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CONVERGENT SUBSEQUENCES OF PARTIAL SUMS OF FOURIER SERIES OF $\varphi(L)$

S. V. KONYAGIN

Department of Mechanics and Mathematics, Moscow State University, 119992 Moscow, Russia
E-mail: konyagin@ok.ru

Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $L(\mathbb{T})$ be the set of all integrable functions $f : \mathbb{T} \to \mathbb{C}$. We associate with a function $f \in L(\mathbb{T})$ its trigonometric Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)e^{ikx}, \quad \hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{T}} f(x)e^{-ikx}dx.$$

For $n \in \mathbb{N}$ define the n-th partial sum of f as

$$S_n(f;x) = \sum_{k=-n}^{n} \hat{f}_k e^{ikx}.$$

Let $\varphi : [0,\infty) \to [0,\infty)$ be a nonconstant convex function. Denote

$$\varphi(L) = \left\{ f \in L(\mathbb{T}) : \int_{\mathbb{T}} \varphi(|f(x)|)dx < \infty \right\}.$$

By $C_1, C_2, \dots$ we denote absolute positive constants.

The paper is motivated by Ul'yanov's question: does there exist a sequence $\{N_j\}$ such that for every function $f \in L(\mathbb{T})$ there is an increasing sequence $\{n_j\}$ such that $n_j \le N_j$ for all j and $S_{n_j}(f) \to f$ almost everywhere? Note that existence of a nonrestricted sequence $\{n_j\}$ with almost everywhere convergence $S_{n_j}(f) \to f$ follows from the classical theorem of Kolmogorov [K]. On the other hand, for any increasing sequence $\{n_j\}$ of positive integers there exists a real function $f \in L(\mathbb{T})$ such that $S_{n_j}(f)$ diverges almost everywhere [G] or even everywhere [T]. Ul'yanov's problem is still open. However, for an Orlicz function space not coinciding with $L(\mathbb{T})$ a sequence $\{N_j\}$ depending on the space does exist.

Theorem 1. *If $\varphi(u)/u \to \infty$ as $u \to \infty$ then there exists a sequence $\{N_j\}$ such that for every function $f \in \varphi(L)$ there is an increasing sequence $\{n_j\}$ such that $n_j \le N_j$ for all j and $S_{n_j}(f) \to f$ almost everywhere.*

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Without loss of generality we can assume that

$$\forall u \geq 0 \quad \phi(u) \geq u, \quad \int_{\mathbb{T}} \varphi(|f(x)|)dx \leq 1. \tag{1}$$

The main part of Theorem 1 is the following lemma.

LEMMA 1. *There exists a sequence* $\{N_j\}$ $(j \geq 0)$ *such that for every* $\varepsilon > 0$ *there is a sequence* $\{n_j\}$ *such that* $N_{j-1} < n_j \leq N_j$ *for all* j *and* $S_{n_j}(f) \to f$ *on a complement to a subset of* $\mathbb{T}$ *of measure less than* ε*.*

Theorem 1 easily follows from Lemma 1. Indeed, if $\{n_j\} = \{n_j\}(\varepsilon)$ is a sequence from the lemma, then there are $j(\varepsilon)$ and a set $E(\varepsilon) \in \mathbb{T}$ such that

$$|E(\varepsilon)| < 2\varepsilon \tag{2}$$

and

$$\forall j \geq j(\varepsilon), x \in \mathbb{T} \setminus E(\varepsilon) \quad |f(x) - S_{n_j}(f;x)| \leq \varepsilon. \tag{3}$$

We can assume that $j(2^{-\nu-1}) \geq j(2^{-\nu})$ for all $\nu \in \mathbb{N}$. Take n_j arbitrary for $j < \nu(1)$, $n_j = n_j(2^{-\nu})$ for $j(2^{-\nu}) \leq j < j(2^{-\nu-1})$,

$$E = \bigcap_J \bigcup_{j \geq J} E_j.$$

Then we have $|E| = 0$ and $S_{n_j}(f;x) \to f(x)$ for $x \in \mathbb{T} \setminus E$.

By Mf we denote Hardy–Littlewood's maximal function of f:

$$f(x) = \sup_{y<x<z} \frac{1}{z-y} \int_y^z |f(x)|dx.$$

Let $M > 0$ and

$$E_1 = \{x \in \mathbb{T} : \, Mf(x) > M\}.$$

Then E_1 is an open set. Note that, by (1),

$$\int_{\mathbb{T}} |f(x)|dx \leq 1. \tag{4}$$

Using the weak-type $(1,1)$ inequality for Mf (see, for example, [D, p. 31]), we get $|E_1| \leq 2/M$. We can write E_1 as a union of disjoint intervals

$$E_1 = \bigcup_\mu (y_\mu, z_\mu).$$

Denote

$$E_2 = \bigcup_\mu (2y_\mu - z_\mu, 2z_\mu - y_\mu).$$

Then $|E_2| \leq 3|E_1| \leq 6/M$. Hence, if $M \geq 7/\varepsilon$, then

$$|E_2| < \varepsilon.$$

Our aim is to construct appropriate sequences $\{N_j\}$, $\{n_j\}$ such that

$$S_{n_j}(f;x) \to f(x) \quad \text{almost everywhere on} \quad \mathbb{T} \setminus E_2. \tag{5}$$

We use the well-known Calderón–Zygmund decomposition of f[CZ]. Let $g(x) = f(x)$ for $x \in \mathbb{T} \setminus E_1$ and

$$g(x) = \frac{1}{z_\mu - y_\mu} \int_{y_\mu}^{z_\mu} f(x)dx$$

for $x \in (y_\mu, z_\mu) \subset E_1$. It is easy to see that $|g(x)| \leq M$ almost everywhere. Indeed, if $x \in (y_\mu, z_\mu)$ and we assume $|g(x)| > M$, then

$$\int_{y_\mu}^{z_\mu+\delta} |f(x)|dx > M(z_\mu + \delta - y_\mu)$$

for some $\delta > 0$, and hence $Mf(z_\mu) > M$, but this is impossible. Further, almost everywhere on $\mathbb{T} \setminus E_1$ we have

$$|g(x)| = |f(x)| \leq Mf(x) \leq M.$$

Therefore, since g is essentially bounded, by Carleson's theorem [C] $S_n(g) \to g$ almost everywhere, and (5) is equivalent to

(6) $$S_{n_j}(f - g; x) \to f(x) - g(x) = 0 \quad \text{almost everywhere on} \quad \mathbb{T} \setminus E_2.$$

By the way, we have proved that for any μ we have

(7) $$\int_{y_\mu}^{z_\mu} |g(x)|dx \leq \int_{y_\mu}^{z_\mu} |f(x)|dx \leq M|z_\mu - y_\mu|.$$

Applying Jensen's inequality to a convex function φ we get

(8) $$\int_{y_\mu}^{z_\mu} \varphi(|g(x)|)dx \leq \int_{y_\mu}^{z_\mu} \varphi(|f(x)|)dx.$$

First, we construct sequences of positive numbers $\{L_\nu\} \to \infty$ and $\{\delta_\nu\} \to 0$. We take $L_\nu \geq 1$ so that

(9) $$\phi(L_\nu)/L_\nu \geq \nu.$$

Let $\delta_1 = 1/2$. If δ_ν has been chosen, define

(10) $$\delta_{\nu+1} = \delta_\nu^{11}/L_\nu^2.$$

We define $\{N_j\}$ to be

(11) $$N_0 = 1, \quad N_j = [1/\delta_{j^4}]\,(j \geq 1).$$

We may assume that $\varepsilon < 1$. Since $|E_2| < \varepsilon$, the length of any interval (y_μ, z_μ) is less than $1/3$. Now for any $\nu \geq 1$ we define

$$f_\nu(x) = \begin{cases} f(x), & x \in (y_\mu, z_\mu), \delta_{\nu+1} < z_\mu - y_\mu \leq \delta_\nu, \\ 0, & \text{otherwise,} \end{cases}$$

$$g_\nu(x) = \begin{cases} g(x), & x \in (y_\mu, z_\mu), \delta_{\nu+1} < z_\mu - y_\mu \leq \delta_\nu, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly,

(12) $$f - g = \sum_\nu (f_\nu - g_\nu).$$

For any $j \ge 2$ we have, by (4),

$$\sum_{\nu=(j-1)^4}^{j^4-1} \int_{\mathbb{T}} |f_\nu(x)|dx \le \int_{\mathbb{T}} |f(x)|dx \le 1.$$

Therefore, there exists $\nu_0 \in [(j-1)^4, j^4)$ such that

$$\int_{\mathbb{T}} |f_{\nu_0}(x)|dx \le 1/(j^4 - (j-1)^4) < 1/j^3. \tag{13}$$

By (7), also

$$\int_{\mathbb{T}} |g_{\nu_0}(x)|dx < 1/j^3. \tag{14}$$

Denote

$$h_1 = \sum_{\nu<\nu_0} (f_\nu - g_\nu), \qquad h_2 = f_{\nu_0} - g_{\nu_0}, \qquad h_3 = \sum_{\nu>\nu_0} (f_\nu - g_\nu).$$

Identity (12) can be rewritten as

$$f - g = h_1 + h_2 + h_3. \tag{15}$$

By (13) and (14),

$$\int_{\mathbb{T}} |h_2(x)|dx < 2/j^3.$$

Therefore, using [K], we obtain that for any n there exists $F_{j,2} \subset \mathbb{T}$ such that

$$|F_{j,2}| \le 1/j^2, \quad |S_n(h_2; x)| \le C_1/j \ (x \in \mathbb{T} \setminus F_{j,2}). \tag{16}$$

Now, let us consider partial sums of the function h_1. We shall show that it is possible to choose $n_j \in (N_{j-1}, N_j]$ such that $|S_{n_j}(h_1)|$ will be small on a large subset of $\mathbb{T} \setminus E_2$. First, we deduce from (8) and convexity of φ that

$$\int_{\mathbb{T}} \varphi(|h_1(x)/2|)dx \le \frac{1}{2}\left(\int_{\mathbb{T}} \varphi(|f(x)|)dx + \int_{\mathbb{T}} \varphi(|g(x)|)dx\right) \le \int_{\mathbb{T}} \varphi(|f(x)|)dx \le 1. \tag{17}$$

Let

$$h_1 = h_{1,1} + h_{1,2}, \tag{18}$$

where $h_{1,1}(x) = h_1(x)$ for $|h_1(x)| \ge 2L_{\nu_0}$ and $h_{1,1}(x) = 0$ otherwise. We estimate partial Fourier sums of the function $h_{1,1}$ in the same way as for the function h_2. By (9) and (17),

$$\int_{\mathbb{T}} |h_{1,1}(x)|dx = 2\int_{\mathbb{T}} |h_{1,1}(x)/2|dx \le \frac{2}{\nu_0} \int_{\mathbb{T}} |\varphi(h_{1,1}(x)/2)|dx$$
$$\le \frac{2}{(j-1)^4} \int_{\mathbb{T}} \varphi(|h_1(x)/2|)dx \le \frac{32}{j^4}.$$

Therefore, using [K] again, we obtain that for any n there exists $F_{j,1,1} \subset \mathbb{T}$ such that

$$|F_{j,1,1}| \le 1/j^2, \quad |S_n(h_{1,1}; x)| \le C_2/j^2 \quad (x \in \mathbb{T} \setminus F_{j,1,1}). \tag{19}$$

Now, let us estimate partial Fourier sums of the function $h_{1,2}$. Using (7) and (1) we have

$$\int_{\mathbb{T}} |h_{1,2}(x)|dx \le \int_{\mathbb{T}} |f(x)|dx + \int_{\mathbb{T}} |g(x)|dx \le 2\int_{\mathbb{T}} |f(x)|dx \le 2.$$

Therefore,

(20) $$\int_{\mathbb{T}} |h_{1,2}(x)|^2 dx \le 4L_{\nu_0}.$$

Fix $x \in \mathbb{T} \setminus E_2$. We use the well-known formula

(21) $$S_n(h_{1,2}; x) = S_{n,1}(x) + S_{n,2}(x),$$

where

$$S_{n,1}(x) = \frac{1}{2\pi} \int_{\mathbb{T}} h_{1,2}(t) \cot\left(\frac{t-x}{2}\right) \sin(n(t-x))dt,$$

$$S_{n,2}(x) = \frac{1}{2\pi} \int_{\mathbb{T}} h_{1,2}(t) \cos(n(t-x))dt.$$

Now, observe that the supposition $h_{1,2}(t) \ne 0$ implies $t \in (y_\mu, z_\mu)$ for some μ with $z_\mu - y_\mu \ge \delta_{\nu_0}$. Also, since $x \in \mathbb{T} \setminus E_2$, we get $x \notin (2y_\mu - z_\mu, 2z_\mu - y_\mu)$. Thus,

$$\left|\cot\left(\frac{t-x}{2}\right)\right| \le \cot(\delta_{\nu_0}/2),$$

and, by (20), we have

$$\int_{\mathbb{T}} \left(h_{1,2}(t) \cot\left(\frac{t-x}{2}\right)\right)^2 \le C_3 L_{\nu_0}/(\delta_{\nu_0})^2.$$

Now, we can use Parseval's identity

(22) $$\sum_n |S_{n,1}(x)|^2 \le \frac{C_3}{4\pi} L_{\nu_0}/(\delta_{\nu_0})^2.$$

For $x \in \mathbb{T} \setminus E_2$ denote

$$\mathcal{N}_1(x) = \{n : |S_{n,1}(x)| > \delta_{\nu_0}\}.$$

By (22), we have

$$|\mathcal{N}_1(x)| \le \frac{C_3}{4\pi} L_{\nu_0}/(\delta_{\nu_0})^4.$$

By integration we get

(23) $$\int_{\mathbb{T}\setminus E_2} |\mathcal{N}_1(x)| dx \le C_3 L_{\nu_0}/(\delta_{\nu_0})^4.$$

Similarly, if we denote

$$\mathcal{N}_2(x) = \{n : |S_{n,2}(x)| > \delta_{\nu_0}\}.$$

then

(24) $$\int_{\mathbb{T}\setminus E_2} |\mathcal{N}_2(x)| dx \le C_3 L_{\nu_0}/(\delta_{\nu_0})^4.$$

Denote

$$N'_j = N_{j-1} + [L_{\nu_0}/(\delta_{\nu_0})^5] + 1.$$

It follows from (23) and (24) that there exists n, $N_{j-1} < n \le N'_j$ such that

(25) $$|F_{j,1,2}| \le 2C_3 \delta_{\nu_0} < 2C_3/N_{j-1},$$

where

$$F_{j,1,2} = F_{j,1,2}(n) = \{x \in \mathbb{T} \setminus E_2 : \, n \in \mathcal{N}_1(x) \cup N_2(x)\}.$$

By the definition of δ_{ν_0+1} and N_j we have $n < [1/\delta_{\nu_0+1}] \le N_j$, and we can take $n_j = n$. So, by (21), we have

$$\forall x \in \mathbb{T} \setminus E_2 \setminus F_{j,1,2} \quad |S_{n_j}(h_{1,2}; x)| \le 2\delta_{\nu_0} < 2/N_{j-1}. \tag{26}$$

Now we will prove that for $n \le N'$ the partial sums $S_n(h_3)$ are uniformly small. Using (7), for any $k \in \mathbb{Z}$ and any μ we have

$$\begin{aligned}\left|\int_{y_\mu}^{z_\mu} (f(x) - g(x))e^{-ikx}dx\right| &= \left|\int_{y_\mu}^{z_\mu} (f(x) - g(x))\left(e^{-ikx} - e^{-iky_\mu}\right)dx\right| \\ &\le |k||z_\mu - y_\mu| \int_{y_\mu}^{z_\mu} |f(x) - g(x)|dx \le 2|k||z_\mu - y_\mu|^2 M.\end{aligned}$$

Therefore,

$$\begin{aligned}|\hat{h}_3(k)| &= \left|\frac{1}{2\pi} \sum_{z_\mu - y_\mu \le \delta_{\nu_0+1}} \int_{y_\mu}^{z_\mu} (f(x) - g(x))dx\right| \le \frac{1}{2\pi} \sum_{z_\mu - y_\mu \le \delta_{\nu_0+1}} 2|k||z_\mu - y_\mu|^2 M \\ &\le \frac{1}{\pi}\delta_{\nu_0+1}|k|M \sum_{z_\mu - y_\mu \le \delta_{\nu_0+1}} |z_\mu - y_\mu| \le 2\delta_{\nu_0+1}|k|M,\end{aligned}$$

and thus for any positive integer n

$$|S_n(h_3; x)| \le \sum_{|k| \le n} |\hat{h}_3(k)| \le 4\delta_{\nu_0+1}n^2 M.$$

In particular,

$$|S_{n_j}(h_3; x)| \le 4\delta_{\nu_0+1}(N_j')^2 M. \tag{27}$$

It follows from the definition that $N_j' \le C_4 L_{\nu_0}/(\delta_{\nu_0})^5$. Consequently, by (10), $\delta_{\nu_0+1}(N_j')^2 \le (C_4)^2\delta_{\nu_0}$, and after combining the last inequality with (27) we obtain

$$|S_{n_j}(h_3; x)| \le C_5 M\delta_{\nu_0} \le C_5 M/N_{j-1}. \tag{28}$$

To finish the proof, we define

$$F_j = F_{j,1,1} \cup F_{j,1,2} \cup F_{j,2}.$$

By (16), (19), and (25),

$$|F_j| \le 2/j^2 + 2C_3/N_{j-1}.$$

Taking into account that, by construction, $\sum_j 1/N_j < \infty$, we get

$$\sum_j |F_j| < \infty. \tag{29}$$

Further, we combine (16), (19), (25), and (28) with (18) and (15). Thus, if $x \in \mathbb{T} \setminus E_2 \setminus F_j$, then

$$|S_{n_j}(f - g; x)| \le C_1/j + C_2/j^2 + (C_5 M + 2)/N_{j-1} \to 0 \quad (j \to \infty). \tag{30}$$

By (29) and (30), $S_{n_j}(f - g; x) \to 0$ almost everywhere on $\mathbb{T} \setminus E_2$. This proves (6) and completes the proof of Lemma 1.

For the whole class $L(\mathbb{T})$ we can construct a sequence $\{N_j\}$ with a weaker property than in Ul'yanov's problem.

THEOREM 2. *There exists a sequence* $\{N_j\}$ *such that for every function* $f \in L(\mathbb{T})$ *there is an increasing sequence* $\{n_j\}$ *such that* $n_j \le N_j$ *for infinitely many* j *and* $S_{n_j}(f) \to f$ *almost everywhere.*

Without loss of generality we can assume that

$$\int_{\mathbb{T}} |f(x)|dx \le 1. \tag{31}$$

The following lemma is the main part of Theorem 2.

LEMMA 2. *There exists a sequence* $\{N_j\}\,(j \ge 0)$ *such that for every* $\varepsilon > 0$ *there is* $S = S(\varepsilon)$ *and also for sufficiently large* j *there are numbers* $N_{j-1} < n_1 < \cdots < n_j \le N_j$ *such that*

$$\max(|S_{n_1}(f;x) - f(x)|, \ldots, |S_{n_j}(f;x) - f(x)|) \le S \tag{32}$$

on a complement to a subset of $\mathbb{T}$ *of measure less than* ε.

Theorem 2 follows easily from Lemma 2. Indeed, let

$$\varepsilon_\mu = 2^{-\mu}, \quad \delta_\mu = \varepsilon_\mu / S(\varepsilon_\mu) \quad (\mu \ge 1).$$

For every μ there exists a trigonometric polynomial P_μ of degree m_μ such that

$$\int_{\mathbb{T}} |f(x) - P_\mu(x)|dx \le \delta_\mu.$$

Denote $g_\mu = f - P_\mu$. By Lemma 2, for any μ there exist $j(\mu)$, $E_\mu \subset \mathbb{T}$, $n_1(\mu), \ldots, n_{j(\mu)}(\mu)$ such that $j(\mu) > j(\mu - 1)$ for $\mu > 1$, $N_{j(\mu)-1} > m_\mu$, $|E_\mu| < \varepsilon_\mu$, $N_{j(\mu)-1} < n_1(\mu) < \cdots < n_{j(\mu)}(\mu) \le N_{j(\mu)}$, and

$$\max_j |S_{n_j(\mu)}(g_\mu; x) - g_\mu(x)| \le \varepsilon_\mu \tag{33}$$

for $x \in \mathbb{T} \setminus E_\mu$. Since $f - g_\mu$ is a trigonometric polynomial of degree less than $n_j(\mu)$ for $j = 1, \ldots, j(\mu)$, (33) can be rewritten as

$$\max_j |S_{n_j(\mu)}(f; x) - f(x)| \le \varepsilon_\mu.$$

We define a sequence $\{n_j\}$ to be the union of the sets $\{n_1(\mu), \ldots, n_{j(\mu)}(\mu)\}$ over $j \ge 1$. Define

$$E = \bigcap_J \bigcup_{\mu \ge J} E_\mu.$$

Then $n_{j(\mu)} \le N_{j(\mu)}$ for all μ, $|E| = 0$, and $S_{n_j}(f; x) \to f(x)$ for all $x \in \mathbb{T} \setminus E$.

In the proof of Lemma 2 we may assume that $\varepsilon < 1$. We define E_1, E_2, g as in the proof of Lemma 1. We consider that $M > 18/\varepsilon$. Then

$$|E_2| < \varepsilon/3. \tag{34}$$

It is enough to prove the existence of appropriate $S, n_1, \ldots, n_j$ such that

$$\max(|S_{n_1}(f - g; x)|, \ldots, |S_{n_j}(f - g; x)|) \le S \tag{35}$$

everywhere on $\mathbb{T} \setminus E$, where

$$E_2 \subset E, \quad |E| < \varepsilon. \tag{36}$$

A sequence $\{\delta_j\}$ will be constructed in the following way. Let $\delta_1 = 1/2$. If δ_ν has been chosen, we consider the set $\mathcal{K}$ of continuous functions $h : \mathbb{T} \to \mathbb{C}$ such that for all $x, y \in \mathbb{T}$

$$|h(x)| \le 1/\delta_\nu, \quad |h(x) - h(y)| \le |x - y|/\delta_\nu^2.$$

$\mathcal{K}$ is a compact subset of $C(\mathbb{T})$. Hence, there is a finite 1-net $\{h_1, \ldots, h_{L_\nu}\}$ for $\mathcal{K}$ (that is, for any $h \in \mathcal{K}$ there is $l \le L_\nu$ such that $\|h - h_l\|_{C(\mathbb{T})} \le 1$). Observe that an 1-net with the same cardinality L_ν exists if the function class is defined on some compact subset of $\mathbb{T}$, since every function can be extended from the subset to $\mathbb{T}$ without change of the uniform norm and the Lipschitzian constant. Define

$$\delta_{\nu+1} = \left(1/\delta_\nu + \nu L_\nu^2\right)^{-2}.$$

We define $\{N_j\}$ to be

$$N_0 = 1, \quad N_j = 1/\delta_{j^4}\, (j \ge 1).$$

(Observe that $1/\delta_\nu$ is an integer for any ν.)

We define f_ν, g_ν, choose $\nu_0 \in [(j-1)^4, j^4)$ for any $j \ge 2$ and further define h_1, h_2, h_3 as in the proof of Lemma 1. We will seek for $n_1, \ldots, n_j$ from the segment $(1/\delta_{\nu_0}, N_j']$ where

$$N_j' = 1/\delta_{\nu_0} + \nu_0 L_{\nu_0}^2 = (\delta_{\nu_0+1})^{-1/2}.$$

Similarly to (16) we prove that for some set $F_{j,2}$ we have

$$|F_{j,2}| \le 1/j, \quad \max(|S_{n_1}(h_2; x)|, \ldots, |S_{n_j}(h_2; x)|) \le C_1/j\, (x \in \mathbb{T} \setminus F_{j,2}), \tag{37}$$

Similarly to (27), we have a uniform estimate

$$\max(|S_{n_1}(h_3; x)|, \ldots, |S_{n_j}(h_3; x)|) \le 4\delta_{\nu_0+1}(N_j')^2 M = 4M. \tag{38}$$

It suffices to estimate partial sums of the function h_1 on $\mathbb{T} \setminus E_2$.

Let us recall some well-known definitions and facts. For any function $h \in L(\mathbb{T})$ define the conjugate function

$$\tilde{h}(x) = \frac{1}{2\pi} \lim_{\delta \to 0} \int_{\delta \le |t| \le \pi} \frac{-h(x+t)}{\tan(t/2)} dt.$$

By the theorem of Lusin and Privalov (see, for example, [Z, 4.3 and 7.1]) this limit exists almost everywhere.

Lemma 2.1. *There exists an absolute constant C_6 such that for any function $h \in L(\mathbb{T})$ and any $\alpha > 0$*

$$\left|\left\{x : |\tilde{h}(x)| > \alpha \int_{\mathbb{T}} |h(t)| dt\right\}\right| < C_6/\alpha.$$

This is the result of [K].

For a positive integer n define a modified Dirichlet kernel of order n:

$$D_n^*(t) = \frac{\sin(nt)}{\tan(t/2)}.$$

Lemma 2.2. *Let*

$$S_n^*(h; x) = \frac{1}{2\pi} \int_{\mathbb{T}} D_n^*(t) h(x+t) dt.$$

Then

$$S_n^*(h;x) = S_n(h;x) - (\hat{h}(n)\exp(inx) + \hat{h}(-n)\exp(-inx))/2.$$

Therefore,

$$|S_n(h;x)| \le |S_n^*(h;x)| + \frac{1}{2\pi}\int_{\mathbb{T}} |h(t)|dt.$$

See the first statement in [Z, 2.3]. The second statement follows immediately from the first one.

Denote $h^{(n)}(x) = h(x)e^{inx}$.

Lemma 2.3. *We have*

$$S_n^*(h;x) = i\exp(-inx)\tilde{h}^{(n)}(x) - i\exp(inx)\tilde{h}^{(-n)}(x) \tag{39}$$

provided that the right-hand side of (39) is defined. Therefore,

$$|S_n^*(h;x)| \le |\tilde{h}^{(n)}(x)| + |\tilde{h}^{(-n)}(x)|.$$

This result is contained in [Z, 7.3].

We will use Lemmas 3–5 for functions $h^{(n)}$, $h = h_1$, and for $x \in \mathbb{T} \setminus E_2$. Note that for such x and $h_1(x+t) \ne 0$ we have $|x-t| \ge \delta = \delta_{\nu_0}$. It is easy to see that

$$|\cot(x/2)| \le 2/\delta, \quad |\cot(x/2) - \cot(y/2)| \le (2/\delta^2 + 1)|x-y| \quad (|x|,|y| \in [\delta,\pi]).$$

Using also that, by (7) and (31),

$$\int_{\mathbb{T}} |h_1(x)|dx \le \int_{\mathbb{T}} |f(x)|dx + \int_{\mathbb{T}} |g(x)|dx \le 2\int_{\mathbb{T}} |f(x)|dx \le 2, \tag{40}$$

we have for any integer n and $x, y \in \mathbb{T} \setminus E_2$

$$|\tilde{h}_1^{(n)}(x)| \le 1/\delta, \quad |\tilde{h}_1^{(n)}(x) - \tilde{h}_1^{(n)}(y)| \le |x-y|/\delta^2.$$

Therefore, by the pigeon-hole principle, for $j \ge 3$ there exist $n_1, \dots, n_j$, $1/\delta_{\nu_0} < n_1 < \cdots < n_j \le N_j'$, such that for all $x \in \mathbb{T} \setminus E_2$ and $\mu = 2, \dots, j$ we have

$$|\tilde{h}_1^{(n_\mu)}(x) - \tilde{h}_1^{(n_1)}(x)| \le 2, \quad |\tilde{h}_1^{(-n_\mu)}(x) - \tilde{h}_1^{(-n_1)}(x)| \le 2.$$

(We have used that $\nu_0 \ge j$ for $j \ge 3$.) Consequently, by Lemmas 2.2 and 2.3, and (40), we get

$$\max(|S_{n_1}(h_1;x)|, \dots, |S_{n_j}(h_1;x)|) \le |\tilde{h}_1^{(n_1)}(x)| + |\tilde{h}_1^{(-n_1)}(x)| + 5. \tag{41}$$

By Lemma 2.1, there is a set $F_{j,1} \subset \mathbb{T}$ such that

$$|F_{j,1}| < \varepsilon/3 \tag{42}$$

and for any $x \in \mathbb{T} \setminus F_{j,1}$ we have

$$|\tilde{h}_1^{(n_1)}(x)| + |\tilde{h}_1^{(-n_1)}(x)| \le C_7/\varepsilon$$

and thus, by (42), for $x \in \mathbb{T} \setminus E_2 \setminus F_{j,1}$

$$\max(|S_{n_1}(h_1;x)|, \dots, |S_{n_j}(h_1;x)|) \le C_7/\varepsilon + 5. \tag{43}$$

Denote

$$E = F_{j,1} \cup F_{j,1} \cup E_2.$$

By (34), (37), and (42), the conditions (36) are satisfied for sufficiently large j. Using (15), (37), (38), and (43), we have

$$\max(|S_{n_1}(f-g;x)|, \dots, |S_{n_j}(f-g;x)|) \le C_1/j + 4M + C_7/\varepsilon + 5$$

for all $x \in \mathbb{T} \setminus E$. This proves (35) and completes the proof of Lemma 2.

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