

EXPONENTIAL ERGODICITY OF SEMILINEAR EQUATIONS DRIVEN BY LÉVY PROCESSES IN HILBERT SPACES

ANNA CHOJNOWSKA-MICHALIK

*Faculty of Mathematics and Computer Science, University of Łódź
Banacha 22, 90-238 Łódź, Poland
E-mail: anmich@math.uni.lodz.pl*

BENIAMIN GOLDYS

*School of Mathematics and Statistics, The University of Sydney
Sydney 2006, Australia
E-mail: beniamin.goldys@sydney.edu.au*

This work is dedicated to Professor Jerzy Zabczyk, our Master and Teacher.

Abstract. We study convergence to the invariant measure for a class of semilinear stochastic evolution equations driven by Lévy noise, including the case of cylindrical noise. For a certain class of equations we prove the exponential rate of convergence in the norm of total variation. Our general result is applied to a number of specific equations driven by cylindrical symmetric α -stable noise and/or cylindrical Wiener noise. We also consider the case of a “singular” Wiener process with unbounded covariance operator. In particular, in the equation with diagonal pure α -stable cylindrical noise introduced by Priola and Zabczyk we generalize results from Priola, Shirikyan, Xu and Zabczyk (2012). In the proof we use an idea of Maslowski and Seidler (1999).

1. Introduction. Let H be a real separable Hilbert space. In this paper we are concerned with the ergodic behaviour of the semilinear stochastic equation

$$\begin{cases} dX_t = (AX_t + F(X_t)) dt + dZ_t, & t > 0, \\ X_0 = x \in H, \end{cases} \quad (1)$$

driven by, possibly cylindrical, Lévy noise Z . We assume that A is a generator of a C_0 -semigroup $(S_t)_{t \geq 0}$ of bounded linear operators on H and $F : H \rightarrow H$ is a Lipschitz mapping. Under the conditions specified in Section 2 equation (1) has a unique solution

2010 *Mathematics Subject Classification*: Primary 60G51, 60H15.

Key words and phrases: Lévy noise, semilinear SPDE, exponential ergodicity, total variation.

The paper is in final form and no version of it will be published elsewhere.

(X_t^x) for every $x \in H$. Our main question is to provide conditions for the exponential convergence of transition measures $\mu_t^x(\cdot) = \mathbb{P}(X_t^x \in \cdot)$ to a unique invariant measure μ in the norm of total variation. This question has been thoroughly investigated in the case when $Z = BW$ with a certain cylindrical Wiener process W and B a linear operator, see for example [5, 6, 9]. In those works more general, non-Lipschitz drifts F were permitted. The two main ingredients for the proofs were the strong Feller property and topological irreducibility of the transition measures μ_t^x .

In this paper we extend the aforementioned approach to equation (1) with general (subject to some natural conditions) Lévy noise Z and bounded Lipschitz drift F . Exponential ergodicity for equation (1) was investigated recently in [12, 13]. In Theorem 2.8 of Section 2 we show that if the semigroup (S_t) is exponentially stable and compact and the measures μ_t^x are strong Feller and irreducible then the convergence to the invariant measure holds in the total variation norm and moreover the rate of convergence is exponential. The proof is based on an idea developed in [10]. In Section 3 we discuss three examples. In Subsection 3.1 we apply our general result to equation (1) with the noise Z being a symmetric α -stable cylindrical noise. In Corollary 3.2 we show that Theorem 2.8 yields a strengthening of the similar result obtained in [12] and of an earlier result in [13]. In the proof we use the fundamental results of Priola and Zabczyk proved in [15]. In Subsection 3.2 we consider Lévy noise with both, the jump part and the Gaussian part, non-vanishing. Finally, in Section 3.3 we consider the stochastic heat equation driven by a mixture of a cylindrical symmetric α -stable noise and a singular Wiener noise with unbounded covariance operator. Section 4 contains proofs related to Section 2.

NOTATION. In what follows, the norm in the Hilbert space H is denoted by $|\cdot|$. The norm in any other Banach space E will be denoted as $|\cdot|_E$.

The Hilbert–Schmidt norm of an operator $K : H \rightarrow H$ will be denoted as $\|K\|_{HS}$.

2. Formulation of the problem and main result. We start with the set of conditions that are assumed to hold throughout the paper. We assume that $Z = (Z_t)_{t \geq 0}$ is a Lévy process defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions and with values in a Hilbert space U , such that $H \subset U$ with continuous and dense embedding. More precisely, we assume that

$$Z_t = ta + Q^{1/2}W_t + Y_t, \quad t \geq 0,$$

where $a \in U$ is a fixed vector, W is a standard cylindrical Wiener process in H , the process Y is a U -valued, pure jump Lévy process, independent of W , and $Q : \text{dom}(Q) \subset H \rightarrow H$ is a selfadjoint and nonnegative operator.

We will assume that the semigroup (S_t) enjoys the following properties.

HYPOTHESIS 2.1.

1. For every $t \in (0, 1]$, the operator S_t has an extension (still denoted by S_t) to a bounded operator $S_t : U \rightarrow H$.
2. We have

$$\int_0^1 |S_s a| ds < \infty. \tag{2}$$

For each $t > 0$ the operator $S_t Q^{1/2}$ extends to a bounded operator on H and

$$\int_0^1 \|S_s Q^{1/2}\|_{HS}^2 ds < \infty. \quad (3)$$

HYPOTHESIS 2.2. For every $t \in (0, 1]$ the stochastic integral

$$\int_0^t S_s dY_s$$

is a well defined H -valued random variable defined as a limit in probability of the corresponding Stieltjes sums (see e.g. [2] and [11]). Moreover, there exists $p > 0$ such that

$$\gamma_p = \sup_{t \in (0, 1]} \mathbb{E} \left| \int_0^t S_s dY_s \right|^p < \infty, \quad (4)$$

and

$$\lim_{t \downarrow 0} \mathbb{E} \left| \int_0^t S_s dY_s \right|^p = 0. \quad (5)$$

The remark below provides a convenient sufficient condition for Hypothesis 2.2 to hold. It is proved in Section 4.

REMARK 2.3. If $U = H$ and $\mathbb{E}|Y_1|^p < \infty$ for a certain $p > 0$ then Hypothesis 2.2 is satisfied with this p .

Finally, we impose the standard Lipschitz condition on the nonlinear term F .

HYPOTHESIS 2.4. The function $F : H \rightarrow H$ is Lipschitz continuous and bounded.

PROPOSITION 2.5. Assume Hypothesis 2.2 and let

$$Y_A(t) = \int_0^t S_{t-s} dY_s, \quad t \geq 0.$$

Then the following hold.

1. For any $t \geq 0$ there exists a constant $C(t, p, A)$ such that

$$\mathbb{E}|Y_A(t)|^p \leq \gamma_p C(t, p, A). \quad (6)$$

2. The process Y_A is continuous in p -th mean.

3. The process Y_A admits a predictable modification.

REMARK 2.6. Clearly, Proposition 2.5 holds for a more general process

$$\begin{aligned} Z_A(t) &= \int_0^t S_{t-s} dZ_s \\ &= \int_0^t S_s a ds + \int_0^t S_{t-s} Q^{1/2} dW_s + \int_0^t S_{t-s} dY_s, \quad t \geq 0, \end{aligned}$$

with

$$\tilde{\gamma}_p = \sup_{t \in (0, 1]} \mathbb{E} \left| \int_0^t S_s dZ_s \right|^p < \infty,$$

by (2), (3) and (4). We will always consider a predictable modification of Z_A .

Let us recall that the process X^x is a mild solution to equation (1) if it is H -valued, predictable and

$$X_t^x = S_t x + \int_0^t S_{t-s} F(X_s^x) ds + Z_A(t), \quad t \geq 0, \quad \mathbb{P}\text{-a.s.} \quad (7)$$

We will use the notation

$$\mu_t^x \quad \text{for the law of the random variable } X_t^x \in H.$$

PROPOSITION 2.7. *Assume Hypotheses 2.1–2.4. Then for every $x \in H$ equation (1) has a unique mild solution X^x . The process X^x is Markov and Feller in H ,*

$$\mathbb{E} |X_t^x|^p < \infty, \quad t \geq 0,$$

and the process X^x is t -continuous in p -th mean.

Now we can formulate the main result of this paper. In the theorem below we denote by $\|\nu\|_{\text{var}}$ the total variation norm of a signed Borel measure ν on H . For the definitions of the strong Feller property and open set irreducibility the reader may consult [4].

THEOREM 2.8. *Assume Hypotheses 2.1–2.4. Moreover assume that the semigroup (S_t) is compact and exponentially stable. Assume also that the solution X to (1) is strongly Feller and open set irreducible. Then there exists a unique invariant measure μ for (1) and there exist constants $\beta > 0$ and $C > 0$ such that*

$$\|\mu_t^x - \mu\|_{\text{var}} \leq C e^{-\beta t} (1 + |x|^p), \quad x \in H, \quad t \geq 0. \quad (8)$$

REMARK 2.9. Let X_t^ν be the solution to (1) starting from an \mathcal{F}_0 -measurable random variable X_0 such that $\text{Law}(X_0) = \nu$. Then inequality (8) extends by standard arguments to a more general one

$$\|\mu_t^\nu - \mu\|_{\text{var}} \leq C e^{-\beta t} \left(1 + \int_H |x|_H^p \nu(dx) \right), \quad t \geq 0, \quad (9)$$

where μ_t^ν is the law of the random variable X_t^ν .

3. Examples

3.1. Model with cylindrical α -stable noise. In this section we apply Theorem 2.8 to equation (1) driven by a cylindrical symmetric α -stable noise Z with $\alpha \in (0, 2)$, introduced by Priola and Zabczyk in [15] and later studied in [12] and [13].

Let $\dim(H) = \infty$ and let $\{e_k : k \geq 1\}$ be a CONS in H . We will identify H with the space l^2 of square-summable sequences. Let (ρ_k) be an arbitrary sequence of positive real numbers. Then we can define the Hilbert space l_ρ^2 of all sequences $x = (x_k)$ such that

$$|x|_\rho^2 = \sum_{k=1}^{\infty} x_k^2 \rho_k^2 < \infty.$$

The equation

$$\begin{cases} dX_t = (AX_t + F(X_t)) dt + dZ_t, & t > 0, \\ X_0 = x \in H, \end{cases} \quad (10)$$

is studied in [15] and in [13] in the space $H = l^2$ under the set of assumptions listed below.

HYPOTHESIS 3.1.

1. Let A be a selfadjoint operator $A : \text{dom}(A) \subset H \rightarrow H$ such that

$$Ae_k = -\lambda_k e_k, \quad k \geq 1,$$

with $0 < \lambda_1 \leq \lambda_2 \leq \dots, \quad \lambda_k \rightarrow \infty.$

2. For $\alpha \in (0, 2)$ we define a cylindrical α -stable process

$$Z(t) = \sum_{k=1}^{\infty} b_k Z_k(t) e_k, \quad t \geq 0,$$

where (Z_k) is a sequence of real-valued symmetric and independent α -stable processes and (b_k) is a sequence of positive reals such that

$$\sum_{k=1}^{\infty} \frac{b_k^\alpha}{\lambda_k} < \infty. \quad (11)$$

3. The mapping $F : H \rightarrow H$ is Lipschitz continuous and bounded.

4. There exist constants $\theta \in (0, 1)$ and $\tilde{c} > 0$ such that

$$b_k \geq \tilde{c} \lambda_k^{-\theta+1/\alpha}, \quad k \geq 1.$$

Let for a fixed $\sigma \geq \frac{1}{\alpha}$,

$$U_\sigma = l_{\rho(\sigma)}^2, \quad \rho(\sigma) = (\lambda_k^{-\sigma}).$$

Clearly, H is continuously and densely imbedded into U_σ .

We shall show that Hypothesis 3.1 implies the assumptions of Proposition 2.7 and of Theorem 2.8.

First, note that Z is a U_σ -valued Lévy process. Indeed, by Proposition 3.3 in [15], Z is a Lévy process with values in l_ρ with $\rho = (\rho_k)$ such that

$$\sum_{k=1}^{\infty} b_k^\alpha \rho_k^\alpha < \infty,$$

hence with values in U_σ by (11).

Next, the operator A with the domain $\text{dom}(A) = l_\lambda^2$, where $\lambda = (\lambda_k)$, generates on H a symmetric compact semigroup (S_t) with

$$S_t e_k = e^{-\lambda_k t}, \quad k \geq 1, \quad t \geq 0.$$

In particular,

$$\|S_t\| = e^{-\lambda_1 t}, \quad t \geq 0. \quad (12)$$

Therefore,

$$\text{the additional assumptions on } (S_t) \text{ in Theorem 2.8 hold.} \quad (13)$$

Fix $t > 0$ and let $x = (x_k) \in H \subset U_\sigma$. Then

$$|S_t x|_H^2 = \sum_{k=1}^{\infty} e^{-2\lambda_k t} x_k^2 = \sum_{k=1}^{\infty} \lambda_k^{2\sigma} e^{-2\lambda_k t} \frac{x_k^2}{\lambda_k^{2\sigma}}. \quad (14)$$

Since $\lambda_k \nearrow \infty$,

$$c(t) = \sup_k (\lambda_k^\sigma e^{-\lambda_k t}) < \infty$$

and (14) yields

$$\|S_t\|_{U_\sigma \rightarrow H} \leq c(t),$$

and the proof of part 1. of Hypothesis 2.1 is complete.

Since $a = 0$ and $Q = 0$ all the conditions of Hypothesis 2.1 are satisfied.

Next, if parts 1. and 2. of Hypothesis 3.1 hold then it follows from Proposition 4.4 of [15] that for any $t > 0$ the stochastic integral $\int_0^t S_s dZ_s$ takes values in H , hence the first part of Hypothesis 2.2 is satisfied and additionally for any $p \in (0, \alpha)$

$$\mathbb{E} \left| \int_0^t S_s dZ_s \right|^p \leq \tilde{c}_p \left(\sum_{k=1}^{\infty} b_k^\alpha \frac{1 - e^{-\alpha \lambda_k t}}{\alpha \lambda_k} \right)^{p/\alpha}, \quad (15)$$

where the constant \tilde{c}_p depends on p only. We note that the last estimate is an immediate consequence of Theorem 4.6 in [15]. Note also that by (11) the right hand side of (15) is finite uniformly in $t > 0$, hence our condition (4) is satisfied with

$$\gamma_p \leq \tilde{c}_p \left(\frac{1}{\alpha} \right)^{p/\alpha} \left(\sum_{k=1}^{\infty} \frac{b_k^\alpha}{\lambda_k} \right)^{p/\alpha}.$$

Moreover, using the Dominated Convergence we find that the right hand side of (15) tends to 0 as $t \downarrow 0$ which yields our condition (5). Finally, all the conditions of Hypothesis 2.2 hold. Hypothesis 2.4 is identical with part 3. of Hypothesis 3.1.

Thus we have proved that under assumptions 1.–3. of Hypothesis 3.1, our Hypotheses 2.1–2.4 are satisfied and thereby as an immediate consequence of Proposition 2.7 and Theorems 5.5 and 5.7 in [15] we obtain

COROLLARY 3.2. *Assume that Hypothesis 3.1 holds. Then for every $x \in H$ there exists a unique mild solution X^x to equation (10) which is an H -valued Markov process, irreducible and strong Feller. Moreover, for any $p \in (0, \alpha)$*

$$\mathbb{E} |X_t^x|^p < \infty, \quad t \geq 0.$$

Finally, taking into account (13) and (12) we note that by Hypothesis 3.1 all the assumptions of our Theorem 2.8 are satisfied and we obtain immediately the following strengthening of Theorem 2.8 in [12] and of Theorems 2.2 and 2.3 in [13].

COROLLARY 3.3. *Under Hypothesis 3.1 there exists a unique invariant measure for (10) and for any $p \in (0, \alpha)$ there exist constants $\beta > 0$ and $C > 0$ such that*

$$\|\mu_t^x - \mu\|_{\text{var}} \leq C e^{-\beta t} (1 + |x|_H^p), \quad t \geq 0, \quad x \in H. \quad (16)$$

REMARK 3.4. Ascertainment (16) is proved in Theorem 2.8 in [12] under Hypothesis 3.1 and an additional assumption that for some $\epsilon > 0$

$$\sum_{k=1}^{\infty} \frac{b_k^\alpha}{\lambda_k^{1-\alpha\epsilon}} < \infty. \quad (17)$$

Let us note that (17) ensures that the solution X to (10) takes values in a linear subspace compactly imbedded into H , while (11) of Hypothesis 3.1 is equivalent to the fact that X is H -valued only (comp. Remark 2.3 in [12]).

Corollary 3.3 strengthens also Theorems 2.2 and 2.3 from an earlier paper [13]. In that paper additionally to Hypothesis 3.1 the nonlinearity F must be sufficiently small and only weak convergence to the invariant measure μ with exponential rate is obtained.

3.2. Linear equation driven by Lévy noise with Gaussian component. In this section we consider equation (1) with $F = 0$, i.e.

$$\begin{cases} dX_t = (AX_t + a) dt + Q^{1/2} dW_t + dY_t, & t > 0 \\ X_0 = x \in H, \end{cases} \quad (18)$$

under Hypotheses 2.1 and 2.2. By (3) the operator $Q_t : H \rightarrow H$ defined as

$$Q_t x = \int_0^t S_s Q S_s^* x ds, \quad x \in H,$$

is nonnegative and trace class for every $t > 0$. We will also assume the following

HYPOTHESIS 3.5.

$$S_t(H) \subset Q_t^{1/2}(H), \quad t > 0.$$

Let us recall that Hypothesis 3.5 holds if and only if the solution X^{OU} (Gaussian Ornstein–Uhlenbeck process), corresponding to equation (18) with $Y = 0$ and $a = 0$, is strong Feller (see Section 9.4.1 of [3]). Therefore, if Hypothesis 3.5 holds then the solution X to (18) is strong Feller as well by Lemma 20.1 in [14].

We note also that by Hypothesis 3.5 and the strong continuity of the semigroup (S_t) the set

$$Q_t^{1/2}(H) = \bigcup_{s \in (0,t]} Q_s^{1/2}(H)$$

is dense in H for every $t > 0$. Hence, $N(S_t x, Q_t) = \mathcal{L}aw(X_t^{OU,x})$ has a support on the whole of H for every $t > 0$. Therefore X_t^x is irreducible for every $t > 0$ and $x \in H$ by Lemma 20.1 in [14].

Thus we obtain the following immediate consequence of Theorem 2.8.

COROLLARY 3.6. *Assume Hypotheses 2.1, 2.2 and 3.5 and assume moreover that the semigroup (S_t) is exponentially stable. Then equation (18) has a unique invariant measure μ that satisfies (8).*

In particular, invoking Remark 2.3 we obtain

COROLLARY 3.7. *Let $U = H$ and $\mathbb{E}|Y_1|^p < \infty$ for certain $p > 0$. Moreover, assume (3) and Hypothesis 3.5 and that the semigroup (S_t) is exponentially stable. Then equation (18) has a unique invariant measure μ that satisfies (8).*

3.3. Heat equation with Lévy noise with singular Gaussian component. In this section we will apply Theorem 2.8 to the stochastic heat equation

$$\begin{cases} dX_t(\xi) = \left(\frac{\partial^2 X_t}{\partial \xi^2}(\xi) + a(\xi) \right) dt + Q^{1/2} dW_t(\xi) + dY_t(\xi), & \xi \in (0, \pi), t > 0 \\ X_0(\xi) = x(\xi), & \xi \in [0, \pi] \\ X_t(0) = X_t(\pi) = 0, & t > 0, \end{cases} \quad (19)$$

with an unbounded operator $Q = (-\frac{\partial^2}{\partial \xi^2})^{\delta/2}$, vector $a \in U$, and a cylindrical α -stable process Y . It is well known that equation (19) can be rewritten as a version of an abstract equation (18). Indeed, let $H = L^2(0, \pi)$ and

$$A = \frac{\partial^2}{\partial \xi^2}, \quad \text{dom}(A) = H^2(0, \pi) \cap H_0^1(0, \pi).$$

Then A has eigenfunctions

$$e_k(\xi) = \sqrt{\frac{2}{\pi}} \sin k\xi, \quad k \geq 1,$$

and eigenvalues

$$-\lambda_k = -k^2, \quad k \geq 1.$$

We assume that for a certain $\delta \in \mathbb{R}$

$$Q = \sum_{k=1}^{\infty} q_k e_k \otimes e_k, \quad q_k = k^\delta. \quad (20)$$

Then for every $t > 0$,

$$\sup_k \sqrt{\frac{\lambda_k}{q_k}} e^{-\lambda_k t} < \infty,$$

and therefore, by Proposition 9.30 in [3], Hypothesis 3.5 holds. Note that (3) holds for A and Q defined above iff

$$\delta < 1. \quad (21)$$

Indeed, (21) is satisfied iff

$$\sum_{k=1}^{\infty} \frac{q_k}{\lambda_k} < \infty,$$

which is equivalent to (3), comp. Proposition 9.30 in [3]. In particular, for $\delta \in (0, 1)$ the operator $Q = (-A)^{\delta/2}$ in (20) is unbounded but condition (3) holds.

Let

$$Y(t) = \sum_{k=1}^{\infty} b_k Y_k(t) e_k, \quad (22)$$

where (Y_k) is a sequence of real-valued, independent, symmetric and α -stable processes with $\alpha \in (0, 2)$. We will assume that the sequence of nonnegative real numbers (b_k) enjoys the following property.

The set

$$\mathcal{K} = \{k \in \mathbb{N} : b_k > 0\}$$

is infinite and there exists $\gamma \in \mathbb{R}$ such that

$$b_k = k^\gamma, \quad k \in \mathcal{K}. \quad (23)$$

Observe that if

$$\gamma < \frac{1}{\alpha}, \quad (24)$$

then condition (11) holds and by results in Section 3.1 Hypothesis 2.2 is satisfied for $p < \alpha$. In particular, if $0 \leq \gamma < \frac{1}{\alpha}$ then the process Y defined by (22) and (23) is a cylindrical noise taking values in the space $U_{1/\alpha}$ (see Section 3.1). Note that if $\mathcal{K} \neq \mathbb{N}$ then the process Y is degenerate. Let

$$a = \sum_{k=1}^{\infty} a_k e_k \in U_{1/2},$$

that is

$$\sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty. \quad (25)$$

Then for every $t \geq 0$

$$\int_0^t |S_s a|^2 ds < \infty,$$

and (2) follows. Finally, as a consequence of Corollary 3.6 we obtain the following

COROLLARY 3.8. *Under assumptions (20)–(25) equation (19) has a unique stationary distribution μ and for any $p \in (0, \alpha)$ exponential estimate (16) holds.*

4. Proofs

Proof of Remark 2.3. If $p > 1$, then the process $(Y_t - \mathbb{E}Y_t)_{t \geq 0}$ is an (\mathcal{F}_t) -martingale and (4) and (5) follow easily.

Let $0 < p \leq 1$. It has been proved in [8], see also [7] for a more general result, that $\mathbb{E}|Y_1|^p < \infty$ iff

$$\int_{|x| \geq 1} |x|^p M(dx) < \infty, \quad (26)$$

where M is the Lévy measure of Y . We can represent

$$Y_t = Y_t^1 + Y_t^2,$$

where Y^1 is a Lévy process corresponding to the Lévy measure M restricted to the ball B_1 and Y^2 is a Lévy process corresponding to $M^{(2)}$, the Lévy measure M restricted to $H \setminus B_1$, and moreover Y^1 and Y^2 are independent. Therefore (4) is satisfied iff (4) holds for Y^2 . By [11] we have

$$\int_0^t S_s dY_s^2 = \int_0^t \int_{H \setminus B_1} S_s x N(ds, dx),$$

where N is the Poisson measure corresponding to Y^2 , in particular N takes nonnegative integer values and hence $\mathbb{E}|N(B)|^p \leq \mathbb{E}|N(B)| = M^{(2)}(B)$ for $B \in \mathcal{B}(H)$. Therefore

as in [1] we obtain the estimate:

$$\begin{aligned} \mathbb{E} \left| \int_0^t S_s dY_s^2 \right|^p &\leq \int_0^t \int_{H \setminus B_1} |S_s x|^p M^{(2)}(dx) ds \\ &\leq \left(\int_0^t \|S_s\|^p ds \right) \left(\int_{H \setminus B_1} |x|^p M(dx) \right) < +\infty, \end{aligned}$$

where the last inequality follows from (26) and (4), and (5) hold. ■

Proof of Proposition 2.5. We will consider only the case of $p \in (0, 1)$ which is less standard. The proof for $p \geq 1$ is similar. We recall that for $p \in (0, 1)$ the space $L^p(\Omega, H)$ is a linear complete metric space with the distance

$$d_p(\xi, \eta) = \mathbb{E} |\xi - \eta|^p.$$

Proof of 1. For $u > 0, v \geq 0$ we have (e.g. [2])

$$\int_v^{u+v} S_{u+v-s} dY_s \stackrel{D}{=} \int_0^u S_{u-s} dY_s \stackrel{D}{=} \int_0^u S_s dY_s, \quad (27)$$

where $\stackrel{D}{=}$ stands for the equality of probability distributions. Hence, invoking the fact that for certain constants $C_A \geq 1$ and δ

$$\|S_t\| \leq C_A e^{\delta t}, \quad t \geq 0,$$

we find that for a fixed $t > 0$

$$\begin{aligned} \mathbb{E} |Y_A(t)|^p &= \mathbb{E} \left| \int_0^t S_s dY_s \right|^p \\ &\leq \sum_{k=0}^{[t]-1} \mathbb{E} \left| \int_k^{k+1} S_s dY_s \right|^p + \|S_{[t]}\|^p \mathbb{E} \left| \int_0^{t-[t]} S_s dY_s \right|^p \\ &\leq \gamma_p C_A^p \sum_{k=0}^{[t]} e^{p\delta k} < \infty, \end{aligned} \quad (28)$$

and (6) follows with

$$C(t, p, A) = C_A^p \sum_{k=0}^{[t]} e^{p\delta k}.$$

In (28) estimates follow from the triangle inequality and (27).

Proof of 2. Fix $t \geq 0$. Then for $h > 0$

$$Y_A(t+h) - Y_A(t) = \int_t^{t+h} S_{t+h-s} dY_s + \int_0^t (S_{t+h-s} - S_{t-s}) dY_s.$$

Hence, by triangle inequality and (27)

$$\begin{aligned} \mathbb{E} |Y_A(t+h) - Y_A(t)|^p &\leq \mathbb{E} \left| \int_0^h S_s dY_s \right|^p + \mathbb{E} |S_h Y_A(t) - Y_A(t)|^p \\ &= J_1(h) + J_2(t, h), \end{aligned}$$

and by (5) of Hypothesis 2.2,

$$\lim_{h \downarrow 0} J_1(h) = 0. \quad (29)$$

Concerning $J_2(t, h)$ observe that for \mathbb{P} -a.e. ω

$$\lim_{h \downarrow 0} S_h Y_A(t, \omega) = Y_A(t, \omega),$$

by the strong continuity of the semigroup (S_t) . Moreover, for $h \in (0, 1]$

$$\begin{aligned} |(S_h - I)Y_A(t, \omega)| &\leq (\|S_h\| + 1)|Y_A(t, \omega)| \\ &\leq C_A(2 + e^\delta)|Y_A(t, \omega)|, \end{aligned}$$

Therefore, by part 1. of the proposition and the Dominated Convergence we obtain

$$\lim_{h \downarrow 0} J_2(t, h) = 0, \quad (30)$$

and the right-continuity of Y_A in p -th mean follows. Similarly, to prove left-continuity, fix $t > 0$ and let $0 < h < t \wedge 1$. Then, using (27) we obtain

$$\begin{aligned} \mathbb{E} |Y_A(t) - Y_A(t-h)|^p &= \mathbb{E} \left| (S_h - I)Y_A(t-h) + \int_{t-h}^t S_{t-s} dY_s \right|^p \\ &\leq J_1(h) + \mathbb{E} \left| (S_h - I) \int_0^{t-h} S_s dY_s \right|^p \\ &= J_1(h) + J_3(t, h), \end{aligned}$$

and

$$\begin{aligned} J_3(t, h) &\leq \mathbb{E} \left| (S_h - I) \int_0^t S_s dY_s \right|^p + \|S_h - I\|^p \mathbb{E} \left| \int_{t-h}^t S_s dY_s \right|^p \\ &\leq J_2(t, h) + C_A^p(2 + e^\delta)^p J_1(h). \end{aligned}$$

Therefore, by (29) and (30)

$$\lim_{h \downarrow 0} J_3(t, h) = 0$$

and part 2. of Proposition 2.5 follows.

Proof of 3. The Chebyshev inequality and part 2. of Proposition 2.5 yield the stochastic continuity of the process Y_A . Since Y_A is (\mathcal{F}_t) -adapted, part 3. follows from Proposition 3.6 of [3]. ■

Proof of Proposition 2.7. We follow the proof of Theorem 5.4 in [15]. Recall that we can choose a predictable version of the process Z_A . Uniqueness of solutions to (1) is an easy consequence of the Gronwall Lemma. To prove existence observe first that under Hypothesis 2.4, for fixed measurable function $f : [0, T] \rightarrow H$ and $x \in H$, the deterministic equation

$$y(t) = S_t x + \int_0^t S_{t-s} F(y(s) + f(s)) ds$$

has a unique continuous solution $y : [0, T] \rightarrow H$, hence by part 3. of Proposition 2.5 there exists a continuous H -valued and (\mathcal{F}_t) -adapted process (V_t) such that \mathbb{P} -a.s.

$$V_t^x = S_t x + \int_0^t S_{t-s} F(V_s^x + Z_A(s)) ds, \quad t \geq 0.$$

Consequently, the process $X_t^x = V_t^x + Z_A(t)$, $t \geq 0$, is a predictable solution to (7). By Proposition 2.5 and Remark 2.6 we have $\mathbb{E} |X_t^x|^p < \infty$. The Markov property is proved in Theorem 5.4 of [15] and the Feller property follows from the Gronwall Lemma. ■

Proof of Theorem 2.8.

Step 1. We show first that there exist positive constants p, κ, c_1, c_2 such that

$$\mathbb{E} |X_t^x|^p \leq c_1 |x|^p e^{-\kappa pt} + c_2, \quad t \geq 0, x \in H. \quad (31)$$

To this end we note that by Hypothesis 2.4

$$C_F = \sup_{x \in H} |F(x)| < \infty,$$

and since the semigroup (S_t) is exponentially stable, there exist constants $C_A, \kappa > 0$ such that

$$\|S_t\| \leq C_A e^{-\kappa t}, \quad t \geq 0.$$

Therefore, using Remark 2.6 and (28) for $Z_A(t)$ we obtain

$$\begin{aligned} \mathbb{E} |X_t^x|^p &\leq |S_t x|^p + \mathbb{E} \left(\int_0^t \|S_{t-s}\| \cdot |F(X_s^x)| ds \right)^p + \tilde{\gamma}_p C_A^p \sum_{j=0}^{[t]} e^{-\kappa pj} \\ &\leq C_A^p e^{-\kappa pt} |x|^p + C_A^p C_F^p \kappa^{-p} + \tilde{\gamma}_p C_A^p (1 - e^{-\kappa p})^{-1}, \end{aligned}$$

and (31) follows.

Step 2. In order to prove the existence of an invariant measure for equation (1) we will use the method of [4] and [13] to show that the family $\{\mu_t^0 : t \geq 1\}$ is tight. Indeed, for $t \geq 1$

$$\begin{aligned} X_t^0 &= S_1 X_{t-1}^0 + \int_{t-1}^t S_{t-s} F(X_s^0) ds + \int_{t-1}^t S_{t-s} dZ_s \\ &= X_{1,t} + X_{2,t} + X_{3,t}. \end{aligned}$$

Fix arbitrary $\epsilon \in (0, 1)$. Since for any $t \geq 1$

$$X_{3,t} \stackrel{D}{=} \int_0^1 S_s dZ_s,$$

there exists a compact set $K_3 = K_3(\epsilon) \subset H$ such that

$$\mathbb{P}(X_{3,t} \in K_3) \geq 1 - \frac{\epsilon}{2}, \quad \text{for all } t \geq 1.$$

Next, by (31)

$$\sup_{t \geq 0} \mathbb{E} |X_t^0|^p \leq c_2 < \infty,$$

hence by the Chebyshev inequality there exists $r = r_\epsilon > 0$ such that

$$\mathbb{P}(X_t^0 \in B_r) \geq 1 - \frac{\epsilon}{2}, \quad \text{for all } t \geq 0,$$

where B_r stands for the closed ball $B_r = \{x \in H : |x| \leq r\}$. Since $S_1 : H \rightarrow H$ is compact, the set

$$K_1 = K_1(\epsilon) = S_1(B_r),$$

is compact in H and

$$\mathbb{P}(S_1 X_{t-1}^0 \in K_1) \geq \mathbb{P}(X_{t-1}^0 \in B_r) \geq 1 - \frac{\epsilon}{2},$$

for all $t \geq 1$. Finally, since (S_t) is a compact semigroup, by Lemma 6.1.4 in [4] the operator $G : L^2([0, 1]; H) \rightarrow H$ defined as

$$Gf = \int_0^1 S_{1-s}f(s) ds$$

is compact. Then, invoking the boundedness of F we find that there exists a compact set K_2 such that

$$X_{2,t} \in K_2, \quad \text{for all } t \geq 1.$$

Let

$$\tilde{K} = K_1 + K_2 + K_3 = \{x_1 + x_2 + x_3 : x_i \in K_i\}.$$

Clearly, \tilde{K} is a compact subset of H and

$$\mathbb{P}(X_t^0 \notin \tilde{K}) \leq \sum_{i=1}^3 \mathbb{P}(X_{i,t} \notin K_i) \leq \epsilon, \quad t \geq 1.$$

Therefore, the set $\{\mu_t^0 : t \geq 1\}$ is tight and an invariant measure for (1) exists by the Krylov–Bogolyubov Theorem (e.g. Theorem 16.2 of [11]).

Step 3. To show (8) we will use the following Theorem 12.1 proved in [6].

THEOREM 4.1. *If an H -valued Markov process (X_t^x) is strongly Feller and irreducible, admits an invariant measure μ , satisfies (31) and*

$$(\forall r > 0 \exists T > 0 \exists \text{compact } K \subset H) \inf_{x \in B_r} \mu_T^x(K) > 0, \quad (32)$$

then (8) holds.

Since the process (X_t^x) is strongly Feller and irreducible, it remains to show (32). Fix $r > 0$ and $T > 0$. Then there exists a compact $K \subset H$ such that $\mu_T^0(K) > \frac{1}{2}$. By the strong Feller property all the measures in the family $\{\mu_T^x : x \in H\}$ are equivalent, hence

$$\mu_T^x(K) > 0, \quad x \in H. \quad (33)$$

To prove (32) we use a nice observation of [10]: since B_r is compact in the weak topology τ_w of H , (32) will follow from (33) and the fact that the function

$$(B_r, \tau_w) \ni x \mapsto \mu_T^x(K) \in \mathbb{R} \quad \text{is continuous.} \quad (34)$$

It remains to prove (34). To this end, take a sequence $(x_n) \subset B_r$ such that $x_n \xrightarrow{w} x$. Since S_t is compact for any $t > 0$ we have

$$\lim_{n \rightarrow \infty} |S_t(x_n - x)| = 0.$$

Next, by the Lipschitz continuity of F we obtain

$$|X_t^{x_n} - X_t^x| \leq |S_t(x_n - x)| + L \int_0^t \|S_{t-s}\| \cdot |X_s^{x_n} - X_s^x| ds,$$

hence for a certain constant $L_1 > 0$ the Gronwall Lemma yields

$$|X_t^{x_n} - X_t^x| \leq |S_t(x_n - x)| + L_1 \int_0^t |S_s(x_n - x)| ds,$$

and thereby, using Dominated Convergence we obtain

$$\lim_{n \rightarrow \infty} |X_t^{x_n} - X_t^x| = 0, \quad \text{uniformly in } \omega.$$

Now, by the Dominated Convergence

$$P_t f(X_t^{x_n}) = \mathbb{E} f(X_t^{x_n}) \longrightarrow \mathbb{E} f(X_t^x) = P_t f(x) \quad (35)$$

for any $t > 0$ and any $f \in C_b(H)$. Putting in (35) $t = \frac{T}{2}$, and $f = P_t I_K \in C_b(H)$ by the strong Feller property, we find that

$$\mu_T^{x_n}(K) = P_T I_K(x_n) \longrightarrow P_T I_K(x) = \mu_T^x(K),$$

which implies (34) since (B_r, τ_w) is metrizable. ■

Acknowledgments. This work was partially supported by the ARC grant DP120101886 and by the Polish Ministry of Science and Higher Education grant “Stochastic equations in infinite dimensional spaces” N N201 419039.

References

- [1] Z. Brzeźniak, J. Zabczyk, *Regularity of Ornstein–Uhlenbeck processes driven by a Lévy white noise*, Potential Anal. 32 (2010), 153–188.
- [2] A. Chojnowska-Michalik, *On processes of Ornstein–Uhlenbeck type in Hilbert space*, Stochastics 21 (1987), 251–286.
- [3] G. Da Prato, J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Encyclopedia Math. Appl. 44, Cambridge Univ. Press, Cambridge 1992.
- [4] G. Da Prato, J. Zabczyk, *Ergodicity for Infinite-Dimensional Systems*, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge 1996.
- [5] B. Goldys, B. Maslowski, *Lower estimates of transition densities and bounds on exponential ergodicity for stochastic PDE’s*, Ann. Probab. 34 (2006), 1451–1496.
- [6] B. Goldys, B. Maslowski, *Exponential ergodicity for stochastic reaction-diffusion equations*, in: Stochastic Partial Differential Equations and Applications VII, Lect. Notes Pure Appl. Math. 245, Chapman Hall/CRC, Boca Raton 2006, 115–131.
- [7] Z. J. Jurek, J. Smalara, *On integrability with respect to infinitely divisible measures*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 29 (1981), 179–185.
- [8] V. M. Kruglov, *On infinitely divisible distributions in Hilbert space* (in Russian), Mat. Zametki 16 (1974), 585–594; English transl.: Math. Notes 16 (1974), 940–946.
- [9] B. Maslowski, *On probability distributions of solutions of semilinear stochastic evolution equations*, Stochastics Stochastics Rep. 45 (1993), 17–44.
- [10] B. Maslowski, J. Seidler, *On sequentially weakly Feller solutions to SPDE-s*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 10 (1999), 69–78.
- [11] S. Peszat, J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise*, Encyclopedia Math. Appl. 113, Cambridge Univ. Press, Cambridge 2007.
- [12] E. Priola, A. Shirikyan, L. Xu, J. Zabczyk, *Exponential ergodicity and regularity for equations with Lévy noise*, Stochastic Process. Appl. 122 (2012), 106–133.
- [13] E. Priola, L. Xu, J. Zabczyk, *Exponential mixing for some SPDEs with Lévy noise*, Stoch. Dyn. 11 (2011), 521–534.
- [14] E. Priola, J. Zabczyk, *Harmonic functions for generalized Mehler semigroups*, in: Stochastic Partial Differential Equations and Applications VII, Lect. Notes Pure Appl. Math. 245, Chapman Hall/CRC, Boca Raton 2006, 243–256.
- [15] E. Priola, J. Zabczyk, *Structural properties of semilinear SPDEs driven by cylindrical stable processes*, Probab. Theory Relat. Fields 149 (2011), 97–137.