SCHUR AND OPERATOR MULTIPLIERS

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Abstract. The present article is a survey of known results on Schur and operator multipliers. It starts with the classical description of Schur multipliers due to Grothendieck, followed by a discussion of measurable Schur multipliers and a generalisation of Grothendieck's Theorem due to Peller. Thereafter, a non-commutative version of Schur multipliers, called operator multipliers and introduced by Kissin and Schulman, is discussed, and a characterisation extending the description in the commutative case is presented. Finally, multidimensional versions of Schur and operator multipliers are considered. The article contains a brief discussion of some applications of Schur multipliers, including double operator integrals and multipliers of group algebras.

Schur multipliers were introduced by Schur in the early 20th century and have since then found a considerable number of applications in Analysis and enjoyed an intensive development. Apart from the beauty of the subject in itself, sources of interest in them were connections with Perturbation Theory, Harmonic Analysis, the Theory of Operator Integrals and others. Advances in the quantisation of Schur multipliers were recently made in [29]. The aim of the present article is to summarise part of the ideas and results in the theory of Schur and operator multipliers. We start with the classical Schur multipliers defined by Schur and their characterisation by Grothendieck, and make our way through measurable multipliers studied by Peller and Spronk, operator multipliers defined by Kissin and Shulman and, finally, multidimensional Schur and operator mul-

tipliers developed by Juschenko and the authors. We point out connections of the area with Harmonic Analysis and the Theory of Operator Integrals.

1. Classical Schur multipliers. For a Hilbert space H, let $\mathcal{B}(H)$ be the collection of all bounded linear operators acting on H equipped with its operator norm $\|\cdot\|_{\mathrm{op}}$. We denote by ℓ^2 the Hilbert space of all square summable complex sequences. With an operator $A \in \mathcal{B}(\ell^2)$, one can associate a matrix $(a_{i,j})_{i,j\in\mathbb{N}}$ by letting $a_{i,j} = (Ae_j, e_i)$, where $\{e_i\}_{i\in\mathbb{N}}$ is the standard orthonormal basis of ℓ^2. The space M_∞ of all matrices obtained in this way is a subspace of the space $M_\mathbb{N}$ of all complex matrices indexed by $\mathbb{N} \times \mathbb{N}$. It is easy to see that the correspondence between $\mathcal{B}(\ell^2)$ and M_∞ is one-to-one.

Any function $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ gives rise to a linear transformation S_φ acting on $M_\mathbb{N}$ and given by $S_\varphi((a_{i,j})_{i,j}) = (\varphi(i,j)a_{i,j})_{i,j}$. In other words, $S_\varphi((a_{i,j})_{i,j})$ is the entry-wise product of the matrices $(\varphi(i,j))_{i,j}$ and $(a_{i,j})_{i,j}$, often called **Schur product**. The function φ is called a **Schur multiplier** if S_φ leaves the subspace M_∞ invariant. We denote by $S(\mathbb{N}, \mathbb{N})$ the set of all Schur multipliers.

Let φ be a Schur multiplier. Then the correspondence between $\mathcal{B}(\ell^2)$ and M_∞ gives rise to a mapping (which we denote in the same way) on $\mathcal{B}(\ell^2)$. We first note that S_φ is necessarily bounded in the operator norm. This follows from the Closed Graph Theorem; indeed, suppose that $A_k \to 0$ and $S_\varphi(A_k) \to B$ in the operator norm, for some elements $A_k, B \in \mathcal{B}(\ell^2)$, $k \in \mathbb{N}$. Letting $(a_{i,j}^k)$ and $(b_{i,j})$ be the corresponding matrices of A_k and B, we have that $a_{i,j}^k = (A_k e_j, e_i) \to_{k\to\infty} 0$, for each $i, j \in \mathbb{N}$. But then $\{\varphi(i,j)a_{i,j}^k\}_{k\in\mathbb{N}}$ converges to both $b_{i,j}$ and 0, and since this holds for every $i, j \in \mathbb{N}$, we conclude that $B = 0$. Let $\|\varphi\|_{\mathrm{m}}$ denote the norm of S_φ as a bounded operator on $\mathcal{B}(\ell^2)$; we call $\|\varphi\|_{\mathrm{m}}$ the multiplier norm of φ.

It now follows that $S(\mathbb{N}, \mathbb{N}) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N})$. Indeed, if $\{E_{i,j}\}_{i,j\in\mathbb{N}}$ is the canonical matrix unit system in $\mathcal{B}(\ell^2)$ then we have that $|\varphi(i,j)| = \|S_\varphi(E_{i,j})\|_{\mathrm{op}} \le \|\varphi\|_{\mathrm{m}}$, for all $i, j \in \mathbb{N}$. It is trivial to verify that $S(\mathbb{N}, \mathbb{N})$ is a subalgebra of $\ell^\infty(\mathbb{N} \times \mathbb{N})$ when the latter is equipped with the usual pointwise operations; moreover, it can be shown that $S(\mathbb{N}, \mathbb{N})$ is a semi-simple commutative Banach algebra when equipped with the norm $\|\cdot\|_{\mathrm{m}}$. We note in passing that, since φ is a bounded function, the restriction of S_φ to the class $\mathcal{C}_2(\ell^2)$ of all Hilbert-Schmidt operators is bounded when $\mathcal{C}_2(\ell^2)$ is equipped with its Hilbert-Schmidt norm $\|\cdot\|_2$, and its norm as an operator on $\mathcal{C}_2(\ell^2)$ is equal to $\|\varphi\|_\infty$. This follows from the fact that if $A \in \mathcal{C}_2(\ell^2)$ and $(a_{i,j})_{i,j} \in M_\infty$ is the matrix corresponding to A then $\|A\|_2 = (\sum_{i,j} |a_{i,j}|^2)^{1/2}$.

Another immediate observation is the fact that if $(a_{i,j})_{i,j} \in M_\infty$ and $a \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ is the function given by $a(i,j) = a_{i,j}$, then a is a Schur multiplier and $\|a\|_{\mathrm{m}} \le \|a\|_{\mathrm{op}}$. To see this, let $b \in M_\infty$ and $B \in \mathcal{B}(\ell^2)$ be the operator corresponding to b. If $A \in \mathcal{B}(\ell^2)$ is the operator corresponding to a, we have by general operator theory that the norm of the operator $A \otimes B \in \mathcal{B}(\ell^2 \otimes \ell^2)$ is equal to $\|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$. Identifying $\ell^2 \otimes \ell^2$ with $\ell^2(\mathbb{N} \times \mathbb{N})$, and letting $P \in \mathcal{B}(\ell^2 \otimes \ell^2)$ be the projection on the closed linear span of $\{e_i \otimes e_i\}_{i\in\mathbb{N}}$, we see that the matrix $(a_{i,j}b_{i,j})_{i,j}$ corresponds to the operator $P(A \otimes B)P$. Thus, $\|S_a(B)\|_{\mathrm{op}} = \|P(A \otimes B)P\|_{\mathrm{op}} \le \|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$, and the claim follows.

We summarise the above inclusions:

$$M_\infty \subseteq S(\mathbb{N}, \mathbb{N}) \subseteq \ell^\infty(\mathbb{N} \times \mathbb{N}).$$

Both of them are strict: for the first one, take the constant function $\mathbf{1}$ taking the value 1 on $\mathbb{N} \times \mathbb{N}$. Obviously, $\mathbf{1}$ is a Schur multiplier (in fact, $S_{\mathbf{1}}$ is the identity transformation) but does not belong to M_∞ since the rows and columns of the matrices in M_∞ are square summable. The fact that the second inclusion is strict is much more subtle. An example of a function which belongs to $\ell^\infty(\mathbb{N} \times \mathbb{N})$ but not to $S(\mathbb{N}, \mathbb{N})$ is the characteristic function χ_Δ of the set $\Delta = \{(i, j) : j \leq i\}$, see for example [12].

The question that arises is: "which" functions are Schur multipliers? The following description was obtained by Grothendieck [20]:

THEOREM 1.1. *Let $\varphi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$. The following are equivalent:*

(i) *φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} < C$;*
(ii) *There exist families $\{a_k\}_{k\in\mathbb{N}}, \{b_k\}_{k\in\mathbb{N}} \in \ell^\infty$ such that*

$$\sup_{i\in\mathbb{N}} \sum_{k=1}^\infty |a_k(i)|^2 < C, \quad \sup_{j\in\mathbb{N}} \sum_{k=1}^\infty |b_k(j)|^2 < C$$

and

$$\varphi(i, j) = \sum_{k=1}^\infty a_k(i) b_k(j), \quad \text{for all } i, j \in \mathbb{N}.$$

Suppose that we are given two *finite* families $\{a_k\}_{k=1}^N, \{b_k\}_{k=1}^N \subseteq \ell^\infty$ and let $\varphi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ be the function given by $\varphi(i, j) = \sum_{k=1}^N a_k(i) b_k(j)$. For $a \in \ell^\infty$, let $D_a \in \mathcal{B}(\ell^2)$ be the operator whose matrix has the sequence a down its main diagonal and zeros everywhere else. An easy computation shows that φ is a Schur multiplier and that, in fact,

$$S_\varphi(T) = \sum_{k=1}^N D_{a_k} T D_{b_k}, \quad T \in \mathcal{B}(\ell^2).$$

The transformations on $\mathcal{B}(\ell^2)$ obtained in this way belong to the important class of *elementary operators*. The norm of this operator, and hence $\|\varphi\|_{\mathrm{m}}$, is bounded by

$$\Big(\sup_{i\in\mathbb{N}} \sum_{k=1}^N |a_k(i)|^2 \sup_{j\in\mathbb{N}} \sum_{k=1}^N |b_k(j)|^2 \Big)^{1/2}.$$

In fact, for $f, g \in \ell^2$ we have

$$|(S_\varphi(T)f, g)| = \Big| \Big(\sum_{k=1}^N D_{a_k} T D_{b_k} f, g \Big) \Big| \leq \sum_{k=1}^N |(T D_{b_k} f, D_{a_k}^* g)|$$

$$\leq \sum_{k=1}^N \|T D_{b_k} f\| \|D_{a_k}^* g\|$$

$$\leq \Big(\sum_{k=1}^N \|T\|^2 \|D_{b_k} f\|^2 \Big)^{1/2} \Big(\sum_{k=1}^N \|D_{a_k}^* g\|^2 \Big)^{1/2}$$

$$= \|T\| \Big(\sum_{k=1}^{N} D_{b_k}^* D_{b_k} f, f \Big)^{1/2} \Big(\sum_{k=1}^{N} D_{a_k} D_{a_k}^* g, g \Big)^{1/2}$$

$$\leq \|T\| \Big\| \sum_{k=1}^{N} D_{b_k}^* D_{b_k} \Big\|^{1/2} \Big\| \sum_{k=1}^{N} D_{a_k} D_{a_k}^* \Big\|^{1/2} \|f\| \|g\|, \tag{1}$$

which proves the claim.

If φ is an arbitrary Schur multiplier and $\{a_k\}_{k \in \mathbb{N}}$ and $\{b_k\}_{k \in \mathbb{N}}$ are families arising in Grothendieck's description, then letting φ_N be the function given by $\varphi_N(i,j) = \sum_{k=1}^{N} a_k(i) b_k(j)$, $N \in \mathbb{N}$, we have that $\varphi_N \to \varphi$ pointwise and, moreover (using again Grothendieck's theorem), the norms $\|\varphi_N\|_{\mathrm{m}}$ are uniformly bounded in N. Thus, the characterisation of Grothendieck can be viewed as a uniform approximation result for Schur multipliers by "elementary" Schur multipliers.

There is another convenient formulation of Theorem 1.1 which links the subject of Schur multipliers to Operator Space Theory. Namely, the space of Schur multipliers can be identified with the extended Haagerup tensor product $\ell^\infty \otimes_{\mathrm{eh}} \ell^\infty$. For the time being, let us take this statement as the definition of the space $\ell^\infty \otimes_{\mathrm{eh}} \ell^\infty$; more on this will be said later.

Truncation of matrices has been important in applications. Suppose that we are given a matrix — either of finite size, or an element of M_∞. To truncate it along a certain subset $\kappa \subseteq \mathbb{N} \times \mathbb{N}$ means to replace it by the matrix that has the same entries on the subset κ and zeros everywhere else. Obviously, this is precisely the operation of Schur-multiplying a matrix $a \in M_\infty$ by χ_κ. Thus, truncation along κ is a well-defined (and automatically bounded) transformation on $\mathcal{B}(\ell^2)$ if and only if χ_κ is a Schur multiplier. The Schur multipliers of this form are precisely the idempotent elements of the Banach algebra $S(\mathbb{N}, \mathbb{N})$.

It is easy to exhibit idempotent Schur multipliers: if $\{\alpha_k\}_{k=1}^\infty$ and $\{\beta_k\}_{k=1}^\infty$ are families of pairwise disjoint subsets of $\mathbb{N}$ and $\kappa = \cup_{k=1}^\infty \alpha_k \times \beta_k$ then

$$S_{\chi_\kappa}(T) = \sum_{k=1}^{\infty} P_k T Q_k, \quad T \in \mathcal{B}(\ell^2),$$

where P_k (resp. Q_k) is the projection onto the closed span of $\{e_i : i \in \alpha_k\}$ (resp. $\{e_i : i \in \beta_k\}$), and hence $\|\chi_\kappa\|_{\mathrm{m}} \leq 1$. Since $S(\mathbb{N}, \mathbb{N})$ is an algebra with respect to pointwise addition and multiplication, the function χ_κ is a Schur multiplier for every set κ belonging to the subset ring generated by the sets of the above form. It is natural to ask whether the elements of this algebra are all idempotent Schur multipliers. Although the answer to this question is not known (in fact, the question is one of the difficult open problems in the area), a related result was recently established by Davidson and Donsig [13]. Let us call a subset $\kappa \subseteq \mathbb{N} \times \mathbb{N}$ a **Schur bounded pattern** if every function $\varphi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ supported on κ is a Schur multiplier. Obviously, if κ is a Schur bounded pattern then χ_κ is a Schur multiplier, and not vice versa (just take $\kappa = \mathbb{N} \times \mathbb{N}$). The aforementioned theorem reads as follows:

THEOREM 1.2. *Let $\kappa \subseteq \mathbb{N} \times \mathbb{N}$. The following are equivalent:*

(i) *κ is a Schur bounded pattern;*

(ii) there exist sets $\kappa_r, \kappa_c \subseteq \mathbb{N} \times \mathbb{N}$ and a number $N \in \mathbb{N}$ such that κ_r (resp. κ_c) has at most N entries in every row (resp. column) and $\kappa = \kappa_r \cup \kappa_c$.

In fact, the Davidson-Donsig results cover the more general case of non-negative functions $\varphi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ which are "hereditary Schur multipliers" (the terminology is ours) in the sense that if $\psi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ and $|\psi| \leq \varphi$ then ψ is a Schur multiplier.

Once there is a complete characterisation of Schur multipliers, it is natural to ask for a description of certain special classes of Schur multipliers. Say that a Schur multiplier φ is compact if the mapping $S_\varphi : \mathcal{B}(\ell^2) \to \mathcal{B}(\ell^2)$ is a compact operator. The following result was established by Hladnik in [24]:

THEOREM 1.3. *A Schur multiplier φ is compact if and only if a representation can be chosen for φ as in Theorem 1.1 (ii), such that $\{a_k\}_{k \in \mathbb{N}}$, $\{b_k\}_{k \in \mathbb{N}} \subseteq c_0$ and the series $\sum_{k=1}^\infty |a_k|^2$ and $\sum_{k=1}^\infty |b_k|^2$ are uniformly convergent.*

Positivity is another natural property that a Schur multiplier may (or may not) have: a Schur multiplier φ is called positive if $S_\varphi(A)$ is a positive operator for every positive operator A. The following holds:

THEOREM 1.4. *A Schur multiplier φ is positive if and only if there exists $\{a_k\}_{k \in \mathbb{N}} \subseteq \ell^\infty$ such that $\sup_{i \in \mathbb{N}} \sum_{k=1}^\infty |a_k(i)|^2 < \infty$ and $\varphi(i, j) = \sum_{k=1}^\infty a_k(i) \, \overline{a_k(j)}$, for all $i, j \in \mathbb{N}$.*

2. Schur multipliers over measure spaces.

If $\varphi \in \ell^\infty(\mathbb{N} \times \mathbb{N})$ then the operator on the Hilbert space $\ell^2(\mathbb{N} \times \mathbb{N})$ of multiplication by φ is bounded. If we equip $\ell^2(\mathbb{N} \times \mathbb{N})$ with the norm arising from its identification with the space $\mathcal{C}_2(\ell^2)$ of all Hilbert-Schmidt operators on ℓ^2, then it is easy to see that φ is a Schur multiplier if and only if this multiplication operator is bounded in the operator norm. This approach is useful because it allows us to study Schur multipliers in a more general setting. To describe this setting, let (X, μ) and (Y, ν) be standard σ-finite measure spaces. We equip $X \times Y$ with the product measure $\mu \times \nu$. The space $L^2(X \times Y)$ can be canonically identified with the space $\mathcal{C}_2(L^2(X), L^2(Y))$ of all Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$: if $f \in L^2(X \times Y)$, let T_f be the Hilbert-Schmidt operator given by

$$T_f \xi(y) = \int_X f(x, y) \xi(x) d\mu(x), \quad \xi \in L^2(X).$$

For $f \in L^2(X \times Y)$, let $\|f\|_{\text{op}}$ be the operator norm of T_f.

Now let $\varphi \in L^\infty(X \times Y)$. The operator $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ of multiplication by φ is bounded in the L^2-norm (its norm is equal to $\|\varphi\|_\infty$). If S_φ is moreover bounded in $\|\cdot\|_{\text{op}}$, that is, if there exists $C > 0$ such that $\|\varphi f\|_{\text{op}} \leq C\|f\|_{\text{op}}$ for every $f \in L^2(X \times Y)$, then we call φ a **Schur μ, ν-multiplier** (or simply a Schur multiplier if the measures are clear from the context). If $X = Y = \mathbb{N}$ is equipped with the counting measure, this new notion reduces to the one described in the previous section.

We note that the property of a function φ to be or not to be a Schur multiplier depends only on the values of the function up to a *null with respect to the product measure* set.

If $\varphi \in L^\infty(X \times Y)$ is a Schur multiplier then the mapping S_φ extends by continuity to a mapping on $\mathcal{K}(L^2(X), L^2(Y))$; after taking its second dual we arrive at a bounded weak* continuous linear transformation (which we denote in the same way) on $\mathcal{B}(L^2(X), L^2(Y))$.

The multiplier norm $\|\varphi\|_{\mathrm{m}}$ of φ is defined as the operator norm of S_φ. We denote by $S(X, Y)$ the set of all Schur multipliers. Clearly, $S(X, Y)$ is a subalgebra of $L^\infty(X \times Y)$ (with respect to pointwise multiplication).

Let $\mathcal{D}_1$ (resp. $\mathcal{D}_2$) be the multiplication masa of $L^\infty(X)$ (resp. $L^\infty(Y)$). We denote by M_a the element of $\mathcal{D}_1$ corresponding to the element $a \in L^\infty(X)$; we use a similar notation for the operators in $\mathcal{D}_2$. If $f \in L^2(X \times Y)$, $a \in L^\infty(X)$ and $b \in L^\infty(Y)$, then $M_b T_f M_a = T_{f(a \otimes b)}$, where $(a \otimes b)(x, y) = a(x)b(y)$, $x \in X$, $y \in Y$. It follows that S_φ is a $\mathcal{D}_2, \mathcal{D}_1$-bimodule map in the sense that $S_\varphi(BTA) = BS_\varphi(T)A$, for all $T \in \mathcal{B}(L^2(X), L^2(Y))$, $A \in \mathcal{D}_1$ and $B \in \mathcal{D}_2$.

A characterisation similar to the one of Grothendieck also holds in the measurable setting: the following result was established by Peller [31] (see also Spronk's paper [38]).

THEOREM 2.1. *Let $\varphi \in L^\infty(X \times Y)$. The following are equivalent:*

(i) *φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} < C$;*
(ii) *there exist families $\{a_k\}_{k \in \mathbb{N}} \subseteq L^\infty(X)$ and $\{b_k\}_{k \in \mathbb{N}} \subseteq L^\infty(Y)$ such that*

$$\operatorname*{esssup}_{x \in X} \sum_{k=1}^\infty |a_k(x)|^2 < C, \qquad \operatorname*{esssup}_{y \in Y} \sum_{k=1}^\infty |b_k(y)|^2 < C$$

and

$$\varphi(x, y) = \sum_{k=1}^\infty a_k(x)b_k(y), \quad \text{for almost all } (x, y) \in X \times Y.$$

We outline the proof of the theorem using results of Smith [37] and Haagerup [21]; this proof relies on the notion of complete boundedness whose mention is deliberately omitted for the time being but will be discussed in detail later.

Sketch of proof. Assume $(X, \mu) = (Y, \nu)$ and φ is a Schur μ, μ-multiplier with $\|\varphi\|_{\mathrm{m}} < C$. Then S_φ can be extended to a bounded $\mathcal{D}, \mathcal{D}$-bimodule map on $\mathcal{K}(L^2(X))$, where $\mathcal{D}$ is the multiplication masa of $L^\infty(X)$. By [37, Theorem 2.1, Theorem 3.1], there exist sequences $\{a_k\}_{k=1}^\infty$ and $\{b_k\}_{k=1}^\infty$ in $L^\infty(X)$ such that $\|\sum_{k=1}^\infty M_{b_k} M_{b_k}^*\| \|\sum M_{a_k}^* M_{a_k}\| = \|\varphi\|_{\mathrm{m}}^2$ and, for all $T \in \mathcal{K}(L^2(X))$,

$$S_\varphi(T) = \sum_{k=1}^\infty M_{b_k} T M_{a_k}, \tag{2}$$

where the sum on the right hand side converges in the strong operator topology. Let $\psi \in L^\infty(X \times X)$ be the function given by $\psi(x, y) = \sum_{k=1}^\infty a_k(x)b_k(y)$. For all $f \in L^2(X \times X)$ we have $S_\varphi(T_f) = T_{\varphi f}$ and $\sum_{k=1}^\infty M_{b_k} T_f M_{a_k} = T_{\psi f}$. We obtain that $\varphi(x, y)f(x, y) = \psi(x, y)f(x, y)$ for every $f \in L^2(X \times X)$ and this implies that $\varphi(x, y) = \psi(x, y)$ for almost all $(x, y) \in X \times X$.

To obtain the converse statement one should first note that for a function of the form $\varphi(x, y) = \sum_{k=1}^\infty a_k(x)b_k(y)$ we have that the operator $S_\varphi(T)$ is given by (2), for all $T \in \mathcal{C}_2(L^2(X))$. To complete the proof, one can apply estimation arguments similar to (1). ∎

Certain notions pertinent to Measure Theory that were initially introduced by Arveson [1] and later developed in [18] play an important role in the study of measurable Schur

multipliers. A subset $E \subseteq X \times Y$ is called **marginally null** if $E \subseteq (M \times Y) \cup (X \times N)$ for some null sets $M \subseteq X$ and $N \subseteq Y$. Two measurable sets $E, F \subseteq X \times Y$ are called **marginally equivalent** if their symmetric difference is marginally null. A measurable subset $\kappa \subseteq X \times Y$ is called ω-**open** if κ is marginally equivalent to a subset of the form $\cup_{i=1}^{\infty} \alpha_i \times \beta_i$, where $\alpha_i \subseteq X$ and $\beta_i \subseteq Y$ are measurable; κ is called ω-**closed** if its complement is ω-open. The collection of all ω-open subsets of $X \times Y$ is an ω-**topology**; that is, a family of sets closed under finite intersections and countable unions, containing the empty set and the set $X \times Y$. The described ω-topology plays an important role in the theory of masa-bimodules (see [18]). The morphisms of ω-topological spaces are ω-continuous mappings – a function $f : X \times Y \to \mathbb{C}$ is ω-**continuous** if $f^{-1}(U)$ is an ω-open set for every open subset $U \subseteq \mathbb{C}$. The following was shown in [29]:

PROPOSITION 2.2. *If φ is a μ, ν-multiplier then there exists an ω-continuous function ψ such that φ and ψ coincide almost everywhere.*

From this result, we immediately obtain that if χ_κ is an idempotent Schur multiplier, where $\kappa \subseteq X \times Y$ then κ is differs by a null set from a subset that is both ω-closed and ω-open. This fact was pointed out in [27].

The predual of $\mathcal{B}(L^2(X), L^2(Y))$ can be naturally identified with the projective tensor product $L^2(X) \hat{\otimes} L^2(Y)$. Suppose that $h \in L^\infty(X) \hat{\otimes} L^2(Y)$ and let $h = \sum_{k=1}^{\infty} f_k \otimes g_k$ be an associated series for h, where $\sum_{k=1}^{\infty} \|f_k\|_2^2 < \infty$ and $\sum_{k=1}^{\infty} \|g_k\|_2^2 < \infty$. These conditions easily imply that the formula

$$h(x, y) = \sum_{k=1}^{\infty} f_k(x) g_k(y), \quad (x, y) \in X \times Y,$$

defines a function (which we denote again by h). If the f_k's or the g_k's are replaced by some equivalent functions (with respect to the measures μ and ν) then the function h will change only on a marginally null set.

Let $\Gamma(X, Y)$ be the set of all functions h defined as above. Another useful result of Peller [31] identifies $S(X, Y)$ with the multiplier algebra of $\Gamma(X, Y)$. Before its formulation, let us agree to say that two measurable functions are equivalent if they agree almost everywhere.

PROPOSITION 2.3. *A function $\varphi \in L^\infty(X \times Y)$ is a Schur multiplier if and only if φh is equivalent to a function in $\Gamma(X, Y)$ for every $h \in \Gamma(X, Y)$.*

Proof. For $k \in L^2(X \times Y)$ and $h \in \Gamma(X, Y)$, one has $\langle T_k, T_h \rangle = \int k h d(\mu \times \nu)$. It follows that $h \in \Gamma(X, Y)$ if and only if

$$\exists C > 0 \text{ with } \left| \int k h d(\mu \times \nu) \right| \leq C \|T_k\|_{\mathrm{op}} \text{ for all } k \in L^2(X \times Y).$$

Now we have that if φ is a Schur multiplier then

$$\left| \int \varphi h k d(\mu \times \nu) \right| = |\langle S_\varphi(T_k), T_h \rangle| \leq |\varphi|_{\mathrm{m}} \|T_h\|_1 \|T_k\|_{\mathrm{op}},$$

and that this happens if and only if φh is equivalent to an element of $\Gamma(X, Y)$.

Conversely, if φh is equivalent to an element of $\Gamma(X, Y)$ for all $h \in \Gamma(X, Y)$, then by the Closed Graph Theorem, the mapping $T_h \mapsto T_{\varphi h}$ on $\mathcal{C}_1(L^2(X), L^2(Y))$ is bounded and

$$|\langle S_\varphi(T_k), T_h\rangle| = \left|\int \varphi h k d(\mu \times \nu)\right| = |\langle T_k, T_{\varphi h}\rangle| \le C\|T_k\|_{\mathrm{op}}\|T_h\|_1.$$

Thus, $\|S_\varphi(T_k)\|_{\mathrm{op}} \le C\|T_k\|_{\mathrm{op}}$. ∎

Measurable Schur multipliers are closely related to the theory of double operator integrals developed by Birman and Solomyak in a series of papers [3], [4], [5] (see also the survey paper [6]). We describe here briefly this connection. Let $E(\cdot)$ and $F(\cdot)$ be spectral measures defined on measure spaces X and Y, and taking values in the projection lattices of Hilbert spaces H and K, respectively. We fix a scalar valued measure μ (resp. ν) on X (resp. Y), equivalent to E (resp. F). A **double operator integral** is a formal expression of the form

$$\mathcal{I}_\varphi^{E,F}(T) = \int \varphi(x, y) dE(x) T dF(y), \tag{3}$$

where φ is an essentially bounded function defined on $X \times Y$, and $T \in \mathcal{B}(K, H)$. A precise meaning can be given to (3) as follows. Let G be the (unique) spectral measure defined on the product $X \times Y$ of the measure spaces X and Y and taking values in the projection lattice of the Hilbert space $\mathcal{C}_2(K, H)$, given on measurable rectangles by $G(\alpha \times \beta)(T) = E(\alpha) T F(\beta)$. For $\varphi \in L^\infty(X \times Y, \mu \times \nu)$ and $T \in \mathcal{C}_2(K, H)$, let

$$\mathcal{I}_\varphi^{E,F}(T) \stackrel{def}{=} \left(\int_{X \times Y} \varphi(x, y) dG(x, y)\right)(T).$$

Spectral Theory gives a precise meaning to the right hand side of the above expression, a well-defined element of $\mathcal{C}_2(K, H)$. Note that $\mathcal{I}_\varphi^{E,F}(T)$ depends only on the equivalence class (with respect to product measure) of the function φ. For some functions φ, there may exist a constant $C > 0$ such that

$$\|\mathcal{I}_\varphi^{E,F}(T)\|_{\mathrm{op}} \le C\|T\|_{\mathrm{op}}; \tag{4}$$

in this case the mapping $\mathcal{I}_\varphi^{E,F}$ extends uniquely to a bounded mapping on $\mathcal{K}(K, H)$ and, after taking second duals, to a bounded mapping on $\mathcal{B}(K, H)$.

Suppose for a moment that the spectral measures E and F are multiplicity free, that is, the (abelian) von Neumann algebras generated by their ranges are maximal. It is easy to note that in this case a function φ satisfies the inequality (4) for some constant $C > 0$ if and only if φ is a Schur multiplier.

If E possesses non-trivial multiplicity then one can decompose the Hilbert space H as a direct integral $H = \int_X H(x) d\mu(x)$ in such a way that the elements of the abelian von Neumann algebra generated by $E(\cdot)$ are precisely the diagonal operators. Similarly, $K = \int_Y K(y) d\nu(y)$. The space $\mathcal{C}_2(K, H)$ possesses a decomposition as $\mathcal{C}_2(K, H) = \int_{X \times Y} \mathcal{C}_2(K(y), H(x)) d\mu \times \nu(x, y)$. Thus, every element T of $\mathcal{C}_2(K, H)$ gives rise to an operator valued "kernel" $(T(x, y))_{x,y}$. We have that $\mathcal{I}_\varphi^{E,F}(T)$ is the operator with corresponding kernel $(\varphi(x, y) T(x, y))_{x,y}$. In this way, double operator integrals may again be realised as measurable Schur multipliers.

One application of double operator integrals is to perturbation theory, in particular, to the study of operators of the form $h(A) - h(B)$ depending on the properties of $A - B$,

where A, B are selfadjoint operators and h is a function defined on an interval containing the spectra of A and B.

Assume $h(x)$ is a uniformly Lipschitz function on $\mathbb{R}$, and let

$$\check{h}(x,y) = \frac{h(x) - h(y)}{x - y}.$$

This function is well-defined and bounded outside the diagonal $\{(x,x) : x \in \mathbb{R}\}$. Since the diagonal has measure zero, $\check{h}(x,y)$ is well defined up to a null set. Thus, extending it in an arbitrary fashion to a bounded function defined on the whole of $\mathbb{R} \times \mathbb{R}$ always yields the same element of $L^\infty(\mathbb{R} \times \mathbb{R})$ which we again denote by $\check{h}$.

THEOREM 2.4 ([6]). *Let $E(\cdot)$, $F(\cdot)$ be the spectral measures of A and B respectively. Suppose that $\check{h}$ satisfies (4). Then*

$$h(A) - h(B) = \mathcal{I}_{\check{h}}^{E,F}(A - B)$$

and hence

$$\|h(A) - h(B)\| \leq C\|A - B\|.$$

The property of $\check{h}$ being a Schur multiplier is closely related to a kind of "operator smoothness" of h. Let α be a compact set in $\mathbb{R}$. A continuous function h on α is called Operator Lipschitz on α if there is $D > 0$ such that

$$\|h(A) - h(B)\| \leq D\|A - B\|$$

for all selfadjoint operators A, B with spectra in α.

PROPOSITION 2.5. *Let I be a compact set. A function $\check{h}$ is a Schur multiplier on $I \times I$ if and only if h is Operator Lipschitz on I.*

The proof of this result is based on the result by Kissin and Shulman [28] that for all compact sets I of $\mathbb{R}$ a function h is Operator Lipschitz on I if and only if h is commutator bounded, that is, there exists $D > 0$ such that for any selfadjoint A with spectrum in I and any bounded operator X, the inequality

$$\|h(A)X - Xh(A)\| \leq D\|AX - XA\| \tag{5}$$

holds. Note also that if A is the multiplication operator $(Ag)(x) = xg(x)$ on $L^2(I, \mu)$ and $X = T_k$, where $k(x,y) = (x - y)k_1(x,y)$ for some $k_1 \in L^2(I \times I, \mu \times \mu)$, then (5) is equivalent to

$$\|S_{\check{h}}(T_k)\| \leq D\|T_k\|.$$

For other applications, in particular to differentiation of functions of selfadjoint operators, we refer the reader to [6], and for more results on applications of Schur multipliers to operator inequalities to [23].

As the connection with double operator integrals shows, the purpose of introducing measurable Schur multipliers is not limited to studying the notion in the greatest possible generality. We now further illustrate this by describing a connection with Harmonic Analysis. Let G be a locally compact group which we assume for technical simplicity to be σ-compact. We let $L^p(G)$, $p = 1, 2, \infty$, be the corresponding function spaces with

respect to the (left) Haar measure. By $\lambda : G \to \mathcal{B}(L^2(G))$ we denote the left regular representation of G; thus, $\lambda_s f(t) = f(s^{-1}t)$, $s, t \in G$, $f \in L^2(G)$. We recall that the Fourier algebra $A(G)$ of G is the space of all "matrix coefficients of G in its left regular representation", that is,

$$A(G) = \{s \to (\lambda_s \xi, \eta) : \xi, \eta \in L^2(G)\}.$$

If G is commutative, then $A(G)$ is the image of $L^1(\hat{G})$ under Fourier transform, where $\hat{G}$ is the dual group of G. The Fourier algebra of general locally compact groups was introduced and studied (along with other objects pertinent to Non-commutative Harmonic Analysis) by Eymard in [19]. It is a commutative regular semi-simple Banach algebra of continuous functions vanishing at infinity and has G as its spectrum. Moreover, its Banach space dual is isometric to the von Neumann algebra $\mathrm{VN}(G)$ of G, that is, the weakly closed subalgebra of $\mathcal{B}(L^2(G))$ generated by the operator λ_s, $s \in G$. The duality between these two spaces is given by the formula $\langle \lambda_x, f \rangle = f(x)$.

A function $\varphi \in L^\infty(G)$ is called a multiplier of $A(G)$ if $\varphi f \in A(G)$ for every $f \in A(G)$. The identification of the multipliers of $A(G)$ (which, as is easy to see, form a function algebra on G) has received a considerable attention in the literature. A classical result in this direction (see [35]) states that if G is abelian then φ is a multiplier of $A(G)$ if and only if it is the Fourier transform of a complex regular Borel measure on $\hat{G}$.

Up to date, a satisfactory characterisation of the multiplier algebra of $A(G)$ is not known. There is, however, a neat and useful characterisation of the subalgebra $M_{cb}A(G)$ of **completely bounded multipliers of** $A(G)$ introduced in [10]. In order to define these multipliers, we need the notions of an operator space and a completely bounded map.

An **operator space** is a complex vector space $\mathcal{X}$ for which a norm $\| \cdot \|_n$ is given on the space $M_n(\mathcal{X})$ of all n by n matrices with entries in $\mathcal{X}$ satisfying the following conditions, called Ruan's axioms:

(R1) $\|x \oplus y\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$, for all $m, n \in \mathbb{N}$ and all $x \in M_n(\mathcal{X})$ and $y \in M_m(\mathcal{X})$;

(R2) $\|\alpha \cdot x \cdot \beta\|_m \leq \|\alpha\|\|\beta\|\|x\|_n$, for all $x \in M_n(\mathcal{X})$, $\alpha \in M_{m,n}(\mathbb{C})$, $\beta \in M_{n,m}(\mathbb{C})$ and all $m, n \in \mathbb{N}$.

In property (R2), we have denoted by $\cdot$ the natural left action of $M_{m,n}$ on $M_n(\mathcal{X})$ as well as the natural right action of $M_{n,m}$ on $M_n(\mathcal{X})$. (Here $M_{k,l}$ is the space of all k by l complex matrices.) By $\|\alpha\|$ we mean the norm of the matrix $\alpha \in M_{m,n}$ when considered as an operator from $\mathbb{C}^n$ into $\mathbb{C}^m$ (where $\mathbb{C}^n$ and $\mathbb{C}^m$ are equipped with the ℓ^2-norm).

If $\mathcal{X}$ and $\mathcal{Y}$ are operator spaces and $\phi : \mathcal{X} \to \mathcal{Y}$ is a linear map, then one may consider the maps $\phi^{(n)} : M_n(\mathcal{X}) \to M_n(\mathcal{Y})$ defined by applying the mapping ϕ entry-wise. The map ϕ is called **completely bounded** if each $\phi^{(n)}$, $n \in \mathbb{N}$, is bounded, and $\sup_{n \in \mathbb{N}} \|\phi^{(n)}\| < \infty$. It is called a **complete isometry** if $\phi^{(n)}$ is an isometry for every $n \in \mathbb{N}$. Since complete isometries preserve all the given structure of an operator space, they constitute the right notion of an isomorphism in the category of operator spaces.

If $\mathcal{X}$ is an operator space and $\mathcal{X}^*$ is its dual Banach space, then the space $M_n(\mathcal{X}^*)$ can be naturally identified with the space $CB(\mathcal{X}, M_n(\mathbb{C}))$ of all completely bounded

linear maps from $\mathcal{X}$ into $M_n(\mathbb{C})$. If we equip $M_n(\mathcal{X}^*)$ with the norm coming from this identification, then the obtained family of norms turns $\mathcal{X}^*$ into an operator space.

If $\mathcal{X} \subseteq \mathcal{B}(H)$ for some Hilbert space H, then $M_n(\mathcal{X})$ is naturally embedded into $\mathcal{B}(H^n)$ and hence inherits its operator norm. The family of norms obtained in this way satisfies Ruan's axioms. Ruan's representation theorem asserts that every operator space is completely isometric to a subspace of $\mathcal{B}(H)$, for some Hilbert space H.

After this very brief introduction of the basic notions of Operator Space Theory, we can return to multipliers. From the last paragraph it follows that every von Neumann algebra, in particular $\mathrm{VN}(G)$, is an operator space. Hence its dual, and therefore its predual $A(G)$ (which is a subspace of its dual) possesses a canonical operator space structure. A multiplier φ of $A(G)$ is called completely bounded if the mapping $f \to \varphi f$ is completely bounded. The following was established by Bożejko and Fendler in [9] (see also [33]):

THEOREM 2.6. *If $\varphi \in L^\infty(G)$, let $\tilde{\varphi} \in L^\infty(G \times G)$ be the function given by $\tilde{\varphi}(s,t) = \varphi(s^{-1}t)$. A function $\varphi \in L^\infty(G)$ belongs to $M_{cb}A(G)$ if and only if the function $\tilde{\varphi}$ is a Schur multiplier with respect to the left Haar measure.*

3. Going non-commutative. Let (X, μ) and (Y, ν) be standard (σ-finite) measure spaces. It is immediate from the definitions that if $\varphi, \psi \in S(X, Y)$ then $S_\varphi S_\psi = S_{\varphi\psi}$. Thus, $S_\varphi S_\psi = S_\psi S_\varphi$ for every $\varphi, \psi \in S(X, Y)$; in other words, the collection of all mappings $\{S_\varphi : \varphi \in S(X, Y)\}$ is commutative. In view of the contemporary trend in Functional Analysis to seek non-commutative versions of "classical" notions and results [7], [16], [30], [34], it is natural to ask whether there is a natural non-commutative, or "quantised" version of Schur multipliers. This question was pursued by Kissin and Shulman in [29], and is the topic of this section.

For a Hilbert space H, we write H^{d} for the (Banach space) dual of H. There exists a conjugate-linear surjective isometry $\partial : H \to H^{\mathrm{d}}$ given by $\partial(x)(y) = (y, x)$, $x, y \in H$.

Let H and K be Hilbert spaces and $\theta : H \otimes K \to \mathcal{C}_2(H^{\mathrm{d}}, K)$ be the natural surjective isometry from the Hilbert space tensor product $H \otimes K$ onto the space $\mathcal{C}_2(H^{\mathrm{d}}, K)$ of all Hilbert-Schmidt operators from H^{d} into K given by $\theta(x \otimes y)(z^{\mathrm{d}}) = (x, z)y$. This identification allows us to equip $H \otimes K$ with an "operator" norm: if $\xi \in H \otimes K$, let $\|\xi\|_{\mathrm{op}} = \|\theta(\xi)\|_{\mathrm{op}}$. We call an element $\varphi \in \mathcal{B}(H \otimes K)$ a **concrete operator multiplier** if there exists a constant $C > 0$ such that $\|\varphi\xi\|_{\mathrm{op}} \leq C\|\xi\|_{\mathrm{op}}$, for every $\xi \in H \otimes K$. We call the smallest possible constant C with this property the concrete multiplier norm of φ. It follows from this definition that the set $\mathfrak{M}(H, K)$ of all concrete operator multipliers on $H \otimes K$ is a subalgebra of $\mathcal{B}(H \otimes K)$. It is also immediate that if $H = L^2(X, \mu)$ and $K = L^2(Y, \nu)$ for some standard measure spaces (X, μ) and (Y, ν) and φ is the multiplication operator on $L^2(X \times Y) = L^2(X) \otimes L^2(Y)$ corresponding to a function $\tilde{\varphi} \in L^\infty(X \times Y)$ then φ is a concrete operator multiplier if and only if $\tilde{\varphi}$ is a Schur μ, ν-multiplier. Thus, the algebra $\mathfrak{M}(H, K)$ contains $S(X, Y)$ as a commutative subalgebra. Note that there are "many" commutative subalgebras of $\mathfrak{M}(H, K)$ of this form, one for each realisation of H and K as L^2-spaces over some standard measure spaces.

In the commutative theory, a special attention has been paid to the case where the measure spaces (X, μ) and (Y, ν) are regular Borel spaces of complete metrisable topologies, and the multipliers φ are continuous functions on $X \times Y$. The non-commutative expression of continuous functions is given in terms of C*-algebras. Therefore, it is natural to extend the setting of concrete operator multipliers given above as follows. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C*-algebras and $\pi : \mathcal{A} \to \mathcal{B}(H_\pi)$ and $\rho : \mathcal{B} \to \mathcal{B}(H_\rho)$ are *-representations. It is well-known that there exists a unique *-representation $\pi \otimes \rho$ of the minimal tensor product $\mathcal{A} \otimes \mathcal{B}$ of $\mathcal{A}$ and $\mathcal{B}$ on $H_\pi \otimes H_\rho$. An element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is called a π, ρ-**multiplier** [29] if $(\pi \otimes \rho)(\varphi)$ is a concrete operator multiplier. We let $\|\varphi\|_{\pi,\rho}$ be the concrete operator multiplier norm of $(\pi \otimes \rho)(\varphi)$.

Let $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$ be the set of all π, ρ-multipliers. It is immediate that $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$ is a subalgebra of $\mathcal{A} \otimes \mathcal{B}$ containing the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$. To see the last statement, note that if $a \in \mathcal{A}$ and $b \in \mathcal{B}$ then $\theta((\pi \otimes \rho)(a \otimes b)(\xi)) = \rho(b)\theta(\xi)\pi(a)^{\mathrm{d}}$, for every $\xi \in H_\pi \otimes H_\rho$; we hence have that $\|a \otimes b\|_{\pi,\rho} \leq \|a\|\|b\|$.

We let $\mathfrak{M}(\mathcal{A}, \mathcal{B}) = \cap_{\pi,\rho}\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$ where the intersection is taken over all representations π of $\mathcal{A}$ and ρ of $\mathcal{B}$. The elements of $\mathfrak{M}(\mathcal{A}, \mathcal{B})$ are called **universal multipliers**. By the previous paragraph, every element of the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ is a universal multiplier. It is not difficult to see that if $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ then $\|\varphi\|_{\mathrm{m}} \stackrel{def}{=} \sup_{\pi,\rho} \|\varphi\|_{\pi,\rho}$ is finite; we call $\|\varphi\|_{\mathrm{m}}$ the (universal) multiplier norm of φ.

Two immediate questions arise:

(a) How does the algebra $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$ depend on π and ρ?

(b) Is there a characterisation of its elements extending Grothendieck-Peller's characterisation of Schur multipliers?

It was observed by Kissin and Shulman [29] that (a) is related to the notion of approximate equivalence of representations due to Voiculescu [41] and its extension, the approximate sub-ordinance introduced by Hadwin [22]. We recall these notion here. Let π and π' be $*$-representations of a C^*-algebra $\mathcal{A}$ on Hilbert spaces H and H', respectively. We say that π' is *approximately subordinate* to π and write $\pi' \stackrel{a}{\ll} \pi$ if there is a net $\{U_\lambda\}$ of isometries from H' to H such that

$$\|\pi(a)U_\lambda - U_\lambda \pi'(a)\| \to 0 \text{ for all } a \in \mathcal{A}. \tag{6}$$

The representations π' and π are said to be *approximately equivalent* if the operators U_λ can be chosen to be unitary; in this case we write $\pi' \stackrel{a}{\sim} \pi$.

The following result was established in [29]:

THEOREM 3.1 (Comparison Theorem). *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and π, π' (resp. ρ, ρ') be representations of $\mathcal{A}$ (resp. $\mathcal{B}$). Suppose that $\pi' \stackrel{a}{\ll} \pi$ and $\rho' \stackrel{a}{\ll} \rho$. Then $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B}) \subseteq \mathfrak{M}^{\pi',\rho'}(\mathcal{A}, \mathcal{B})$. Moreover, if $\varphi \in \mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$ then $\|\varphi\|_{\pi',\rho'} \leq \|\varphi\|_{\pi,\rho}$.*

In particular, if $\pi' \stackrel{a}{\sim} \pi$ and $\rho' \stackrel{a}{\sim} \rho$ then $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B}) = \mathfrak{M}^{\pi',\rho'}(\mathcal{A}, \mathcal{B})$ and $\|\varphi\|_{\pi',\rho'} = \|\varphi\|_{\pi,\rho}$ for every $\varphi \in \mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$.

We note that, by [22], $\pi' \stackrel{a}{\ll} \pi$ if and only if

$$\operatorname{rank} \pi'(a) \leq \operatorname{rank} \pi(a), \quad \text{for every } a \in \mathcal{A}.$$

Theorem 3.1 has some interesting consequences about measurable and classical Schur multipliers [29]:

COROLLARY 3.2. *Let X and Y be locally compact Hausdorff spaces with countable bases, and let μ and μ' (resp. ν and ν') be σ-finite Borel measures on X (resp. Y). Suppose that $\operatorname{supp}\mu' \subseteq \operatorname{supp}\mu$ and $\operatorname{supp}\nu' \subseteq \operatorname{supp}\nu$. Then every μ,ν-multiplier in $C_0(X \times Y)$ is also a μ',ν'-multiplier.*

In particular, if $\operatorname{supp}\mu = X$ and $\operatorname{supp}\nu = Y$ then an element $\varphi \in C_0(X \times Y)$ is a μ,ν-multiplier if and only if φ is a classical Schur multiplier on $X \times Y$.

There is a version of the last result for functions φ that are not required to be continuous [29]:

THEOREM 3.3. *Let (X, μ) and (Y, ν) be standard σ-finite measure spaces and let $\varphi \in L^\infty(X \times Y)$ be an ω-continuous function. The following are equivalent:*

(i) φ is a μ,ν-multiplier;

(ii) there exist null sets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ such that the restriction $\tilde{\varphi}$ of φ to $(X \setminus X_0) \times (Y \setminus Y_0)$ is a classical Scur multiplier.

Moreover, if (i) holds then the sets X_0 and Y_0 can be chosen in such a way that the μ,ν-multiplier norm of φ equals the classical Schur multiplier norm of $\tilde{\varphi}$.

We now address Question (b) above concerning the characterisation of operator multipliers. At the moment, no such characterisation is known for the classes $\mathfrak{M}^{\pi,\rho}(\mathcal{A}, \mathcal{B})$. The reason lies in the lack of complete boundedness, which we now explain. Suppose that $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$ are concrete C*-algebras, and take an element $\varphi \in \mathfrak{M}^{\mathrm{id},\mathrm{id}}(\mathcal{A}, \mathcal{B})$, where id denotes the identity representations of $\mathcal{A}$ and $\mathcal{B}$ on H and K, respectively. Since φ is a concrete operator multiplier on $H \otimes K$, we have that the mapping $S_\varphi : \mathcal{C}_2(H^{\mathrm{d}}, K) \to \mathcal{C}_2(H^{\mathrm{d}}, K)$ given by $S_\varphi(\theta(\xi)) = \theta(\varphi\xi)$, has a canonical extension to a bounded mapping (denoted in the same way) $S_\varphi : \mathcal{K}(H^{\mathrm{d}}, K) \to \mathcal{K}(H^{\mathrm{d}}, K)$. By passing to second duals, we arrive at a bounded mapping $S_\varphi^{**} : \mathcal{B}(H^{\mathrm{d}}, K) \to \mathcal{B}(H^{\mathrm{d}}, K)$. In general, however, the mappings S_φ and S_φ^{**} need not be completely bounded. If φ is assumed to lie in the smaller class $\mathfrak{M}(\mathcal{A}, \mathcal{B})$ of universal multipliers, then the mappings S_φ and S_φ^{**} turn out to be completely bounded. This can be seen by considering the n-fold ampliations $\mathrm{id}^{(n)}$ of the identity representations of $\mathcal{A}$ and $\mathcal{B}$. In fact, the following statement [25] shows that in order to decide whether an element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is a universal multiplier, it suffices to check that it is an $\mathrm{id}^{(n)}, \mathrm{id}^{(n)}$-multiplier for all $n \in \mathbb{N}$. The proof of this result uses the Comparison Theorem 3.1.

PROPOSITION 3.4. *Let $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$. An element $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is a universal multiplier if and only if the (weak* continuous) mapping S_φ^{**} is completely bounded.*

The structure of normal (that is, weak* continuous) completely bounded maps on $\mathcal{B}(K, H)$ is well understood. We recall a well-known result of Haagerup [21]: a mapping $\Phi : \mathcal{B}(K, H) \to \mathcal{B}(K, H)$ is normal and completely bounded if and only if there exist families $\{a_k\}_{k=1}^\infty \subseteq \mathcal{B}(H)$ and $\{b_k\}_{k=1}^\infty \subseteq \mathcal{B}(K)$ such that the series $\sum_{k=1}^\infty a_k a_k^*$ and

$\sum_{k=1}^{\infty} b_k^* b_k$ are weak* convergent and

$$\Phi(x) = \sum_{k=1}^{\infty} a_k x b_k, \quad \text{for all } x \in \mathcal{B}(K, H).$$

Thus, one may associate with every normal completely bounded map Φ on $\mathcal{B}(K, H)$ a formal series $\sum_{k=1}^{\infty} a_k \otimes b_k$ where the families $\{a_k\}_{k=1}^{\infty} \subseteq \mathcal{B}(H)$ and $\{b_k\}_{k=1}^{\infty} \subseteq \mathcal{B}(K)$ are assumed to satisfy the above convergence conditions. Two such formal series are identified if the corresponding mappings are equal. The collection of all such series is known as the weak* (or the extended) Haagerup tensor product of $\mathcal{B}(H)$ and $\mathcal{B}(K)$ and denoted by $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$. In fact, $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$ can be viewed as a certain weak completion of the algebraic tensor product $\mathcal{B}(H) \odot \mathcal{B}(K)$ (see [8] where this tensor product was introduced).

The weak* Haagerup tensor product can be defined for any pair of dual operator spaces $\mathcal{X}^*$ and $\mathcal{Y}^*$ and is the dual operator space of the **Haagerup tensor product** $\mathcal{X} \otimes_{\mathrm{h}} \mathcal{Y}$ of $\mathcal{X}$ and $\mathcal{Y}$. The latter is the tensor product that linearises completely bounded bilinear mappings. These are defined as follows: Suppose that $\phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ is a bilinear mapping, where $\mathcal{Z}$ is another operator space. One may define the mappings $\phi^{(n)} : M_n(\mathcal{X}) \times M_n(\mathcal{Y}) \to M_n(\mathcal{Z})$, $n \in \mathbb{N}$, by $\phi^{(n)}((x_{i,j}), (y_{k,l})) = (\sum_{k=1}^{n} \phi(x_{i,k}, y_{k,j})_{i,j}$. The mapping ϕ is called completely bounded if $\|\phi\|_{cb} \overset{def}{=} \sup_{n \in \mathbb{N}} \|\phi^{(n)}\| < \infty$. The operator space $\mathcal{X} \otimes_{\mathrm{h}} \mathcal{Y}$ has the property that for every completely bounded bilinear mapping $\phi : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ the linearised mapping $\tilde{\phi}$ is completely bounded as a map from $\mathcal{X} \otimes_{\mathrm{h}} \mathcal{Y}$ into $\mathcal{Z}$ with the same cb norm. We note that we will introduce later a generalisation of the above notion of complete boundedness to multilinear maps.

The correspondence between elements of $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$ and normal completely bounded mappings on $\mathcal{B}(K, H)$ is bijective; if $u \in \mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$, we write Φ_u for the corresponding mapping. The space $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$ is an operator space in its own right (indeed, we have the completely isometric identification $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K) = (\mathcal{C}_1(H) \otimes_{\mathrm{h}} \mathcal{C}_1(K))^*)$, and the norm $\|u\|_{\mathrm{eh}}$ of an element u is equal to the completely bounded norm of Φ_u.

The extended Haagerup tensor product can be defined for every pair of operator spaces [17]. To do this, we follow the approach given in [38]. Let $\mathcal{E} \subseteq \mathcal{B}(H)$ and $\mathcal{F} \subseteq \mathcal{B}(K)$ be *norm closed* subspaces. Then the extended Haagerup tensor product $\mathcal{E} \otimes_{\mathrm{eh}} \mathcal{F}$ of $\mathcal{E}$ and $\mathcal{F}$ is the subspace

$$\{u \in \mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K) : \mathrm{id} \otimes \omega(u) \in \mathcal{E}, \tau \otimes \mathrm{id}(u) \in \mathcal{F}, \ \forall \ \omega \in \mathcal{B}(K)_*, \tau \in \mathcal{B}(H)_*\}.$$

Here, $\mathrm{id} \otimes \omega$ (resp. $\tau \otimes \mathrm{id}$) is the left (resp. right) slice map from $\mathcal{B}(H) \otimes_{\mathrm{eh}} \mathcal{B}(K)$ into $\mathcal{B}(H)$ (resp. $\mathcal{B}(K)$) along the functional ω (resp. τ). The fact that these maps are well-defined needs a justification that we omit.

The extended Haagerup tensor product of operator spaces is functorial: if $f : \mathcal{E} \to \mathcal{E}'$ and $g : \mathcal{F} \to \mathcal{F}'$ are completely bounded maps then there exists a unique completely bounded map $f \otimes g : \mathcal{E} \otimes_{\mathrm{eh}} \mathcal{F} \to \mathcal{E}' \otimes_{\mathrm{eh}} \mathcal{F}'$ given on elementary tensors by $(f \otimes g)(a \otimes b) = f(a) \otimes g(b)$. Moreover, if f and g are complete isometries then so is $f \otimes g$ [17].

We now return to universal multipliers. Recall that we have fixed two C*-algebras $\mathcal{A} \subseteq \mathcal{B}(H)$ and $\mathcal{B} \subseteq \mathcal{B}(K)$. Suppose that $\varphi \in \mathcal{A} \otimes \mathcal{B}$ is a universal multiplier. By

Proposition 3.4, the map $S_\varphi^{**} : \mathcal{B}(H^{\mathrm{d}}, K) \to \mathcal{B}(H^{\mathrm{d}}, K)$ is completely bounded. Thus, by Haagerup's result described above, there exists an element $u \in \mathcal{B}(K) \otimes_{\mathrm{eh}} \mathcal{B}(H^{\mathrm{d}})$ such that $S_\varphi^{**} = \Phi_u$. It was shown in [26] that, in fact, u lies in the extended Haagerup tensor product $\mathcal{B} \otimes_{\mathrm{eh}} \mathcal{A}^{\mathrm{d}}$, and that it does not depend on the concrete representations of the C*-algebras $\mathcal{A}$ and $\mathcal{B}$ that we started with. More precisely, the following "symbolic calculus" result holds. (We denote by $\mathcal{A}^o$ the *opposite* C*-algebra of $\mathcal{A}$ which coincides with $\mathcal{A}$ as an involutive normed linear space but is equipped with the product $a \circ b = ba$. For a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ we let $\pi^{\mathrm{d}} : \mathcal{A}^o \to \mathcal{B}(H^{\mathrm{d}})$ be the representation given by $\pi^{\mathrm{d}}(a^o) = \pi(a)^{\mathrm{d}}$.)

THEOREM 3.5 (Symbolic Calculus for universal multipliers). *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. There exists an injective homomorphism $\varphi \to u_\varphi$ from $\mathfrak{M}(\mathcal{A}, \mathcal{B})$ into $\mathcal{B} \otimes_{\mathrm{eh}} \mathcal{A}^o$ with the following universal property: if $\pi : \mathcal{A} \to \mathcal{B}(H)$ and $\rho : \mathcal{B} \to \mathcal{B}(K)$ are *-representations then*

$$S_{\pi \otimes \rho(\varphi)}^{**} = \Phi_{\rho \otimes \pi^{\mathrm{d}}(u_\varphi)}.$$

Moreover, $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$, and $u_{a \otimes b} = b \otimes a^o$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$.

We call the element u_φ the **symbol** of the universal multiplier φ. The term "symbolic calculus" was first used in the context of Schur multipliers by Katavolos and Paulsen [27] where they explored the correspondence between a measurable Schur multiplier φ and the mapping S_φ.

Suppose now that $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$. The corresponding symbol u_φ has an associated series $\sum_{i=1}^\infty b_i \otimes a_i^o$, where $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$, $i \in \mathbb{N}$. Let $u_N = \sum_{i=1}^N b_i \otimes a_i^o$ and $\varphi_N = \sum_{i=1}^N a_i \otimes b_i \in \mathcal{A} \odot \mathcal{B}$, $N \in \mathbb{N}$. By Symbolic Calculus, $u_{\varphi_N} = u_N$; moreover, $u_\varphi = \text{w}^*\text{-}\lim_{N \to \infty} u_N$. We also have that $\|\varphi_N\|_{\mathrm{m}} = \|u_N\|_{\mathrm{eh}} \le \|u_\varphi\|_{eh}$ for all $N \in \mathbb{N}$. These observations can be used to prove the next result which is the appropriate generalisation of Grothendieck's and Peller's theorems. Before its formulation, we note that it is easy to see that, for an element $v \in \mathcal{B} \odot \mathcal{A}^o$, we have

$$\|v\|_{\mathrm{eh}} = \inf \left\{ \Big\| \sum_{i=1}^k d_i d_i^* \Big\|^{\frac{1}{2}} \Big\| \sum_{i=1}^k c_i c_i^* b \Big\|^{\frac{1}{2}} : v = \sum_{i=1}^k d_i \otimes c_i^o \right\}.$$

For en element $\psi = \sum_{i=1}^k c_i \otimes d_i \in \mathcal{A} \odot \mathcal{B}$, we let $\|\psi\|_{\mathrm{ph}} = \| \sum_{i=1}^k d_i \otimes c_i^o \|_{\mathrm{eh}}$.

THEOREM 3.6 (Characterisation Theorem). *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\varphi \in \mathcal{A} \otimes \mathcal{B}$. The following statements are equivalent:*

(i) $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ and $\|\varphi\|_{\mathrm{m}} < C$;

(ii) there exists a net $\{\varphi_\alpha\} \subseteq \mathcal{A} \odot \mathcal{B}$ such that $\|\varphi_\alpha\|_{\mathrm{ph}} < C$ for all α and $(\pi \otimes \rho)(\varphi_\alpha) \to_\alpha (\pi \otimes \rho)(\varphi)$ weakly, for every pair π, ρ of irreducible representations of $\mathcal{A}$ and $\mathcal{B}$.

In the case $\mathcal{A} = C(X)$ and $\mathcal{B} = C(Y)$ are (unital) C*-algebras (X and Y being compact Hausdorff spaces), we obtain the following fact as a consequence of the Characterisation Theorem, which shows that it does extend Theorem 2.1 to the non-commutative case: If μ and ν are regular Borel measures on X and Y, respectively, then a function $\varphi \in C(X \times Y)$ is a μ, ν-multiplier if and only if there exist families $\{a_i\}_{i=1}^\infty \subseteq C(X)$ and

$\{b_i\}_{i=1}^{\infty} \subseteq C(Y)$ such that, if $\varphi_k \in C(X \times Y)$ is given by $\varphi_k(x, y) = \sum_{i=1}^{k} a_i(x)b_i(y)$, $(x, y) \in X \times Y$, then $\sup_{k \in \mathbb{N}} \|\varphi_k\|_{\mathrm{m}} < \infty$ and $\varphi_k \to \varphi$ pointwise $\mu \times \nu$-almost everywhere.

We now turn our attention to a subclass of universal multipliers; in order to define it we recall a notion of compactness of completely bounded maps introduced by Saar in [36]. Let $\mathcal{E}$ and $\mathcal{F}$ be operator spaces and $\Phi : \mathcal{E} \to \mathcal{F}$ be a completely bounded map. One calls Φ **completely compact** if for every $\epsilon > 0$ there exists a finite dimensional subspace $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\mathrm{dist}(\Phi^{(n)}(x), M_n(\mathcal{F}_0)) < \epsilon$, for every x in the unit ball of $M_n(\mathcal{X})$, and for every $n \in \mathbb{N}$. Clearly, every completely compact map is compact.

The following characterisation of completely compact maps on $\mathcal{K}(H)$ was first obtained by Saar in [36] and later in [26] using a different method.

THEOREM 3.7. *A completely bounded map* $\Phi : \mathcal{K}(H) \to \mathcal{K}(H)$ *is completely compact if and only if there exist sequences* $\{a_i\}, \{b_i\} \subseteq \mathcal{K}(H)$ *such that the series* $\sum_{i=1}^{\infty} b_i b_i^*$ *and* $\sum_{i=1}^{\infty} a_i^* a_i$ *are norm convergent and*

$$\Phi(x) = \sum_{i=1}^{\infty} b_i x a_i, \qquad \text{for all } x \in \mathcal{K}(H).$$

We call an element $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ a **compact** (resp. **completely compact**) multiplier if there exist faithful representations π and ρ of $\mathcal{A}$ and $\mathcal{B}$, respectively, such that the mapping $S_{\pi \otimes \rho(\varphi)}^{**}$ is compact (resp. completely compact). It is clear that every completely compact multiplier is compact. It was shown in [26] that an element $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ is (completely) compact if and only if the mapping S_{π_a, ρ_a}^{**} is (completely) compact, where π_a (resp. ρ_a) is the reduced atomic representation of $\mathcal{A}$ (resp. $\mathcal{B}$). This is rather natural to expect: let us recall a result of Ylinen concerning the compact elements of a C*-algebra. An element $a \in \mathcal{A}$ is called **compact** if the mapping $x \to axa$ on $\mathcal{A}$ is compact. Ylinen [42] showed that an element $a \in \mathcal{A}$ is compact if and only if there exists a faithful representation π of $\mathcal{A}$ such that $\pi(a)$ is a compact operator; moreover, this happens if and only if $\pi_a(a)$ is a compact operator.

Let us denote by $\mathcal{K}(\mathcal{A})$ the set of all compact elements of $\mathcal{A}$; it is well-known that $\mathcal{K}(\mathcal{A})$ is a closed two sided ideal of $\mathcal{A}$. By virtue of Ylinen's result, $\mathcal{K}(\mathcal{A})$ is *-isomorphic to a C*-algebra of compact operators, and is hence *-isomorphic to a c_0-direct sum of the form $\oplus_{j \in J}^{c_0} \mathcal{K}(H_j)$, for some index set J and some Hilbert spaces H_j, $j \in J$.

In the theorem that follows, we view the Haagerup tensor product $\mathcal{E} \otimes_{\mathrm{h}} \mathcal{F}$ of two operator spaces $\mathcal{E}$ and $\mathcal{F}$ as sitting completely isometrically in their extended Haagerup tensor product $\mathcal{E} \otimes_{\mathrm{eh}} \mathcal{F}$.

THEOREM 3.8. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be C*-algebras and* $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$. *The following statements are equivalent:*

(i)   *φ is a completely compact multiplier;*
(ii) *$u_\varphi \in \mathcal{K}(\mathcal{B}) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}^o)$;*
(iii) *there exists a sequence* $\{\varphi_k\}_{k=1}^{\infty} \subseteq \mathcal{K}(\mathcal{A}) \odot \mathcal{K}(\mathcal{B})$ *such that* $\|\varphi - \varphi_k\|_{\mathrm{m}} \to_{k \to \infty} 0$.

Of course, it is natural to ask what happens if φ is only assumed to be a compact multiplier. One can show [26, Proposition 7.1] that if $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ is a compact multiplier then $u_\varphi \in \mathcal{K}(\mathcal{B}) \otimes_{\mathrm{eh}} \mathcal{K}(\mathcal{A}^o)$; however, the converse statement fails. We do not have at

present a complete characterisation of the compact universal multipliers; however, the following "automatic complete compactness" result holds:

THEOREM 3.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Assume that $\mathcal{K}(\mathcal{A}) \simeq \oplus_{i \in J_1}^{c_0} M_{n_i}$ and $\mathcal{K}(\mathcal{B}) \simeq \oplus_{j \in J_2}^{c_0} M_{m_j}$, where $\sup_{i \in J_1} n_i$ and $\sup_{j \in J_2} m_j$ are both finite. Then every compact multiplier $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ is automatically completely compact.*

We now see that Theorem 3.8 generalises Hladnik's description of compact Schur multipliers (Theorem 1.3) since in this case, by Theorem 3.9, every compact multiplier is automatically completely compact.

In the case that $\sup_{i \in J_1} n_i$ and $\sup_{j \in J_2} m_j$ are both infinite, we were able to exhibit in [26] a compact multiplier $\varphi \in \mathfrak{M}(\mathcal{A}, \mathcal{B})$ that is not completely compact. The construction is based on an example of Saar [36] of a compact completely bounded map on $\mathcal{K}(H)$ which is not completely compact.

In the last result that we mention in this section, an answer is provided to the question of when every universal multiplier is automatically compact. It is not surprising that finite dimensionality is crucial for this to happen.

THEOREM 3.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. The following statements are equivalent:*

(i) every element of $\mathfrak{M}(\mathcal{A}, \mathcal{B})$ is a compact multiplier;

(ii) either $\mathcal{A}$ is finite dimensional and $\mathcal{K}(\mathcal{B}) = \mathcal{B}$ or $\mathcal{B}$ is finite dimensional and $\mathcal{A} = \mathcal{K}(\mathcal{A})$.

The proof is based, in particular, on a result of Varopoulos [40] showing that if X and Y are infinite compact Hausdorff spaces then there exists a sequence $(f_i)_{i=1}^{\infty} \subseteq C(X) \otimes_{\mathrm{h}} C(Y)$ such that $\sup_{i \in \mathbb{N}} \|f_i\|_{\mathrm{h}} < \infty$ which converges uniformly to a function $f \in C(X \times Y) \setminus C(X) \otimes_{\mathrm{h}} C(Y)$. By the Characterisation Theorem 3.6, such an f must belong to $\mathfrak{M}(C(X), C(Y))$.

4. Going multidimensional.
In the present section we discuss a multidimensional version of Schur and operator multipliers that was introduced in [25] and later developed in [26].

If $R_1, \ldots, R_{n+1}$ are rings, M_i is a R_i-left and R_{i+1}-right module for each $i = 1, \ldots, n$, and M is an R_1-left and R_{n+1}-right module, a multilinear map $\Phi : M_1 \times \cdots \times M_n \to M$ is called $R_1, \ldots, R_{n+1}$-modular (or simply modular if $R_1, \ldots, R_{n+1}$ are clear from the context) if

$$\Phi(a_1 m_1 a_2, m_2 a_3, \ldots, m_n a_{n+1}) = a_1 \Phi(m_1, a_2 m_2, a_3 m_3, \ldots, a_n m_n) a_{n+1},$$

for all $m_i \in M_i$ $(i = 1, \ldots, n)$ and $a_j \in R_j$ $(j = 1, \ldots, n+1)$.

A multilinear Schur product was introduced by Effros and Ruan [15] as a multilinear map $T : \underbrace{M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C})}_{r} \to M_n(\mathbb{C})$ which is $\underbrace{D_n, \ldots, D_n}_{r+1}$-modular, where D_n is the algebra of all diagonal matrices in $M_n(\mathbb{C})$.

It is not difficult to see that any such mapping T has the form

$$T(a^r, \ldots, a^1)_{i,j} = \sum_{(k_1, \ldots, k_{r-1})} A_{i,j}^{k_{r-1} \ldots k_1} a_{i,k_{r-1}}^r a_{k_{r-1}, k_{r-2}}^{r-1} \ldots a_{k_1 j}^1,$$

where $a^i = (a^i_{k,l})_{k,l}$ and, given the usual matrix units $e_{i,j} \in M_n(\mathbb{C})$,

$$A^{k_{r-1}\ldots k_1}_{i,j} = T(e_{i,k_{r-1}}, e_{k_{r-1},k_{r-2}}\ldots e_{k_1,j})_{i,j}.$$

The following theorem gives a characterisation of all bounded multilinear Schur products.

THEOREM 4.1. *Suppose* $T : \underbrace{M_n(\mathbb{C}) \times \cdots \times M_n(\mathbb{C})}_{r} \to M_n(\mathbb{C})$ *is a multilinear Schur product map. Then the following are equivalent:*

(i) the linearisation of T *is a contraction for the Haagerup norm;*

(ii) there exists a Hilbert space H, $2n$ *contractions* $a_1(j) \in \mathcal{B}(H,\mathbb{C})$, $a_r(i) \in \mathcal{B}(\mathbb{C},H)$, $i,j = 1,\ldots,n$ *and* n^{r-1} *contractions* $a_l(k) \in \mathcal{B}(H)$, $l = 2,\ldots,r-1$, $k = 1,\ldots,n$ *such that*

$$A^{k_{r-1}\ldots k_1}_{i,j} = a_r(i)a_{r-1}(k_{r-1})\ldots a_2(k_1)a_1(j).$$

The theorem was proved in [15] for a complete contraction T. A generalisation of Smith's result [37, Theorem 2.1] to the mutlidimensional setting, [25, Lemma 3.3], giving that any D_n, D_n-modular contraction T is a complete contraction allows us to formulate the statement in this generality. More about completely bounded multilinear maps will be said later.

We now introduce multidimenisonal measurable Schur multipliers following [25].

Let (X_i, μ_i), $i = 1,\ldots,n$, be standard σ-finite measure spaces. For notational convenience, integration with respect to μ_i will be denoted by dx_i. Let

$$\Gamma(X_1,\ldots,X_n) = L^2(X_1 \times X_2) \odot L^2(X_2 \times X_3) \odot \cdots \odot L^2(X_{n-1} \times X_n),$$

where each product $X_i \times X_{i+1}$ is equipped with the corresponding product measure.

We identify the elements of $\Gamma(X_1,\ldots,X_n)$ with functions on

$$X_1 \times X_2 \times X_2 \times \cdots \times X_{n-1} \times X_{n-1} \times X_n$$

in the obvious fashion and equip $\Gamma(X_1,\ldots,X_n)$ with two norms; one is the projective norm $\|\cdot\|_{2,\wedge}$, where each of the L^2-spaces is equipped with its L^2-norm, and the other is the Haagerup tensor norm $\|\cdot\|_{\mathrm{h}}$, where the L^2-spaces are given their opposite operator space structure arising from the identification of $L^2(X \times Y)$ with the class of Hilbert-Schmidt operators from $L^2(X)$ into $L^2(Y)$ given by $f \mapsto T_f$.

For each $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$, we consider a linear map S_φ defined on $\Gamma(X_1,\ldots,X_n)$ and taking values in $L^2(X_1 \times X_n)$: for a tensor $f_1 \otimes \cdots \otimes f_{n-1} \in \Gamma(X_1,\ldots,X_n)$, we set $S_\varphi(f_1 \otimes \cdots \otimes f_{n-1})(x_1,x_n)$ to be equal to

$$\int_{X_2 \times \cdots \times X_{n-1}} \varphi(x_1,\ldots,x_n)f_1(x_1,x_2)f_2(x_2,x_3)\ldots f_{n-1}(x_{n-1},x_n)dx_2\ldots dx_{n-1}. \quad (7)$$

One can show that S_φ is bounded from $(\Gamma(X_1,\ldots,X_n), \|\cdot\|_{2,\wedge})$ into $(L^2(X_1 \times X_n), \|\cdot\|_2)$. Moreover, any multilinear bounded map $S : L^2(X_1 \times X_2) \times L^2(X_2 \times X_3) \times \cdots \times L^2(X_{n-1} \times X_n) \to L^2(X_1 \times X_n)$ which is $L^\infty(X_1),\ldots L^\infty(X_n)$-modular is given by (7) in analogy with Effros-Ruan's multilinear Schur product.

If, moreover, S_φ is bounded from $(\Gamma(X_1,\ldots,X_n), \|\cdot\|_{\mathrm{h}})$ into $(L^2(X_1 \times X_n), \|\cdot\|_{\mathrm{op}})$, that is, if there exists $C > 0$ such that $\|S_\varphi(F)\|_{\mathrm{op}} \le C\|F\|_{\mathrm{h}}$, for all $F \in \Gamma(X_1,\ldots,X_n)$,

then we say that φ is a *Schur $\mu_1, \ldots, \mu_n$-multiplier* or simply a Schur multiplier, if the measures are clear from the context. The smallest constant C with this property will be denoted by $\|\varphi\|_{\mathrm{m}}$.

We note also that if $H_i = L^2(X_i)$, $\mathcal{D}_i = \{M_\psi : \psi \in L^\infty(X_i)\}$, $i = 1, \ldots, n$, and

$$\hat{S}_\varphi : \mathcal{C}_2(H_1, H_2) \times \cdots \times \mathcal{C}_2(H_{n-1}, H_n) \to \mathcal{C}_2(H_1, H_n)$$

is the map defined by $\hat{S}_\varphi(T_{f_1}, \ldots, T_{f_{n-1}}) = T_{S_\varphi(f_1, \ldots, f_{n-1})}$, then if $a_i \in L^\infty(X_i)$, $i = 1, \ldots, n$, and $\varphi(x_1, \ldots, x_n) = a_1(x_1) \ldots a_n(x_n)$, we obtain

$$\hat{S}_\varphi(T_{f_1}, \ldots, T_{f_{n-1}}) = M_{a_n} T_{f_{n-1}} M_{a_{n-1}} \ldots T_{f_1} M_{a_1}.$$

The next theorem generalizes Theorem 2.1 to the multidimensional case giving a characterisation of all Schur multipliers in $L^\infty(X_1 \times \ldots \times X_n)$.

THEOREM 4.2. *Let $\varphi \in L^\infty(X_1 \times \cdots \times X_n)$. The following are equivalent:*

(i) φ is a Schur multiplier and $\|\varphi\|_{\mathrm{m}} < C$;

(ii) there exist essentially bounded functions $a_1 : X_1 \to M_{\infty, 1}$, $a_n : X_n \to M_{1, \infty}$ and $a_i : X_i \to M_\infty$, $i = 2, \ldots, n-1$, such that, for almost all $x_1, \ldots, x_n$ we have

$$\varphi(x_1, \ldots, x_n) = a_n(x_n) a_{n-1}(x_{n-1}) \ldots a_1(x_1) \quad and \quad \operatorname*{esssup}_{x_i \in X_i} \prod_{i=1}^{n} \|a_i(x_i)\| < C.$$

The proof is based on the fact that if φ is a Schur multiplier then $\hat{S}_\varphi$ gives rise to a multilinear map from $\mathcal{B}(H_{n-1}, H_n) \times \cdots \times \mathcal{B}(H_1, H_2)$ into $\mathcal{B}(H_1, H_n)$ which is completely bounded, normal and $\mathcal{D}_n, \ldots, \mathcal{D}_1$-modular, and a characterisation of such maps given by Christensen and Sinclair [11].

The space of all functions satisfying condition (ii) of Theorem 4.2 can be identified with the extended Haagerup tensor product $L^\infty(X_1) \otimes_{\mathrm{eh}} \ldots \otimes_{\mathrm{eh}} L^\infty(X_n)$ which will be discussed later.

In the same way two dimensional Schur multipliers are related to double operator integrals, multidimensional Schur multipliers are related to multiple operator integrals studied recently by Peller in [32]. This notion is important due to its application to the study of higher order differentiability of functions of operators.

To define multiple operator integrals we fix spectral measures $E_1(\cdot), \ldots, E_n(\cdot)$ on $X_1, \ldots, X_n$, respectively. Let $\mu_1, \ldots, \mu_n$ be scalar measures equivalent to $E_1, \ldots, E_n$, respectively. Consider the space of all functions φ for which there exists a measure space $(\mathcal{T}, \nu)$ and measurable functions g_i on $X_i \times \mathcal{T}$ such that

$$\varphi(x_1, \ldots, x_n) = \int_{\mathcal{T}} g_1(x_1, t) \ldots g_n(x_n, t) d\nu(t), \tag{8}$$

for almost all $x_1, \ldots, x_n$, where

$$\int_{\mathcal{T}} \|g_1(\cdot, t)\|_\infty \ldots \|g_n(\cdot, t)\|_\infty d\nu(t) < \infty.$$

The space is called the integral projective tensor product of $L^\infty(X_1), \ldots, L^\infty(X_n)$ and denoted by $L^\infty(X_1) \hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n)$.

In the case $n = 2$ this space coincides with the space of all Schur multipliers by [31]. For $n > 2$ we can only show that the space consists of Schur multipliers.

For $\varphi \in L^\infty(X_1) \hat{\otimes}_i \ldots \hat{\otimes}_i L^\infty(X_n)$ and $(n-1)$-tuple $(T_1, \ldots, T_{n-1})$ of bounded operators Peller defines a multiple operator integral by

$$I_\varphi((T_1, \ldots, T_{n-1}) = \int_{\mathcal{T}} \left(\int_{X_1} g_1(x_1, t) dE_1(x_1) \right) T_1 \left(\int_{X_2} g_2(x_2, t) dE_2(x_2) \right) T_2$$

$$\ldots T_{n-1} \left(\int_{X_n} g_n(x_n, t) dE_n(x_n) \right) d\nu(t).$$

If the spectral measures are multiplicity free and $T_1, \ldots, T_{n-1}$ are Hilbert-Schmidt operators with respective kernels $f_1, \ldots, f_{n-1}$ then one can easily see that $I_\varphi(T_1, \ldots, T_n)$ is a Hilbert-Schmidt operator with kernel $S_\varphi(f_1 \otimes \ldots \otimes f_n)$.

Like measurable Schur multipliers, operator multipliers defined by Kissin and Shulman can be generalised to the multidimensional setting.

Let $H_1, \ldots, H_n$ be Hilbert spaces (n is even) and let $H = H_1 \otimes \cdots \otimes H_n$. We define a Hilbert space $HS(H_1, \ldots, H_n)$ isometrically isomorphic to H: we let $HS(H_1, H_2) = \mathcal{C}_2(H_1^d, H_2)$, and by induction define

$$HS(H_1, \ldots, H_n) = \mathcal{C}_2(HS(H_2, H_3)^d, HS(H_1, H_4, \ldots,)).$$

Let $\theta : H \otimes K \to \mathcal{C}_2(H^d, K)$ be the natural surjective isometry from the product of Hilbert spaces $H \otimes K$ to the Hilbert space $\mathcal{C}_2(H^d, K)$ of Hilbert-Schmidt operators defined in Section 3. We extend this map by induction to the multidimensional case to get a map $\theta : H \to HS(H_1, \ldots, H_n)$ by letting

$$\theta(\xi_{2,3} \otimes \xi) = \theta(\theta(\xi_{2,3}) \otimes \theta(\xi)),$$

where $\xi_{2,3} \in H_2 \otimes H_3$ and $\xi \in H_1 \otimes H_4 \otimes \cdots \otimes H_n$.

Let $\Gamma(H_1, \ldots, H_n) = (H_1 \otimes H_2) \odot (H_2^d \otimes H_3^d) \odot \cdots \odot (H_{n-1} \otimes H_n)$ equipped with the Haagerup norm $\| \cdot \|_h$ where $H_i \otimes H_{i+1}$ is given the operator space structure opposite to the one arising from the embedding $\theta : H_i \otimes H_{i+1} \hookrightarrow \mathcal{B}(H_i^d, H_{i+1})$ (and similarly for $(H_i \otimes H_{i+1})^d = H_i^d \otimes H_{i+1}^d$).

Fix $\varphi \in B(H)$. We define a mapping $S_\varphi : \Gamma(H_1, \ldots, H_n) \to \mathcal{B}(H_1^d, H_n)$ as follows: if $\zeta \in \Gamma(H_1, \ldots, H_n)$ is an elementary tensor, namely,

$$\zeta = \xi_{1,2} \otimes \eta_{2,3}^d \otimes \xi_{3,4} \otimes \cdots \otimes \xi_{n-1,n},$$

we let

$$S_\varphi(\zeta) = \theta(\varphi(\xi_{1,2} \otimes \cdots \otimes \xi_{n-1,n})(\theta(\eta_{2,3}^d)) \ldots (\theta(\eta_{n-2,n-1}^d)).$$

Using the natural identification, we consider S_φ as a map from $\mathcal{C}_2(H_1^d, H_2) \odot \ldots \odot \mathcal{C}_2(H_{n-1}^d, H_n)$ into $\mathcal{B}(H_1^d, H_n)$ which in particular satisfies the following:

$$S_{a_1 \otimes \ldots \otimes a_n}(T_1 \otimes \cdots \otimes T_{n-1}) = a_n T_{n-1} \ldots a_3^d T_2 a_2 T_1 a_1^d$$

for any $a_1 \otimes \ldots \otimes a_n \in \mathcal{B}(H)$.

For odd n, a similar definition can be given by "adding" to $H_1, \ldots, H_n$ the one-dimensional Hilbert space $\mathbb{C}$. For technical simplicity from now on we restrict our attention to the case of even n and refer the reader to [25, 26] for the general case.

We call φ a **concrete operator multiplier** if there exists $C > 0$ such that

$$\|S_\varphi(\zeta)\|_{op} \leq C \|\zeta\|_h, \quad \text{for all } \zeta \in \Gamma(H_1, \ldots, H_n).$$

As in the two dimensional case, we want to specify classes of "continuous" operator multipliers. Let $\mathcal{A}_i$ be a C*-algebra and $\pi_i : \mathcal{A}_i \to B(H_i)$ be a representation, $i = 1, \ldots, n$. An element $\varphi \in \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ is called an operator $\pi_1, \ldots, \pi_n$-*multiplier* if $(\pi_1 \otimes \cdots \otimes \pi_n)(\varphi)$ is a concrete operator multiplier. We denote the set of all $(\pi_1, \ldots, \pi_n)$-multipliers by $\mathfrak{M}_{\pi_1, \ldots, \pi_n}$ and denote by $\|\varphi\|_{\pi_1, \ldots, \pi_n}$ the concrete multiplier norm.

In analogy with the two-dimensional case we say that φ is a *universal operator multiplier* if it is a $\pi_1, \ldots, \pi_n$-multiplier for all choices of $\pi_1, \ldots, \pi_n$. In this case,

$$\|\varphi\|_{\mathrm{m}} \stackrel{def}{=} \sup \|\varphi\|_{\pi_1, \ldots, \pi_n} < \infty.$$

By $\mathfrak{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ we will denote the set of all universal operator multipliers. For $n = 2$ one obtains the notion of universal operator multipliers introduced by Kissin and Shulman and discussed in Section 3. Moreover, natural analogs of the Comparision Theorem 3.1 and Corollary 3.2 hold in the new multidimensional setting.

As one may expect, multidimensional operator multipliers are related to completely bounded multilinear maps. We now recall the definition and some related notions.

Let $\mathcal{E}, \mathcal{E}_1, \ldots, \mathcal{E}_n$ be closed subspaces of $\mathcal{B}(H), \mathcal{B}(H_1), \ldots, \mathcal{B}(H_n)$, respectively. We denote by $\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n$ the algebraic tensor product of $\mathcal{E}_1, \ldots \mathcal{E}_n$. Let $a_k = (a_{i,j}^k) \in M_{m_k, m_{k+1}}(\mathcal{E}_k)$, $k = 1, \ldots, n$. We denote by

$$a^1 \odot \cdots \odot a^n \in M_{m_1, m_{n+1}}(\mathcal{E}_1 \odot \cdots \odot \mathcal{E}_n) \tag{9}$$

the matrix whose i, j-entry is

$$\sum_{i_2, \ldots, i_n} a_{i,i_2}^1 \otimes a_{i_2,i_3}^2 \otimes \cdots \otimes a_{i_n,j}^n.$$

Let $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ be a multilinear map and

$$\Phi^{(m)} : M_m(\mathcal{E}_1) \times M_m(\mathcal{E}_2) \times \cdots \times M_m(\mathcal{E}_n) \to M_m(\mathcal{E})$$

be the multilinear map given by

$$\Phi^{(m)}(a^1, \ldots, a^n)_{i,j} = \sum_{i_2, \ldots, i_n} \Phi(a_{i,i_2}^1, a_{i_2,i_3}^2, \ldots, a_{i_n,j}^n),$$

where $a^k = (a_{i,j}^k) \in M_m(\mathcal{E}_k)$, $1 \le i, j \le m$. The map Φ is called completely bounded if there exists $C > 0$ such that for all $m \in \mathbb{N}$ and all elements $a^k \in M_m(\mathcal{E}_k)$, $k = 1, \ldots, n$, we have

$$\|\Phi^{(m)}(a^1, \ldots, a^n)\| \le C \|a^1\| \ldots \|a^n\|.$$

Every completely bounded multilinear map $\Phi : \mathcal{E}_1 \times \cdots \times \mathcal{E}_n \to \mathcal{E}$ gives rise to a completely bounded linear map from the Haagerup tensor product $\mathcal{E}_1 \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{E}_n$ into $\mathcal{E}$.

The *extended Haagerup tensor product* $\mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$ is defined in [17] as the space of all normal (in each variable) completely bounded maps $u : \mathcal{E}_1^* \times \cdots \times \mathcal{E}_n^* \to \mathbb{C}$. It was shown in [17] that if $u \in \mathcal{E}_1 \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{E}_n$ then there exist index sets $J_1, J_2, \ldots, J_{n-1}$ and matrices $a^1 = (a_{1,s}^1) \in M_{1,J_1}(\mathcal{E}_1), a^2 = (a_{s,t}^2) \in M_{J_1,J_2}(\mathcal{E}_2), \ldots, a^n = (a_{t,1}^n) \in M_{J_{n-1},1}(\mathcal{E}_n)$ such that if $f_i \in \mathcal{E}_i^*$, $i = 1, \ldots, n$, then

$$\langle u, f_1 \otimes \cdots \otimes f_n \rangle \stackrel{def}{=} u(f_1, \ldots, f_n) = \langle a^1, f_1 \rangle \ldots \langle a^n, f_n \rangle, \tag{10}$$

where $\langle a^k, f_k \rangle = \left(f_k(a_{s,t}^k) \right)_{s,t}$ and the product of the (possibly infinite) matrices in (10) is defined to be the limit of the sums

$$\sum_{i_1 \in F_1, \ldots, i_{n-1} \in F_{n-1}} f_1(a_{1,i_1}^1) f_2(a_{i_1,i_2}^2) \ldots f_n(a_{i_{n-1},1}^n)$$

along the net $\{(F_1 \times \cdots \times F_{n-1}) : F_j \subseteq J_j \text{ finite}, 1 \le j \le n-1\}$.

We identify u with the matrix product $a^1 \odot \cdots \odot a^n$; two elements $a^1 \odot \cdots \odot a^n$ and $\tilde{a}^1 \odot \cdots \odot \tilde{a}^n$ coincide if $\langle a^1, f_1 \rangle \ldots \langle a^n, f_n \rangle = \langle \tilde{a}^1, f_1 \rangle \ldots \langle \tilde{a}^n, f_n \rangle$ for all $f_i \in \mathcal{E}_i^*$, $i = 1, \ldots, n$. Moreover,

$$\|u\|_{\mathrm{eh}} = \inf\{\|a^1\| \ldots \|a^n\| : u = a^1 \odot \cdots \odot a^n\}.$$

There is a natural bijection γ between the extended Haagerup tensor product $\mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ and the space of multilinear normal completely bounded maps from $\mathcal{B}(H_2, H_1) \times \ldots \times \mathcal{B}(H_n, H_{n-1})$ to $\mathcal{B}(H_n, H_1)$ given as follows: if $u = A_1 \odot \cdots \odot A_n \in \mathcal{B}(H_1) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_n)$ then

$$\gamma(u)(T_1, \ldots, T_{n-1}) = A_1(T_1 \otimes I)A_2 \ldots A_{n-1}(T_{n-1} \otimes I)A_n,$$

for all $T_i \in \mathcal{B}(H_{i+1}, H_i)$, $i = 1, \ldots, n-1$. This is due to Christensen and Sinclair [11].

The connection of multilinear completely bounded maps with universal multidimensional operator multipliers arises as follows. Let $\mathcal{A}_i$ be a C^*-algebra, $i = 1, \ldots, n$, and $\varphi \in \mathfrak{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. Then the map S_φ is completely bounded for the opposite operator space structures, and hence has a completely bounded extension to a map

$$\Phi_\varphi : \left(\mathcal{K}(H_{n-1}^{\mathrm{d}}, H_n) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_1^{\mathrm{d}}, H_2), \|\cdot\|_{\mathrm{h}} \right) \to \left(\mathcal{K}(H_1^{\mathrm{d}}, H_n), \|\cdot\|_{\mathrm{op}} \right)$$

given by

$$\Phi_\varphi(T_{n-1} \otimes \cdots \otimes T_1) = S_\varphi(T_1 \otimes \cdots \otimes T_{n-1}).$$

Thus, Φ_φ^{**} is a completely bounded normal map, and the $(\mathcal{A}_n', (\mathcal{A}_{n-1}^{\mathrm{d}})', \ldots, (\mathcal{A}_1^{\mathrm{d}})')$-modularity of Φ_φ allows then to define a symbol of a multidimensional universal multiplier:

THEOREM 4.3. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be C*-algebras and $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. There exists a unique element $u_\varphi \in \mathcal{A}_n \otimes_{\mathrm{eh}} \mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2 \otimes_{\mathrm{eh}} \mathcal{A}_1^o$ with the property that if π_i is a representation of $\mathcal{A}_i$ for $i = 1, \ldots, n$ then the map $\Phi_{\pi_1 \otimes \cdots \otimes \pi_n(\varphi)}$ coincides with the restriction of $\gamma(\pi_n \otimes_{\mathrm{eh}} \pi_{n-1}^{\mathrm{d}} \otimes_{\mathrm{eh}} \ldots \otimes_{\mathrm{eh}} \pi_1^{\mathrm{d}}(u_\varphi))$.*

The map $\varphi \to u_\varphi$ is linear and if $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$ then $u_{a_1 \otimes \cdots \otimes a_n} = a_n \otimes a_{n-1}^o \otimes \cdots \otimes a_1^o$. Moreover, $\|\varphi\|_{\mathrm{m}} = \|u_\varphi\|_{\mathrm{eh}}$.

The notion of completely compact map, completely compact and compact multipliers have natural extensions to the mutlidimensional case. Namely, if $\mathcal{Y}, \mathcal{X}_1, \ldots, \mathcal{X}_n$ are operator spaces and $\Phi : \mathcal{X}_1 \times \cdots \times \mathcal{X}_n \to \mathcal{Y}$ is a completely bounded multilinear map, we call Φ *completely compact* if for each $\epsilon > 0$ there exists a finite dimensional subspace $F \subseteq \mathcal{Y}$ such that

$$\mathrm{dist}(\Phi^{(m)}(x_1, \ldots, x_n), M_m(F)) < \epsilon,$$

for all $x_i \in M_m(\mathcal{X}_i)$, $\|x_i\| \le 1$, $i = 1, \ldots, n$, and all $m \in \mathbb{N}$. We denote by $CC(\mathcal{X}_1 \times \cdots \times \mathcal{X}_n, \mathcal{Y})$ the space of all completely bounded completely compact multilinear maps from $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$ into $\mathcal{Y}$.

Let $\mathcal{A}_i$ be a C^*-algebra, $i = 1, \ldots, n$. We define $\mathfrak{M}_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ (resp. $\mathfrak{M}_c$ ($\mathcal{A}_1$, $\ldots, \mathcal{A}_n$) and $\mathfrak{M}_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$) as the set of all $\varphi \in \mathfrak{M}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ such that there exist faithful representations $\pi_1, \ldots, \pi_n$ of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ with the property that if $\pi = \pi_1 \otimes \cdots \otimes \pi_n$ then $\Phi_{\pi(\varphi)}$ is completely compact (resp. $\Phi_{\pi(\varphi)}$ is compact and the range of $\Phi_{\pi(\varphi)}$ is a finite dimensional space of finite-rank operators).

Let

$$\mathcal{K}_{\mathrm{h}} \overset{\mathrm{def}}{=} \mathcal{K}(H_2, H_1) \otimes_{\mathrm{h}} \cdots \otimes_{\mathrm{h}} \mathcal{K}(H_n, H_{n-1}).$$

Saar's result (Theorem 3.7) has the following generalisation:

THEOREM 4.4. *The operator space* $\mathcal{E} \overset{\mathrm{def}}{=} \mathcal{K}(H_1) \otimes_{\mathrm{h}} (\mathcal{B}(H_2) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{B}(H_{n-1})) \otimes_{\mathrm{h}} \mathcal{K}(H_n)$ *is isometrically isomorphic to* $\mathcal{F} \overset{\mathrm{def}}{=} CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ *via the restriction of* γ *to* $\mathcal{E}$.

This leads to the following characterisation of completely compact universal multipliers in terms of their symbols:

THEOREM 4.5. *Let* $\mathcal{A}_1, \ldots, \mathcal{A}_n$ *be* C^*-*algebras and* $\varphi \in M(\mathcal{A}_1, \ldots, \mathcal{A}_n)$. *The following are equivalent:*

(i) $\varphi \in M_{cc}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$;
(ii) $u_\varphi \in \mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} (\mathcal{A}_{n-1}^o \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} \mathcal{A}_2) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1^o)$
(iii) *there exists a net* $\{\varphi_\alpha\}_\alpha \subseteq M_{ff}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *such that* $\|\varphi_\alpha - \varphi\|_{\mathrm{m}} \to_\alpha 0$.

The important point to note here is that there is no direct correspondence to the two-dimensional case: the space $CC(\mathcal{K}_{\mathrm{h}}, \mathcal{K}(H_n, H_1))$ is not isometrically isomorphic to $\mathcal{K}(H_1) \otimes_{\mathrm{h}} \mathcal{K}(H_2) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} \mathcal{K}(H_n)$, and the symbol of a completely compact map maybe an element of a space bigger than $\mathcal{K}(\mathcal{A}_n) \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_{n-1}^o) \otimes_{\mathrm{h}} \ldots \otimes_{\mathrm{h}} \mathcal{K}(\mathcal{A}_1^o)$, contrary to what one may expect by following the analogy with the two-dimensional case.

The property of the set of universal completely compact multipliers in $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ to coincide with the set of compact universal multipliers depends only on the structure of compact elements of $\mathcal{A}_1$ and $\mathcal{A}_n$ and not on $\mathcal{A}_k$, $k = 2, \ldots, n - 1$. The conditions on $\mathcal{K}(\mathcal{A}_1)$ and $\mathcal{K}(\mathcal{A}_n)$ are exactly the ones given in Theorem 3.9.

We shall end the section by describing an application of multidimensional Schur multipliers to abstract harmonic analysis. Let G be a locally compact, σ-compact group and let $A(G)$ be the Fourier algebra of G. We recall that λ denotes the left regular representation of G on $L^2(G)$. Since $A(G)$ is the predual of the von Neumann algebra $\mathrm{VN}(G)$, it possesses a canonical operator structure. Therefore we can define the multidimensional Fourier algebra as follows:

$$A^n(G) = \underbrace{A(G) \otimes_{\mathrm{eh}} \cdots \otimes_{\mathrm{eh}} A(G)}_{n}.$$

Since $\mathrm{VN}(G)$ is generated by λ_x, $x \in G$, it follows from the definition that the elements of $A^n(G)$ can be identified with functions $f \in L^\infty(G^n)$ given by $f(x_n, \ldots, x_1) = \Phi(\lambda_{x_n}, \ldots, \lambda_{x_1})$, where $\Phi : \mathrm{VN}(G) \times \cdots \times \mathrm{VN}(G) \to \mathbb{C}$ is a normal completely bounded multilinear map.

For $f \in A(G)$ we define a map $\theta : A(G) \to A^n(G)$ by

$$\theta(f)(x_n, \ldots, x_1) = f(x_n \ldots x_1).$$

We call a function $\varphi \in L^\infty(G^n)$ an *n-multiplier* of $A(G)$ if $\varphi\theta(f) \in A^n(G)$ whenever $f \in A(G)$. We call φ a *completely bounded n-multiplier* if the mapping $f \mapsto \varphi\theta(f)$ is completely bounded. The following characterisation, which is a multidimensional version of Theorem 2.6, was established in [39].

THEOREM 4.6. *Let $\varphi \in L^\infty(G^n)$. The following are equivalent:*

(i) *φ is a completely bounded n-multiplier;*
(ii) *the function $\tilde\varphi \in L^\infty(G^{n+1})$ given by*

$$\tilde\varphi(x_1, \ldots, x_n) = \varphi(x_{n+1}^{-1}x_n, \ldots, x_2^{-1}x_1)$$

is a Schur multiplier with respect to the left Haar measure.

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