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MUSIELAK-ORLICZ SPACES AND PREDICTION PROBLEMS

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Dedicated to the memory of Professor Władysław Orlicz

Abstract. By a harmonizable sequence of random variables we mean the sequence of Fourier coefficients of a random measure M:

$$X_n(M) = \int_0^1 e^{2\pi nis} M(ds) \quad (n = 0, \pm 1, \ldots)$$

The paper deals with prediction problems for sequences $\{X_n(M)\}$ for isotropic and atomless random measures M. The crucial result asserts that the space of all complex-valued M-integrable functions on the unit interval is a Musielak-Orlicz space. Hence it follows that the problem for $\{X_n(M)\}$ $(n = 0, \pm 1, \ldots)$ to be deterministic is in fact an extremal problem of Szegö's type for Musielak-Orlicz spaces in question. This leads to a characterization of deterministic sequences $\{X_n(M)\}$ $(n = 0, \pm 1, \ldots)$ in terms of random measures M.

1. Random measures and harmonizable sequences. A function M defined on the σ-algebra of all Borel subsets of the unit interval I whose values are complex random variables is called a *random measure* if

(i) for every sequence $E_1, E_2, \ldots$ of disjoint Borel sets

$$M\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sum_{n=1}^{\infty} M(E_n),$$

where the series converges with probability 1,

(ii) for every sequence $E_1, E_2, \ldots$ of disjoint Borel sets the random variables $M(E_1)$, $M(E_2), \ldots$ are independent.

The theory of random measures was developed by A. Prékopa in [15, 16] and [17]. For further results see [8], [22], [3] and [4].

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A random measure M is said to be *atomless* if $M(\{a\}) = 0$ with probability 1 for every one-point set $\{a\}$. Moreover a random measure M is said to be *isotropic* if for every orthogonal transformation U of the complex plane and every Borel subset E of the unit interval I the random variables $M(E)$ and $UM(E)$ have the same probability distribution. In particular, isotropic random measures are *symmetric*, i.e. for every Borel set E the random variables $M(E)$ and $-M(E)$ are identically distributed. All random measures under consideration in the sequel will tacitly be assumed to be atomless and isotropic. In particular for every Borel set E the random variable $M(E)$ has an infinitely divisible distribution and its characteristic function can be written in the form

$$\varphi_{M(E)}(t) = \exp\left(\int_0^\infty (J_0(x|t|) - 1)\frac{1+x^2}{x^2}\mu_M(E, dx)\right), \tag{1.1}$$

where J_0 is the Bessel function

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin n) dn,$$

$\mu_M(E, \cdot)$ is a finite non-negative Borel measure on the positive half line R_+, $t \in R^2$ and $|t|^2 = (t, t)$. Moreover, for every Borel subset A of R_+ the set-function $\mu_M(\cdot, A)$ is a non-negative atomless Borel measure on I.

In the sequel we shall identify random variables which are equal with probability 1. Given a random measure M, we say that a Borel set E is an *M-null set* if $M(A) = 0$ for all Borel subsets A of E. Relations valid except of an M-null set are said to be valid M-almost everywhere.

By a *harmonizable sequence of random variables* we mean the sequence of Fourier coefficients of a random measure M, i.e. the sequence

$$X_n(M) = \int_0^1 e^{2\pi ins} M(ds) \quad (n = 0, \pm 1, \ldots).$$

It is clear that the Fourier coefficients $\{X_n(M)\}$ determine the random measure M uniquely.

A sequence $\{X_n(M)\}$ $(n = 0, \pm 1, \ldots)$ of random variables is called *strictly stationary*, or, briefly, *stationary*, if for every system $m, n_1, n_2, \ldots, n_k$ of integers the multivariate distribution of the random variables

$$X_{n_1+m}, X_{n_2+m}, \ldots, X_{n_k+m}$$

does not depend upon m. One can prove the following result ([20],Theorem 4.1): *A sequence $\{X_n(M)\}$ of Fourier coefficients is stationary if and only if the random measure M is isotropic. In this case the probability distribution of $\{X_n(M)\}$ is completely determined by the set-function $\mu_M(\cdot, \cdot)$.*

The concept of the integral with respect to a random measure was introduced in [16] (the unconditional integral) and in [22]. We shall quote the basic definition, which is an adaptation of Dunford's definition of the integral with respect to a measure whose values belong to a Banach space ([7], Chapter IV).

If f is a complex-valued Borel simple function on I, i.e.

$$f = \sum_{j=1}^{n} c_j 1_{E_j},$$

where c_j are complex numbers and 1_{E_j} denote the indicators of Borel sets E_j, then the integral on every Borel set E of f with respect to the random measure M is defined by the formula

$$\int_E f(s)M(ds) = \sum_{j=1}^{n} c_j M(E_j \cap E).$$

Further, a complex-valued Borel function g on I is said to be *M-integrable* if there exists a sequence $\{g_n\}$ of simple Borel functions such that

(a) the sequence $\{g_n\}$ converges to g M-almost everywhere on I,
(b) for every Borel set E the sequence $\{\int_E g_n(s)M(s)\}$ converges in probability.

Now, by definition, the integral $\int_E g(s)M(ds)$ is the limit in probability of the sequence $\{\int_E g_n(s)M(ds)\}$.

Let $L(M)$ be the set of all complex-valued M-integrable functions on I. We indentify functions which are equal M-almost everywhere. The space $L(M)$ is a complete linear metric space under usual addition and scalar multiplication with a non-homogeneous norm defined by the formula

$$\|f\|_M = \left\| \int_I f(s)\, M(ds) \right\|,$$

where $\|X\|$ denotes the Fréchet norm of the random variable X i.e. the expectation $E(|x|/(1+|x|))$ (see [22] and [21]). It should be noted that the convergence of a sequence of functions in $L(M)$ is equivalent to the convergence in probability of the sequence of their M-integrals. Moreover, the set of all Borel simple functions on I is dense in $L(M)$.

2. Sequences admitting a prediction. Given a stationary sequence of random variables $\{X_n\}$, by $[X_n]$ and $[X_n : n \le k]$ we shall denote the linear spaces closed with respect to the convergence in probability spanned by all random variables X_n and by random variables X_n with $n \le k$ respectively. To each stationary sequence $\{X_n\}$ there corresponds a shift transformation $TX_n = X_{n+1}$, $(n = 0, \pm 1, \ldots)$ which can be extended to an invertible linear transformation T on $[X_n]$. Of course, the transformation T preserves the probability distribution.

A concept of prediction for stationary sequences which need not have a finite variance was introduced in [19]. In this paper we restrict ourselves to symmetric sequences. In this case 0 is the only constant belonging to $[X_n]$.

We say that a stationary symmetric sequence $\{X_n\}$ *admits a prediction* if there exists a continuous linear operator A_0 from $[X_n]$ onto $[X_n : n \le 0]$ such that

(i) $A_0 X = X$ whenever $X \in [X_n : n \le 0]$,
(ii) if for every $Y \in [X_n : n \le 0]$ the random variables $X \in [X_n]$ and Y are independent, then $A_0 X = 0$,

(iii) for every $X \in [X_n]$ and $Y \in [X_n : n \leq 0]$ the random variables $X - A_0 X$ and Y are independent.

The random variable $A_0 X$ can be regarded as a linear prediction of X based on the full past of the sequence $\{X_n\}$ up to time 0. An optimality criterion is given by (iii). In what follows the operator A_0 will be called a *predictor* based on the past of the sequence $\{X_n\}$ up to time 0.

It should be noted that Gaussian stationary sequences with zero mean always admit a prediction. This follows from the fact that in this case the concepts of independence and orthogonality are equivalent and, moreover, the square-mean convergence and the convergence in probability are equivalent. Therefore the predictor A_0 is simply the best linear least squares predictor, i.e. the orthogonal projector from $[X_n]$ onto $[X_n : n \leq 0]$.

The predictor A_0 and the shift T induced by $\{X_n\}$ determine the predictor A_k based on the full past of $\{X_n\}$ up to time k by means of the formula $A_k = T^k A_0 T^{-k}$.

A stationary sequence $\{X_n\}$ admitting a prediction is called *deterministic* if $A_0 X = X$ for every $X \in [X_n]$. Further, a stationary sequence $\{X_n\}$ admitting a prediction is called *completely non-deterministic* if for every $X \in [X_n]$ we have

$$\lim_{k \to -\infty} A_k X = 0.$$

It is very easy to prove that every stationary sequence admitting a prediction can be decomposed into a deterministic and a completely non-deterministic components ([19], Theorem 1). Moreover, each stationary harmonizable sequence admitting a prediction is the sum of two independent stationary harmonizable sequences admitting a prediction, one completely non-deterministic and the other deterministic ([20], Theorem 4.2). Thus the study of stationary harmonizable sequences admitting a prediction is reduced to the study of deterministic and completely non-deterministic stationary harmonizable sequences.

We note that the condition $[X_n] = [X_n : n \leq 0]$ characterizes deterministic sequences $\{X_n\}$. Therefore, the structure of the space $[X_n]$ plays a key role in our considerations. In the next section we shall quote some auxiliary concepts and a characterization of the space $L(M)$. Hence a complete description of the space $[X_n]$ will follow.

3. Musielak-Orlicz spaces. Given a finite measure ν defined on Borel subsets of the unit interval I with $\nu(I) > 0$, we take a function Φ defined on $I \times R_+$ and satisfying the following conditions:

(3.1) $\Phi(t, 0) = 0$ and $\Phi(t, x) > 0$ for $x > 0$ and ν-almost all t,

(3.2) $\Phi(t, x)$ is a continuous non-decreasing function of x for every $t \in I$,

(3.3) $\Phi(t, x)$ is Borel measurable as a function of t for every $t \in I$,

(3.4) $\int_I \Phi(t, 1)\nu(dt) < \infty$,

(3.5) (the Δ_2-condition) there exists a positive constant c such that $\Phi(t, 2x) \leq c\,\Phi(t, x)$ for all x and ν-almost all t.

Throughout this paper we identify functions equal ν-almost everywhere. Let f be a complex-valued Borel function on I. It is easily seen that $\Phi(t, |f(t)|)$ is also a Borel function on I. We define a modular ρ by means of the formula

$$\rho(f) = \int_I \Phi(t, |f(t)|)\, \nu(dt).$$

Let $L_\Phi(\nu)$ be the set of all complex-valued Borel functions f on I for which $\rho(f)$ is finite. The set $L_\Phi(\nu)$ is a linear space over the complex field under usual addition and scalar multiplication. Moreover, it becomes a complete linear metric space under the non-homogeneous norm

$$\|f\| = \inf\{a : a > 0,\ \rho(a^{-1}f) \le a\}.$$

The space $L_\Phi(\nu)$ with this norm was introduced and investigated by J. Musielak and W. Orlicz in [14] and will be called a *Musielak-Orlicz space*. From (3.4) it follows that all bounded Borel functions on I belong to $L_\Phi(\nu)$. Moreover, the set of all Borel simple functions is dense in $L_\Phi(\nu)$.

In this paper two linear metric spaces $(Y, \|\ \|_1)$ and $(Y, \|\ \|_2)$ will be treated as identical if the convergences in both norms $\|\ \|_1$ and $\|\ \|_2$ are equivalent. In particular, if

$$a\,\Phi(t,x) \le \Psi(t,x) \le b\,\Psi(t,x)$$

for some positive numbers a and b, ν-almost all t and sufficiently large x, then $L_\Phi(\nu) = L_\Psi(\nu)$. Moreover, if $\beta(t) > 0$ for $t \in I$, $\int_I \beta(s)\nu(ds) < \infty$, $\Phi(t,x) = \Psi(t,x)/\beta(t)$ and $\lambda(E) = \int_E \beta(s)\nu(ds)$, then $L_\Psi(\nu) = L_\Phi(\nu)$. Therefore, without loss of generality, we may always assume that

$$\Phi(t,1) = 1 \quad \text{for} \quad t \in I. \tag{3.6}$$

Let K be the class of all pairs (Φ, ν) satisfying conditions (3.1)–(3.5) such that the measure ν is atomless and for ν-almost all t the function $\Phi(t, \sqrt{x})$ is concave on R_+.

Given a random measure M we denote by $\mu_M(\cdot,\cdot)$ the corresponding set-function appearing in formula (1.1). Put

$$\nu_M(E) = \mu_M(E, R_+)$$

for every Borel subset E of I. It is obvious that all measures $\mu_M(\cdot, A)$ are absolutely continuous with respect to the measure ν_M. Consequently, by the Radon–Nikodym Theorem,

$$\mu_M(E, [0,x)) = \int_E g_M(s,x)\, \nu_M(ds),$$

where $0 \le g_M(s,x) \le 1$ and the function $g_M(\cdot, x)$ is Borel measurable on I. Moreover, we may assume, without loss of generality, that the function $g_M(s,\cdot)$ is monotone non-decreasing and continuous to the left on R_+. Put

$$\Phi_M(t,x) = \int_{1/x}^{\infty} \frac{g_M(t,u)}{u^3}du \quad (t \in I, x \in R_+).$$

By a simple calculation we have

$$\Phi_M(t, \sqrt{x}) = \frac{1}{2}\int_0^x g_M\left(t, \frac{1}{\sqrt{u}}\right) du$$

and, consequently, $(\Phi_M, \nu_M) \in K$.

We shall lean heavily on the following representation of the space $L(M)$ of M-integrable functions, which provides a tool for investigating random harmonizable sequences ([20], Theorem 3.1).

THEOREM 3.1. *For every random measure M we have the relations $(\Phi_M, \nu_M) \in K$ and $L(M) = L_{\Phi_M(\nu_M)}$.*

The converse implication is also true.

THEOREM 3.2. *For every pair $(\Phi, \nu) \in K$ there exists a random measure M such that $L(M) = L_\Phi(\nu)$.*

Proof. Let $(\Phi, \nu) \in K$. Without loss of generality we may assume that condition (3.6) holds. Put

$$\Phi(t, \sqrt{x}) = \int_0^x q(t,u)du \quad (t \in I),$$

where $q(t, .)$ is a non-negative monotone non-increasing function. Setting $r(t, u) = q(t, u)$ for $u > 1$ and $r(t, u) = 1$ for $0 \le u \le 1$ we get a non-negative monotone non-increasing function $r(t, .)$. Moreover, the function

$$\Psi(t, x) = \int_0^{x^2} r(t,u)du \tag{3.7}$$

fulfils the condition $\Phi(t, x) = \Psi(t, x)$ for $x \ge 1$ and $(\Psi, \nu) \in K$. Consequently,

$$L_\Psi(\nu) = L_\Phi(\nu). \tag{3.8}$$

Now we shall prove that there exists a random measure M fulfilling the condition

$$\mu_M(E, [0, x)) = \int_E r(s, x^{-2})\nu(ds). \tag{3.9}$$

In fact, for the set-function (3.9) there exists a separable stochastic process with independent increments such that the characteristic function of the increment $X(b) - X(a)$ is given by the expression

$$\exp\left(\int_0^\infty (J_0(x|t| - 1)\frac{1+x^2}{x^2}\mu_M([a,b), dx)\right),$$

(see [6], p. 61 and 418). Setting $M(\bigcup_{j=1}^n [a_j, b_j)) = \sum_{j=1}^n (X(b_j) - X(a_j))$ for disjoint intervals $[a_j, b_j)$ $(j = 1, 2, \ldots, n)$ we get a random set function which, by Prékopa's Theorems ([15], p.227, 243) can be extended to a random measure M defined on Borel subset of I. Further, from (3.9) we get $\nu_M = \nu$ and $g_M(t, x) = r(t, x^{-2})$ which, by (3.7) yields the equality

$$\Phi_M(t, x) = \frac{1}{2}\Psi(t, x).$$

Consequently, $L_{\Phi_M}(\nu_M) = L_\Psi(\nu)$ and, by (3.8) and Theorem 3.1, $L(M) = L_\Phi(\nu)$. The theorem is thus proved. ■

In attempting to visualize these representation theorems we shall give some examples.

EXAMPLE 3.1. We say that M is a *random Poisson measure* if there exists a finite Borel measure $\beta(\cdot,\cdot)$ on $I \times R_+$ such that

$$\mu_M(E, dx) = \frac{x^2}{1+x^2}\beta(E, dx).$$

Integrating by parts it is easy to verify that

$$\begin{aligned}\int_I \Phi_M(t,x)\nu_M(dt) &= \int_{1/2}^{\infty}\int_I g_M(t,u)\nu_M(dt)\frac{du}{u^3}\\ &= \frac{1}{2}\int_0^{\infty}\frac{\min(x^2u^2,1)}{1+u^2}\beta(I,du) \le \beta(I, R_+)\end{aligned}$$

for every $x \in R_+$. Consequently, $\Phi_M(t,\cdot)$ are bounded for ν-almost every $t \in I$.

EXAMPLE 3.2. Given $p \ge 0$ and an atomless measure ν on I we put

$$\mu_M(E,dx) = 2\lambda(E)p^{-p}e^p x(1+e^p x^{-2})^{-2}\log^{p-1}(e^p + x^{-2})\log(1+e^p x^{-2})dx.$$

Then $\nu_M = \lambda$ and

$$\Phi_M(t,x) = e^p p^{-p} 2^{-1}(1+p)^{-1}(\log^{1+p}(e^p + x^2) - p^{1+p}).$$

EXAMPLE 3.3. Let λ be an atomles measure on I and

$$\mu_M(E,dx) = 2\lambda(E)x(1+ex^2)^{-2}(\log\log(e+x^{-2}) + 1 - \log^{-1}(e+x^{-2})).$$

Then

$$\Phi_M(r,x) = \frac{e}{2}\log(e+x^2)\log\log(e+x^2).$$

EXAMPLE 3.4. Given $0 < p < 2$ and an atomless measure ν we put

$$\mu_M(E,dx) = \beta\nu(E)\frac{x^{1-p}}{1+x^2}dx,$$

where $\beta = \frac{2}{\pi}\sin\frac{p\pi}{2}$. Here we have $\nu_M = \nu$ and the measure μ_M corresponds to a p-stable random measure M with the characteristic function $\varphi_{M(E)}(t) = \exp(-\nu(E)|t|^p)$. It is easy to check that

$$a\,x^p \le \Phi_M(t,x) \le b\,x^p \quad (t \in I, x \in R_+)$$

for some positive constants a and b. Thus $L(M) = L^p(\nu)$.

Consider a stationary harmonizable sequence $\{X_n(M)\}$ $(n = 0, \pm 1, \ldots)$ corresponding to a random measure M. It is easy to verify that the mapping

$$X_n(M) \to e^{2\pi n i s} \quad (n = 0, \pm 1, \ldots, s \in I) \tag{3.10}$$

can be extended in a natural way to an isomorphism between $[X_n(M)]$ and $L(M)$. Moreover,

$$[X_n(M)] = \left\{\int_I f(s)M(ds) : f \in L(M)\right\}$$

and by Theorem 3.1, formula (3.10) defines a natural isomorphism from $[X_n(M)]$ onto the Musielak-Orlicz space $L_{\Phi_M}(\nu_M)$. It is evident that the sequence $\{X_n(M)\}$ is deterministic if and only if

$$X_0(M) \in [X_n(M) : n \le -1].$$

Denoting by $\| \ \|$ the norm in $L_{\Phi_M}(\nu_M)$, we infer that $\{X_n(M)\}$ is deterministic if and only if

$$\inf \left\| 1 + \sum_{k=1}^{n} a_k e^{-2\pi kis} \right\| = 0, \tag{3.11}$$

where the infimum is taken over all complex numbers $a_1, a_2, \ldots, a_n$ and $n = 1, 2, \ldots$ Since $\|f\| = \|\overline{f}\|$, we observe that (3.11) is equivalent to the relation

$$\inf \left\| 1 + \sum_{k=1}^{n} a_k e^{2\pi kis} \right\| = 0.$$

A solution of this extremal problem of Szegö's type can be regarded as a generalisation of the famous Kolmogorov-Krein criterion for L^p-spaces ([9, 10]). This question will be discussed in the next section.

4. An extremal problem for Musielak-Orlicz spaces. Given a Borel measure ν on I by ν_c we shall denote the absolutely continuous component of ν with respect to the Lebesgue measure and by $d\nu_c/dt$ a Borel measurable version of its Radon-Nikodym density function. For any pair (Φ, ν) satisfying conditions (3.1)–(3.6) we introduce auxiliary functions $\lambda_{\Phi,\nu}$ and $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ by means of the formulas

$$\Lambda_{\Phi,\nu}(t, x) = \sup \left\{ \frac{\log y}{\Phi(t, y)} \left(\frac{d\nu_c}{dt} \right)^{-1} : y \geq x \right\},$$
$$\Omega_{\Phi,\nu,n}(t) = \inf\{x : \Lambda_{\Phi,\nu}(t, x) \leq n, x \geq 1\},$$

where the infimum of an empty set is defined as ∞. It is clear that all these functions are Borel measurable and $1 \leq Q_{\Phi,\nu,n}(t) \leq \infty$ $(n = 1, 2, \ldots)$.

The following generalization of the Kolmogorov-Krein criterion was proved in [20] (Theorem 1.1).

THEOREM 4.1. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space with the norm $\| \ \|$. The equation*

$$\inf \left\| 1 + \sum_{k=1}^{n} a_k e^{2\pi kit} \right\| = 0, \tag{4.1}$$

where the infimum is taken over all complex numbers $a_1, a_2, \ldots, a_n$ and $n = 1, 2, \ldots$, holds if and only if no function

$$\log \Omega_{\Phi,\nu,n} \quad (n = 1, 2, \ldots)$$

is Lebesgue integrable over I.

Now we shall quote some particular cases of this theorem. Given a number $b > 1$, we say that a function Φ satisfies the Λ_b-*condition* if there exists a constant $e_b > 1$ and a positive number x_0 such that

$$\Phi(t, x)e_b \leq \Phi(t, bx)$$

for all $t \in I$ and $x \geq x_0$ (see [13]).

THEOREM 4.2. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space satisfying the Λ_b-condition for some constant $b > 1$. Then equation (4.1) holds if and only if $\log \frac{d\nu_c}{dt}$ is not Lebesgue integrable over I.*

Proof. From the Λ_b-condition it follows that there are positive constants c_1 and p such that

$$c_1 x^p \leq \Phi(t,x)$$

for sufficiently large x and all $t \in I$ (see [13], 124). Further, from the Δ_2-condition (3.5) it follows that there are positive constants c_2 and q such that

$$\Phi(t,x) \leq c_2 x^q$$

for sufficiently large x and ν-almost all t. Consequently, we can find a positive number $x_0 > 1$ such that

$$c_1 x^p \leq \frac{\Phi(t,x)}{\log x} \leq c_2 x^q \tag{4.2}$$

for all $x \geq x_0$ and ν-almost all t. Hence in particular it follows that

$$\lim_{x\to\infty} \Lambda_{\Phi,\nu}(t,x) = 0$$

ν-almost everywhere. Consequently, the functions $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ are finite ν-almost everywhere.

Suppose first that the Lebesgue measure is not absolutely continuous with respect to the measure ν. Then $\frac{d\nu_c}{dt}$ vanishes on a set of positive Lebesgue measure. Consequently, all functions $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ and the function $\log \frac{d\nu_c}{dt}$ are infinite on a set of positive Lebesgue measure, which, by Theorem 4.1, implies our assertion.

Now suppose that the Lebesgue measure is absolutely continuous with respect to the measure ν. Then the functions $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ are finite almost everywhere in the sense of the Lebesgue measure. Moreover, inequality (4.2) holds also for all $x \geq x_0$ and for almost all t in the sense of Lebesgue measure. Put

$$F_n = \{t : x_0 < \Omega_{\Phi,\nu,n}(t) < \infty\} \quad (n = 1, 2, \ldots).$$

It is very easy to verify that both functions $\log \frac{d\nu_c}{dt}$ and $\log \Omega_{\Phi,\nu,n}(t)$ are Lebesgue integrable over $I \subset F_n$. Moreover, for all $t \in F_n$ we have the formula

$$\log \Omega_{\Phi,\nu,n}(t) \left(\frac{d\nu_c}{dt}\right)^{-1} = n\,\Phi(t, \Omega_{\Phi,\nu,n}(t)).$$

Hence and from (4.2) we get the inequality

$$nc_1 \Omega^p_{\Phi,\nu,n}(t) \leq \left(\frac{d\nu_c}{dt}\right)^{-1} \leq nc_2 \Omega^q_{\Phi,\nu,n}(t)$$

for almost all t from F_n in the sense of the Lebesgue measure. Consequently, the function $\log \frac{d\nu_c}{dt}$ and all the functions $\log \Omega_{\Phi,\nu,n}$ simultaneously are not Lebesgue integrable over I which, by Theorem 4.1, completes the proof. ■

THEOREM 4.3. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space satisfying the condition*

$$\lim_{x\to\infty} \frac{\Phi(t,x)}{\log x} = 0 \tag{4.3}$$

on a set of positive Lebesgue measure. Then equation (4.1) is fulfilled.

Proof. It is very easy to verify that $\Omega_{\Phi,\nu,n}(t) = \infty$ for all t from I satisfying (4.3) and the inequality $\frac{d\nu_c}{dt} < \infty$. Since the density function $\frac{d\nu_c}{dt}$ is finite almost everywhere with respect to the Lebesgue measure, we conclude that no function $\log \Omega_{\Phi,\nu,n}$ is Lebesgue integrable over I which, by Theorem 4.1, gives formula (4.1). ■

In the same way one can prove the following theorems.

THEOREM 4.4. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space satisfying for some positive numbers a and b the condition*

$$a \le \frac{\Phi(t,x)}{\log x} \le b$$

for $x \ge x_0$ and almost all t in the sense of the Lebesgue measure. Then equation (4.1) holds if and only if $\operatorname{ess\,inf} \frac{d\nu_c}{dt} = 0$.

THEOREM 4.5. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space. If there are positive numbers a, b, p and x_0 such that*

$$a \le \frac{\Phi(t,x)}{\log^{1+p} x} \le b$$

for $x \ge x_0$ and almost all t in the sense of the Lebesgue measure, then equation (4.1) holds if and only if

$$\int_I \left(\frac{d\nu_c}{dt}\right)^{-1/p} dt = \infty.$$

THEOREM 4.6. *Let $L_\Phi(\nu)$ be a Musielak-Orlicz space. If there are positive numbers a, b and x_0 such that*

$$a \le \frac{\Phi(t,x)}{\log x \log\log x} \le b,$$

for $x \ge x_0$ and almost all t in the sense of the Lebesgue measure, then equation (4.1) holds if and only if

$$\int_I \exp\left\{n^{-1}\left(\frac{d\nu_c}{dt}\right)^{-1}\right\} dt = \infty$$

for all positive integers n.

5. Deterministic harmonizable sequences. We proceed now to a description of stationary harmonizable sequences $\{X_n(M)\}$ in terms of probabilistic characteristics of the random measure M. We recall that to every random measure M there corresponds a Borel measure ν_M on I and a function Φ_M on $I \times R_+$ and the pair (Φ_M, ν_M) determines the sequence of functions $\Omega_{\Phi,\nu,n}$ $(n = 1, 2, \ldots)$ on I. We already know that the sequence $\{X_n(M)\}$ is deterministic if and only if equation (4.1) holds in $L_{\Phi_M}(\nu_M)$. Consequently, Theorem 4.1 yields the following characterization of deterministic sequences.

THEOREM 5.1. *A stationary harmonizable sequence $\{X_n(M)\}$ is deterministic if and only if no function $\log \Omega_{\Phi_M,\nu_M,n}$ $(n = 1, 2, \ldots)$ is Lebesgue integrable over I.*

We illustrate this theorem by some examples.

EXAMPLE 5.1. Comparing Example 3.1 and Theorem 4.3 we conclude that stationary harmonizable sequences $\{X_n(M)\}$ induced by random Poisson measures M are always deterministic.

EXAMPLE 5.2. Taking into account Example 3.4 and Theorem 4.2 we infer that a stationary harmonizable sequence $\{X_n(M)\}$ corresponding to a p-stable random measure M with $0 < p < 2$ and $\nu_M = \nu$ is deterministic if and only if $\log \frac{d\nu_c}{dt}$ is not Lebesgue integrable over I.

EXAMPLE 5.3. Consider a stationary harmonizable sequence $\{X_n(M)\}$ corresponding to the measure M appearing in Example 3.2 with $p = 0$ and $\nu_M = \nu$. By Theorem 4.4 this sequence is deterministic if and only if $\operatorname{ess\,inf} \frac{d\nu_c}{dt} = 0$.

EXAMPLE 5.4. Taking a stationary harmonizable sequence $\{X_n(M)\}$ corresponding to the measure M appearing in Example 3.2 with $p > 0$ and $\nu_M = \nu$ we infer, by Theorem 4.5, that $\{X_n(M)\}$ is deterministic if and only if

$$\int_I \left(\frac{d\nu_c}{dt}\right)^{-1/p} dt = \infty.$$

EXAMPLE 5.5. Let M be the random measure described by Example 3.3 with $\nu_M = \nu$. Applying Theorem 4.6 we conclude that the sequence $\{X_n(M)\}$ is deterministic if and only if

$$\int_I \exp\left\{n^{-1}\left(\frac{d\nu_c}{dt}\right)^{-1}\right\} dt = \infty$$

for all positive integers n.

6. Completely non-deterministic harmonizable sequences. First we shall quote a continuous analogue of the Bernstein-Darmois Theorem ([1], [5]), which is a main tool in the study of completely non-deterministic sequences. For homogeneous random measures this problem was discussed in [11], [18] and [21]. The following theorem was proved in [20] (Theorem 2.1).

THEOREM 6.1. *Let f and g be M-integrable functions with respect to a random measure. If the random variables $\int_I f(s)M(ds)$ and $\int_I g(s)M(ds)$ are independent, then for every Borel subset E of the set $\{s : f(s)g(s) \neq 0\}$ the random variable $M(E)$ is Gaussian.*

Here the degenerate case $M(E) = 0$ is also treated as the Gaussian one. Further, a random measure M is said to be *Gaussian* if for every Borel subset E of I the random variable $M(E)$ is Gaussian. If in addition $M(I)$ does not vanish with probability 1 we have $L(M) = L^2(\nu_M)$. The classical characterization of completely non-deterministic wide sense stationary sequences ([6], Chapter XII, 4) implies the following statement.

THEOREM 6.2. *Let M be a Gaussian random measure. The sequence $\{X_n(M)\}$ is completely non-deterministic if and only if either $M \equiv 0$ with probability 1 or the measure ν_M is absolutely continuous with respect to the Lebesgue measure and $\log \frac{d\nu_M}{dt}$ is Lebesgue integrable over I.*

A complete description of stationary harmonizable completely non-deterministic sequences is given by the following theorem.

THEOREM 6.3. *A stationary harmonizable sequence $\{X_n(M)\}$ is completely non-deterministic if and only if either $M \equiv 0$ with probability 1 or the measure M is Gaussian, ν_M is absolutely continuous with respect to the Lebesgue measure and $\log \frac{d\nu_M}{dt}$ is Lebesgue integrable over I.*

Proof. By Theorem 6.2 it suffices to prove that the measure M is Gaussian provided $\{X_n(M)\}$ is completely non-deterministic.

Let A_k be the predictor based on the full past of $X_n(M)$ up to time k. Since

$$[X_n(M)] = \left\{ \int_I f(s)M(ds) : f \in L(M) \right\},$$

we have the formula

$$A_k X_0(M) = \int_I f_k(s)M(ds)$$

where $f_k \in L(M)$. Setting

$$E_k = \{s : f_k(s) \neq 1\},$$

we get the formula

$$A_k X_0(M) = \int_{E_k} f_k(s)M(ds) + M(I \setminus E_k).$$

Of course, the random variables $M(I \setminus E_k)$ and $\int_{E_k} f_k(s)M(ds)$ are independent and symmetrically distributed. Consequently, the relation

$$\lim_{k\to-\infty} A_k X_0(M) = 0$$

implies the relation

$$\lim_{k\to-\infty} M(I \setminus E_k) = 0. \tag{6.1}$$

By the definition of predictors the random variables $X_0(M) - A_k X_0(M)$ and $X_k(M)$ are independent. In other words, the integrals

$$\int_I (1 - f_k(s))M(ds) \quad \text{and} \quad \int_I e^{2\pi kis}M(ds)$$

are independent. Since both integrands are different from 0 on E_k, we infer, by Theorem 6.1, that the random $M(E_k)$ is Gaussian. Hence and from (6.1) it follows that $M(I)$, being the limit in probability of Gaussian random variables $M(E_k)$, is Gaussian too. By Cramér's Theorem ([12], p. 271), M is a Gaussian random measure, which completes the proof. ■

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