}$.

Thus, $x^*x = 0$ and $x = 0$. Therefore, $\mathcal{A}_r(J) = (0)$ and hence $J = E$. Similarly,

$$E = K \oplus \mathcal{A}_l(K)^*, \ K \in \mathcal{L}_r. \tag{3.3}$$

Thus, for any $I \in \mathcal{L}_l$, $E = \mathcal{A}_r(I) \oplus (\mathcal{A}_l(\mathcal{A}_r(I)))^*$ and $E = \mathcal{A}_r(I)^* \oplus \mathcal{A}_l(\mathcal{A}_r(I))$. Since $\mathcal{A}_r(I)^* \cap I = (0)$ (see (3.2)) and $I \subseteq \mathcal{A}_l(\mathcal{A}_r(I))$, we get that $\mathcal{A}_l(\mathcal{A}_r(I)) = I$ for all $I \in \mathcal{L}_l$. Similarly, $\mathcal{A}_r(\mathcal{A}_l(K)) = K$ for all $K \in \mathcal{L}_r$.

2) $\Rightarrow$ 3) : The mapping

$$\perp_l := * \circ \mathcal{A}_r : \mathcal{L}_l \to \mathcal{L}_l \ \text{ with } \ I \mapsto I^{\perp_l} := \mathcal{A}_r(I)^* \tag{3.4}$$

defines on E a left complementor. By the very definitions and the duality of E, we get

$$I^{\perp_l \perp_l} = \mathcal{A}_l(\mathcal{A}_l(I^*)^*) = \mathcal{A}_l(\mathcal{A}_r(I)) = I,$$

for all $I \in \mathcal{L}_l$. Thus, by (3.2) and the fact that $\perp_l$ reverses the inclusion, we finally get that $(E, \perp_l)$ is a left complemented algebra. Likewise, $(E, \perp_r)$ with

$$K^{\perp_r} := \mathcal{A}_l(K)^*, \ K \in \mathcal{L}_r \tag{3.5}$$

is a right complemented algebra.

3) $\Rightarrow$ 2) : For any $I \in \mathcal{L}_l$, $I = [\mathcal{A}_l(I^*)]^{\perp_l} = \mathcal{A}_l(\mathcal{A}_r(I^{**})) = \mathcal{A}_l(\mathcal{A}_r(I))$. Similarly, $K = \mathcal{A}_r(\mathcal{A}_l(K))$ for all $K \in \mathcal{L}_r$.

2) $\Rightarrow$ 1) : It is obvious. ∎

Theorem 3.2. *Every Hausdorff annihilator locally C^*-algebra admits a dual pair of complementors.*

Proof. By Theorem 3.1, an algebra, as in the statement, is dual and complemented with left and right complementors given by (3.4) and (3.5). Thus, for any $I \in \mathcal{L}_l$,

$$I^{\perp_l} = \mathcal{A}_l(\mathcal{A}_r(I^{\perp_l})) = \mathcal{A}_l([\mathcal{A}_l(\mathcal{A}_r(I))]^*) = \mathcal{A}_l([\mathcal{A}_r(I)]^{\perp_r}).$$

Hence $\perp_l$ is derived from $\perp_r$. It follows from Proposition 2.11 that the pair $(\perp_l, \perp_r)$ is dual. ∎

The existence of a continuous involution in a left or right complemented algebra forces it to be a complemented algebra, as the following result shows. A similar proof holds by interchanging "left" by "right". See also [16: p. 818].

Proposition 3.3. *Every left complemented algebra with a continuous involution is a complemented algebra.*

Proof. Let $(E, \perp_l)$ be a topological algebra as in the statement and $*$ its involution. For $J \in \mathcal{L}_r$, the left ideal $J^* \equiv I$ is closed and hence the mapping $\perp_r$ given by

$$J^{\perp_r} := (I^{\perp_l})^*, \ J \in \mathcal{L}_r, \tag{3.6}$$

where $J = I^*$, $I \in \mathcal{L}_l$, is meaningful. Since $E = I \oplus I^{\perp_l}$, $E = J \oplus J^{\perp_r}$. Moreover, $J^{\perp_r \perp_r} = [(I^{\perp_l})^*]^{\perp_r}$. So, if we let $K = (I^{\perp_l})^*$, then $K \in \mathcal{L}_r$ and hence $[(I^{\perp_l})^*]^{\perp_r} = [(I^{\perp_l})^{\perp_l}]^* = J$. Namely, $\perp_r$ is reflexive. On the other hand, if $J_1 \subseteq J_2$ in $\mathcal{L}_r$, then $J_1^* \subseteq J_2^*$. Since $J_1 = I_1^*$, $J_2 = I_2^*$ for some $I_1, I_2 \in \mathcal{L}_l$, we get in turn $I_1 \subseteq I_2$ and $J_2^{\perp_r} \subseteq J_1^{\perp_r}$, which completes the proof. ∎

Lemma 3.4. *Let $(E, \perp_l, \perp_r)$ be a complemented algebra with an involution $*$ such that $K^{\perp_l} := \mathcal{A}_r(K)^*$, $K \in \mathcal{L}_l$ and $J^{\perp_r} := \mathcal{A}_l(J)^*$, $J \in \mathcal{L}_r$. Then E is a dual algebra.*

Proof. For K in $\mathcal{L}_l$, $K^{\perp_l} := \mathcal{A}_r(K)^* = \mathcal{A}_l(K^*)$. Namely, $K^{\perp_l}$ is a left annihilator ideal. Thus (cf. also [14: p. 96, Section 8]), $K^{\perp_l} = \mathcal{A}_l(\mathcal{A}_r(K^{\perp_l}))$. Moreover, $K = \mathcal{A}_l(K^*)^{\perp_l} := [\mathcal{A}_r(\mathcal{A}_l(K^*))]^* = \mathcal{A}_l(\mathcal{A}_l(K^*)^*) = \mathcal{A}_l(\mathcal{A}_r(K))$. Likewise, $J = \mathcal{A}_r(\mathcal{A}_l(J))$ for all $J \in \mathcal{L}_r$. ■

From Corollary 2.10 in [11: p. 209] every closed right ideal in a locally C^*-algebra E has a (self-adjoint) left approximate identity. We do not know if there are closed right ideals in E having a right approximate identity. In the positive direction, we have (ii) in the next proposition.

Proposition 3.5. *Let E be a Hausdorff annihilator locally C^*-algebra. Then the following hold true.*

(i) Every closed 2-sided ideal in E is a dual complemented algebra with a dual pair of complementors.

(ii) Every closed right ideal in E which contains a right (self-adjoint) approximate identity, with respect to itself, is a dual complemented algebra with a dual pair of complementors.

Proof. (i) Let J be an ideal as in the statement. By Theorem 2.7 in [11: p. 209], J is self-adjoint and hence by Theorem 3.2 (see also its proof), $J^{\perp_r} = \mathcal{A}_l(J)^* = \mathcal{A}_r(J) = \mathcal{A}_l(J) = \mathcal{A}_r(J)^* = J^{\perp_l}$. Thus, $E = J \oplus \mathcal{A}_l(J) = J \oplus \mathcal{A}_r(J)$ and hence $\mathcal{L}_l(J) \subseteq \mathcal{L}_l$, $\mathcal{L}_r(J) \subseteq \mathcal{L}_r$. Therefore, since E is a dual algebra, J is a dual algebra, as well (apply a proof analogous to that of Lemma 4.11 in [6: p. 160]. We note that the duality of J is also justified by Lemma 3.4). Moreover, J is a complemented algebra, under the induced complementors from E, given by

$$N^{c_l} = \mathcal{A}_r(N)^* \cap J,\ N \in \mathcal{L}_l(J) \quad \text{and} \quad M^{c_r} = \mathcal{A}_l(M)^* \cap J,\ M \in \mathcal{L}_r(J).$$

See also Theorem 3.1 in [8: p. 3727]. By the very definitions and the duality of J, we get for any $N \in \mathcal{L}_l(J)$,

$$N^{c_l} = \mathcal{A}_r(N)^* \cap J = \mathcal{A}_l^J(\mathcal{A}_r^J(\mathcal{A}_l^J(N^*))) = \mathcal{A}_l^J(\mathcal{A}_l(\mathcal{A}_r^J(N)^*) \cap J) = \mathcal{A}_l^J(\mathcal{A}_r^J(N)^{c_r}).$$

Thus, the complementor c_l is derived from c_r, which by Proposition 2.11, is derived from c_l.

(ii) An ideal, as in the statement, is by [8: p. 3723, Lemma 2.2] self-adjoint and hence 2-sided. The assertion now follows from (i). ■

There are some classes of topological algebras E (with involution) for which a minimal right ideal has the form xE with x a non-zero (self-adjoint) idempotent element in E (see [5: p. 1183, Lemma 4.3], [6: p. 152, Theorem 3.4 and p. 153, Theorem 3.6] and [7: p. 144, Theorem 3.1]). In Proposition 3.7 below, we give a framework in which a dual left complementor exists. For this we need the next lemma.

Lemma 3.6. *Let E be a topological algebra and J a (closed) right ideal in E of the form $J = xE$ with $x \in \mathcal{I}d(E)$. Consider the assertions:*

1) J is a right complemented algebra.

2) For $K \in \mathcal{L}_r(J) - \{(0)\}$, there exists $K' \in \mathcal{L}_r(J) - \{(0)\}$ such that $J = K \oplus K'$ and $K = yE$, $K' = zE$ with $y \in K \cap \mathcal{I}d(E)$, $z \in K' \cap \mathcal{I}d(E)$, $yz = zy = 0$.

3) $\mathcal{L}_r(J) \subseteq \mathcal{L}_r$.

Then 1) $\Rightarrow$ 2) $\Rightarrow$ 3)*. Also,* 3) $\Rightarrow$ 1)*, if E is moreover a right complemented algebra.*

Proof. 1) $\Rightarrow$ 2) : See [8: p. 3728, Lemma 3.7].

2) $\Rightarrow$ 3) : Let K be a closed right ideal in J. Then $K = yE$ with $y \in K \cap \mathcal{I}d(E)$. Thus, $KE = yEE \subseteq K$.

3) $\Rightarrow$ 1) : [ibid. p. 3728, Theorem 3.1]. ∎

PROPOSITION 3.7. *Let $(E, (p_\alpha)_{\alpha \in A})$ be a Hausdorff annihilator locally C^*-algebra. Suppose there is a (closed) minimal right ideal J in E, of the form $J = xE$ with $x \in \mathcal{I}d(E)$ and self-adjoint, so that all $I \in \mathcal{L}_r(J)$ commute with x. Then J is a right dual and complemented algebra with a dual left complementor.*

Proof. By Theorem 3.2, E admits a dual pair of complementors $(\perp_l, \perp_r)$ given by (3.4) and (3.5). Since $E = J \oplus J^{\perp_r}$, we get for any $I \in \mathcal{L}_r(J)$, $IE \subseteq IJ + I\mathcal{A}_l(J)^* \subseteq I + I(1-x)E$, where $(1-x)E = \{t - xt : t \in E\}$. If $z \in I$, $z = xz$, so that $xz(\omega - x\omega) = 0$ for all $\omega \in E$. Thus $I(1-x)E = (0)$ and hence $IE \subseteq I$. Therefore, $\mathcal{L}_r(J) \subseteq \mathcal{L}_r$ and (J, c_r) is a right complemented algebra with c_r given by $N^{c_r} := \mathcal{A}_l(N)^* \cap J$, $N \in \mathcal{L}_r(J)$. Moreover, J is a left complemented algebra with a left complementor c_l induced by $\perp_l$ as follows:

$$K^{c_l} := \mathcal{A}_r(\overline{< K >_l})^* \cap J,\ K \in \mathcal{L}_l(J).$$

Here $\overline{< K >_l}$ denotes the closed left ideal of E, generated by K (see also [8: p. 3727, Theorem 3.1; see also its proof]). It is easily seen that $\mathcal{A}_r(K) = \mathcal{A}_r(\overline{< K >_l})$, so that $K^{c_l} = \mathcal{A}_r(K)^* \cap J$, $K \in \mathcal{L}_l(J)$. Since $E = \overline{< K >_l} \oplus \overline{< K >_l}^{\perp_l}$, $J = \overline{< K >_l} \cap J + \mathcal{A}_r(K)^* \cap J$. We claim that $\overline{< K >_l} \cap J = K$. We only have to prove that $\overline{< K >_l} \cap J \subseteq K$. Thus, if $z \in \overline{< K >_l} \cap J$, $xz = z = \lim_\delta z_\delta$ with $(z_\delta)_{\delta \in \Delta}$ a net in $< K >_l$ (the left ideal of E, generated by K). Therefore, $z_\delta = \sum_{i=1}^n \lambda_i^\delta u_i^\delta + \sum_{j=1}^m \omega_j^\delta v_j^\delta$, $u_i^\delta, v_j^\delta \in K$, $\omega_j^\delta \in E$ and $\lambda_i^\delta \in \mathbb{C}$, $1 \leq i \leq n$, $1 \leq j \leq m$, $\delta \in \Delta$. Hence

$$xz_\delta = \sum_{i=1}^n \lambda_i^\delta x u_i^\delta + \sum_{j=1}^m x\omega_j^\delta x v_j^\delta. \tag{3.7}$$

Since the ideal xE is minimal, it follows, by the Gel'fand-Mazur theorem, that $xEx = \mathbb{C}$ within an isomorphism of topological algebras (see [6: p. 155, Theorem 3.11; see also its proof] as well as [12: p. 52, Lemma 3.1]). Therefore, $x\omega_j^\delta x = \mu_j^\delta x$, $1 \leq j \leq m$, $\delta \in \Delta$. So (see also (3.7)), $xz_\delta = \sum_{i=1}^n \lambda_i^\delta x u_i^\delta + \sum_{j=1}^m \mu_j^\delta x v_j^\delta \in K$, which proves the assertion. The above argument shows that $J = K + K^{c_l}$. Now, if $t \in K$ and $tK^* = (0)$, then $tt^* = 0$ and hence $t = 0$. Therefore, $J = K \oplus K^{c_l}$. Denote by $\mathcal{A}_l^J$, $\mathcal{A}_r^J$ the left and right annihilators on J. Then, for $K_1 \subseteq K_2$ in $\mathcal{L}_l(J)$, we get

$$K_2^{c_l} := \mathcal{A}_l^J(K_2^*) \subseteq \mathcal{A}_l^J(K_1^*) = K_1^{c_l}.$$

Moreover, for any $I \in \mathcal{L}_l(J)$, $I^{c_l c_l} = \mathcal{A}_l^J(\mathcal{A}_l^J(I^*)^*) = \mathcal{A}_l^J(\mathcal{A}_r^J(I))$. Therefore, $I \subseteq I^{c_l c_l}$. Since $J = I \oplus I^{c_l} = I^{c_l} \oplus I^{c_l c_l}$ it follows $I = I^{c_l c_l}$. Thus, (J, c_l, c_r) is a complemented algebra. Moreover, J *is a right dual algebra*: Let R be a (non-zero) closed right ideal in J. Then, by Lemma 3.6, $R = yE$ for some $y \in \mathcal{I}d(E) \cap R$. Routine computations show that $\mathcal{A}_l(yJ) = J(1-y) \equiv \{j - jy : j \in J\}$. Thus,

$$\mathcal{A}_r^J(\mathcal{A}_l^J(R)) = \mathcal{A}_r^J(J(1-y)) = \mathcal{A}_r^J(\mathcal{A}_l(yJ)). \tag{3.8}$$

Moreover, $(yx)^2 = y^2x = yx$. If $yx = 0$, $yxy = 0$ or $y^2 = 0$, a contradiction. Therefore, yx is a non-zero idempotent element in E and hence yJ is a closed right ideal in E. Moreover, $\mathcal{A}_r(\mathcal{A}_l(yJ)) = yJ$, from which, in connection with (3.8), we get $\mathcal{A}_r^J(\mathcal{A}_l^J(R)) \subseteq yJ \subseteq RJ \subseteq R$ and thus $\mathcal{A}_r^J(\mathcal{A}_l^J(R)) = R$. It follows, for any $N \in \mathcal{L}_r(J)$,

$$N^{c_r} = \mathcal{A}_l(N)^* \cap J = \mathcal{A}_r^J(N^*) = \mathcal{A}_r^J(\mathcal{A}_l^J(\mathcal{A}_r^J(N^*))) = \mathcal{A}_r^J(\mathcal{A}_l^J(N)^{c_l}). \ \blacksquare$$

THEOREM 3.8. *Let E be a proper Hausdorff orthocomplemented locally convex H^*-algebra. Then*

(i) E has a dual pair of complementors.

(ii) Every closed 2-sided ideal in E is a dual algebra and admits a dual pair of complementors.

(iii) Every closed left ideal in E which is closed with respect to adjoints is a dual algebra and admits a dual pair of complementors.

Proof. (i) By [4: p. 457, Lemma 3.3 and p. 458, Lemma 3.7 and Theorem 3.9] E is a dual, complemented algebra with complementors the orthomaps $\perp_l$, $\perp_r$ (see also (1.10)), so that

$$I^{\perp_l} = \mathcal{A}_r(I)^*, \ I \in \mathcal{L}_l \ \text{ and } \ J^{\perp_r} = \mathcal{A}_l(J)^*, \ J \in \mathcal{L}_r. \tag{3.9}$$

Now, for any $I \in \mathcal{L}_l$, $\mathcal{A}_r(I)^{\perp_r} = \mathcal{A}_r(\mathcal{A}_r(I)^*) = \mathcal{A}_r(I^{\perp_l})$. Moreover, by Theorem 2.12 and [4: p. 452, Theorem 1.2], $(\perp_l, \perp_r)$ is a dual pair of complementors.

(ii) Let I be a closed 2-sided ideal in E. By [4: p. 457, Lemma 3.4], I is self-adjoint and hence it is a proper, Hausdorff, locally convex H^*-algebra [ibid. p. 453, Lemma 1.4]. Moreover, $E = I \oplus \mathcal{A}_l(I) = I \oplus \mathcal{A}_r(I)$. Thus, $\mathcal{L}_l(I) \subseteq \mathcal{L}_l$ and $\mathcal{L}_r(I) \subseteq \mathcal{L}_r$ (see also (3.9) and [8: p. 3725, Theorem 2.9]). So, the mappings $p : \mathcal{L}_l(I) \to \mathcal{L}_l(I)$ and $q : \mathcal{L}_r(I) \to \mathcal{L}_r(I)$ given by $J^p := J^{\perp_l} \cap I = \mathcal{A}_r(J)^* \cap I$, $J \in \mathcal{L}_l(I)$ and $K^q := K^{\perp_r} \cap I = \mathcal{A}_l(K)^* \cap I$, $K \in \mathcal{L}_r(I)$ (see (3.9) and [8: p. 3727, Theorem 3.1]) are meaningful and define a left (resp. right) complementor on I. Moreover, for any $J \in \mathcal{L}_l(I)$, $\mathcal{A}_r^I(J)^q = \mathcal{A}_r(\mathcal{A}_r(J)^* \cap I) \cap I = \mathcal{A}_r^I(J^p)$. The last part of the assertion follows now from Theorem 2.12 and [4: p. 458, Theorem 3.9].

(iii) The assertion follows from (ii), since any ideal, as in the statement, is actually 2-sided. $\blacksquare$

4. $\perp_l$ and $\perp_r$-projections. A structure theorem. Let $(E, \perp_l)$ be a left complemented algebra and x an idempotent element (projection) in E. If $(Ex)^{\perp_l} = E(1-x)$, x is called a *$\perp_l$-projection.* If x is, in particular, minimal, it is called a *minimal $\perp_l$-projection.* Analogous notions are given for a right complemented algebra $(E, \perp_r)$ (cf. also [17: p. 4]). Since x is idempotent, we have the Peirce decompositions $E = Ex \oplus E(1-x)$ and $E = xE \oplus (1-x)E$. Thus, *in a strictly left* (resp. *right*) *complemented algebra* (viz. $I^{\perp_l}$ is unique, such that $E = I \oplus I^{\perp_l}$, $I \in \mathcal{L}_l$; resp. "on the right"; see [3: p. 963]) *all projections are* $\perp_l$ (resp. $\perp_r$)*-projections.* Moreover, by [6: p. 153, Theorem 3.6 and p. 155, Corollary 3.10], *in a semisimple annihilator Q'-algebra a $\perp_l$- projection is minimal if and only if it is a primitive element* (viz. it cannot be expressed as the sum of two orthogonal idempotents).

Concerning the next result see also [17: p. 50, Lemma 3.1]. This is also true, by interchanging "left" and "right".

LEMMA 4.1. *Let $(E, \perp_l)$ be a semisimple annihilator left complemented Q'-algebra. Then every non-zero left ideal I contains a minimal $\perp_l$-projection. If I is closed and $(x_i)_{i\in\Lambda}$ is the family of minimal $\perp_l$-projections in I, then $I = \overline{\sum_i Ex_i}$.*

Proof. By [6: p. 153, Theorem 3.6 and p. 154, Corollary 3.7], I contains a minimal left ideal, say J, which is closed and thus minimal closed. Therefore, $J^{\perp_l}$ is a maximal closed left ideal, which is regular (see [6: p. 152, Theorem 3.4]). Thus, by [8: p. 3729, Theorem 3.9], J contains an idempotent element, say x, so that $J = Ex$ and $J^{\perp_l} = E(1-x)$. Thus x is a $\perp_l$-projection, which moreover, by [6: p. 154, Theorem 3.9], is primitive or equivalently minimal (cf. also the comments preceding Lemma 4.1) and this completes the first part of the assertion.

Now, if $I \in \mathcal{L}_l$, then $K \equiv \overline{\sum_{i\in\Lambda} Ex_i} \subseteq I$. If $\omega \in I$, then $\omega = y + z$ with $y \in K$, $z \in K^{\perp_l}$. If $z \neq 0$, then $z = \omega - y \in I \cap K^{\perp_l}$ and hence $I \cap K^{\perp_l} \neq (0)$. Thus, the left ideal $I \cap K^{\perp_l}$ contains a minimal $\perp_l$-projection, say z_0. But then $z_0 \notin K \subseteq I$, which is a contradiction. So, $z = 0$ and hence $I \subseteq K$, which completes the proof. ■

PROPOSITION 4.2. *Let $(E, \perp_l, \perp_r)$ be a preannihilator complemented algebra, such that the pair $(\perp_l, \perp_r)$ is dual. Then an element $x \in \mathcal{I}d(E)$ is a $\perp_l$-projection if and only if it is a $\perp_r$-projection.*

Proof. If x is a $\perp_l$-projection, then $(Ex)^{\perp_l} = E(1-x)$. Since $(\perp_l, \perp_r)$ is dual, E is a dual algebra and $[\mathcal{A}_r(I)]^{\perp_r} = \mathcal{A}_r(I^{\perp_l})$ for all $I \in \mathcal{L}_l$ (see Theorem 2.12). Thus $(xE)^{\perp_r} = [\mathcal{A}_r(\mathcal{A}_l(xE))]^{\perp_r} = \mathcal{A}_r(E(1-x)^{\perp_l}) = (1-x)E$. Therefore, x is a $\perp_r$-projection. A similar argument establishes the reverse implication. ■

Topological algebras, as in Lemma 4.1, do have $\perp_l$ and $\perp_r$-projections (see also the comment preceding Lemma 4.1). So, we get the next result, that generalizes Proposition 3.3 in [16: p. 817] stated for semisimple, annihilator, bicomplemented Banach algebras (which are dual ones; [ibid. p. 816, Remark]).

COROLLARY 4.3. *Proposition 4.2 holds true for any semisimple annihilator complemented Q'_l-algebra.*

Based on Lemma 4.1, we obtain a structure theorem, employing fewer minimal 1-sided ideals than those given through the socle. Namely, we have the next

THEOREM 4.4. *Let $(E, \perp_l)$ be a semisimple annihilator left complemented Q'-algebra. Then*

(i) E contains a maximal family $(x_i)_{i\in\Lambda}$ of mutually orthogonal, minimal $\perp_l$-projections.

(ii) $E = \overline{\sum_{i\in\Lambda} Ex_i} = \overline{\sum_{i\in\Lambda} x_iE}$.

Proof. (i) By [8: p. 3730, Lemma 3.12], E contains a maximal (closed) regular left ideal, say M. By Lemma 4.1, there exist $x \in M$ and $y \in M^{\perp_l}$, minimal $\perp_l$-projections, so that $(Ex)^{\perp_l} = E(1-x)$ and $(Ey)^{\perp_l} = E(1-y)$. Therefore, $E(1-x)^{\perp_l} = Ex \subseteq M$ and $M^{\perp_l} \subseteq E(1-x)$. Hence $y = \omega - \omega x$ for some $\omega \in E$, so that $yx = 0$. Likewise, $xy = 0$. Namely, x, y are orthogonal. If we get the subsets of the set of all orthogonal mimimal $\perp_l$-projections of E, we obtain an "inductive" (partially) ordered set, with respect to inclusion. So, by Zorn's Lemma, we get the required maximal family.

(ii) Consider the closed left ideal $I \equiv \overline{\sum_{i\in\Lambda} Ex_i}$. Obviously, $I \neq (0)$. So, if I is proper, $I^{\perp_l} \neq (0)$. Thus (Lemma 4.1), $I^{\perp_l}$ contains a minimal $\perp_l$-projection, say z. Then $z \neq x_i$ for all $i \in \Lambda$. Otherwise, $z = x_j$ for some $j \in \Lambda$. Thus $z \in Ex_j \subseteq I$ and hence $z = 0$, that is a contradiction. Now, $zx_i = x_i z = 0$, $i \in \Lambda$ (cf. the proof of (i)), which contradicts the maximality of the family $(x_i)_{i\in\Lambda}$. This finishes the proof of the first equality of the assertion. Set $J = \bigcap_{i\in\Lambda} E(1-x_i)$, then $Ex_i = E(1-x_i)^{\perp_l} \subseteq J^{\perp_l}$ and hence $E = J^{\perp_l}$. Consider the closed right ideal $L = \overline{\sum_{i\in\Lambda} x_i E}$. If $z \in \mathcal{A}_l(L)$, then $zx_i = 0$ and hence $z = z - zx_i \in E(1-x_i)$, $i \in \Lambda$. Therefore, $\mathcal{A}_l(L) \subseteq J = E^{\perp_l} = (0)$ and hence $E = L$. ■

Corollary 4.5. *Let E be a Hausdorff annihilator locally C^*-algebra. Then E contains a maximal family $(x_i)_{i\in\Lambda}$ of mutually orthogonal, minimal, self-adjoint $\perp_l$ (equivalently $\perp_r$)-projections with respect to the dual pair of complementors $(\perp_l, \perp_r)$ defined by (3.4) and (3.5). Moreover,*

$$E = \overline{\sum_{i\in\Lambda} Ex_i} = \overline{\sum_{i\in\Lambda} x_i E}.$$

Proof. By Theorems 3.1 and 3.2, E is a dual, complemented algebra with the dual pair of complementors $(\perp_l, \perp_r)$ given by (3.4) and (3.5). Moreover, E is semisimple and by the argument in the proof of (i) in Theorem 4.4, $(Ex)^{\perp_l} = E(1-x)$ with x a minimal $\perp_l$-projection in E. Thus, see also (3.4), $\mathcal{A}_r(Ex)^* = E(1-x)$ from which we get in turn $\mathcal{A}_l(x^*E) = E(1-x^*) = E(1-x)$, $\mathcal{A}_r(E(1-x^*)) = \mathcal{A}_r(E(1-x))$, $x^*E = xE$. Thus, for any $z \in E$, $xz = x^*xz$, so that $(x - x^*x)z = 0$ for all $z \in E$ and hence $x = x^*x = x^*$. So, by an analogous proof as in Theorem 4.4, we get a maximal family, say $(x_i)_{i\in\Lambda}$, of mutually orthogonal, minimal, self-adjoint $\perp_l$ (equivalently $\perp_r$)-projections. The assertion now follows from Proposition 4.2 and the proof of (ii) in Theorem 4.4. ■

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