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CONDITIONAL MARKOV CHAINS – CONSTRUCTION AND PROPERTIES

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Abstract. In this paper we study finite state conditional Markov chains (CMCs). We give two examples of CMCs, one which admits intensity, and another one, which does not admit an intensity. We also give a sufficient condition under which a doubly stochastic Markov chain is a CMC. In addition we provide a method for construction of conditional Markov chains via change of measure.

1. Introduction. Conditional Markov chains (CMCs) are important generalizations of processes such as doubly stochastic Poisson processes or indicator processes of stopping times.

In this paper we continue study of finite state conditional Markov chains that was originated in [4]. Here we introduce a modified definition of the conditional Markov property, which is less general than Definition 11.3.1 used in Chapter 11.3 in [4]. However, the definition of conditional Markov property proposed in this paper is well suited for study

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of Markov consistency properties for conditional Markov chains and study of Markov copulae for conditional Markov chains that will be conducted in the follow-up papers [2], [3].

In Section 2 we introduce our concept of CMC. We give two examples of $(\mathbb{F}, \mathbb{G})$-CMC, one which does not have the intensity, and one with the intensity. We give a sufficient condition under which a doubly stochastic Markov chain is a CMC. In Section 3 we provide a systematic construction of a CMC with a specified generator (intensity) process. Our construction proceeds via a change of probability measure and allows for intensity processes that are more general than the ones considered in [4]. Moreover we are able to construct CMC which is adapted to filtration $\mathbb{G}$, given in advance, provided that there exists a $\mathbb{G}$-adapted time homogeneous Markov chain with values in S.

2. Conditional Markov chain and its intensity. Let $T > 0$ be a fixed, finite time horizon. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an underlying complete probability space, which is endowed with two filtrations, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$, that are assumed to satisfy the usual conditions. Typically, processes considered in this paper are defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and are restricted to the time interval $[0, T]$. In addition, we fix a finite set S, and we denote by d the cardinality of S. Without loss of generality we take $S = \{1, 2, 3, \dots, d\}$.

Definition 2.1. An S-valued, $\mathbb{G}$-adapted càdlàg process X is called *an* $(\mathbb{F}, \mathbb{G})$*-conditional Markov chain* if for every $x_1, \dots, x_k \in S$ and for every $0 \le t \le t_1 \le \ldots \le t_k \le T$ it satisfies the following property

$$\mathbb{P}\big(X_{t_k} = x_k, \dots, X_{t_1} = x_1 \,|\, \mathcal{F}_t \vee \mathcal{G}_t\big) = \mathbb{P}\big(X_{t_k} = x_k, \dots, X_{t_1} = x_1 \,|\, \mathcal{F}_t \vee \sigma(X_t)\big). \tag{1}$$

Remark 2.2.

(i) We will call filtration $\mathbb{G}$ the *base* filtration, and $\mathbb{F}$ the *reference* filtration.

(ii) If $\mathbb{G}$ is independent of $\mathbb{F}$, then the above definition reduces to the case of a classical $\mathbb{G}$-Markov chain. In other words, a classical $\mathbb{G}$-Markov chain is an $(\mathbb{F}, \mathbb{G})$-conditional Markov chain for the reference filtration independent of the base filtration.

(iii) If $\mathbb{F} \supseteq \mathbb{G}$ then any càdlàg process X with state space S is an $(\mathbb{F}, \mathbb{G})$-conditional Markov chain. Thus the concept of the $(\mathbb{F}, \mathbb{G})$-conditional Markov chain is worth studying only if $\mathbb{F} \not\supseteq \mathbb{G}$.

In what follows we shall write $(\mathbb{F}, \mathbb{G})$-CMC, for short, in place of $(\mathbb{F}, \mathbb{G})$-conditional Markov chain.

From now on, we assume that $\mathbb{P}(X_0 = x_0) = 1$ for some $x_0 \in S$.

2.1. Intensity of an $(\mathbb{F}, \mathbb{G})$-CMC. Let X be an $(\mathbb{F}, \mathbb{G})$-CMC. For each $x \in S$ we define the corresponding state indicator process of X,

$$H^x_t := \mathbb{1}_{\{X_t = x\}}, \quad t \in [0, T]. \tag{2}$$

Accordingly, we define a column vector $H_t = (H^x_t)^\top_{x\in S}$, where $\top$ denotes transposition. Similarly, for $x, y \in S,\ x \neq y$, we define process H^{xy} that counts the number of transitions from x to y,

$$H^{xy}_t := \#\{u \le t : X_{u-} = x \text{ and } X_u = y\} = \int_{]0,t]} H^x_{u-}\, dH^y_u, \quad 0 \le t \le T.$$

The following definition generalizes the concept of the generator matrix (or intensity matrix) of a Markov chain.

Definition 2.3. We say that an $\mathbb{F}$-progressively measurable (matrix valued) process $\Lambda_t = [\lambda_t^{xy}]_{x,y\in S}$ such that

$$\lambda_t^{xy} \geq 0, \quad \forall x, y \in S,\ x \neq y, \qquad \sum_{y\in S} \lambda_t^{xy} = 0, \quad \forall x \in S, \tag{3}$$

is an $\mathbb{F}$-*intensity matrix process for* X, if the process M defined as

$$M_t = H_t - \int_0^t \Lambda_u^\top H_u \, du, \quad t \in [0, T], \tag{4}$$

is an $\mathbb{F} \vee \mathbb{G}$-(local) martingale (with values in $\mathbb{R}^d$).

The example of CMC admitting $\mathbb{F}$-intensity will be given later (see Example 2.10). In the case of classical Markov chains with finite state space, intensity matrix may not exist if the matrix of transition probabilities is not differentiable (e.g. when X is not quasi-left continuous, see Rogers and Williams [11, III.11]). In the case of $(\mathbb{F}, \mathbb{G})$-CMC the situation is similar. That is, there exist $(\mathbb{F}, \mathbb{G})$-CMCs that do not admit $\mathbb{F}$-intensities. We illustrate this possibility by means of the following example:

Example 2.4. Suppose that $(\Omega, \mathcal{A}, \mathbb{P})$ supports a standard, real valued Brownian motion W, and a random variable E with unit exponential distribution[1] and independent from W. Define a nonnegative process γ by the formula

$$\gamma_t := \sup_{u\in[0,t]} W_u, \quad t \in [0, T].$$

By definition, γ is an increasing and continuous process. It is well known (cf. Section 1.7 in [6]) that trajectories of γ are not absolutely continuous with respect to the Lebesgue measure on real line.

Next, define a process X by

$$X_t := \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0,$$

where

$$\tau := \inf\{t > 0 : \gamma_t > E\}.$$

Note that, for any $t \in [0, T]$ we have

$$\mathbb{P}(\tau > t \,|\, \mathcal{F}_t^W) = \mathbb{P}(E \geq \gamma_t \,|\, \mathcal{F}_t^W) = \mathbb{E}\big(\mathbb{P}(E \geq \gamma_t \,|\, \mathcal{F}_\infty^W) \,|\, \mathcal{F}_t^W\big) = e^{-\gamma_t}.$$

This means that the process γ is the $\mathbb{F}^W$-hazard process of τ (cf. e.g. [4]). It is well known (see e.g. [4, Prop. 5.1.3]) that the process $\widehat{M}$ given as

$$\widehat{M}_t = \mathbb{1}_{\{\tau\leq t\}} - \int_{]0,t\wedge\tau]} d\gamma_u = X_t - \int_{]0,t]} (1 - X_t)\, d\gamma_u \tag{5}$$

is an $\mathbb{F}^W \vee \mathbb{F}^X$ martingale.

To show that X is an $(\mathbb{F}^W, \mathbb{F}^X)$-CMC we first note that state 1 is an absorbing state for X. Thus, for every $x_1, \ldots, x_k \in S$ and for every $0 \leq t \leq t_1 \leq \ldots \leq t_k \leq T$

[1] That is, E is exponentially distributed with mean 1.

and $A = \{X_{t_k} = x_k, \dots, X_{t_1} = x_1\}$ we have

$$\mathbb{P}(A \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) = \begin{cases} 0 & \text{if } \exists k < n,\ x_k = 1,\ x_{k+1} = 0, \\ \mathbb{P}(X_{t_1} = 1 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) & \text{if } \forall k = 1, \dots, n,\ x_k = 1, \\ \mathbb{P}(X_{t_n} = 0 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) & \text{if } \forall k = 1, \dots, n,\ x_k = 0, \\ \mathbb{P}(X_{t_k} = 0, X_{t_{k+1}} = 1 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) & \text{if } \exists k < n,\ x_k = 0,\ x_{k+1} = 1. \end{cases} \tag{6}$$

Using, once again, the fact that state 1 is an absorbing state for X, we conclude that

$$\begin{aligned} \mathbb{P}(X_u = 1, X_v = 0 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) &= \mathbb{1}_{\{X_t = 0\}} \mathbb{P}(X_u = 1, X_v = 0 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) \\ &= \mathbb{1}_{\{X_t = 0\}} \mathbb{P}(X_u = 1, X_v = 0 \,|\, \mathcal{F}_t^W \vee \sigma(X_t)), \\ \mathbb{P}(X_u = 1 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) &= \mathbb{1}_{\{X_t = 1\}} + \mathbb{1}_{\{X_t = 0\}} \mathbb{P}(X_u = 1 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) \\ &= \mathbb{1}_{\{X_t = 1\}} + \mathbb{1}_{\{X_t = 0\}} \mathbb{P}(X_u = 1 \,|\, \mathcal{F}_t^W \vee \sigma(X_t)), \end{aligned}$$

for $t \le v \le u \le T$. Hence

$$\begin{aligned} \mathbb{P}(X_u = 1, X_v = 0 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) &= \mathbb{P}(X_u = 1, X_v = 0 \,|\, \mathcal{F}_t^W \vee \sigma(X_t)), \\ \mathbb{P}(X_u = 1 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) &= \mathbb{P}(X_u = 1 \,|\, \mathcal{F}_t^W \vee \sigma(X_t)), \\ \mathbb{P}(X_u = 0 \,|\, \mathcal{F}_t^W \vee \mathcal{F}_t^X) &= \mathbb{P}(X_u = 0 \,|\, \mathcal{F}_t^W \vee \sigma(X_t)). \end{aligned}$$

This and (6) imply condition (1), which proves that X is an $(\mathbb{F}^W, \mathbb{F}^X)$-CMC.

In order to see that X does not admit an $\mathbb{F}^W$-intensity matrix let us consider the (vector) indicator process $H = (H^0, H^1)$ associated with X:

$$H_t^0 = (1 - X_t), \quad H_t^1 = X_t.$$

Martingale property of $\widehat{M}$ given by (5) yields that the processes

$$M_t^0 = H_t^0 - \int_{]0,t]} H_t^0 \, d(-\gamma_u) = 1 - \widehat{M}_t, \tag{7}$$

$$M_t^1 = H_t^1 - \int_{]0,t]} H_t^0 \, d\gamma_u = \widehat{M}_t, \tag{8}$$

are $(\mathbb{F}^W, \mathbb{F}^X)$ martingales. Upon denoting $\widetilde{M} = (M^0,\, M^1)^\top$, we see that equalities (7) and (8) can be written as

$$\widetilde{M}_t = H_t - \int_0^t (d\Gamma_t)^\top H_t, \tag{9}$$

where

$$\Gamma_t := \begin{bmatrix} -\gamma_t & \gamma_t \\ 0 & 0 \end{bmatrix}.$$

Now, suppose that there exists an $\mathbb{F}$-intensity matrix Λ for X, so that process M given as in (4) is an $\mathbb{F}^W \vee \mathbb{F}^X$ martingale. This implies that a process N given as

$$N_t = \widetilde{M}_t - M_t = \int_0^t (d\Gamma_u)^\top H_u - \int_0^t \Lambda_u^\top H_u \, du$$

is an $\mathbb{F}^W \vee \mathbb{F}^X$ martingale. Since N is a continuous process of finite variation, it must be constant, so that $N_t = N_0 = 0$. This is a contradiction because this would imply that

$$\int_0^t (d\Gamma_u)^\top H_u = \int_0^t \Lambda_u^\top H_u \, du \quad \text{for } t \in [0, T],$$

which is impossible as Γ is not absolutely continuous with respect to Lebesgue measure on the real line. We conclude that process X does not admit an $\mathbb{F}$-intensity matrix. This completes the example.

In the rest of this paper we restrict ourselves to CMCs which admit $\mathbb{F}$-intensity.

2.1.1. *Intensity of an $(\mathbb{F},\mathbb{G})$-CMC and compensators of counting processes H^{xy}.* The $\mathbb{F}$-intensity matrix of an $(\mathbb{F},\mathbb{G})$-CMC X is related to the $\mathbb{F}\vee\mathbb{G}$-compensators of processes H^{xy}, $x,y\in S$, $x\neq y$. In fact, we have the following result, which is a special case of more general lemma proven in [9, Lemma 4.3],

THEOREM 2.5. *Let X be an $(\mathbb{F},\mathbb{G})$-CMC.*

1) *Suppose that X admits an $\mathbb{F}$-intensity matrix process Λ. Then for every $x,y\in S$, $x\neq y$, the process M^{xy} given as*
$$M_t^{xy} = H_t^{xy} - \int_0^t H_u^x \lambda_u^{xy}\,du, \quad t\geq 0, \tag{10}$$
is an $\mathbb{F}\vee\mathbb{G}$ local martingale.
2) *Suppose that we are given a family of nonnegative $\mathbb{F}$-progressively measurable processes λ^{xy}, $x,y\in S$, $x\neq y$, such that for every $x,y\in S$, $x\neq y$, the process M^{xy} given in* (10) *is an $\mathbb{F}\vee\mathbb{G}$ local martingale. Then, the matrix valued process $\Lambda_t=[\lambda_t^{xy}]_{x,y\in S}$, with diagonal elements defined as*
$$\lambda^{xx} = -\sum_{y\in S, y\neq x}\lambda^{xy}, \quad x\in S,$$
is an $\mathbb{F}$-intensity matrix of X.

The above theorem sheds light on the issue of existence of $\mathbb{F}$-intensities. We see that the $\mathbb{F}$-intensity may not exist since $\mathbb{F}\vee\mathbb{G}$-compensators of H^{xy} may not be absolutely continuous with respect to Lebesgue measure. In fact, fixing $y\in S$, if at least one of the processes H^{xy}, for $x\in S$, $x\neq y$, does not admit the $\mathbb{F}\vee\mathbb{G}$-compensator that is absolutely continuous with respect to Lebesgue measure, then the process $H^y = H_0^y + \sum_{x\in S,x\neq y}(H^{xy}-H^{yx})$ does not admit the $\mathbb{F}\vee\mathbb{G}$-compensator that is absolutely continuous with respect to Lebesgue measure. On the other hand, absolute continuity of $\mathbb{F}\vee\mathbb{G}$-compensators of all processes H^{xy}, for $x\in S$, $x\neq y$, is not sufficient for existence of $\mathbb{F}$-intensity. This is due to the fact that the density of $\mathbb{F}\vee\mathbb{G}$ compensator is, in general, $\mathbb{F}\vee\mathbb{G}$-adapted, whereas by definition the $\mathbb{F}$-intensity is only $\mathbb{F}$-adapted.

2.2. $(\mathbb{F},\mathbb{G})$-CMC as a pure jump semimartingale. It is important to note that an $(\mathbb{F},\mathbb{G})$-CMC X admitting $\mathbb{F}$-intensity process Λ can be viewed as a pure jump semimartingale,[2] with values in S, whose corresponding random jump measure μ defined by (cf. Jacod [7])
$$\mu(\omega,dt,dz) = \sum_{n\geq 1}\delta_{(T_n(\omega),X_{T_n(\omega)}(\omega))}(dt,dz)\,\mathbb{1}_{\{T_n(\omega)<T\}},$$

[2]We adhere to the standard convention that semimartingale processes (taking values in finite dimensional spaces) are càdlàg.

where

$$T_n := \inf \left\{ t : T_{n-1} < t \leq T,\ X_t \neq X_{T_{n-1}} \right\} \wedge T, \quad T_0 = 0,$$

has the $\mathbb{F} \vee \mathbb{G}$ predictable projection (or compensating measure) given as

$$\nu(dt, dz) = \sum_{y \in S} \delta_y(dz) \Big(\sum_{x \in S \setminus \{y\}} H_t^x \lambda_t^{xy} \Big)\, dt = \sum_{y \in S} \delta_y(dz) \Big(\sum_{x \in S \setminus \{y\}} \mathbb{1}_{\{X_t = x\}} \lambda_t^{xy} \Big)\, dt. \quad (11)$$

So the problem of construction of an $(\mathbb{F}, \mathbb{G})$-CMC with an $\mathbb{F}$-intensity (matrix) process Λ is equivalent to the problem of construction any $\mathbb{G}$-adapted, S-valued pure jump semimartingale with the $\mathbb{F} \vee \mathbb{G}$ compensator ν given by (11), and additionally satisfying condition (1).

Remark 2.6. With a slight abuse of terminology, we shall refer to a $\mathbb{G}$-adapted, S-valued pure jump semimartingale X with the $\mathbb{F} \vee \mathbb{G}$ compensator ν given by (11), as to a $\mathbb{G}$-adapted, S-valued pure jump semimartingale admitting the $\mathbb{F}$-intensity process Λ. In particular, this also means that the process M corresponding to X as in (4) is an $\mathbb{F} \vee \mathbb{G}$-(local) martingale, and, in addition, that Theorem 2.5 is true for such a process X.

2.3. DSMC that is a CMC. An important family of jump processes, so called doubly stochastic Markov chains (DSMC), was introduced in [9]. Here we provide a sufficient condition under which a doubly stochastic Markov chain is a CMC.

Definition 2.7. A $\mathbb{G}$-adapted càdlàg process $X = (X_t)_{t \in [0,T]}$ is called an $(\mathbb{F}, \mathbb{G})$-*doubly stochastic Markov chain with state space S* if there exists a stochastic matrix valued random field $P(s,t) = (p_{xy}(s,t))_{x,y \in S}$ for $0 \leq s \leq t \leq T$ such that

1) $P(s, \cdot)$ is $\mathbb{F}$-progressively measurable,
2) for any $0 \leq s \leq t \leq T$ and every $x, y \in S$ we have

$$\mathbb{P}(X_t = y \,|\, \mathcal{F}_T \vee \mathcal{G}_s) \mathbb{1}_{\{X_s = x\}} = \mathbb{1}_{\{X_s = x\}} p_{xy}(s,t). \quad (12)$$

For brevity we write $(\mathbb{F}, \mathbb{G})$-DSMC, for the "$(\mathbb{F}, \mathbb{G})$-doubly stochastic Markov chain".

In order to state the next result we need the following definition.

Definition 2.8. We say that a filtration $\mathbb{F}$ is $\mathbb{P}$-*immersed* in a filtration $\mathbb{G}$ if $\mathbb{F} \subset \mathbb{G}$ and if every $(\mathbb{P}, \mathbb{F})$-local martingale is a $(\mathbb{P}, \mathbb{G})$-local martingale.

Proposition 2.9. *Assume that X is an $(\mathbb{F}, \mathbb{G})$-DSMC and that $\mathbb{F}$ is immersed in $\mathbb{F} \vee \mathbb{G}$, then X is an $(\mathbb{F}, \mathbb{G})$-CMC.*

Proof. Let us fix arbitrary $x_1, \ldots, x_k \in S$ and $0 \leq t < t_1 < \ldots < t_k \leq T$, and let us define a set A by

$$A = \{X_{t_k} = x_k, \ldots, X_{t_1} = x_1\}.$$

We need to show that

$$\mathbb{P}(A \,|\, \mathcal{F}_t \vee \mathcal{G}_t) = \mathbb{P}(A \,|\, \mathcal{F}_t \vee \sigma(X_t)). \quad (13)$$

To this end note that by Lemma 3.1 in [9] we have

$$\mathbb{P}(A \,|\, \mathcal{F}_T \vee \mathcal{G}_t)\mathbb{1}_{\{X_t=x\}} = \mathbb{1}_{\{X_t=x\}} p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}). \tag{14}$$

The tower property of conditional expectation and (14) imply

$$\begin{aligned}\mathbb{P}(A \,|\, \mathcal{F}_t \vee \mathcal{G}_t) &= \mathbb{E}\Big(\sum_{x\in S} \mathbb{E}(\mathbb{1}_A \,|\, \mathcal{F}_T \vee \mathcal{G}_t)\mathbb{1}_{\{X_t=x\}} \,\Big|\, \mathcal{F}_t \vee \mathcal{G}_t\Big) \\ &= \mathbb{E}\Big(\sum_{x\in S} \mathbb{1}_{\{X_t=x\}} p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) \,\Big|\, \mathcal{F}_t \vee \mathcal{G}_t\Big) \\ &= \sum_{x\in S} \mathbb{1}_{\{X_t=x\}} \mathbb{E}\Big(p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) \,\Big|\, \mathcal{F}_t \vee \mathcal{G}_t\Big).\end{aligned}$$

Now, using the assumption about immersion of $\mathbb{F}$ in $\mathbb{F} \vee \mathbb{G}$ and [10, Proposition 9.1.1], we obtain

$$\mathbb{P}(A \,|\, \mathcal{F}_t \vee \mathcal{G}_t) = \sum_{x\in S} \mathbb{1}_{\{X_t=x\}} \mathbb{E}\Big(p_{x,x_1}(t,t_1) \prod_{n=1}^{k-1} p_{x_n,x_{n+1}}(t_n,t_{n+1}) \,\Big|\, \mathcal{F}_t\Big).$$

This ends the proof. ∎

EXAMPLE 2.10 (Time changed discrete Markov chain). Consider a process $\bar{C}$, which is a discrete time Markov chain with values in $S = \{1,\ldots,K\}$ with transition probability matrix P, and a process N, which is a Cox process with càdlàg $\mathbb{F}$-intensity process $\tilde{\lambda}$. From Jakubowski and Niewęgłowski [8, Theorem 7 and 9] we know that under assumption that the processes $(\bar{C}_k)_{k\geq 0}$ and $(N_t)_{t\in[0,T]}$ are independent and conditionally independent given $\mathcal{F}_T$, the process

$$C_t := \bar{C}_{N_t}$$

is an $(\mathbb{F}, \mathbb{F}^C)$-DSMC. Thus, by Proposition 3.4 in [9] and Proposition 2.9, the process C is an $(\mathbb{F}, \mathbb{F}^C)$-CMC. Moreover the $\mathbb{F}$-intensity Λ of C is given by

$$\lambda_t^{xy} = (P-I)_{x,y}\tilde{\lambda}_t.$$

3. CMC—construction via change of measure. Let $\Lambda_t = [\lambda_t^{xy}]_{x,y\in S}$ be an $\mathbb{F}$-progressively measurable, matrix valued process on $(\Omega, \mathcal{A})$, satisfying (3) and such that the set $A^{xy}(\omega) := \{0 \leq t \leq T : \lambda_{t-}^{xy}(\omega) \neq \lambda_t^{xy}(\omega)\}$ has Lebesgue measure zero for every $x, y \in S$, $x \neq y$, and all $\omega \in \Omega$. In this section we provide a method for constructing an $(\mathbb{F}, \mathbb{G})$-CMC, with this Λ as the $\mathbb{F}$-intensity matrix.

THEOREM 3.1. *Let $(\Omega, \mathcal{A}, \mathbb{Q})$ be a probability space such that filtrations $\mathbb{F}$ and $\mathbb{G}$ are independent under $\mathbb{Q}$. Let $(X_t)_{0\leq t\leq T}$ be a time homogeneous $\mathbb{G}$-Markov chain with state space S and with the infinitesimal generator $A = [a^{xy}]_{x,y\in S}$ whose off-diagonal entries are strictly positive. Suppose that processes κ^{xy} defined, for all $x, y \in S$, $x \neq y$, by*

$$\kappa_t^{xy} = \frac{\lambda_{t-}^{xy}}{a^{xy}} - 1, \quad t \in [0,T],$$

are such that random variable η *given as*

$$\eta = \prod_{x,y\in S: x\neq y} \exp\Big(-\int_0^T H^x_{u-} a^{xy} \kappa^{xy}_u \, du\Big) \prod_{0<u\leq T} (1+\kappa^{xy}_u \Delta H^{xy}_u),$$

satisfies $\mathbb{E}_{\mathbb{Q}}\eta = 1$*. Then, under the probability measure* $\mathbb{P}$ *defined by*

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_T\vee\mathcal{G}_T} = \eta \tag{15}$$

the process X *is an* $(\mathbb{F},\mathbb{G})$*-CMC under* $\mathbb{P}$ *with an* $\mathbb{F}$*-intensity matrix process* Λ*.*

Proof. It needs to be shown that for every $x_1,\dots,x_k \in S$ and every $0\leq s\leq t_1\leq \ldots \leq t_k$

$$\mathbb{P}(X_{t_k}=x_k,\dots,X_{t_1}=x_1 \,|\, \mathcal{F}_s\vee\mathcal{G}_s) = \mathbb{P}(X_{t_k}=x_k,\dots,X_{t_1}=x_1 \,|\, \mathcal{F}_s\vee\sigma(X_s)), \tag{16}$$

and that the $\mathbb{R}^d$ valued process M defined by (cf. (4))

$$M_t = H_t - \int_0^t \Lambda^\top_u H_u \, du, \quad t\in[0,T], \tag{17}$$

is an $\mathbb{F}\vee\mathbb{G}$-(local) martingale under $\mathbb{P}$.

Step 1. Proof of (16).

Let us define a Doob martingale $\eta_t,\ t\in[0,T]$, by

$$\eta_t = \mathbb{E}_{\mathbb{Q}}(\eta \,|\, \mathcal{F}_t\vee\mathcal{G}_t).$$

Then, we have

$$\eta_t = \prod_{x,y\in S: x\neq y} \exp\Big(-\int_0^t H^x_{u-} a^{xy} \kappa^{xy}_u \, du\Big) \prod_{0<u\leq t} (1+\kappa^{xy}_u \Delta H^{xy}_u).$$

We note that, for $0\leq s\leq t$,

$$\eta_t = \eta_s L_{s,t},$$

where

$$L_{s,t} := \prod_{x,y\in S: x\neq y} \exp\Big(-\int_s^t H^x_{u-} a^{xy} \kappa^{xy}_u \, du\Big) \prod_{s<u\leq t} (1+\kappa^{xy}_u \Delta H^{xy}(u)).$$

This implies that on the set $\{\eta_s>0\}$

$$\frac{\eta_t}{\eta_s} = L_{s,t}\,.$$

From now on we fix arbitrary $x_1,\dots,x_k\in S$ and $0\leq s<t_1<\ldots<t_k\leq T$, and we define

$$A := \{X_{t_k}=x_k,\dots,X_{t_1}=x_1\}.$$

Then by the Bayes formula we have

$$\mathbb{P}(A \,|\, \mathcal{F}_s\vee\mathcal{G}_s) = \mathbb{1}_{\{\eta_s>0\}} \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_A \frac{\eta_T}{\eta_s} \,\Big|\, \mathcal{F}_s\vee\mathcal{G}_s\Big) \quad \mathbb{P}\text{-a.s.}$$

Since $\mathbb{P}(\eta_s>0)=1$, we may conclude that

$$\mathbb{P}(A \,|\, \mathcal{F}_s\vee\mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_A \frac{\eta_T}{\eta_s} \,\Big|\, \mathcal{F}_s\vee\mathcal{G}_s\Big) \quad \mathbb{P}\text{-a.s.} \tag{18}$$

Now, equality (16) will follow from (19) below, which is true in view of Lemma 4.1,

$$\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A L_{s,T} \,|\, \mathcal{F}_s \vee \mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A L_{s,T} \,|\, \mathcal{F}_s \vee \sigma(X_s)) \quad \mathbb{P}\text{-a.s.} \tag{19}$$

Indeed, from (18) and (19), we have

$$\mathbb{P}(A \,|\, \mathcal{F}_s \vee \mathcal{G}_s) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A L_{s,T} \,|\, \mathcal{F}_s \vee \sigma(X_s)) \quad \mathbb{P}\text{-a.s.} \tag{20}$$

Thus, in view of (20),

$$\begin{aligned}\mathbb{P}(A \,|\, \mathcal{F}_s \vee \sigma(X_s)) &= \mathbb{E}_{\mathbb{P}}(\mathbb{P}(A \,|\, \mathcal{F}_s \vee \mathcal{G}_s) \,|\, \mathcal{F}_s \vee \sigma(X_s)) \\ &= \mathbb{E}_{\mathbb{P}}(\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A L_{s,T} \,|\, \mathcal{F}_s \vee \sigma(X_s)) \,|\, \mathcal{F}_s \vee \sigma(X_s)) \\ &= \mathbb{E}_{\mathbb{Q}}(\mathbb{1}_A L_{s,T} \,|\, \mathcal{F}_s \vee \sigma(X_s)) = \mathbb{P}(A \,|\, \mathcal{F}_s \vee \mathcal{G}_s),\end{aligned}$$

which is (16).

Step 2. M defined by (17) is an $\mathbb{F} \vee \mathbb{G}$-local martingale under $\mathbb{P}$.
Since X is a $\mathbb{G}$ Markov chain we know that, for every $x, y \in S$, $x \neq y$, the process M^{xy} given by

$$M_t^{xy} = H_t^{xy} - \int_0^t H_u^x a^{xy}\, du, \quad 0 \le t \le T,$$

is a $\mathbb{G}$ martingale under $\mathbb{Q}$. In view of independence under $\mathbb{Q}$ of the filtrations $\mathbb{F}$ and $\mathbb{G}$, process X is also an $\mathbb{F} \vee \mathbb{G}$ Markov chain, and thus, each process M^{xy} is also an $\mathbb{F} \vee \mathbb{G}$ martingale under $\mathbb{Q}$. Applying Girsanov's theorem (see Brémaud [5, Thm. VI.T3]) we conclude that, for each $x, y \in S$, $x \neq y$, the $\mathbb{F} \vee \mathbb{G}$-compensator of H^{xy} under $\mathbb{P}$ has density with respect to the Lebesgue measure given as

$$H_t^x a^{xy}(1 + \kappa_t^{xy}) = H_t^x a^{xy}\Big(1 + \frac{\lambda_{t-}^{xy}}{a^{xy}} - 1\Big) = H_t^x \lambda_{t-}^{xy}, \quad 0 \le t \le T.$$

Thus

$$\widetilde{M}_t^{xy} = H_t^{xy} - \int_0^t H_u^x \lambda_{u-}^{xy}\, du, \quad 0 \le t \le T,$$

is an $\mathbb{F} \vee \mathbb{G}$-martingale under $\mathbb{P}$. Given our technical assumption that set A^{xy} has Lebesgue measure zero, we see that, for $0 \le t \le T$,

$$\widetilde{M}_t^{xy} = H_t^{xy} - \int_0^t H_u^x \lambda_u^{xy}\, du, \quad 0 \le t \le T. \tag{21}$$

Thus according to Remark 2.6 we can use Theorem 2.5 to conclude that we can take Λ as a version of $\mathbb{F}$-intensity of X. ∎

4. Appendix

LEMMA 4.1. *Let $\mathbb{F}$ and $\mathbb{G}$ be independent under $\mathbb{P}$, and let X be a $\mathbb{G}$-Markov chain. Let us fix $s \in [0, T]$, and suppose that η, ξ are bounded random variables such that $\eta \in \sigma(X_u,\ s \le u \le T)$, and $\xi \in \mathcal{F}_T$. Then we have*

$$\mathbb{E}(\eta\xi \,|\, \mathcal{F}_s \vee \mathcal{G}_s) = \mathbb{E}(\eta\xi \,|\, \mathcal{F}_s \vee \sigma(X_s)). \tag{22}$$

Proof. We have

$$\begin{aligned}\mathbb{E}(\eta\xi \mid \mathcal{F}_s \vee \mathcal{G}_s) &= \mathbb{E}(\mathbb{E}(\eta\xi \mid \mathcal{F}_\infty \vee \mathcal{G}_s) \mid \mathcal{F}_s \vee \mathcal{G}_s) = \mathbb{E}(\xi\mathbb{E}(\eta \mid \mathcal{F}_\infty \vee \mathcal{G}_s) \mid \mathcal{F}_s \vee \mathcal{G}_s)\\ &= \mathbb{E}(\xi\mathbb{E}(\eta \mid \mathcal{G}_s) \mid \mathcal{F}_s \vee \mathcal{G}_s) = \mathbb{E}(\xi\mathbb{E}(\eta \mid \sigma(X_s)) \mid \mathcal{F}_s \vee \mathcal{G}_s)\\ &= \mathbb{E}(\xi \mid \mathcal{F}_s \vee \mathcal{G}_s)\mathbb{E}(\eta \mid \sigma(X_s)) = \mathbb{E}(\xi \mid \mathcal{F}_s)\mathbb{E}(\eta \mid \sigma(X_s)),\end{aligned}$$

where the third equality follows from independence of $\mathbb{F}$ and $\mathbb{G}$, the fourth one from $\mathbb{G}$-Markov property of X, and the sixth one follows from the independence of $\mathbb{F}$ and $\mathbb{G}$, as well. On the other hand we have

$$\begin{aligned}\mathbb{E}(\eta\xi \mid \mathcal{F}_s \vee \sigma(X_s)) &= \mathbb{E}(\mathbb{E}(\eta\xi \mid \mathcal{F}_\infty \vee \sigma(X_s)) \mid \mathcal{F}_s \vee \sigma(X_s))\\ &= \mathbb{E}(\xi\mathbb{E}(\eta \mid \mathcal{F}_\infty \vee \sigma(X_s)) \mid \mathcal{F}_s \vee \sigma(X_s))\\ &= \mathbb{E}(\xi\mathbb{E}(\eta \mid \sigma(X_s)) \mid \mathcal{F}_s \vee \sigma(X_s)) = \mathbb{E}(\xi \mid \mathcal{F}_s \vee \sigma(X_s))\mathbb{E}(\eta \mid \sigma(X_s))\\ &= \mathbb{E}(\xi \mid \mathcal{F}_s)\mathbb{E}(\eta \mid \sigma(X_s)),\end{aligned}$$

where the third equality follows from the independence of $\mathbb{F}$ and $\mathbb{G}$. ∎

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