BANACH ALGEBRAS 2009
BANACH CENTER PUBLICATIONS, VOLUME 91
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2010

STRUCTURE THEORY OF HOMOLOGICALLY TRIVIAL AND ANNIHILATOR LOCALLY C^*-ALGEBRAS

ALEXEI YU. PIRKOVSKII

Department of Geometry and Topology, Faculty of Mathematics
State University – Higher School of Economics
Vavilova 7, 117312 Moscow, Russia
E-mail: aupirkovskii@hse.ru, pirkosha@online.ru

YURII V. SELIVANOV

Department of Higher Mathematics
Russian State Technological University (MATI)
Orshanskaya 3, Moscow 121552, Russia
E-mail: selkatya@post.ru

Dedicated to the memory of M. A. Naĭmark on the centenary of his birth

Abstract. We study the structure of certain classes of homologically trivial locally C^*-algebras. These include algebras with projective irreducible Hermitian A-modules, biprojective algebras, and superbiprojective algebras. We prove that, if A is a locally C^*-algebra, then all irreducible Hermitian A-modules are projective if and only if A is a direct topological sum of elementary C^*-algebras. This is also equivalent to A being an annihilator (dual, complemented, left quasi-complemented, or topologically modular annihilator) topological algebra. We characterize all annihilator σ-C^*-algebras and describe the structure of biprojective locally C^*-algebras. Also, we present an example of a biprojective locally C^*-algebra that is not topologically isomorphic to a Cartesian product of biprojective C^*-algebras. Finally, we show that every superbiprojective locally C^*-algebra is topologically *-isomorphic to a Cartesian product of full matrix algebras.

1. Introduction. This paper is devoted to the study of the structure of locally C^*-algebras satisfying various homological triviality conditions. The properties of projectivity

2010 *Mathematics Subject Classification*: Primary 46H20, 46M20; Secondary 46H25, 46M10.
Key words and phrases: topological algebra, locally C^*-algebra, annihilator algebra, dual algebra, complemented algebra, topologically modular annihilator algebra, topological module, Hermitian module, projective module, biprojective algebra, contractible algebra, superbiprojective algebra.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc91-0-17

of topological modules and of biprojectivity or superbiprojectivity of topological algebras will play here a central role. We are concerned with the following questions.

How to characterize, in inner terms of their structure, locally C^*-algebras A such that:

(i) *all irreducible Hermitian A-modules*[1] *are projective?*
(ii) *all Hermitian A-module are projective?*
(iii) *the algebra A is biprojective?*
(iv) *the algebra A is superbiprojective?*

Earlier, similar questions were answered for C^*-algebras [34, 80, 81, 82, 40, 41, 89, 90]. Apart from this, the last two questions were answered for σ-C^*-algebras [67].

The answers to the above questions will be given in Theorems 3.24, 4.21, 5.9 below.

First of all we describe (in Theorem 3.3) the structure of locally C^*-algebras with dense socle, and also the structure of their closed two-sided ideals. Next we show that property (i) characterizes locally C^*-algebras with dense socle or, equivalently, direct topological sums of elementary[2] C^*-algebras. Moreover, the same property is equivalent to A being an annihilator (dual in Kaplansky's sense, complemented, left quasi-complemented, or topologically modular annihilator[3]) topological algebra. Also, we characterize (in Corollary 3.21) projective irreducible Hermitian modules over locally C^*-algebras, and describe (in Theorem 3.30 and Corollary 3.32) annihilator unital locally C^*-algebras and annihilator σ-C^*-algebras. In particular, we show that every annihilator σ-C^*-algebra is topologically *-isomorphic to the Cartesian product of a countable family of annihilator C^*-algebras.

Properties (ii) and (iii) turn out to be equivalent. We describe the structure of biprojective locally C^*-algebras, and present an example of a biprojective locally C^*-algebra that is not topologically isomorphic to a Cartesian product of biprojective C^*-algebras.

Finally, we show that property (iv) characterizes Cartesian products of full matrix algebras. In particular, we establish that all superbiprojective locally C^*-algebras are contractible.

2. Preliminaries. By a *topological algebra* we shall mean a complete Hausdorff locally convex space over $\mathbb{C}$ equipped with a jointly continuous multiplication. No commutativity or existence of an identity is assumed.

The symbol $\widehat{\otimes}$ denotes the complete projective topological tensor product. The *product map* is the continuous linear operator $\pi_A\colon A\widehat{\otimes} A \to A$ uniquely determined by $\pi_A(a\otimes b) = ab$. By a *Fréchet algebra* we mean a metrizable topological algebra.

We recall that a seminorm p on an algebra A is *submultiplicative* if $p(ab) \le p(a)p(b)$ for all $a, b \in A$. If A is a *-algebra, then a *C^*-seminorm* on A is a submultiplicative seminorm p satisfying $p(a^*) = p(a)$ and $p(a^*a) = p(a)^2$ for all $a \in A$. A topological algebra A

[1] See Definition 3.14.

[2] Recall that a C^*-algebra A is called *elementary* if A is isomorphic to the algebra $\mathcal{K}(H)$ of all compact operators on a Hilbert space H.

[3] See Definition 3.23.

is an *Arens-Michael algebra* if the topology on A can be defined by a family of submultiplicative seminorms. A *locally C^*-algebra*[4] is an Arens-Michael *-algebra whose topology is defined by a family of C^*-seminorms. Recall that the term "locally C^*-algebra" is due to A. Inoue [29]. Metrizable locally C^*-algebras are also called *σ-C^*-algebras*[5] [10, 61, 62].

As is known, every locally C^*-algebra is topologically *-isomorphic to an inverse limit of C^*-algebras. σ-C^*-algebras are countable inverse limits of C^*-algebras.

S. J. Bhatt and D. J. Karia obtained an interesting characterization of locally C^*-algebras among Arens-Michael *-algebras:

THEOREM 2.1 (see [14]). *An Arens-Michael *-algebra A is a locally C^*-algebra if and only if there exists a dense *-subalgebra B of A which is a C^*-algebra under some norm and which is continuously embedded in A.*

We recall (cf. [24, Lemma 8.14]) that every locally C^*-algebra A is semisimple, i.e., the Jacobson radical of A is 0. For details and references on locally C^*-algebras, see [26, 61, 62].

Now for any subset S of an algebra A, let $\mathrm{lan}(S)$ and $\mathrm{ran}(S)$ denote the *left* and *right annihilators* of S in A, respectively (see, e.g., [15, §30]). So we have

$$\mathrm{lan}(S) = \{a \in A \mid ab = 0 \text{ for all } b \in S\} \quad \text{and} \quad \mathrm{ran}(S) = \{a \in A \mid ba = 0 \text{ for all } b \in S\}.$$

As is known, $\mathrm{lan}(S)$ is always a left ideal of A, and $\mathrm{ran}(S)$ is always a right ideal of A. Furthermore, the left (respectively, right) annihilator of a left (respectively, right) ideal is a two-sided ideal. Also, if A is a topological algebra, then both $\mathrm{lan}(S)$ and $\mathrm{ran}(S)$ are closed.

Recall that a net $\{e_\nu,\ \nu \in \Lambda\}$ in a topological algebra A is called a *left* (respectively, *right*) *bounded approximate identity* if $e_\nu a \to a$ (respectively, $ae_\nu \to a$) for each $a \in A$ and if the elements e_ν form a bounded set in A. A *bounded approximate identity* is a net which is both a left and a right bounded approximate identity.

We recall that a locally C^*-algebra always has a bounded approximate identity (see [26, Theorem 11.5]). This implies, in particular, that $\mathrm{lan}(A) = \mathrm{ran}(A) = 0$ for every such algebra A.

A topological algebra A is an *annihilator algebra* if, for every closed left ideal J and for every closed right ideal K, we have $\mathrm{ran}(J) = 0$ if and only if $J = A$ and $\mathrm{lan}(K) = 0$ if and only if $K = A$. If $\mathrm{lan}(\mathrm{ran}(J)) = J$ and $\mathrm{ran}(\mathrm{lan}(K)) = K$, then A is called a *dual algebra*. It is obvious that a dual algebra is automatically an annihilator algebra.

Dual algebras were introduced by I. Kaplansky [47], and annihilator algebras were introduced by F. F. Bonsall and A. W. Goldie [16]. The existence of an annihilator Banach algebra which is not dual was first established by B. E. Johnson [45]. His example was commutative and semisimple. Later A. M. Davie [19] gave an example of a topologically simple annihilator Banach algebra which is not dual. However every annihilator C^*-algebra is dual (see [70, Corollary 4.10.26]).

[4] These objects are called b^*-algebras in [6, 21, 7], LMC^*-algebras in [50, 77], pro-C^*-algebras in [99, 61, 62] and multi-C^*-algebras in [39].

[5] Also, metrizable locally C^*-algebras are called F^*-algebras in [18].

Now we recall (see [89] or [43, p. 63]) that the c_0-*sum*, $\bigoplus_0\{A_\nu \mid \nu \in \Lambda\}$, of a family of C^*-algebras A_ν, $\nu \in \Lambda$, is defined to be the set of all functions f defined on Λ such that

(i) $f(\nu) \in A_\nu$ for each $\nu \in \Lambda$, and
(ii) for each $\varepsilon > 0$ the set $\{\nu : \|f(\nu)\| \geq \varepsilon\}$ is finite.

As is known, this set is a C^*-algebra with respect to pointwise operations and the norm $\|f\| = \sup_{\nu\in\Lambda} \|f(\nu)\|$.

Following [20], we say that a C^*-algebra A is *elementary* if A is isometrically *-isomorphic to the algebra $\mathcal{K}(H)$ of all compact operators on a Hilbert space H. We recall (see [20, 4.7.20]) that a C^*-algebra A is annihilator (or, equivalently, dual) if and only if it is isometrically *-isomorphic to the c_0-sum of a family of elementary C^*-algebras.

For details and references on annihilator and dual algebras, see [70, 57, 15, 59].

Let A be a topological algebra and let $\mathcal{M}_\ell$ be the set of all closed left ideals of A. Then A is called a *left quasi-complemented algebra* if there exists a mapping $q\colon J \mapsto J^q$ of $\mathcal{M}_\ell$ into itself having the following properties:

(i) $J \cap J^q = 0$ $(J \in \mathcal{M}_\ell)$;
(ii) $(J^q)^q = J$ $(J \in \mathcal{M}_\ell)$;
(iii) if $J_1 \subset J_2$, then $J_2^q \subset J_1^q$ $(J_1, J_2 \in \mathcal{M}_\ell)$.

The mapping q is called a *left quasi-complementor* on A.

A left quasi-complemented algebra is called a *left complemented algebra* if it satisfies:

(iv) $J + J^q = A$ $(J \in \mathcal{M}_\ell)$.

In this case, the mapping q is called a *left complementor* on A.

Right quasi-complemented algebras and *right complemented algebras* are defined analogously. A left and right complemented algebra is called a *complemented algebra*.

Complemented Banach algebras were introduced by B. J. Tomiuk [95] and have been studied by many authors (see, e.g., [58, 1, 4, 2, 100, 98, 3, 101, 55, 96, 97]). Left (right) quasi-complemented topological algebras were defined by T. Husain and P. K. Wong [44]. They showed that there exist right quasi-complemented algebras which are not right complemented.

Structural properties of left complemented semisimple topological algebras in which every modular maximal left ideal is closed were studied by M. Haralampidou [32].

It is known that a C^*-algebra is complemented if and only if it is dual (see [4]). A similar result is true for left (right) quasi-complemented algebras (see [44]).

Let $\{I_\nu \mid \nu \in \Lambda\}$ be a family of (left, two-sided) ideals in an algebra A. Recall that the smallest (left, two-sided) ideal in A which contains every I_ν is called the *sum* of the ideals I_ν. The sum of the ideals I_ν evidently consists of all finite sums of elements from the ideals I_ν.

If A is a topological algebra, then the closure of the sum of the ideals I_ν is called their *topological sum*. If each I_ν is closed and intersects the topological sum of the remaining ideals in the zero element, then the topological sum is called a *direct topological sum* (C. E. Rickart's terminology, see [70, p. 46]).

We recall that a *minimal left ideal* of an algebra A is a left ideal $J \neq 0$ such that 0 and J are the only left ideals contained in J. Recall also (see, e.g., [15, §30]) that the *left socle* of an algebra A is the sum of all minimal left ideals of A. The *right socle* is similarly defined in terms of right ideals. If the left socle coincides with the right socle, then it is called the *socle* of A and is denoted by $\mathrm{Soc}(A)$.

Recall that an algebra A is *semiprime* if it has no two-sided ideals, $I \neq 0$, with $I^2 = 0$. Since semisimple algebras are semiprime (see, e.g., [15, Proposition 30.5]), we conclude that every locally C^*-algebra is semiprime.

Recall that, if A is semiprime, then the left and right socles of A coincide, and so A has a socle. $\mathrm{Soc}(A)$ is known to be a two-sided ideal of A. Recall that a C^*-algebra A is dual if and only if $\mathrm{Soc}(A)$ is dense in A (see [20, 4.7.20]).

Also, we recall that a *minimal closed two-sided ideal* of a topological algebra is a closed two-sided ideal $I \neq 0$ that contains no closed two-sided ideals other than 0 and I. Finally, we recall that a topological algebra A is *topologically simple* if its only closed two-sided ideals are 0 and A.

The proof of the following lemma repeats the proof of [15, Lemma 32.4].

LEMMA 2.2. *Let A be a semiprime topological algebra, and let I be a two-sided ideal of A. Then $\mathrm{lan}(I) = \mathrm{ran}(I)$, $I \cap \mathrm{lan}(I) = 0$. If, in addition, A is an annihilator algebra, then $I \oplus \mathrm{lan}(I)$ is dense in A.*

If A is a locally C^*-algebra, then even more is true.

LEMMA 2.3 (cf. [57, §25.2] and [33]). *Let A be a locally C^*-algebra.*

(i) *If J is a closed left ideal of A, then $J \cap (\mathrm{ran}(J))^* = 0$ and $J \oplus (\mathrm{ran}(J))^*$ is a closed left ideal of A. If, in addition, A is an annihilator algebra, then $A = J \oplus (\mathrm{ran}(J))^*$.*

(ii) *If K is a closed right ideal of A, then $K \cap (\mathrm{lan}(K))^* = 0$ and $K \oplus (\mathrm{lan}(K))^*$ is a closed right ideal of A. If, in addition, A is an annihilator algebra, then $A = K \oplus (\mathrm{lan}(K))^*$.*

(iii) *If I is a closed two-sided ideal of A, then $I \cap \mathrm{lan}(I) = 0$ and $I \oplus \mathrm{lan}(I)$ is a closed two-sided ideal of A. If, in addition, A is an annihilator algebra, then $A = I \oplus \mathrm{lan}(I)$.*

Proof. (i) Clearly, $(\mathrm{ran}(J))^*$ is a closed left ideal of A. Suppose that $a \in J \cap (\mathrm{ran}(J))^*$. Then $aa^* = 0$, and consequently, for each continuous C^*-seminorm p on A,

$$p(a) = p(aa^*)^{1/2} = 0.$$

Hence $a = 0$, and so $J \cap (\mathrm{ran}(J))^* = 0$.

Now let $L = J \oplus (\mathrm{ran}(J))^*$, and let $a = b + c \in L$, where $b \in J$, $c \in (\mathrm{ran}(J))^*$. Then $ac^* = cc^*$, and therefore

$$p(c)^2 = p(cc^*) \leq p(a)p(c^*)$$

and

$$p(c) \leq p(a), \tag{1}$$

for each continuous C^*-seminorm p on A.

Similarly, we have

$$p(b) \leq p(a), \tag{2}$$

for each continuous C^*-seminorm p on A. It follows easily from these inequalities that the left ideal L is closed.

We prove now that $\operatorname{ran}(L) = 0$. Indeed, suppose that $a \in \operatorname{ran}(L)$. Then $Ja = 0$ and $(\operatorname{ran}(J))^* a = 0$. It follows from the first equality that $a \in \operatorname{ran}(J)$ and $a^* \in (\operatorname{ran}(J))^*$. Hence $a^* a = 0$, and therefore, for each continuous C^*-seminorm p on A,

$$p(a) = p(a^* a)^{1/2} = 0,$$

and so $a = 0$. Thus $\operatorname{ran}(L) = 0$, and we see that, if A is an annihilator algebra, then $L = A$.

(ii) This is similar.

(iii) This follows from (ii) and from the fact that every closed two-sided ideal of a locally C^*-algebra is a *-ideal (see [26, Theorem 11.7]). ∎

The proof of the next lemma repeats the proof of [70, Theorem 2.8.29]; see also [33, Theorem 2.4].

LEMMA 2.4. *Let A be a topological algebra such that* $\operatorname{lan}(A) = \operatorname{ran}(A) = 0$. *Suppose that A is equal to the topological sum of a given family $\{I_\nu \mid \nu \in \Lambda\}$ of its closed two-sided ideals. If each I_ν is an annihilator algebra, then A is an annihilator algebra.*

3. Locally C^*-algebras with projective irreducible modules

3.1. Locally C^*-algebras with dense socle. Let A be a locally C^*-algebra, and let P be the family of all continuous C^*-seminorms on A. For each $p \in P$ we set

$$N_p = \operatorname{Ker} p,$$

where $\operatorname{Ker} p = \{a \in A \mid p(a) = 0\}$. Then each N_p is a two-sided *-ideal of A. The quotient seminorm of p on A/N_p is a C^*-norm. We denote this norm by $\|\cdot\|_p$. The completion of A/N_p with respect to this norm is a C^*-algebra which is denoted by A_p. Following [39], we call the C^*-algebra A_p *concomitant* with A.

Note that every quotient algebra A/N_p is in fact already complete with respect to the norm $\|\cdot\|_p$ (see [7], [77], [62, Corollary 1.2.8]), and so $A_p = (A/N_p, \|\cdot\|_p)$.

PROPOSITION 3.1. *Let A be a locally C^*-algebra, and let P be the family of all continuous C^*-seminorms on A. Then, for every $p \in P$, the quotient topology on A/N_p coincides with the topology determined by the norm $\|\cdot\|_p$.*

Proof. Evidently, the first topology is stronger than the second one. On the other hand, let $q \in P$, and suppose that $q \geq p$ (i.e., $q(a) \geq p(a)$ for all $a \in A$). Then the quotient seminorm of q on A/N_p is equal to the quotient norm of the C^*-norm $\|\cdot\|_q$ on $A_q = A/N_q$, and hence is itself a (complete) C^*-norm. Therefore the latter norm is equal to $\|\cdot\|_p$.

Note that P is directed with the order $p \leq q$ (see [62]). Hence we conclude that, if $q \in P$ is arbitrary, and $\widehat{q}$ is the quotient seminorm of q on A/N_p, then $\|\cdot\|_p \geq \widehat{q}$. The rest is clear. ∎

Let A be an algebra. We recall [15] that a non-zero idempotent $e \in A$ is called *minimal* if eAe is a division algebra. Since an Arens-Michael division algebra is isomorphic to $\mathbb{C}$ (see, e.g., [39, Theorem V.1.7]), we conclude that, for a minimal idempotent e in an Arens-Michael algebra A, we have $eAe = \mathbb{C}e$.

COROLLARY 3.2. *Let A be a topologically simple locally C^*-algebra that contains a minimal idempotent. Then A is isomorphic, as a topological *-algebra, to the C^*-algebra of all compact operators on a Hilbert space.*

Proof. Let $p \in P$, and suppose that $p \neq 0$. Then $\operatorname{Ker} p$ is a closed two-sided ideal of A, and hence $\operatorname{Ker} p = 0$ because A is topologically simple. Thus p is a norm. By Proposition 3.1, the topology on $A = A/N_p$ coincides with the topology determined by the norm $p = \|\cdot\|_p$. So A is isomorphic, as a topological *-algebra, to the C^*-algebra $(A, \|\cdot\|_p)$. Since A contains a minimal idempotent, it follows from [15, Proposition 30.6] that $\operatorname{Soc}(A) \neq 0$. Since A is topologically simple, and $\operatorname{Soc}(A)$ is a two-sided ideal, it follows that $\operatorname{Soc}(A)$ is dense in A. It remains to apply [20, 4.7.20] and [70, Corollary 4.10.20]. ∎

THEOREM 3.3. *Let A be a locally C^*-algebra with dense socle, and let $\{I_\nu \mid \nu \in \Lambda\}$ be the collection of minimal closed two-sided ideals of this algebra; we put $P_\nu = \operatorname{lan}(I_\nu)$ $(= \operatorname{ran}(I_\nu))$ $(\nu \in \Lambda)$. Then the following statements hold.*

(i) *A is the direct topological sum of all the I_ν, and moreover, for each $\nu_1, \nu_2 \in \Lambda$ with $\nu_1 \neq \nu_2$, we have $I_{\nu_1} I_{\nu_2} = 0$.*

(ii) *For each $\nu \in \Lambda$ the algebra I_ν is isomorphic, as a topological *-algebra, to the C^*-algebra $\mathcal{K}(H_\nu)$, where H_ν is a Hilbert space.*

(iii) *Every closed two-sided ideal I of A is the intersection of the ideals P_ν that contain I. In particular, $\bigcap_{\nu \in \Lambda} P_\nu = 0$.*

(iv) *Every closed two-sided ideal I of A is the direct topological sum of the ideals I_ν contained in I. As a consequence, for each ν, $P_\nu = \overline{\bigoplus_{\mu \in \Lambda \setminus \{\nu\}} I_\mu}$.*

(v) *A is an annihilator algebra.*

(vi) *For each ν, $A = I_\nu \oplus P_\nu$. Moreover, let $\psi_\nu \colon A \to A/P_\nu$, $a \mapsto a_\nu$, denote the quotient map. Then $\varphi_\nu \colon I_\nu \hookrightarrow A \xrightarrow{\psi_\nu} A/P_\nu$ is an isometric *-isomorphism of C^*-algebras.*

(vii) *The homomorphism of locally C^*-algebras*
$$\tau \colon A \to \prod_{\nu \in \Lambda} A/P_\nu = \prod_{\nu \in \Lambda} \mathcal{K}(H_\nu), \quad a \mapsto \{a_\nu\},$$
is a continuous embedding with dense range.

(viii) *If I, $I = \overline{\bigoplus_{\nu \in \Lambda_1} I_\nu}$, is an arbitrary closed two-sided ideal of A, then $A = I \oplus J$, where*
$$J = \operatorname{lan}(I) = \overline{\bigoplus_{\nu \in \Lambda \setminus \Lambda_1} I_\nu}.$$
Moreover, the natural homomorphism $\varphi \colon J \hookrightarrow A \to A/I$ is a topological *-isomorphism. In particular, A/I is complete.

(ix) *If I is a closed two-sided ideal of A, then $\operatorname{Soc}(I)$ is dense in I, $\operatorname{Soc}(A/I)$ is dense in A/I, and A is topologically *-isomorphic to the Cartesian product of the algebras I and A/I.*

(x) *If p is a continuous C^*-seminorm on A, then there exists a subset $\Lambda_1 \subset \Lambda$ such that $\operatorname{Ker} p = \operatorname{lan}(J)$, where $J = \overline{\bigoplus_{\nu \in \Lambda_1} I_\nu}$ and J is isomorphic, as a topological *-algebra, to a C^*-algebra (and hence J is the c_0-sum of the family of C^*-algebras $I_\nu = \mathcal{K}(H_\nu)$, $\nu \in \Lambda_1$). Moreover, for each $a \in A$, $p(a) = \sup_{\nu \in \Lambda_1} \left\{ \|a_\nu\|_{\mathcal{K}(H_\nu)} \right\}$.*

(xi) *There exists a* **-homomorphism of locally* C^**-algebras*

$$\omega\colon \bigoplus\nolimits_0 \{\mathcal{K}(H_\nu) \mid \nu \in \Lambda\} \to A$$

such that ω *is a continuous embedding with dense range, and* $\tau \circ \omega$ *is the natural embedding of the* C^**-algebra* $\bigoplus_0 \{\mathcal{K}(H_\nu) \mid \nu \in \Lambda\}$ *into* $\prod_{\nu\in\Lambda} \mathcal{K}(H_\nu)$.

Proof. (i) Note first that, if J is a minimal left ideal of A, then the closed two-sided ideal I generated by J is a minimal closed two-sided ideal of A.

Indeed (cf. [57, §25.3, II]), let K be a closed two-sided ideal of A contained in I. Then $J \cap K$ is a left ideal of A contained in the minimal left ideal J. Therefore either $J \subset K$ or $J \cap K = 0$. In the first case $I \subset K$, and so $K = I$. In the second case

$$K \cdot J \subset J \cap K = 0,$$

and consequently $J \subset \operatorname{ran}(K)$. Since $\operatorname{ran}(K)$ is a closed two-sided ideal of A, we have $I \subset \operatorname{ran}(K)$. Hence $K \subset \operatorname{ran}(K)$, i.e., $K^2 = 0$. Since A is semiprime, we get $K = 0$. Therefore I is a minimal closed two-sided ideal of A.

Thus A coincides not only with the topological sum of all its minimal left ideals but also with the topological sum of all the minimal closed two-sided ideals I_ν, $\nu \in \Lambda$. The latter topological sum is direct, and moreover, for each $\nu_1, \nu_2 \in \Lambda$ with $\nu_1 \neq \nu_2$, we have $I_{\nu_1} I_{\nu_2} = 0$ (cf. [57, the proof of Theorem 5 in §25.3]).

(ii) Let $\nu \in \Lambda$. Note first that the minimal closed two-sided ideal I_ν is a topologically simple locally C^*-algebra.

Indeed, I_ν is a *-ideal by [26, Theorem 11.7]. Thus I_ν is a closed *-subalgebra of A, i.e., a locally C^*-algebra. Since A is the direct topological sum of all the I_ν, it follows easily that I_ν is a topologically simple algebra.

We prove now that I_ν contains a minimal left ideal L of A. It is then obvious that I_ν is generated by L.

Indeed, suppose I_ν contains no minimal left ideals of A, and let J be a minimal left ideal. Then $J \cap I_\nu \subset J$ and $J \cap I_\nu \neq J$, since $J \not\subset I_\nu$. Consequently $J \cap I_\nu = 0$, and therefore $I_\nu \cdot J \subset J \cap I_\nu = 0$. Since J is arbitrary, $I_\nu \cdot \operatorname{Soc}(A) = 0$. Hence

$$(I_\nu)^2 \subset I_\nu \cdot A = I_\nu \cdot \overline{\operatorname{Soc}(A)} = 0.$$

Since A is semiprime, $I_\nu = 0$. So we have a contradiction.

Finally we note that, by [15, Lemma 30.2], L has the form $L = Ae$, where $e \in I_\nu \subset A$ is a minimal idempotent. Since eAe is an Arens-Michael division algebra, we have $eAe = \mathbb{C}e$, and hence $eI_\nu e = \mathbb{C}e$. It remains to apply Corollary 3.2.

(iii) Let I be a closed two-sided ideal of A, and let $a \in \bigcap\{P_\nu : P_\nu \supset I\}$. Let I_μ, $\mu \in \Lambda$, be a minimal closed two-sided ideal of A. Then either $I_\mu \subset I$ or $I_\mu \cap I = 0$. In the first case, $I_\mu a \subset I_\mu \subset I$. In the second case, $I \cdot I_\mu \subset I \cap I_\mu = 0$, and consequently $I \subset \operatorname{lan}(I_\mu) = P_\mu$. Hence $a \in P_\mu$ and $I_\mu a \subset I_\mu \cdot P_\mu = 0 \subset I$. Thus $I_\mu a \subset I$ for each $\mu \in \Lambda$. Since the sum of the ideals I_μ is dense in A, $\overline{Aa} \subset I$. Since A has an approximate identity, we see that $a \in I$.

(iv) Let I be a closed two-sided ideal of A, and let $\Lambda_1 = \{\nu \in \Lambda : I_\nu \subset I\}$ and $J = \overline{\bigoplus_{\nu\in\Lambda_1} I_\nu}$. As we noted in the proof of (iii), $I \subset P_\nu$ for each $\nu \in \Lambda \setminus \Lambda_1$, and hence $J \subset P_\nu$. For each $\nu \in \Lambda_1$ we have $J \not\subset P_\nu$, since otherwise $I_\nu \subset P_\nu$ and $I_\nu = I_\nu \cap P_\nu = 0$.

By (iii), $J = \bigcap_{\nu\in\Lambda\setminus\Lambda_1} P_\nu$. Since I is contained in this intersection, we conclude that $I = J$.

(v) This follows from Lemma 2.4 and from the fact that the Banach algebra of all compact operators on a Hilbert space is an annihilator algebra (see, e.g., [57, §25.6, Theorem 13]).

(vi) This follows from (v), from Lemma 2.3 and from inequalities (1) and (2) (see the proof of Lemma 2.3).

(vii) This follows easily by using (i)–(iv) and (vi).

(viii) This follows by the same argument as in (vi).

(ix) This follows at once from (viii) and (iv).

(x) Note that $\operatorname{Ker} p$ is a closed two-sided ideal of A. By (iv), there exists a subset $\Lambda_0 \subset \Lambda$ such that $\operatorname{Ker} p = \overline{\bigoplus_{\nu\in\Lambda_0} I_\nu}$. We set

$$\Lambda_1 = \Lambda \setminus \Lambda_0.$$

By (viii), we have $A = \operatorname{Ker} p \oplus J$, where

$$J = \operatorname{lan}(\operatorname{Ker} p) = \overline{\bigoplus_{\nu\in\Lambda_1} I_\nu},$$

and moreover the natural homomorphism

$$\varphi\colon J \hookrightarrow A \to A/\operatorname{Ker} p$$

is a topological *-isomorphism of locally C^*-algebras. The rest follows from the fact that, by Proposition 3.1, $A/\operatorname{Ker} p$ is topologically *-isomorphic to the concomitant C^*-algebra $(A_p, \|\cdot\|_p)$.

(xi) This follows easily by using (i), (ii), (vi), (vii) and (x). ∎

The next corollary follows easily from Theorem 3.3.

Corollary 3.4. *Let A be a locally C^*-algebra with dense socle.*

(i) *If, for each continuous C^*-seminorm p on A, the concomitant C^*-algebra A_p has finite spectrum $\widehat{A}_p$, then A is topologically *-isomorphic to the Cartesian product of a family of elementary C^*-algebras.*
(ii) *If, for each continuous C^*-seminorm p on A, the concomitant C^*-algebra A_p is finite-dimensional, then A is topologically *-isomorphic to the Cartesian product of a family of full matrix algebras.*
(iii) *If A has an identity, then A is topologically *-isomorphic to the Cartesian product of a family of full matrix algebras.*

3.2. Some definitions and results concerning topological algebras and modules. Suppose that A is a topological algebra[6] and X is a complete Hausdorff locally convex space which is also a left A-module. Thus there is a bilinear map $(a, x) \mapsto a\cdot x$ from $A \times X$ to X such that $(ab)\cdot x = a\cdot(b\cdot x)$ for $a, b \in A$, $x \in X$. Then X is called a *left topological A-module* if the above module map is jointly continuous. For two such

[6] Recall that by a topological algebra we mean a complete Hausdorff locally convex space equipped with a jointly continuous multiplication.

modules X and Y, an *A-module morphism* from X to Y is a continuous linear operator $\varphi\colon X \to Y$ which is a module homomorphism.

Similar definitions apply to *right topological A-modules* and *topological A-bimodules.* For example, the algebra A is itself a topological A-bimodule with respect to the maps given by the product in A. If X is a left topological A-module and Y is a right topological A-module, then $X \mathbin{\widehat{\otimes}} Y$ is a topological A-bimodule for the products defined by

$$a \cdot (x \otimes y) = a \cdot x \otimes y, \quad (x \otimes y) \cdot a = x \otimes y \cdot a \qquad (a \in A,\ x \in X,\ y \in Y).$$

In particular, $A \mathbin{\widehat{\otimes}} A$ is a topological A-bimodule in this way.

For a left A-module X, we denote by $A \cdot X$ the linear span of the elements of the form $a \cdot x$, where $a \in A$ and $x \in X$; expressions of the type $Y \cdot A$ have a similar meaning for right A-modules Y. We write A^2 for $A \cdot A$. A topological algebra A is *idempotent* if A^2 is dense in A.

A left topological A-module X is *essential* if $A \cdot X$ is dense in X. A left module X over an algebra A is (algebraically) *irreducible* if $A \cdot X \neq 0$ and X contains no non-zero proper submodules. It is obvious that, if X is an irreducible left topological A-module, then it is essential.

Let A_+ denote the unitization of A. Recall that there is the so-called *canonical morphism* $\pi_+\colon A_+ \mathbin{\widehat{\otimes}} X \to X$ (resp., $\pi_+\colon A_+ \mathbin{\widehat{\otimes}} X \mathbin{\widehat{\otimes}} A_+ \to X$) associated with any left topological A-module (resp., topological A-bimodule) X; this morphism is defined by $\pi_+(a \otimes x) = a \cdot x$ (resp., $\pi_+(a \otimes x \otimes b) = a \cdot x \cdot b$), where $a, b \in A_+$, $x \in X$.

Now let us recall some important definitions from the homology theory of topological algebras (see [38, Chapter IV], [39]).

Definition 3.5. A left topological A-module (respectively, topological A-bimodule) is *projective* if the canonical morphism π_+ has a right inverse in the corresponding category.

Definition 3.6. A topological algebra A is *left projective* if the left topological A-module A is projective, and *biprojective* if the topological A-bimodule A is projective.

Right projective topological algebras are defined similarly. Recall that every biprojective topological algebra is left and right projective (see [38, Proposition IV.1.3]).

Recall an important characterization of biprojectivity for topological algebras.

Proposition 3.7 (see [38, 42]). *A topological algebra A is biprojective if and only if the product map $\pi_A\colon A \mathbin{\widehat{\otimes}} A \to A$ has a right inverse in the category of topological A-bimodules.*

In particular, every biprojective topological algebra A is idempotent.

Definition 3.8. A topological algebra A is *contractible* if A is biprojective and has an identity.

The simplest example of a contractible topological algebra is the C^*-algebra $M_n(\mathbb{C})$ of all complex $n \times n$-matrices (the full matrix algebra). J. L. Taylor noticed in 1972 (see [93, p. 181]; for a proof see [84, Lemma 11]) that a topological Cartesian product of full matrix algebras is a contractible algebra.

Given a left topological A-module X, the *reduced module* associated with X is defined by $X_\Pi = A \mathbin{\widehat{\otimes}}_A X$ (see [38]). Here $\widehat{\otimes}_A$ denotes the (complete) projective tensor product of topological modules over A (see [38, Chapter II, §4.1]).

The following proposition is a generalization of the corresponding result on Banach modules over Banach algebras (see [38, Proposition II.3.13]). For Fréchet modules over Fréchet algebras, Proposition 3.9 was essentially also proved in [68].

PROPOSITION 3.9. *Let A be a topological algebra with a left bounded approximate identity, and let X be a left topological A-module. Then the map*

$$\kappa_X \colon X_\Pi = A \mathbin{\widehat{\otimes}}_A X \to X, \quad a \otimes_A x \mapsto a \cdot x,$$

is a topological isomorphism onto $\overline{A \cdot X}$.

Proof. Let $L \subset A \mathbin{\widehat{\otimes}}_A X$ denote the linear span of all elements of the form $a \otimes_A x$ ($a \in A$, $x \in X$), and let $\{e_\nu,\, \nu \in \Lambda\}$ be a left bounded approximate identity in A. If $u = a \otimes_A x$, then we have

$$u = \lim(e_\nu a \otimes_A x) = \lim(e_\nu \otimes_A a \cdot x) = \lim(e_\nu \otimes_A \kappa_X(u)).$$

Therefore

$$u = \lim(e_\nu \otimes_A \kappa_X(u)) \quad \text{for each } u \in L. \tag{3}$$

Given continuous seminorms p_α on A and q_β on X respectively, let $r_{\alpha,\beta}$ denote the corresponding projective tensor seminorm on $A \mathbin{\widehat{\otimes}}_A X$. Then (3) implies that

$$r_{\alpha,\beta}(u) \le \big(\sup p_\alpha(e_\nu)\big)\, q_\beta(\kappa_X(u)) \quad \text{for each } u \in L.$$

By continuity, the same estimate holds for every $u \in A \mathbin{\widehat{\otimes}}_A X$. Hence κ_X is topologically injective[7]. Since the image of κ_X clearly contains $A \cdot X$ and is contained in $\overline{A \cdot X}$, the result follows. ∎

For the next result see [38, Proposition IV.5.2].

PROPOSITION 3.10. *Let A be a biprojective topological algebra, and let X be a left topological A-module. Then the reduced module X_Π is projective.*

From this and from Proposition 3.9 we obtain the next corollary. Earlier, the corresponding result concerning Banach algebras and Banach modules was proved in [34, Theorem 2] (see also [38, Proposition IV.5.3]).

COROLLARY 3.11. *Let A be a biprojective topological algebra with a left bounded approximate identity. Then, for any left topological A-module X, the submodule $\overline{A \cdot X}$ is projective.*

In what follows, given a locally convex space E, we denote by $E^\sim$ the completion of E. For the reader's convenience, we recall a result which is a topological version of [38, Theorem II.3.17].

[7] In other words, it is a homeomorphism onto $\operatorname{Im} \kappa_X$.

Proposition 3.12 ([66, Proposition 3.1]). *Let A be a topological algebra, J a closed left ideal of A_+, and X a right topological A-module. Then there is a topological isomorphism*

$$\alpha\colon X \mathop{\widehat{\otimes}}_A (A_+/J)^\sim \longrightarrow (X/\overline{X\cdot J})^\sim \tag{4}$$

uniquely determined by $x \mathop{\otimes}_A (a+J) \mapsto x\cdot a + \overline{X\cdot J}$.

The following result is a generalization of [82, Lemma 1.1].

Proposition 3.13. *Let A be a biprojective topological algebra, and let J be a closed left ideal of A_+. Then the left topological A-module $(A/\overline{A\cdot J})^\sim$ is projective.*

Proof. By Proposition 3.12, the left topological A-module $(A/\overline{A\cdot J})^\sim$ is topologically isomorphic to $A \mathop{\widehat{\otimes}}_A (A_+/J)^\sim$. Now the result follows from Proposition 3.10. ∎

3.3. Homologically trivial and annihilator locally C^*-algebras. The following class of left topological modules will play a central role in studying homologically trivial locally C^*-algebras.

Definition 3.14 (cf. [71, 72]). Let A be a locally C^*-algebra. A *Hermitian A-module*[8] is an essential left topological A-module H whose underlying topological vector space is a Hilbert space, and, moreover, $\langle a\cdot x, y\rangle = \langle x, a^*\cdot y\rangle$ for all $a\in A$, $x,y\in H$.

Evidently, the above definition means exactly that the continuous representation of A associated with our module is a non-degenerate *-representation.

Let us recall the following result.

Theorem 3.15 ([81, Theorem 5]; see also [89, Theorem 4.40]). *Let A be a C^*-algebra. Then the following conditions are equivalent:*

- (i) *all irreducible left Banach A-modules (or, equivalently*[9]*, all irreducible Hermitian A-modules) are projective;*
- (ii) *A is an annihilator algebra;*
- (iii) *for every closed left ideal J of A, there is a closed left ideal L of A such that $A = J\oplus L$.*

Following [11], we say that an algebra A is a *modular annihilator algebra* if, for every modular maximal left ideal M and every modular maximal right ideal N of A,

- (i) $\operatorname{ran}(M)\neq 0$ and $\operatorname{ran}(A)=0$, and
- (ii) $\operatorname{lan}(N)\neq 0$ and $\operatorname{lan}(A)=0$.

An equivalent formulation when A is semiprime is as follows: A is a modular annihilator algebra if and only if $A/\operatorname{Soc}(A)$ is a radical algebra (see [103, Theorem 3.4]).

Recall from [13] that a significant number of important algebras are modular annihilator algebras. In particular, all semiprime Banach algebras with dense socle, all so-called

[8] Such modules are called non-degenerate star modules in [39] and non-degenerate Hilbert modules in [41]. We prefer to call them "Hermitian modules", as the phrase "Hilbert module" is used in the literature in various senses (see, e.g., [22, 49, 56, 61, 31]).

[9] As is known (see, e.g., [20, 2.9.6 and 2.12.18]), every irreducible left Banach module over a C^*-algebra A is topologically isomorphic to a Hermitian A-module.

compact Banach algebras (see [5]) and some algebras of linear operators are modular annihilator algebras. A well-known result of B. A. Barnes [12, Theorem 4.2] (see also [59, Theorem 8.6.4]) asserts that a semisimple Banach algebra A is modular annihilator if and only if the spectrum of each element $a \in A$ has no non-zero accumulation points.

Recall also [103, Example 4.3] that there exists a primitive (and even topologically simple) modular annihilator Banach algebra which is not an annihilator algebra. However every modular annihilator C^*-algebra is dual [103, Theorem 4.1].

For details and references on modular annihilator algebras, see [59, 60].

Now we recall that a Banach space E is said to have the *approximation property*[10] if the identity operator $\mathbf{1}_E$ can be uniformly approximated on every compact subset K by continuous finite-rank operators (i.e., for every $\varepsilon > 0$ there is a continuous finite-rank operator $T\colon E \to E$ (depending on K and ε) satisfying $\|T(x) - x\| \le \varepsilon$ for all $x \in K$).

Recall also the following result.

THEOREM 3.16 ([89, Theorem 4.39]). *Let A be a semiprime Banach algebra which satisfies at least one of the following conditions:*

(a) *A has the approximation property;*
(b) *all irreducible left Banach A-modules have the approximation property.*

Then all irreducible left Banach A-modules are projective if and only if A is a modular annihilator algebra.

The next result follows immediately from [30, I, Lemma 19] or [48, 43.2(12)].

LEMMA 3.17. *Let E be a complete Hausdorff locally convex space, and let H be a Hilbert space. Then for each $u \in E \mathbin{\widehat{\otimes}} H$, $u \ne 0$, there exist continuous linear functionals $f \in E^*$ and $g \in H^*$ such that $(f \mathbin{\widehat{\otimes}} g)(u) \ne 0$.*

The proof of the next proposition, which uses Lemma 3.17, is analogous to the proof of [82, Lemma 1.4].

PROPOSITION 3.18. *Let A be a topological algebra, and let H be a projective Hermitian A-module. Then for each $x \in H$, $x \ne 0$, there exists an A-module morphism $\chi\colon H \to A$ such that $\chi(x) \ne 0$.*

The next result is [70, Lemma 4.10.1].

LEMMA 3.19. *Let A be an arbitrary *-algebra in which $x^*x = 0$ implies $x = 0$. Then every minimal left ideal of A is of the form Ae, where e is a unique self-adjoint idempotent. A similar result holds for right ideals.*

The next proposition is related to [70, Theorem 4.10.3].

PROPOSITION 3.20. *Let A be a locally C^*-algebra, and let J be a minimal left ideal of A. Then an inner product $\langle x, y\rangle$ can be introduced into J with the following properties:*

[10] This property is discussed in [30, 52, 38, 59]. As is widely known, it was P. Enflo [23] who gave the first example of a Banach space without the approximation property.

(i) *if p is a continuous C^*-seminorm on A, then, for each $x \in J$, we have $p(x) = \lambda\|x\|_0$, where either $\lambda = 0$ or $\lambda = 1$, and $\|x\|_0 = \langle x, x\rangle^{1/2}$, and so J becomes a Hilbert space with the norm $\|x\|_0$;*
(ii) *$\langle ax, y\rangle = \langle x, a^*y\rangle$ for all $a \in A$, $x, y \in J$;*
(iii) *the left regular representation, $a \mapsto T_a$, of A on J is a continuous *-representation relative to $\langle x, y\rangle$;*
(iv) *the Hermitian A-module associated with the left regular representation of A on J is projective.*

Proof. By Lemma 3.19, there exists a self-adjoint idempotent e such that $J = Ae$. Also, by [15, Lemma 30.2], e is a minimal idempotent. Since eAe is an Arens-Michael division algebra, it follows from [39, Theorem V.1.7] that $eAe = \mathbb{C}e$, i.e., eAe consists of scalar multiples of e. Now, if x and y are any two elements of J, then $y^*x \in eAe$ and hence there exists a scalar $\langle x, y\rangle$ such that $y^*x = \langle x, y\rangle e$. We shall prove that $\langle x, y\rangle$ is the desired inner product.

It is evident that $\langle x, y\rangle$ is linear in the first variable. Also,

$$\langle y, x\rangle e = x^*y = (y^*x)^* = (\langle x, y\rangle e)^* = \overline{\langle x, y\rangle} e.$$

Hence $\langle y, x\rangle = \overline{\langle x, y\rangle}$. In particular, we have $\langle x, x\rangle = \overline{\langle x, x\rangle}$, i.e., $\langle x, x\rangle$ is a real number. Since

$$x^*x = \langle x, x\rangle e^*e, \tag{5}$$

it follows from [29, Corollary 2.5] that $\langle x, x\rangle \geq 0$. Moreover, $\langle x, x\rangle = 0$ implies $x^*x = 0$, i.e., $x = 0$. So $\langle x, y\rangle$ is indeed an inner product.

Now, if p is a continuous C^*-seminorm on A, then, in view of (5), we have

$$p(x)^2 = p(x^*x) = \langle x, x\rangle p(e^*e) = (\|x\|_0)^2 p(e)^2, \quad x \in J.$$

Thus, for each $x \in J$, $p(x) = \lambda\|x\|_0$, where $\lambda = p(e)$. Since

$$p(e)^2 = p(e^*e) = p(e),$$

it follows that either $\lambda = 0$ or $\lambda = 1$. So the topology on the closed subspace $J \subset A$ is generated by the inner product norm $\|x\|_0$. Since J is complete, J becomes a Hilbert space.

Next, for $a \in A$ and $x, y \in J$, we have

$$\langle ax, y\rangle e = y^*ax = (a^*y)^*x = \langle x, a^*y\rangle e,$$

and hence

$$\langle ax, y\rangle = \langle x, a^*y\rangle. \tag{6}$$

Moreover, in view of (6), we have

$$\langle T_a x, y\rangle = \langle x, T_{a^*} y\rangle,$$

and thus we obtain $T_{a^*} = T_a^*$, which shows that $a \mapsto T_a$ is a *-representation on J.

Finally, let H be the Hermitian A-module associated with the left regular representation of A on J. Consider the operator

$$\rho\colon H \to A_+ \mathbin{\widehat{\otimes}} H, \quad x \mapsto x \mathbin{\widehat{\otimes}} e,$$

where $x = ae \in H = J \subset A_+$. It is obvious that ρ is an A-module morphism such that $\pi_+ \circ \rho$ is the identity on H, where $\pi_+ \colon A_+ \widehat{\otimes} H \to H$ is the canonical morphism. It follows that H is projective. ∎

The next result characterizes projective irreducible Hermitian modules over a locally C^*-algebra.

COROLLARY 3.21. *Let H be an irreducible Hermitian module over a locally C^*-algebra. Then the following conditions are equivalent:*

(i) *H is projective;*
(ii) *H is topologically isomorphic to a minimal left ideal of A.*

Proof. (i) $\Rightarrow$ (ii) Since H is projective, it follows from Proposition 3.18 that there exists an A-module morphism $\chi\colon H \to A$ such that $\chi(x) \neq 0$ for some $x \neq 0$. Then $\operatorname{Ker}\chi \neq H$ and $\operatorname{Im}\chi \neq 0$. Since H is irreducible, it follows at once that $\operatorname{Ker}\chi = 0$ and $\operatorname{Im}\chi$ is an irreducible submodule of A. The latter implies that $\operatorname{Im}\chi$ is a minimal left ideal of A. Since, by Proposition 3.20, every minimal left ideal of A is a Hilbert space, the rest follows from the Open Mapping Theorem.

(ii) $\Rightarrow$ (i) This follows from Proposition 3.20. ∎

The following result is a part of [59, Proposition 8.4.4].

LEMMA 3.22. *Let A be a semiprime algebra and let e be a minimal idempotent of A. Then $A(1-e)$ is a modular maximal left ideal of A.*

DEFINITION 3.23. Let A be a topological algebra. We call A a *topologically modular annihilator algebra* if, for every closed modular maximal left ideal M and every closed modular maximal right ideal N of A,

(i) $\operatorname{ran}(M) \neq 0$ and $\operatorname{ran}(A) = 0$, and
(ii) $\operatorname{lan}(N) \neq 0$ and $\operatorname{lan}(A) = 0$.

We recall (cf. [39, Chapter VI, §1.2]) that a closed submodule X_0 of a left (right, or two-sided) topological module X over a topological algebra A is said to be *complemented as a topological module* if there exists another closed submodule X_1 of X such that the operator

$$\lambda\colon X_0 \times X_1 \to X, \quad (x, y) \mapsto x + y,$$

is an isomorphism of topological modules.

Now we are in a position to answer the first question posed at the beginning of the paper.

THEOREM 3.24. *Let A be a locally C^*-algebra. Then the following conditions are equivalent:*

(i) *all irreducible Hermitian A-modules are projective;*
(ii) $\operatorname{Soc}(A)$ *is dense in A;*
(iii) *A is the direct topological sum of its minimal closed two-sided ideals each of which is topologically *-isomorphic to an elementary C^*-algebra;*
(iv) *A is an annihilator algebra;*

(v) *for every closed left ideal J of A and for every closed right ideal K of A, we have $A = J \oplus (\mathrm{ran}(J))^*$ and $A = K \oplus (\mathrm{lan}(K))^*$, and moreover J and K are complemented as topological submodules of A;*

(vi) *A is a dual algebra;*

(vii) *A is a complemented algebra;*

(viii) *A is a left quasi-complemented algebra;*

(ix) *for every closed modular maximal left ideal M of A, there is a closed left ideal L of A such that $A = M \oplus L$;*

(x) *the closed modular maximal left ideals of A are precisely the left ideals of the form $A(1-e)$, where e is a minimal idempotent*[11] *of A;*

(xi) *all irreducible barrelled*[12] *left topological A-modules are projective;*

(xii) *A is a topologically modular annihilator algebra.*

Proof. (i) ⇒ (ii) Let $I = \overline{\mathrm{Soc}(A)} \neq A$. Then I is a proper closed two-sided ideal of A. By [26, Theorem 11.7], I is a *-ideal and the quotient algebra A/I equipped with the quotient topology is a C^*-convex algebra[13]. According to [26, Corollary 20.4], every C^*-convex algebra has enough continuous topologically irreducible *-representations to separate its points. Therefore there exists a non-zero continuous topologically irreducible *-representation $T\colon A/I \to \mathcal{B}(H)$ on some Hilbert space H. (Here, as usual, $\mathcal{B}(H)$ is the Banach algebra of all continuous linear operators on H.) Consider the quotient map $\sigma\colon A \to A/I$ and the topologically irreducible *-representation $\widetilde{T} = T \circ \sigma$ of A on H. Then, by [26, Theorem 19.2], $\widetilde{T}\colon A \to \mathcal{B}(H)$ is algebraically irreducible. The associated Hermitian A-module H is irreducible and hence projective. By Proposition 3.18, there exists an A-module morphism $\chi\colon H \to A$ such that $\chi(x) \neq 0$ for some $x \neq 0$. Then $\mathrm{Ker}\,\chi \neq H$ and $\mathrm{Im}\,\chi \neq 0$. Let $J = \mathrm{Im}\,\chi$. Since H is irreducible, it follows that $\mathrm{Ker}\,\chi = 0$ and J is a minimal left ideal of A. Hence $J \subset \mathrm{Soc}(A) \subset I$. Since, obviously, $\widetilde{T}(I) = 0$, we have $I \cdot H = 0$ and

$$J^2 \subset I \cdot J \subset \chi(I \cdot H) = 0.$$

Since the algebra A is semiprime, we get $J = 0$. So we have a contradiction.

(ii) ⇒ (iii) This follows from Theorem 3.3.

(iii) ⇒ (iv) This is essentially Theorem 3.3(v).

(iv) ⇒ (v) This follows from Lemma 2.3 and from inequalities (1) and (2) (see the proof of Lemma 2.3).

(v) ⇒ (vi) Let J be a closed left ideal of A. By condition (v), we have

$$A = J \oplus (\mathrm{ran}(J))^*. \tag{7}$$

Since $\mathrm{ran}(J)$ is a closed right ideal of A, it follows from condition (v) that

$$A = \mathrm{ran}(J) \oplus (\mathrm{lan}(\mathrm{ran}(J)))^*.$$

[11] Obviously, such left ideals can also be characterized as those of the form $\mathrm{lan}(R)$ for some minimal right ideal R of A.

[12] Recall [17] that a locally convex space E is *barrelled* if every closed absorbing absolutely convex subset of E is a neighbourhood of zero. In particular, all Fréchet spaces are barrelled.

[13] In other words, a (not necessarily complete) locally m-convex *-algebra whose topology is defined by a family of C^*-seminorms.

Applying the involution to this equality, we obtain $A = (\mathrm{ran}(J))^* \oplus \mathrm{lan}(\mathrm{ran}(J))$, i.e.,

$$A = \mathrm{lan}(\mathrm{ran}(J)) \oplus (\mathrm{ran}(J))^*. \tag{8}$$

Since $J \subset \mathrm{lan}(\mathrm{ran}(J))$, it follows from (7) and (8) that $\mathrm{lan}(\mathrm{ran}(J)) = J$. The proof of the equality $\mathrm{ran}(\mathrm{lan}(K)) = K$, for every closed right ideal K of A, is similar.

(vi) $\Rightarrow$ (vii) For every closed left ideal J of A, we set $J^q = (\mathrm{ran}(J))^*$. Then it is clear that J^q is a closed left ideal of A and $J^q = \mathrm{lan}(J^*)$. Moreover, we have

$$(J^q)^q = \mathrm{lan}((J^q)^*) = \mathrm{lan}((\mathrm{ran}(J))^{**}) = \mathrm{lan}(\mathrm{ran}(J)) = J,$$

because A is dual by condition (vi).

Now note that A, being a dual algebra, is an annihilator algebra. From this and from Lemma 2.3 we get that $A = J \oplus J^q$ for every closed left ideal J.

Finally, it is evident that, if J_1 and J_2 are closed left ideals and if $J_1 \subset J_2$, then $J_2^q \subset J_1^q$. Thus A is a left complemented algebra with left complementor $q\colon J \mapsto J^q$.

Similarly, it can be proved that A is a right complemented algebra with right complementor $p\colon K \mapsto K^p$, where $K^p = (\mathrm{lan}(K))^*$ for every closed right ideal K.

(vii) $\Rightarrow$ (viii) This is trivial.

(viii) $\Rightarrow$ (ix) Let $q\colon J \mapsto J^q$ be a left quasi-complementor on A. Then, since

$$A^q = A^q \cap A = 0,$$

we have $0^q = (A^q)^q = A$. Now let M be an arbitrary closed modular maximal left ideal of A. Then $M^q \neq 0$, since otherwise

$$M = (M^q)^q = 0^q = A.$$

Since $M + M^q$ is a left ideal which contains M properly, it follows that $M + M^q = A$. Putting $L = M^q$, we get $A = M \oplus L$.

(ix) $\Rightarrow$ (x) By Lemma 3.22, we just need to see that any closed modular maximal left ideal M of A is of the form $A(1-e)$ for a minimal idempotent $e \in A$. By condition (ix), for every such left ideal M there is a closed left ideal L of A such that $A = M \oplus L$. Since the left ideal M is maximal, it follows that L is minimal.

Let u be a right identity for A modulo M. Since $A = M \oplus L$, we can write $u = m + e$, where $m \in M$ and $e \in L$. It is clear that $A(1-e) \subset M$, i.e., e is also a right identity for A modulo M. Let now $b \in L$. Since

$$b - be \in M \cap L = 0,$$

we have $b = be$. It follows that $L = Ae$ and $e^2 = e$. Since L is a minimal left ideal of A, the idempotent e is minimal (see [15, Lemma 30.2]).

Suppose now that M properly includes $A(1-e) = \{a \in A \mid ae = 0\}$. Then there is some $a \in M$ with $ae \neq 0$. Since $a - ae \in M$, we have

$$ae \in M \cap Ae = M \cap L = 0.$$

This contradiction shows that $M = A(1-e)$.

(x) $\Rightarrow$ (xi) Let X be an irreducible barrelled left topological A-module. Taking an arbitrary element $x \in X \setminus \{0\}$ and using [39, Theorem VI.2.8] we get that the set

$$M = \{a \in A \mid a \cdot x = 0\}$$

is a closed modular maximal left ideal of A. By condition (x), we have $M = A(1-e)$, where $e \in A$ is a minimal idempotent. Since, for $a \in A$, we have $a - ae \in M$ and hence

$$a \cdot (x - e \cdot x) = (a - ae) \cdot x = 0,$$

we get that $e \cdot x = x$ (see [39, Proposition VI.2.6(II)]).

Let $L = Ae$. By [15, Proposition 30.6], L is a minimal left ideal of A. It follows from Proposition 3.20 that L is a Hilbert space. Furthermore, it is a projective irreducible Hermitian A-module. Consider the operator

$$\varphi\colon L \to X, \quad a \mapsto a \cdot x.$$

Clearly, φ is an A-module morphism such that

$$\operatorname{Ker} \varphi = M \cap L = 0$$

and $\varphi(e) = x \neq 0$. Since X is irreducible, it follows that $\operatorname{Im} \varphi = X$. Since X is barrelled, the Open Mapping Theorem [73, Chapter VI, Theorem 7 and Corollary 2] implies that φ is a topological isomorphism. Thus X is topologically isomorphic to a projective left topological A-module, and so it is projective.

(xi) $\Rightarrow$ (i) This is trivial, since all Hilbert spaces are barrelled.

(x) $\Rightarrow$ (xii) Let M be a closed modular maximal left ideal of A. Since, by condition (x), we have $M = A(1-e)$, where e is a minimal idempotent of A, it follows that $e \in \operatorname{ran}(M)$. Hence $\operatorname{ran}(M) \neq 0$. Using the involution it is easily seen that every closed modular maximal right ideal N of A has a non-zero left annihilator. Thus A is a topologically modular annihilator algebra.

(xii) $\Rightarrow$ (ix) Let M be a closed modular maximal left ideal of A. By condition (xii), we have $\operatorname{ran}(M) \neq 0$. Define

$$L = (\operatorname{ran}(M))^*.$$

Then L is a non-zero closed left ideal of A. By Lemma 2.3(i), we have $M \cap L = 0$. Since $M \oplus L$ is a left ideal which contains M properly, it follows that $M \oplus L = A$. ∎

Remark 3.25. The implications (iv) $\Rightarrow$ (vi) $\Rightarrow$ (vii) in Theorem 3.24 were proved earlier in [33]. We included a proof for completeness.

Remark 3.26. Within the setting of C^*-algebras, many other conditions equivalent to those listed in Theorem 3.24 were obtained in [27, 28].

From Theorems 3.3 and 3.24 we immediately get the following result.

Proposition 3.27. *Let A be an annihilator locally C^*-algebra, and let I be a closed two-sided ideal of A. Then I and A/I are annihilator locally C^*-algebras, and moreover A is topologically *-isomorphic to the Cartesian product of the algebras I and A/I.*

Propositions 3.1 and 3.27 imply the following corollary.

Corollary 3.28. *Let A be an annihilator locally C^*-algebra, and let P be the family of all continuous C^*-seminorms on A. Then, for each $p \in P$, the concomitant C^*-algebra A_p is annihilator and, as a consequence, A can be represented as an inverse limit of annihilator C^*-algebras.*

Lemma 2.4 and Proposition 3.27 imply the following.

PROPOSITION 3.29. *The Cartesian product $A = \prod_{\nu\in\Lambda} A_\nu$ of a family $\{A_\nu \mid \nu \in \Lambda\}$ of locally C^*-algebras, with the product topology, is an annihilator algebra if and only if all the algebras A_ν are annihilator algebras.*

3.4. Unital, metrizable, and non-unital annihilator locally C^*-algebras. Theorem 3.24 can be strengthened in certain cases. First note that Corollary 3.4, Theorem 3.24 and Proposition 3.29 imply the following result.

THEOREM 3.30. *Let A be a unital locally C^*-algebra. Then A satisfies the equivalent conditions of Theorem 3.24 if and only if A is topologically *-isomorphic to the Cartesian product of a family of full matrix algebras.*

Next we recall that a σ-C^*-algebra is a metrizable locally C^*-algebra (or, equivalently, a locally C^*-algebra whose topology is determined by a countable family of C^*-seminorms). Every σ-C^*-algebra is topologically *-isomorphic to the inverse limit of a sequence

$$A_1 \xleftarrow{\sigma_1} A_2 \xleftarrow{\sigma_2} A_3 \xleftarrow{\sigma_3} \cdots \tag{9}$$

of C^*-algebras and surjective *-homomorphisms $\sigma_n\colon A_{n+1} \to A_n$ (see [61, Section 5]). Moreover, all the natural homomorphisms $A \to A_n$ ($n \in \mathbb{N}$) are also surjective.

THEOREM 3.31. *Let A be an annihilator σ-C^*-algebra. Then A is topologically *-isomorphic to the Cartesian product of a countable family of annihilator C^*-algebras.*

Proof. Choose an inverse system (9) of C^*-algebras such that

$$A \cong \varprojlim A_n.$$

As was noted above, we may assume that all the maps $A_{n+1} \to A_n$ and $A \to A_n$ are onto. By the Open Mapping Theorem, A_n is topologically *-isomorphic to A/I_n, where

$$I_n = \operatorname{Ker}(A \to A_n).$$

It follows from Proposition 3.27 that A_n is an annihilator C^*-algebra for each $n \in \mathbb{N}$. So, by [20, 4.7.20], each A_n is isometrically *-isomorphic to a c_0-sum of elementary C^*-algebras. In particular, the spectrum $\widehat{A}_n$ of A_n is discrete. Now [67, Proposition 5.2] implies that there exists a family $\{B_n \mid n \in \mathbb{N}\}$ of C^*-algebras such that A is topologically *-isomorphic to $\prod_n B_n$. Using Proposition 3.29, we see that each B_n is an annihilator algebra. This completes the proof. ∎

By combining this result with Theorem 3.24 and Proposition 3.29 we get the following.

COROLLARY 3.32. *Let A be a σ-C^*-algebra. Then A satisfies the equivalent conditions of Theorem 3.24 if and only if A is topologically *-isomorphic to the countable Cartesian product $\prod_n A_n$, where each A_n is a C^*-algebra isomorphic to a c_0-sum of elementary C^*-algebras.*

REMARK 3.33. Theorem 3.31 and Corollary 3.32 cannot be extended to arbitrary (i.e., not necessarily metrizable) locally C^*-algebras. Namely, there exists a non-unital annihilator locally C^*-algebra that is not topologically isomorphic to a Cartesian product of annihilator C^*-algebras (see Theorem 4.29 below).

At the same time we have the following result.

PROPOSITION 3.34. *Let A be a non-unital annihilator locally C^*-algebra. Then A is topologically *-isomorphic to the Cartesian product of two annihilator locally C^*-algebras one of which is an infinite-dimensional C^*-algebra.*

Proof. By Theorem 3.24, $\mathrm{Soc}(A)$ is dense in A. Since A is non-unital, it follows from Corollary 3.4(ii) that there exists a continuous C^*-seminorm p on A such that the concomitant C^*-algebra A_p is infinite-dimensional. We set $I = \mathrm{Ker}\, p$. Then I is a closed two-sided ideal of A and, by Proposition 3.1, A/I is topologically *-isomorphic to the C^*-algebra A_p. It remains to apply Proposition 3.27. ■

4. Biprojective locally C^*-algebras

4.1. Examples and general properties of biprojective algebras. We recall that biprojective Banach and topological algebras were introduced by A. Ya. Helemskii [34, 36] and have been studied by many authors (see, e.g., [34, 35, 46, 78, 80, 81, 37, 82, 53, 69, 83, 87, 88, 89, 91, 8, 63, 102, 64, 65, 67, 9, 76]).

The original motivation to study such algebras was the vanishing of their cohomology groups[14], $\mathcal{H}^n(A, X)$, with coefficients in arbitrary topological A-bimodules X for all $n \geq 3$ (see, e.g., [42, Theorem 2.4.21]). The structure of biprojective semisimple Banach algebras with the approximation property is described in [82] (see also [38] and [75]). In [88], the cohomology groups of biprojective Banach algebras are completely computed for arbitrary coefficients. The description of these groups is given in terms of double multipliers and quasi-multipliers of a given bimodule of coefficients. As an application, biprojective Banach algebras are characterized in terms of their cohomology groups. In particular, it is shown (see [88, Theorem 5.9]) that a Banach algebra is biprojective if and only if its one-dimensional cohomology groups with coefficients in bimodules of double multipliers are trivial.

We recall some examples of biprojective algebras.

EXAMPLE 4.1. As was noted after Definition 3.8, the Cartesian product $\prod_{\nu\in\Lambda} M_{n_\nu}(\mathbb{C})$ of any family of full matrix algebras is a contractible topological algebra, i.e, a biprojective algebra with an identity. Moreover, we recall that the Cartesian product of any family of contractible topological algebras is a contractible algebra (see [84, Lemma 11]).

Here it is important to note that a topological algebra A is contractible if and only if its cohomology groups with coefficients in arbitrary topological A-bimodules vanish for all $n \geq 1$ (see [38, Theorem IV.5.8]). For some results concerning contractible algebras, see [93, 94, 51, 79, 81, 38, 84, 74, 25, 91, 64, 67].

EXAMPLE 4.2. The simplest example of a non-contractible biprojective Banach algebra is perhaps the Banach sequence algebra ℓ_1 with coordinatewise multiplication. The Banach sequence algebra c_0 is also biprojective (see [38, Examples IV.5.9–IV.5.10] or [39, Example VII.1.80]).

EXAMPLE 4.3. If G is a compact group, then the group algebras $L^1(G)$ and $C^*(G)$ are biprojective Banach algebras ([34], see also [38]).

[14] The definition of cohomology groups of topological algebras can be found, e.g., in [38].

EXAMPLE 4.4. The c_0-sum, $\bigoplus_0\{M_{n_\nu}(\mathbb{C}) \mid \nu \in \Lambda\}$, of a family of full matrix C^*-algebras is a biprojective C^*-algebra [34, Theorem 3].

Actually, there are no other biprojective C^*-algebras:

THEOREM 4.5 ([80, 81]; see also [89, Theorem 4.62]). *Every biprojective C^*-algebra is isometrically *-isomorphic to the c_0-sum of a family of full matrix algebras. In particular, every commutative biprojective C^*-algebra is isometrically *-isomorphic to a C^*-algebra of the form*[15] *$c_0(\Lambda)$ for some set Λ.*

EXAMPLE 4.6. The Cartesian product of a countable family of biprojective Fréchet algebras is a biprojective Fréchet algebra [67, Proposition 1.15]. As a consequence, the Cartesian product of a countable family of biprojective C^*-algebras is a biprojective σ-C^*-algebra.

Actually, there are no other biprojective σ-C^*-algebras:

THEOREM 4.7 ([67, Theorem 5.3]). *Every biprojective σ-C^*-algebra is topologically *-isomorphic to the countable Cartesian product $\prod_n A_n$, where each A_n is a C^*-algebra isomorphic to a c_0-sum of full matrix algebras.*

EXAMPLE 4.8. Let $\alpha = (\alpha_n)$, $n \in \mathbb{N}$, be a non-decreasing sequence of positive numbers with $\lim_{n\to\infty} \alpha_n = +\infty$. Then the power series spaces $\Lambda_1(\alpha)$ and $\Lambda_\infty(\alpha)$ are biprojective Fréchet algebras under pointwise product (see [63] or [64]). In particular, the algebra s of rapidly decreasing sequences is biprojective.

EXAMPLE 4.9. If G is a compact Lie group, then the Fréchet algebra $\mathcal{E}(G)$ of smooth functions on G with convolution product is biprojective ([86, Example 6]; see also [63]). Note that $\mathcal{E}(G)$ is not a Banach algebra unless G is finite.

EXAMPLE 4.10. If G is a compact Lie group, then the algebra $\mathcal{E}'(G)$ of distributions on G with convolution multiplication is a biprojective topological algebra with an identity ([94, Proposition 7.3], see also [38, Assertion IV.5.30]). In other words, $\mathcal{E}'(G)$ is a contractible algebra. Note that the biprojective Fréchet algebra $\mathcal{E}(G)$ from Example 4.9 is a dense ideal in $\mathcal{E}'(G)$.

EXAMPLE 4.11. Let (E, F) be a pair of complete Hausdorff locally convex spaces, and let $\langle\,\cdot\,,\,\cdot\,\rangle\colon E \times F \to \mathbb{C}$ be a jointly continuous bilinear form that is not identically zero. The space $A = E \mathbin{\widehat{\otimes}} F$ is then a topological algebra with respect to the multiplication defined by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = \langle x_2, y_1\rangle\, x_1 \otimes y_2 \quad (x_i \in E,\ y_i \in F).$$

This algebra is biprojective ([86, Example 4], see also [82], [63] and [64]).

In particular, if E is a Banach space with the approximation property, then the algebra $A = E \mathbin{\widehat{\otimes}} E^*$ is isomorphic to the Banach algebra $\mathcal{N}(E)$ of all nuclear operators on E, and so $\mathcal{N}(E)$ is biprojective.

Actually, there are no other biprojective Banach algebras of the form $\mathcal{N}(E)$:

[15] As usual (see, e.g., [52]), we write $c_0(\Lambda)$ for $\bigoplus_0\{A_\nu \mid \nu \in \Lambda\}$, where $A_\nu = \mathbb{C}$ for each $\nu \in \Lambda$.

THEOREM 4.12 (see [83, Corollaries 1, 3]). *Let E be a Banach space, and let $A = \mathcal{N}(E)$. Then the following conditions are equivalent:*

(i) *A is a biprojective algebra;*
(ii) *$\mathcal{H}^n(A, X) = 0$ for all Banach A-bimodules X and for all $n \geq 3$;*
(iii) *there exists $k \geq 3$ such that $\mathcal{H}^n(A, X) = 0$ for all Banach A-bimodules X and for all $n \geq k$;*
(iv) *E has the approximation property.*

Let us briefly explain why the algebra $\mathcal{N}(E)$ cannot be biprojective unless E has the approximation property. It is known that, for a biprojective Banach algebra A, the operator

$$\kappa_A \colon A \mathop{\widehat{\otimes}}_A A \to A, \quad a \mathop{\otimes}_A b \mapsto ab,$$

is a topological isomorphism (see [88, Corollary 4.2]). Let now $A = \mathcal{N}(E)$, where E is a Banach space. By [83, Lemma 3], the A-bimodule $A \mathop{\widehat{\otimes}}_A A$ is isomorphic to $E \mathbin{\widehat{\otimes}} E^*$, and the operator κ_A can be identified with the so-called *trace homomorphism*

$$\mathrm{Tr} \colon E \mathbin{\widehat{\otimes}} E^* \to \mathcal{N}(E), \quad \mathrm{Tr}(x \otimes f)(y) = \langle y, f \rangle \, x \qquad (x, y \in E, \ f \in E^*),$$

of the dual pair (E, E^*) (see [38]). However, Tr is known to have a non-trivial kernel unless E has the approximation property (see [30, I, Proposition 35]).

Let us also note that, for any Banach space E, the Banach algebra $\mathcal{N}(E)$ has the weaker property of being "quasi-biprojective". For details, see [92].

The following result of O. Yu. Aristov describes the structure of finite-dimensional biprojective algebras.

THEOREM 4.13 (see [9, Theorem 7.1]). *Let A be a finite-dimensional biprojective algebra. Then A is isomorphic to the Cartesian product of finitely many algebras of the type $E \mathbin{\widehat{\otimes}} F$, where $(E, F, \langle \cdot \, , \cdot \rangle)$ is a pair of finite-dimensional spaces together with a non-zero bilinear form.*

We pass from examples to some general facts. Recall that, if A and B are topological algebras, then $A \mathbin{\widehat{\otimes}} B$ is also a topological algebra with multiplication defined by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \qquad (a_1, a_2 \in A, \ b_1, b_2 \in B).$$

By A^{op} we denote the topological algebra *opposite* to A (with the same underlying space as A, but with multiplication $a \circ b$ equal to the "previous" ba). The next proposition follows immediately from Proposition 3.7.

PROPOSITION 4.14. *Let A and B be topological algebras.*

(i) *If A and B are biprojective, then so is $A \mathbin{\widehat{\otimes}} B$.*
(ii) *If A is biprojective, then so is A^{op}.*

The following is a generalization of the corresponding result on Banach algebras (see [82, Lemma 1.3]).

PROPOSITION 4.15. *Let A be a biprojective topological algebra, and let I be a closed two-sided ideal of A. Then $(A/\overline{A \cdot I})^{\sim}$ is a biprojective topological algebra.*

Proof. Set $B = (A/\overline{A \cdot I})^\sim$, and consider the commutative diagram

$$\begin{tikzcd} A \mathbin{\widehat{\otimes}} A \arrow[r, "\pi_A"] \arrow[d, "\sigma \mathbin{\widehat{\otimes}} \sigma"'] & A \arrow[d, "\sigma"] \\ B \mathbin{\widehat{\otimes}} B \arrow[r, "\pi_B"] & B \end{tikzcd}$$

in the category of topological A-bimodules. Here π_A and π_B are the product maps, and σ is the natural operator from A to $(A/\overline{A \cdot I})^\sim$. Using Proposition 3.7, choose an A-bimodule morphism $\rho_A \colon A \to A \mathbin{\widehat{\otimes}} A$ such that $\pi_A \circ \rho_A = \mathbf{1}_A$. We have

$$\rho_A(A \cdot I) \subset (A \mathbin{\widehat{\otimes}} A) \cdot I \subset \operatorname{Ker}(\sigma \mathbin{\widehat{\otimes}} \sigma),$$

and so there exists a unique A-bimodule morphism ρ_B making the diagram

$$\begin{tikzcd} A \mathbin{\widehat{\otimes}} A \arrow[d, "\sigma \mathbin{\widehat{\otimes}} \sigma"'] & A \arrow[l, "\rho_A"'] \arrow[d, "\sigma"] \\ B \mathbin{\widehat{\otimes}} B & B \arrow[l, "\rho_B"'] \end{tikzcd}$$

commutative. Therefore,

$$\pi_B \circ \rho_B \circ \sigma = \pi_B \circ (\sigma \mathbin{\widehat{\otimes}} \sigma) \circ \rho_A = \sigma \circ \pi_A \circ \rho_A = \sigma.$$

Since σ has dense range, this implies that $\pi_B \circ \rho_B = \mathbf{1}_B$. For the same reason, ρ_B is a B-bimodule morphism. The rest follows from Proposition 3.7. ∎

The following result generalizes [67, Corollary 1.17].

COROLLARY 4.16. *Let A be a biprojective topological algebra, and let I be a closed two-sided ideal of A such that I is complemented as a topological submodule of the A-bimodule A. Then I and A/I are biprojective topological algebras.*

Proof. It follows from the hypothesis that there exists another closed two-sided ideal J of A such that $A = I \oplus J$, and moreover the operator

$$\lambda \colon I \times J \to A, \quad (a, b) \mapsto a + b,$$

is a topological isomorphism.

Since A is biprojective, we have $\overline{A^2} = A$. This implies that $\overline{A \cdot I} = I$ and $\overline{A \cdot J} = J$. Hence the topological algebras

$$I \cong A/J = A/\overline{A \cdot J} \quad \text{and} \quad A/I = A/\overline{A \cdot I}$$

are biprojective by Proposition 4.15. ∎

Now suppose that A and B are topological algebras, $\varphi \colon A \to B$ is a continuous homomorphism, and X is a left topological B-module. Then, obviously, the action of B on X induces a continuous action of A on X defined by $a \cdot x = \varphi(a) \cdot x$, where $a \in A$, $x \in X$. In particular, B is a left topological A-module with the action $a \cdot b = \varphi(a)b$, where $a \in A$, $b \in B$; in this case, the operator $\varphi \colon A \to B$ is an A-module morphism.

PROPOSITION 4.17. *Let A and B be topological algebras, and let $\varphi\colon A \to B$ be a continuous homomorphism with dense range. Suppose further that A is biprojective and has either a left or a right bounded approximate identity. Then B is biprojective.*

Proof. Consider the case where A has a left bounded approximate identity. Since A is biprojective, there exists an A-bimodule morphism $\rho\colon A \to A_+ \mathbin{\widehat{\otimes}} A$ such that $\pi_+ \circ \rho = \mathbf{1}_A$, where $\pi_+\colon A_+ \mathbin{\widehat{\otimes}} A \to A$ is the canonical morphism (cf. [39, Proposition VII.1.66]). Set $\rho_1 = \mathbf{1}_B \mathbin{\widehat{\otimes}}_A \rho \mathbin{\widehat{\otimes}}_A \mathbf{1}_B$. Then

$$\rho_1\colon B \mathbin{\widehat{\otimes}}_A A \mathbin{\widehat{\otimes}}_A B \to B \mathbin{\widehat{\otimes}}_A (A_+ \mathbin{\widehat{\otimes}} A) \mathbin{\widehat{\otimes}}_A B$$

is a B-bimodule morphism which is a right inverse for

$$\pi_1\colon B \mathbin{\widehat{\otimes}} (A \mathbin{\widehat{\otimes}}_A B) \to B \mathbin{\widehat{\otimes}}_A (A \mathbin{\widehat{\otimes}}_A B), \quad b \otimes u \mapsto b \otimes_A u \qquad (b \in B,\ u \in A \mathbin{\widehat{\otimes}}_A B).$$

Since A has a left bounded approximate identity and $\varphi\colon A \to B$ has dense range, it follows from Proposition 3.9 that there exists a topological isomorphism

$$\kappa_B\colon A \mathbin{\widehat{\otimes}}_A B \to \overline{A \cdot B} = B$$

uniquely determined by

$$\kappa_B(a \otimes_A b) = a \cdot b = \varphi(a)b \qquad (a \in A,\ b \in B).$$

It is obvious that $\kappa_B\colon A \mathbin{\widehat{\otimes}}_A B \to B$ is a right B-module morphism. Since $\varphi\colon A \to B$ has dense range and, clearly (cf. [54, Chapter XIV, Lemma 1.2]), B has a left bounded approximate identity, we see that the B-bimodule morphism π_1 may be identified with the morphism

$$\widetilde{\pi}_1\colon B \mathbin{\widehat{\otimes}} B \to B \mathbin{\widehat{\otimes}}_B B \cong B, \quad b \otimes c \mapsto bc \qquad (b, c \in B).$$

But π_1 (and hence also $\widetilde{\pi}_1$) has a right inverse B-bimodule morphism. So it follows from Proposition 3.7 that B is biprojective. A similar argument applies in the case where A has a right bounded approximate identity. ∎

The proof of the following proposition repeats the proof of [67, Proposition 1.15].

PROPOSITION 4.18. *The Cartesian product of an arbitrary family of biprojective topological algebras is a biprojective topological algebra.*

The following result generalizes Proposition 4.18.

PROPOSITION 4.19. *Let $A = \varprojlim (A_\lambda,\ \tau_\lambda^\mu\colon A_\mu \to A_\lambda)$ be a reduced inverse limit of topological algebras. Suppose that for each λ there exists an A_λ-bimodule morphism*

$$\rho_\lambda\colon A_\lambda \to A_\lambda \mathbin{\widehat{\otimes}} A_\lambda$$

such that $\pi_{A_\lambda} \circ \rho_\lambda = \mathbf{1}_{A_\lambda}$ and that

$$(\tau_\lambda^\mu \mathbin{\widehat{\otimes}} \tau_\lambda^\mu) \circ \rho_\mu = \rho_\lambda \circ \tau_\lambda^\mu$$

whenever $\lambda \prec \mu$. Then A is biprojective.

Proof. Since projective tensor product commutes with reduced inverse limits (see [48, §41.6]), we may identify $A \mathbin{\widehat{\otimes}} A$ with $\varprojlim (A_\lambda \mathbin{\widehat{\otimes}} A_\lambda,\ \tau_\lambda^\mu \mathbin{\widehat{\otimes}} \tau_\lambda^\mu)$. It follows from the assumption that there exists a well-defined continuous linear operator

$$\rho = \varprojlim \rho_\lambda \colon A \to A \mathbin{\widehat{\otimes}} A.$$

It is readily verified that ρ is an A-bimodule morphism and that $\pi_A \circ \rho = \mathbf{1}_A$. Therefore A is biprojective, as required. ∎

EXAMPLE 4.20. *For every function $s \colon \mathbb{N} \to \mathbb{N}$, consider the set*

$$M_s = \{(i,j) \in \mathbb{N} \times \mathbb{N} \,:\, 1 \le j \le s(i)\},$$

and define

$$A = \Big\{ a = (a_{ij}) \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}} \,:\, \lim_{\substack{(i,j)\to\infty \\ (i,j)\in M_s}} |a_{ij}| = 0, \quad \forall s \in \mathbb{N}^{\mathbb{N}} \Big\}.$$

Clearly, A is a *-subalgebra of $\mathbb{C}^{\mathbb{N}\times\mathbb{N}}$. Moreover, A is a locally C^*-algebra with respect to the family of C^*-seminorms

$$p_s(a) = \sup_{(i,j)\in M_s} |a_{ij}| \qquad (s \in \mathbb{N}^{\mathbb{N}})$$

(cf. [61, Example 5.10]). *This algebra is biprojective.*

Proof. It is easy to see that, for each $s \in \mathbb{N}^{\mathbb{N}}$, the concomitant C^*-algebra $A_s = A/\operatorname{Ker} p_s$ is isomorphic to $c_0(M_s)$, and that the linking maps

$$A_t \to A_s \quad (s \le t)$$

are just the restriction maps from $c_0(M_t)$ to $c_0(M_s)$.

For each $(i,j) \in \mathbb{N} \times \mathbb{N}$, let $e_{ij} \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ denote the function which is 1 at (i,j), and 0 elsewhere. Fix $s \in \mathbb{N}^{\mathbb{N}}$. By [39, Example VII.1.80], there exists an A_s-bimodule morphism $\rho_s \colon A_s \to A_s \mathbin{\widehat{\otimes}} A_s$ such that

$$\rho_s(e_{ij}) = e_{ij} \otimes e_{ij}$$

for each $(i,j) \in M_s$. It is readily verified that the family $\{\rho_s \mid s \in \mathbb{N}^{\mathbb{N}}\}$ satisfies the conditions of Proposition 4.19. This implies that A is biprojective. ∎

4.2. Structure theorems for biprojective locally C^*-algebras. Our next aim is to answer the second and the third questions posed at the beginning of the paper.

THEOREM 4.21. *Let A be a locally C^*-algebra. Then the following conditions are equivalent:*

(i) *A is a biprojective algebra;*
(ii) *all essential left topological A-modules are projective;*
(iii) *all Hermitian A-modules are projective;*
(iv) $\operatorname{Soc}(A)$ *is dense in A, and moreover, for each continuous C^*-seminorm p on A, the concomitant C^*-algebra A_p is biprojective;*
(v) *A is the direct topological sum of its minimal closed two-sided ideals each of which is topologically *-isomorphic to a full matrix C^*-algebra;*
(vi) *A is an annihilator algebra with finite-dimensional minimal closed two-sided ideals;*
(vii) *there exists a dense *-subalgebra B of A which is a biprojective C^*-algebra under some norm and which is continuously embedded in A.*

For the proof of this theorem the following two results will be important.

Theorem 4.22 ([40, 41]). *The following properties of a C^*-algebra A are equivalent:*

(i) *all Hermitian A-modules are projective;*
(ii) *A is isometrically *-isomorphic to the c_0-sum of a family of full matrix C^*-algebras.*

The next lemma follows immediately from [38, Proposition IV.1.7].

Lemma 4.23. *Let A and B be locally C^*-algebras, and let $\varphi\colon A \to B$ be a continuous *-homomorphism with dense range. Suppose further that all Hermitian A-modules are projective. Then all Hermitian B-modules are projective.*

Proof of Theorem 4.21. (i) $\Rightarrow$ (ii) This follows from Corollary 3.11.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (iv) Since all irreducible Hermitian A-modules are projective, it follows from Theorem 3.24 that $\operatorname{Soc}(A)$ is dense in A. Further, let p be a continuous C^*-seminorm on A, and let A_p be the corresponding concomitant C^*-algebra. Using Lemma 4.23, we see that all Hermitian A_p-modules are projective. Applying Theorem 4.22 and taking into account Example 4.4, we conclude that A_p is biprojective.

(iv) $\Rightarrow$ (v) This follows from Theorem 3.3 (see also Theorem 3.24) and Theorem 4.5.

(v) $\Rightarrow$ (vi) This follows from Theorem 3.24.

(vi) $\Rightarrow$ (vii) This follows from Theorem 3.24, Theorem 3.3(xi), and Example 4.4.

(vii) $\Rightarrow$ (i) This follows from Proposition 4.17. ∎

Proposition 4.24. *Let A be a biprojective locally C^*-algebra. Then the following statements hold.*

(i) *For each closed left ideal J of A, J and A/J are projective left topological A-modules. Moreover, J is complemented as a topological submodule of the left A-module A.*
(ii) *For each closed right ideal K of A, K and A/K are projective right topological A-modules. Moreover, K is complemented as a topological submodule of the right A-module A.*
(iii) *For each closed two-sided ideal I of A, I and A/I are projective topological A-bimodules. Moreover, I is complemented as a topological submodule of the A-bimodule A.*

Proof. (i) It follows from Theorem 4.21 that A is an annihilator algebra, and that $\operatorname{Soc}(A)$ is dense in A. Now Theorem 3.24 implies that J is complemented as a topological submodule of the left A-module A. Hence J and A/J are retracts (i.e., A-module direct summands) of the module A. Since the biprojective algebra A is left projective, we see that J and A/J are projective left topological A-modules (see [38, Proposition III.1.16]).

(ii) and (iii) These are similar. ∎

The next result follows easily from Theorems 3.3 and 4.21 (see also Proposition 4.24 and Corollary 4.16).

Proposition 4.25. *Let A be a biprojective locally C^*-algebra, and let I be a closed two-sided ideal of A. Then I and A/I are biprojective locally C^*-algebras, and moreover A is topologically *-isomorphic to the Cartesian product of the algebras I and A/I.*

Propositions 3.1 and 4.25 (see also Theorem 4.21) imply the following.

COROLLARY 4.26. *Let A be a biprojective locally C^*-algebra, and let P be the family of all continuous C^*-seminorms on A. Then, for each $p \in P$, the concomitant C^*-algebra A_p is biprojective and, as a consequence, A can be represented as an inverse limit of biprojective C^*-algebras.*

Applying Theorems 4.21 and 3.30 and taking into account Example 4.1, we get the following theorem.

THEOREM 4.27. *Let A be a unital locally C^*-algebra. Then the following conditions are equivalent:*

(i) *A is a biprojective (or, equivalently, contractible) algebra;*
(ii) *all essential (or, equivalently*[16]*, unital) left topological A-modules are projective;*
(iii) *all Hermitian A-modules are projective;*
(iv) *all irreducible Hermitian A-modules are projective;*
(v) $\mathrm{Soc}(A)$ *is dense in A;*
(vi) *A is an annihilator algebra;*
(vii) *A is topologically *-isomorphic to the Cartesian product of a family of full matrix algebras.*

REMARK 4.28. The equivalence of conditions (i), (ii) and (vii) of Theorem 4.27 was proved earlier in [25, Theorem 3.3].

In [67, Corollary 5.4], we proved that every biprojective σ-C^*-algebra is topologically *-isomorphic to the Cartesian product of a family of biprojective C^*-algebras. In the same paper, we have raised the following question (see [67, Question 5.5]):

Can this result be extended to arbitrary (i.e., non-metrizable) locally C^*-algebras?

Now we show that the answer to this question is negative. Namely, we present an example of a non-unital biprojective locally C^*-algebra that is not topologically isomorphic to a Cartesian product of (biprojective) C^*-algebras.

THEOREM 4.29. *There exists a biprojective locally C^*-algebra that is not topologically isomorphic to a Cartesian product of C^*-algebras.*

Proof. Consider the biprojective locally C^*-algebra A defined in Example 4.20. Assume towards a contradiction that this algebra is topologically isomorphic to the Cartesian product $\prod_{i\in I} A_i$ of a family of C^*-algebras A_i, $i \in I$. Since A is biprojective, it follows from Proposition 4.25 that A_i is biprojective for each $i \in I$. Since A is commutative, it follows from Theorem 4.5 that each algebra A_i, $i \in I$, is isometrically *-isomorphic to a C^*-algebra of the form $c_0(\Lambda_i)$ for some set Λ_i. So A is topologically isomorphic to the algebra $\prod_{i\in I} c_0(\Lambda_i)$.

Denote the latter algebra by B. Note that A is not metrizable, because the set $\mathbb{N}^{\mathbb{N}}$ of all functions $s\colon \mathbb{N} \to \mathbb{N}$ does not have a countable cofinal subset (cf. [61, Example 5.10]). Therefore B is not metrizable either, and so I is uncountable. For each $i \in I$, let

$$p_i\colon B \to c_0(\Lambda_i)$$

[16] Clearly, every essential left topological module over a unital topological algebra is unital.

denote the natural projection onto the ith factor. Then it is easy to check (cf. [39, Proposition V.1.8]) that every non-zero continuous character on B has the form

$$b \mapsto p_i(b)(\nu)$$

for some $i \in I$ and for some $\nu \in \Lambda_i$. Therefore there is no countable separating set of continuous characters on B. On the other hand, the characters

$$\{a \mapsto a_{ij} \mid (i,j) \in \mathbb{N} \times \mathbb{N}\}$$

obviously separate the points of A. The resulting contradiction shows that A is not topologically isomorphic to B. This completes the proof. ∎

Thus Theorem 4.7 and [67, Corollary 5.4] cannot be generalized to arbitrary locally C^*-algebras. Nevertheless we have the following result.

PROPOSITION 4.30. *Let A be a non-unital biprojective locally C^*-algebra. Then A is topologically *-isomorphic to the Cartesian product of two biprojective locally C^*-algebras one of which is an infinite-dimensional C^*-algebra.*

Proof. By Theorem 4.21, A is an annihilator algebra. Now Proposition 3.34 implies that A is topologically *-isomorphic to the Cartesian product of two annihilator locally C^*-algebras one of which is an infinite-dimensional C^*-algebra. The rest follows from Proposition 4.25. ∎

5. Superbiprojective locally C^*-algebras. Here our aim is to answer the fourth question posed in the beginning of the paper.

The following result is well known (cf. [85, Corollary 8], see also [91, Corollary 2.4.3]).

THEOREM 5.1. *Let A be a biprojective topological algebra. Then the following conditions are equivalent:*

(i) $\mathcal{H}^2(A, X) = 0$ *for all topological A-bimodules X;*
(ii) $\mathcal{H}^2(A, A \widehat{\otimes} A) = 0$*;*
(ii) *the operator*

$$\Delta\colon A \widehat{\otimes} A \to (A_+ \widehat{\otimes} A) \oplus (A \widehat{\otimes} A_+), \quad a \otimes b \mapsto (a \otimes b,\, a \otimes b),$$

has a left inverse A-bimodule morphism.

DEFINITION 5.2 (cf. [90] and [67]). A topological algebra A is said to be *superbiprojective* if A is biprojective and satisfies the equivalent conditions of Theorem 5.1.

It is clear that every contractible topological algebra is superbiprojective.

The proof of the next proposition repeats, in essence, the proof of [78, Theorem 1].

PROPOSITION 5.3. *Let A be a biprojective topological algebra. If A has a left or a right identity, then A is superbiprojective.*

EXAMPLE 5.4. Let F be a complete Hausdorff locally convex space with $\dim F > 1$. Then every continuous linear functional $f \in F^*$, $f \neq 0$, defines on F the structure of a topological algebra, denoted by $S_f(F)$, with multiplication given by $ab = f(a)b$. It is easily seen that $S_f(F)$ is a superbiprojective algebra with a left identity ([86, Example 3], see also [78, Theorem 1] and [64, Example 2.6]). Clearly, $S_f(F)$ is not contractible. Note that $S_f(F)$ is a particular case of the algebras $E \widehat{\otimes} F$ considered in Example 4.11. Indeed, it suffices to set $E = \mathbb{C}$ and to define $\langle \lambda, x\rangle = \lambda f(x)$ for $\lambda \in \mathbb{C}$ and $x \in F$.

The following result generalizes [67, Proposition 2.8].

PROPOSITION 5.5. *Let A be a superbiprojective topological algebra, and let I be a closed two-sided ideal of A such that I is complemented as a topological submodule of the A-bimodule A. Then I and A/I are superbiprojective topological algebras.*

Proof. Using Corollary 4.16, we see that I and A/I are biprojective topological algebras. Also, it follows from the hypothesis that there is a closed two-sided ideal J of A such that $A = I \oplus J$, and the operator $\lambda\colon I \times J \to A$, $(a, b) \mapsto a + b$, is a topological isomorphism. Moreover, as we noted in the proof of Corollary 4.16, $\overline{A \cdot I} = I$ and $\overline{A \cdot J} = J$. Now Proposition 3.13 implies that the left topological A-modules

$$I \cong A/J = A/\overline{A \cdot J} \quad \text{and} \quad A/I = A/\overline{A \cdot I}$$

are projective. Similarly, $I \cong A/J$ and A/I are projective in the category of right topological A-modules. Therefore the natural projections of topological algebras $A \to A/J$ and $A \to A/I$ satisfy the conditions of [63, Theorem 5.2], and so the algebras $I \cong A/J$ and A/I are superbiprojective. ∎

By combining this result with Proposition 4.24(iii) we get the following.

COROLLARY 5.6. *Let A be a superbiprojective locally C^*-algebra, and let I be a closed two-sided ideal of A. Then I and A/I are superbiprojective locally C^*-algebras.*

This result, together with Proposition 3.1, implies the following.

COROLLARY 5.7. *Let A be a superbiprojective locally C^*-algebra, and let P be the family of all continuous C^*-seminorms on A. Then, for each $p \in P$, the concomitant C^*-algebra A_p is superbiprojective.*

Now we recall a result concerning C^*-algebras.

THEOREM 5.8 (see [89, Corollary 4.64] and [90]). *Let A be a superbiprojective C^*-algebra. Then A is topologically *-isomorphic to the Cartesian product of finitely many full matrix algebras. In particular, A is finite-dimensional and contractible.*

Everything is now ready for the proof of the main result of this section. The following theorem generalizes [67, Theorem 5.6 and Corollary 5.7].

THEOREM 5.9. *Let A be a locally C^*-algebra. Then the following conditions are equivalent:*

(i) *A is a superbiprojective algebra;*
(ii) *A is topologically *-isomorphic to the Cartesian product of a family of full matrix algebras;*
(iii) *A is a contractible algebra.*

Proof. (i) $\Rightarrow$ (ii) It follows from the hypothesis and from Theorem 4.21 that A is biprojective, and that $\mathrm{Soc}(A)$ is dense in A. On the other hand, Corollary 5.7 and Theorem 5.8 imply that, for each continuous C^*-seminorm p on A, the concomitant C^*-algebra A_p is finite-dimensional. Now the result follows from Corollary 3.4(ii).

(ii) $\Rightarrow$ (iii) This follows from [84, Lemma 11] and [39, Proposition VII.1.73]. (See also Example 4.1.)

(iii) $\Rightarrow$ (i) This is trivial. ∎

Recall that the equivalence of conditions (ii) and (iii) in Theorem 5.9 was proved earlier in [25, Theorem 3.3] (see Remark 4.28).

We now recall an important definition (see [38, 42]).

DEFINITION 5.10. Let A be a topological algebra. The least n for which $\mathcal{H}^m(A, X) = 0$ for all topological A-bimodules X and all $m > n$, or ∞ if there is no such n, is called the *homological bidimension* of A, and is denoted by $\mathrm{db}\, A$.

Clearly, $\mathrm{db}\, A = 0$ means exactly that A is contractible. Also, a biprojective topological algebra A always has $\mathrm{db}\, A \le 2$, and it has $\mathrm{db}\, A \le 1$ if and only if it is superbiprojective.

EXAMPLE 5.11. If G is an infinite compact Lie group, then the Fréchet algebra $\mathcal{E}(G)$ (see Example 4.9) is superbiprojective and $\mathrm{db}\, \mathcal{E}(G) = 1$ [63, Corollary 5.4].

From Theorem 5.9 we immediately get the following corollaries.

COROLLARY 5.12. *Let A be a non-unital biprojective locally C^*-algebra. Then* $\mathrm{db}\, A = 2$ *and* $\mathcal{H}^2(A, A \widehat{\otimes} A) \ne 0$.

COROLLARY 5.13. *In the class of biprojective locally C^*-algebras the homological bidimension* $\mathrm{db}\, A$ *can take only the values* 0 *and* 2.

Acknowledgements. This paper is based on a lecture of the second author delivered at the 19th International Conference on Banach Algebras held at Będlewo, July 14–24, 2009. The support for the meeting by the Polish Academy of Sciences, the European Science Foundation under the ESF-EMS-ERCOM partnership, and the Faculty of Mathematics and Computer Science of the Adam Mickiewicz University at Poznań is gratefully acknowledged.

The research was partially supported by the RFBR grant 08-01-00867. The first author was also partially supported by the Ministry of Education and Science of Russia (programme “Development of the scientific potential of the Higher School”, grant no. 2.1.1/2775), and by the President of Russia grant MK-1173.2009.1. We are grateful to D. P. Blecher, M. Fragoulopoulou, M. Haralampidou, A. Ya. Helemskii, and G. Racher for useful discussions. Also, we are grateful to the referee for his/her valuable comments.

References

[1] F. E. Alexander, *On complemented and annihilator algebras*, Glasgow Math. J. 10 (1969), 38–45.

[2] F. E. Alexander, *Representation theorems for complemented algebras*, Trans. Amer. Math. Soc. 148 (1970), 385–398.

[3] F. E. Alexander, *Some counter-examples on annihilator, dual, and complemented A^*-algebras*, J. London Math. Soc. (2) 8 (1974), 735–740.

[4] F. E. Alexander and B. J. Tomiuk, *Complemented B^*-algebras*, Trans. Amer. Math. Soc. 137 (1969), 459–480.

[5] J. C. Alexander, *Compact Banach algebras*, Proc. London Math. Soc. (3) 18 (1968), 1–18.

[6] G. R. Allan, *On a class of locally convex algebras*, Proc. London Math. Soc. (3) 17 (1967), 91–114.

[7] C. Apostol, *b^*-algebras and their representation*, J. London Math. Soc. (2) 3 (1971), 30–38.

[8] O. Yu. Aristov, *Biprojective algebras and operator spaces*, J. Math. Sci. (New York) 111 (2002), no. 2, 3339–3386.

[9] O. Yu. Aristov, *On the structure of biprojective Banach algebras with a non-trivial radical*, Izv. Ross. Akad. Nauk Ser. Mat. 72 (2008), no. 6, 53–84 (in Russian); English transl.: Izv. Math. 72 (2008), no. 6, 1111–1140.

[10] W. Arveson, *The harmonic analysis of automorphism groups*, in: Proc. Symposia Pure Math. 38, Part 1, Amer. Math. Soc., Providence, RI, 1982, 199–269.

[11] B. A. Barnes, *Modular annihilator algebras*, Canad. J. Math. 18 (1966), 566–578.

[12] B. A. Barnes, *On the existence of minimal ideals in a Banach algebra*, Trans. Amer. Math. Soc. 133 (1968), 511–517.

[13] B. A. Barnes, *Examples of modular annihilator algebras*, Rocky Mountain J. Math. 1 (1971), 657–665.

[14] S. J. Bhatt and D. J. Karia, *An intrinsic characterization of pro-C^*-algebras and its applications*, J. Math. Anal. Appl. 175 (1993), 68–80.

[15] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin, 1973.

[16] F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3) 4 (1954), 154–167.

[17] N. Bourbaki, *Topological Vector Spaces*, Chapters 1–5, translated from the French by H. G. Eggleston and S. Madan, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987.

[18] R. M. Brooks, *On representing F^*-algebras*, Pacific J. Math. 39 (1971), 51–69.

[19] A. M. Davie, *A counterexample on dual Banach algebras*, Bull. London Math. Soc. 5 (1973), 79–80.

[20] J. Dixmier, *Les C^*-Algèbres et leurs Représentations*, Gauthier-Villars, Paris, 1969.

[21] P. G. Dixon, *Generalized B^*-algebras*, Proc. London Math. Soc. 21 (1970), 693–715.

[22] R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Pitman Research Notes in Math. 217, Longman, New York, 1989.

[23] P. Enflo, *A counterexample to the approximation problem in Banach spaces*, Acta Math. 130 (1973), 309–317.

[24] M. Fragoulopoulou, *Symmetric Topological *-Algebras. Applications*, Schriftenreihe Math. Inst. Univ. Münster, 3. Serie, 9, Münster, 1993.

[25] M. Fragoulopoulou, *Structure of contractible locally C^*-algebras*, Proc. Amer. Math. Soc. 129 (2001), 2889–2896.

[26] M. Fragoulopoulou, *Topological Algebras with Involution*, North-Holland Mathematics Studies 200, Elsevier, Amsterdam, 2005.

[27] M. Frank, *Characterizing C^*-algebras of compact operators by generic categorical properties of Hilbert C^*-modules*, J. K-Theory 2 (2008), 453–462.
[28] M. Frank and K. Sharifi, *Adjointability of densely defined closed operators and the Magajna-Schweizer theorem*, J. Operator Theory 63 (2010), 271–282.
[29] A. Inoue, *Locally C^*-algebras*, Mem. Faculy Sci. Kyushu Univ. (Ser. A) 25 (1971), 197–235.
[30] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[31] T. O. Gulinskiy, *Projectivity of Hilbert modules over uniform algebras*, Math. Proc. Cambridge Philos. Soc. 135 (2003), 269–275.
[32] M. Haralampidou, *Structure theorems for complemented topological algebras*, Boll. Un. Mat. Ital. B (7) 7 (1993), 961–971.
[33] M. Haralampidou, *Dual complementors in topological algebras*, in: Topological Algebras, their Applications, and Related Topics, Banach Center Publ. 67, Polish Acad. Sci., Warsaw, 2005, 219–233.
[34] A. Ya. Helemskii, *A certain method of computing and estimating the global homology dimension of Banach algebras*, Mat. Sb. (N.S.) 87 (129) (1972), 122–135 (in Russian); English transl.: Math. USSR Sb. 16 (1972), 125–138.
[35] A. Ya. Helemskii, *The global dimension of a functional Banach algebra is different from one*, Funkcional. Anal. i Priložen. 6 (1972), no. 2, 95–96 (in Russian); English transl.: Funct. Anal. Appl. 6 (1972), 166–168.
[36] A. Ya. Helemskii, *Homological characteristics of Banach algebras*, Diss. Doktor Fiz.-Mat. Nauk, Moscow, 1973 (in Russian).
[37] A. Ya. Helemskii, *Lower values that admit the global homology dimension of functional Banach algebras*, Trudy Sem. Petrovsk. 3 (1978), 223–242 (in Russian); English transl.: Amer. Math. Soc. Transl. 124 (1984), 75–96.
[38] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Kluwer Academic Publishers, Dordrecht, 1989.
[39] A. Ya. Helemskii, *Banach and Locally Convex Algebras*, Clarendon Press, Oxford University Press, New York, 1993.
[40] A. Ya. Helemskii, *The structure of a C^*-algebra and its supply of projective Hilbert modules*, Uspekhi Mat. Nauk 52 (1997), no. 4, 221–222 (in Russian); English transl.: Russian Math. Surveys 52 (1997), no. 4, 834–835.
[41] A. Ya. Helemskii, *Projective homological classification of C^*-algebras*, Comm. Algebra 26 (1998), 977–996.
[42] A. Ya. Helemskii, *Homology for the algebras of analysis*, in: Handbook of Algebra, Vol. 2, North-Holland, Amsterdam, 2000, 151–274.
[43] A. Ya. Helemskii, *Lectures and Exercises on Functional Analysis*, Translations of Mathematical Monographs 233, Amer. Math. Soc., Providence, RI, 2006.
[44] T. Husain and P. K. Wong, *Quasi-complemented algebras*, Trans. Amer. Math. Soc. 174 (1972), 141–154.
[45] B. E. Johnson, *A commutative semisimple annihilator Banach algebra which is not dual*, Bull. Amer. Math. Soc. 73 (1967), 407–409.
[46] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras*, Amer. J. Math. 94 (1972), 685–698.
[47] I. Kaplansky, *Dual rings*, Ann. of Math. (2) 49 (1948), 689–701.
[48] G. Köthe, *Topological Vector Spaces II*, Springer-Verlag, New York-Berlin, 1979.

[49] E. C. Lance, *Hilbert C^*-Modules*, Cambridge University Press, Cambridge, 1995.

[50] G. Lassner, *Über Realisierungen gewisser *-Algebren*, Math. Nachr. 52 (1972), 161–166.

[51] M. J. Liddell, *Separable topological algebras, I*, Trans. Amer. Math. Soc. 195 (1974), 31–59.

[52] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I. Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 92, Springer-Verlag, Berlin-New York, 1977.

[53] Z. A. Lykova, *On conditions for projectivity of Banach algebras of completely continuous operators*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 34 (1979), no. 4, 8–13 (in Russian); English transl.: Moscow Univ. Math. Bull. 34 (1979), no. 4, 6–11.

[54] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam, 1986.

[55] B. D. Malviya, *A note on complemented Banach *-algebras*, J. Reine Angew. Math. 278/279 (1975), 191–194.

[56] V. M. Manuilov and E. V. Troitsky, *Hilbert C^*-Modules*, Translations of Mathematical Monographs 226, Amer. Math. Soc., Providence, RI, 2005.

[57] M. A. Naĭmark, *Normed Algebras*, Wolters-Noordhoff Series of Monographs and Textbooks on Pure and Applied Mathematics, Wolters-Noordhoff Publishing, Groningen, 1972.

[58] A. Olubummo, *Complemented Banach algebras*, Canad. J. Math. 16 (1964), 149–150.

[59] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras*, Vol. 1, Cambridge University Press, Cambridge, 1994.

[60] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras*, Vol. 2, Cambridge University Press, Cambridge, 2001.

[61] N. C. Phillips, *Inverse limits of C^*-algebras*, J. Operator Theory 19 (1988), 159–195.

[62] N. C. Phillips, *Inverse limits of C^*-algebras and applications*, in: Operator Algebras and Applications, Vol. 1, London Math. Soc. Lecture Note Ser. 135, Cambridge University Press, Cambridge, 1988, 127–186.

[63] A. Yu. Pirkovskii, *Biprojective topological algebras of homological bidimension* 1, J. Math. Sci. (New York) 111 (2002), 3476–3495.

[64] A. Yu. Pirkovskii, *Homological bidimension of biprojective topological algebras and nuclearity*, Acta Univ. Oulu Ser. A Sci. Rerum Natur. 408 (2004), 179–196.

[65] A. Yu. Pirkovskii, *Biprojectivity and biflatness for convolution algebras of nuclear operators*, Canad. Math. Bull. 47 (2004), 445–455.

[66] A. Yu. Pirkovskii, *Flat cyclic Fréchet modules, amenable Fréchet algebras, and approximate identities*, Homology Homotopy Appl. 11 (2009), 81–114.

[67] A. Yu. Pirkovskii and Yu. V. Selivanov, *Homologically trivial Fréchet algebras*, in: Topological Algebras and Applications, Contemp. Math. 427, Amer. Math. Soc., Providence, RI, 2007, 367–387.

[68] C. P. Podara, *On strictly flat Fréchet modules*, in: Topological Algebras and Applications, Contemp. Math. 427, Amer. Math. Soc., Providence, RI, 2007, 389–399.

[69] G. Racher, *On amenable and compact groups*, Monatsh. Math. 92 (1981), 305–311.

[70] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, Princeton, N.J., 1960.

[71] M. A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. 13 (1974), 176–257.

[72] M. A. Rieffel, *Morita equivalence for C^*-algebras and W^*-algebras*, J. Pure Appl. Algebra 5 (1974), 51–96.

[73] A. P. Robertson and W. Robertson, *Topological Vector Spaces*, Cambridge University Press, 1964.

[74] V. Runde, *The structure of contractible and amenable Banach Algebras*, in: Banach Algebras '97 (Blaubeuren), de Gruyter, Berlin, 1998, 415–430.
[75] V. Runde, *Lectures on Amenability*, Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin, 2002.
[76] V. Runde, *Biflatness and biprojectivity of the Fourier algebra*, Arch. Math. (Basel) 92 (2009), no. 5, 525–530.
[77] K. Schmüdgen, *Über LMC^*-Algebren*, Math. Nachr. 68 (1975), 167–182.
[78] Yu. V. Selivanov, *The values assumed by the global dimension in certain classes of Banach algebras*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 30 (1975), no. 1, 37–42 (in Russian); English transl.: Moscow Univ. Math. Bull. 30 (1975), no. 1, 30–34.
[79] Yu. V. Selivanov, *Banach algebras of small global dimension zero*, Uspekhi Mat. Nauk 31 (1976), no. 2, 227–228 (in Russian).
[80] Yu. V. Selivanov, *Biprojective Banach algebras, their structure, cohomology, and relation with nuclear operators*, Funkcional. Anal. i Priložen. 10 (1976), no. 1, 89–90 (in Russian); English transl.: Funct. Anal. Appl. 10 (1976), no. 1, 78–79.
[81] Yu. V. Selivanov, *Certain questions of the homological classification of Banach algebras*, Diss. Kandidat Fiz.-Mat. Nauk, Moscow, 1977 (in Russian).
[82] Yu. V. Selivanov, *Biprojective Banach algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 5, 1159–1174 (in Russian); English transl.: Math. USSR Izv. 15 (1980), no. 2, 387–399.
[83] Yu. V. Selivanov, *Homological characterizations of the approximation property for Banach spaces*, Glasgow Math. J. 34 (1992), 229–239.
[84] Yu. V. Selivanov, *Fréchet algebras of global dimension zero*, in: Algebra (Krasnoyarsk, 1993), de Gruyter, Berlin, 1996, 225–236.
[85] Yu. V. Selivanov, *Weak homological bidimension and its values in the class of biflat Banach algebras*, Extracta Math. 11 (1996), 348–365.
[86] Yu. V. Selivanov, *Biprojective topological algebras* (talk in Sheffield, 12 November 1996), Unpublished manuscript, 1996.
[87] Yu. V. Selivanov, *Homological dimensions of tensor products of Banach algebras*, in: Banach Algebras '97 (Blaubeuren), de Gruyter, Berlin, 1998, 441–459.
[88] Yu. V. Selivanov, *Cohomological characterizations of biprojective and biflat Banach algebras*, Monatsh. Math. 128 (1999), 35–60.
[89] Yu. V. Selivanov, *Coretraction problems and homological properties of Banach algebras*, in: Topological Homology, Nova Sci. Publ., Huntington, New York, 2000, 145–199.
[90] Yu. V. Selivanov, *Superbiprojective and superbiflat Banach algebras*, preprint, Odense, 2001.
[91] Yu. V. Selivanov, *The cohomology of Banach and related algebras*, Diss. Doktor Fiz.-Mat. Nauk, Moscow, 2002 (in Russian).
[92] Yu. V. Selivanov, *Classes of Banach algebras of global dimension infinity*, in: Banach Algebras and Their Applications, Contemp. Math. 363, Amer. Math. Soc., Providence, RI, 2004, 321–333.
[93] J. L. Taylor, *Homology and cohomology for topological algebras*, Advances in Math. 9 (1972), 137–182.
[94] J. L. Taylor, *A general framework for a multi-operator functional calculus*, Advances in Math. 9 (1972), 183–252.
[95] B. J. Tomiuk, *Structure theory of complemented Banach algebras*, Canad. J. Math. 14 (1962), 651–659.

[96] B. J. Tomiuk, *Dual complementors on Banach algebras*, Proc. Amer. Math. Soc. 103 (1988), 815–822.
[97] B. J. Tomiuk, *Multipliers on complemented Banach algebras*, Proc. Amer. Math. Soc. 115 (1992), 397–404.
[98] B. J. Tomiuk and P. K. Wong, *Annihilator and complemented Banach *-algebras*, J. Austral. Math. Soc. 13 (1972), 47–66.
[99] D. Voiculescu, *Dual algebraic structures on operator algebras related to free products*, J. Operator Theory 17 (1987), 85–98.
[100] P. K. Wong, *Continuous complementors on B^*-algebras*, Pacific J. Math. 33 (1970), 255–260.
[101] P. K. Wong, *A note on annihilator and complemented Banach algebras*, J. Austral. Math. Soc. 18 (1974), 474–481.
[102] P. J. Wood, *The operator biprojectivity of the Fourier algebra*, Canad. J. Math. 54 (2002), 1100–1120.
[103] B. Yood, *Ideals in topological rings*, Canad. J. Math. 16 (1964), 28–45.