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ERGODIC CONTROL OF LINEAR STOCHASTIC EQUATIONS IN A HILBERT SPACE WITH FRACTIONAL BROWNIAN MOTION

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Abstract. A linear-quadratic control problem with an infinite time horizon for some infinite dimensional controlled stochastic differential equations driven by a fractional Brownian motion is formulated and solved. The feedback form of the optimal control and the optimal cost are given explicitly. The optimal control is the sum of the well known linear feedback control for the associated infinite dimensional deterministic linear-quadratic control problem and a suitable prediction of the adjoint optimal system response to the future noise. Some examples of controlled stochastic partial differential equations that satisfy the problem formulation are given.

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1. Introduction. In this paper an ergodic control problem is formulated and solved that is described by a linear stochastic differential equation in a Hilbert space driven by a fractional Brownian motion (fBm) and an ergodic (long run average) quadratic cost functional.

Some infinite dimensional linear-quadratic (LQ) control problems (and the corresponding infinite dimensional Riccati equations) have been studied extensively almost simultaneously with the development of techniques and methods for solving and understanding the concept of solutions for stochastic evolution equations. Some notable early papers are [Zab], [DI], [I], [F11] and [F12] and in particular for infinite dimensional Riccati equation the papers by Lasiecka and Triggiani (cf. e.g. [LTa] and the references therein). In these stochastic models some stochastic perturbations of white Gaussian noise type (in time) were considered and this work was later extended in many directions, e.g. to linear stochastic adaptive boundary and distributed control (cf. [DGP] or [DMP1]).

Only a few results seem to be available so far for stochastic control in the case when the driving process is not white in time, especially if it is a fractional Brownian motion. In the scalar case the LQ problem has been studied in the papers by Kleptsyna, Le Breton and Viot (cf. [K1] for finite time horizon and [K2] for infinite time horizon LQ control problems for a fractional Brownian motion noise with the Hurst parameter in the interval $(\frac{1}{2}, 1)$) and in [DP] for the multidimensional case for an arbitrary square integrable noise process with continuous sample paths. The classical finite time horizon LQ problem for stochastic fractional Brownian motion (fBm) driven equations in Hilbert spaces has been treated in [DMP]. This result is applicable to distributed as well as to boundary control of parabolic stochastic controlled systems.

The present paper may be viewed as an extension of the current authors' results in [DMP] to the ergodic control problem. Under natural stabilizability and detectability conditions an optimal control in the feedback form and an expression for the optimal cost are given explicitly. Unlike in the Markov case, the optimal feedback control contains a suitable prediction of the adjoint optimal system response to the future noise. The proof is based on a careful analysis of asymptotic properties of such predictors.

The paper is divided into four sections. Sections 2 and 3 contain the formulation of the problem and the main result that are formulated and solved for a general control system. Section 4 is devoted to applications of the main result to some specific cases: a finite dimensional equation, a stochastic parabolic system and a stochastic wave equation.

It is a pleasure for the authors to dedicate this work to Jerzy Zabczyk who introduced some of them to this area and whose work has been a great inspiration for them for a long time.

2. Problem formulation. Consider the infinite dimensional controlled linear stochastic equation

$$dX(t) = (AX(t) + Bu(t))\,dt + dB_H(t) \tag{1}$$

$$X(0) = x \tag{2}$$

where $x \in V$, $X(t) \in V$, V is an infinite dimensional real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. The process $(B_H(t), t \geq 0)$ is a V-valued fractional

Brownian motion with the Hurst parameter $H \in (\frac{1}{2}, 1)$ and having the incremental covariance $\widetilde{Q}$ where $\widetilde{Q}$ is trace class ($\mathrm{Tr}(\widetilde{Q}) < \infty$) so that

$$\mathbb{E}\langle B_H(t), x\rangle \langle B_H(s), y\rangle = \frac{1}{2}\langle \widetilde{Q}x, y\rangle (t^{2H} + s^{2H} - |t-s|^{2H}). \tag{3}$$

for $x, y \in V$. The operator $A : \mathrm{Dom}(A) \to V$ with $\mathrm{Dom}(A) \subset V$ is a linear, densely defined operator on V which is the infinitesimal generator of a strongly continuous semigroup $(S(t), t \geq 0)$. Let $\overline{\mathcal{U}} = (U, \langle\cdot,\cdot\rangle_U, |\cdot|_U)$ be another Hilbert space, the state space of controls, and assume that $B \in \mathcal{L}(U, V)$. Furthermore consider the family of admissible controls, $\mathcal{U}$, defined as follows

$$\mathcal{U} = \Big\{ u : \mathbb{R}_+ \times \Omega \to U, u \text{ is progressively measurable,} \\ \mathbb{E}\int_0^T |u(t)|_U^2 \, dt < \infty \text{ for all } T > 0 \Big\}$$

where the progressive measurability is understood with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}(t), t \geq 0))$ where $(\mathcal{F}(t), t \geq 0)$ is the natural filtration induced by the fractional Brownian motion $(B_H(t), t \geq 0)$.

The solution of the equation (1) is defined as the mild solution, that is,

$$X(t) = S(t)x + \int_0^t S(t-s)Bu(t)\, dt + \int_0^t S(t-s)\, dB_H(t) \tag{4}$$

for $t \geq 0$ and it is known that with the above assumptions there is one and only one V-continuous solution to (1) (cf. [DZ1], [DMP01] and the references therein). Now the cost functional is defined for the control problem. Let J_T be given as follows

$$J_T(x, u) := \frac{1}{2}\int_0^T \big(|LX(s)|^2 + \langle Ru(s), u(s)\rangle_U\big)\, ds \tag{5}$$

where $L \in \mathcal{L}(V)$, $R \in \mathcal{L}(U)$, R is self-adjoint and invertible. The control problem is to minimize the following ergodic cost

$$\limsup_{T\to\infty} \frac{1}{T}\, \mathbb{E}J_T(x, u). \tag{6}$$

The following standard conditions are imposed.

(A1) There are $K \in \mathcal{L}(V)$, $M_K > 0$, and $\omega_K > 0$ such that

$$|e^{(A+KL)t}|_{\mathcal{L}(V)} \leq M_K e^{-\omega_K t} \tag{7}$$

for all $t > 0$ (detectability).

(A2) There are $F \in \mathcal{L}(V, U)$, $M_F > 0$, and $\omega_F > 0$ such that

$$|e^{(A+BF)t}|_{\mathcal{L}(V)} \leq M_F e^{-\omega_F t} \tag{8}$$

for all $t > 0$ (stabilizability).

3. Main result. In this section the ergodic control problem described by (1)–(6) is solved. The following result on the solution of the algebraic Riccati equation for the control problem here is used ([Za], Theorem 4.4 or [CZ], Theorem 6.2.7).

Proposition 3.1. *If* (A1)–(A2) *are satisfied then there is a unique self-adjoint nonnegative operator $P \in \mathcal{L}(V)$ satisfying the algebraic Riccati equation*

$$\langle Px, Ay\rangle + \langle Ax, Py\rangle + \langle L^* Lx, y\rangle - \langle R^{-1}B^* Px, B^* Py\rangle = 0 \tag{9}$$

for all $x, y \in \mathrm{Dom}(A)$. Moreover the strongly continuous semigroup $(\Phi(t), t \geq 0)$ generated by $A_P = A - BR^{-1}B^*P$ is exponentially stable, that is

$$|\Phi(t)|_{\mathcal{L}(V)} \leq M_P e^{-\tilde{\omega} t} \tag{10}$$

for some constants $M_P > 0$ and $\widetilde{\omega} > 0$.

Let $\Psi(t) = \Phi^*(t)$ be the adjoint semigroup of $(\Phi(t)), t \geq 0)$ that is generated by A_P^*. It is well known [DMP01] that the stochastic integral

$$\varphi_T(t) = \int_t^T \Psi(s-t) P\, dB_H(s) \tag{11}$$

for $t \in [0, T]$ is a well defined centered Gaussian process in $L^p(\Omega \times (0,T), V)$ for each $p \in [1, \infty)$. Define V_T and W as

$$V_T(t) = \mathbb{E}\big[\varphi_T(t) \,|\, \mathcal{F}(t)\big] \tag{12}$$

and

$$W(t) = \mathbb{E}\big[\varphi(t) \,|\, \mathcal{F}(t)\big] \tag{13}$$

where

$$\varphi(t) = \int_t^\infty \Psi(s-t) P\, dB_H(s). \tag{14}$$

By (10) the semigroup $(\Psi(t), t \geq 0)$ is exponentially stable so it is easy to see that $(\varphi(t), t \geq 0)$ and $(W(t), t \geq 0)$ are well-defined processes with values in $L^p(\Omega \times (0,T); V)$ for each $T > 0$ and $1 \leq p < \infty$. The following theorem is the main result of this section.

Theorem 3.2. *Let* (A1)–(A2) *be satisfied and let $u \in \mathcal{U}$ be a control satisfying*

$$\lim_{T\to\infty} \frac{1}{T} \mathbb{E}\langle PX^u(T), X^u(T)\rangle = 0 \tag{15}$$

where $(X^u(T), T \in [0, \infty))$ is the solution to (1) *with the control $u \in \mathcal{U}$. Then*

$$\limsup_{T\to\infty} \frac{1}{T} \mathbb{E} J_T(x, u) \geq J_\infty \tag{16}$$

where

$$J_\infty := \limsup_{T\to\infty} \frac{-1}{2T} \mathbb{E} \int_0^T |R^{-1/2} B^* W(s)|_U^2\, ds + \int_0^\infty \mathrm{Tr}(\widetilde{Q} P \Phi(t)) \phi_H(r)\, dr \tag{17}$$

for each $x \in V$ where $\phi_H(r) = H(2H-1)|r|^{2H-2}$, $r \in \mathbb{R}$. Moreover, the feedback control $\hat{u}(t) = -R^{-1}B^*(PX^{\hat{u}}(t) + V(t))$ is admissible, satisfies the condition (15) *and*

$$\limsup_{T\to\infty} \frac{1}{T} \mathbb{E} J_T(x, \hat{u}) = J_\infty \tag{18}$$

for each $x \in V$. Thus $\hat{u}$ is an optimal ergodic control and J_∞ is the optimal cost for the ergodic control problem (1)–(6).

Prior to the proof of this theorem, some auxiliary results are stated and verified.

Lemma 3.3. *If* (A1)–(A2) *are satisfied,* $u \in \mathcal{U}$ *and* $T > 0$ *is fixed then*

$$\begin{aligned}\mathbb{E}J_T(x,u) = \frac{1}{2}\langle Px, x\rangle + \frac{1}{2}\,\mathbb{E}\int_0^T \big(|R^{1/2}(u(s) + R^{-1}B^*PX(s) + R^{-1}B^*V_T(s))|_U^2\big)\,ds \\ - \frac{1}{2}\,\mathbb{E}\langle PX(T), X(T)\rangle - \frac{1}{2}\,\mathbb{E}\int_0^T |\,R^{-1/2}B^*V_T(s)\,|_U^2\,ds \\ + \int_0^T\int_0^s \mathrm{Tr}(\widetilde{Q}P\Phi(s-r))\phi_H(s-r)\,dr\,ds\end{aligned} \tag{19}$$

where V_T *is given by* (12).

Proof. The equality (19) is very similar to the "fundamental control equality" for the corresponding control problem on a finite time horizon that is solved in [DMP] (cf. Theorem 4.6). The proof for this lemma is lengthy and technical, but it is only a variant of the proof in [DMP]. The only difference is that instead of the solution of an algebraic Riccati equation here, in [DMP] the solution of a differential Riccati equation on a finite time interval is considered. Thus only the main ideas of the proof are sketched here. For $\lambda > 0$ sufficiently large let $R(\lambda) = \lambda(I\lambda - A)^{-1}$ and consider the Yosida approximations $x_\lambda = R(\lambda)x$, $B(\lambda) = R(\lambda)B$. Let $(\Psi_\lambda(t), t \geq 0)$ be the semigroup generated by $A^* - PB(\lambda)R^{-1}B^*(\lambda)$ and let $\varphi_{T,\lambda}$ be defined as follows

$$\varphi_{T,\lambda}(t) = \int_t^T \Psi_\lambda(s-t)P\,dB_H(t) \tag{20}$$

Furthermore let X_λ be the solution of the approximating system

$$dX_\lambda(t) = (AX_\lambda(t) + B(\lambda)u(t))\,dt + R(\lambda)\,dB_H(t) \tag{21}$$

$$X_\lambda(0) = x_\lambda. \tag{22}$$

The processes X_λ and $\varphi_{T,\lambda}$ are further approximated by smooth processes $X_{\lambda,n}$ and $\varphi_{T,\lambda,n}$ respectively, by considering polygonal approximations of the fractional Brownian motion B_H in (20) and (21). By taking the differentials of the processes $(\langle PX_{\lambda,n}(t), X_{\lambda,n}\rangle, t \geq 0)$ and $(\langle \varphi_{T,\lambda,n}, X_{\lambda,n}(t)\rangle, t \geq 0)$, it follows after letting $n \to \infty$ that

$$\begin{aligned}&J_{T,\lambda}(x_\lambda, u) - \frac{1}{2}\langle Px_\lambda, x_\lambda\rangle - \mathbb{E}\langle \varphi_{T,\lambda}(0), x_\lambda\rangle \\ &= \frac{1}{2}\,\mathbb{E}\int_0^T \big[|R^{-1/2}(Ru(s) + B^*(\lambda)PX_\lambda(s) + B^*(\lambda)\varphi_{T,\lambda}(s))|_U^2 \\ &\quad - |R^{-1/2}B^*(\lambda)\varphi_\lambda(s)|_U^2\big]\,ds \\ &\quad - \frac{1}{2}\,\mathbb{E}\langle PX_\lambda(T), X_\lambda(T)\rangle + \int_0^T\int_0^s \mathrm{Tr}(\widetilde{Q}P\Phi_\lambda(s-r)\phi_H(r-s)\,dr\,ds,\end{aligned} \tag{23}$$

where $\Phi_\lambda(t) := e^{(A - B(\lambda)R^{-1}B^*(\lambda)P)t}$ and after letting $\lambda \to \infty$ it follows that

$$\begin{aligned}J_T(x,u) = \frac{1}{2}\langle Px, x\rangle + \frac{1}{2}\,\mathbb{E}\int_0^T [|R^{1/2}(u(s) + R^{-1}B^*PX(s) + R^{-1}B^*\varphi_T(s))|_U^2 \\ - |R^{-1/2}B^*\varphi_T(s)|_U^2]\,ds - \frac{1}{2}\,\mathbb{E}\langle PX(T), X(T)\rangle \\ + \int_0^T\int_0^s \mathrm{Tr}(\widetilde{Q}P\Phi(s-r)\phi_H(r-s)\,dr\,ds,\end{aligned} \tag{24}$$

Finally, by expressing φ_T as the sum of V_T and the process orthogonal to V_T in the space that is orthogonal to the space of $(\mathcal{F}(t))$-progressively measurable processes in $L^2((0,T)\times\Omega;U)$, (19) is obtained (cf. [DMP], Theorems 4.4 and 4.6 for details). ∎

LEMMA 3.4. *If* (A1)–(A2) *are satisfied then there exist some constants* $\alpha>0$ *and* $\beta>0$ *such that*

$$\mathbb{E}|V_T(t)-W(t)|^2 \le \alpha e^{-\beta(T-t)}. \tag{25}$$

Furthermore,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \left(\mathbb{E}|V_T(t)-W(t)|^2\right)^q dt = 0 \tag{26}$$

and

$$\mathbb{E}|V_T(t)|^{2q} \le C_q \tag{27}$$

for each $q>0$, $t\in[0,T]$ *and some constants* C_q *that are independent of* $T>0$ *where* V_T *is given by* (12) *and* W *is given by* (13).

Proof. By (10) it follows that

$$|\Psi(t)|_{\mathcal{L}(V)} \le M_P e^{-\tilde{\omega}t} \tag{28}$$

for all $t>0$ and some $M_P>0$.

It follows that

$$\begin{aligned}
&\mathbb{E}|V_T(t)-W(t)|^2\\
&= \mathbb{E}\left|\mathbb{E}\Big[\int_t^T \Psi(s-t)P\,dB_H(s)\,|\,\mathcal{F}(t)\Big] - \mathbb{E}\Big[\int_t^\infty \Psi(s-t)P\,dB_H(s)\,|\,\mathcal{F}(t)\Big]\right|^2\\
&\le \mathbb{E}\left|\int_T^\infty \Psi(s-t)P\,dB_H(s)\right|^2\\
&\le \int_T^\infty\int_T^\infty |\Psi(s-t)P|_{\mathcal{L}(V)}|\Psi(r-t)P|_{\mathcal{L}(V)}\phi_H(s-r)\,ds\,dr.
\end{aligned}$$

Now by (28) there are constants $c_1>0$ and $c_2>0$ such that

$$\begin{aligned}
\mathbb{E}|V_T(t)-W(t)|^2 &\le c_1\int_T^\infty\int_T^\infty e^{-\tilde{\omega}(s-t)}e^{-\tilde{\omega}(r-t)}\phi_H(s-r)\,ds\,dr\\
&= c_1\int_{T-t}^\infty\int_{T-t}^\infty e^{-\tilde{\omega}s}e^{-\tilde{\omega}r}\phi_H(s-r)\,ds\,dr\\
&\le c_2|e^{-\tilde{\omega}\cdot}|^2_{L^2(T-t,\infty)} = c_2\,\frac{1}{2\widetilde{\omega}}\,e^{-2\tilde{\omega}(T-t)}
\end{aligned} \tag{29}$$

where the continuous embedding of the space $L^2(T-t,\infty)$ defined as

$$\Big\{f:(T-t,\infty)\to\mathbb{R}:\ \|f\|_{\mathcal{H}_\infty} = \int_{T-t}^\infty |f(r)|\,|f(s)|\phi_H(r-s)\,dr\,ds<\infty\Big\}$$

into $\mathcal{H}_\infty$ (cf. [PT]) is used (the above inequality can be proved also directly without using the embedding argument). This implies (25). Furthermore, it follows that for any $q>0$ there is a suitable constant $c>0$ such that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T(\mathbb{E}|V_T(t)-W(t)|^2)^q\,dt \le \lim_{T\to\infty}\frac{c}{T}\int_0^T \exp(-2\widetilde{\omega}q(T-t))\,dt = 0 \tag{30}$$

This result completes the proof of (26). The proof of (27) is similar to the above.

$$\begin{aligned}\mathbb{E}|V_T(t)|^2 &\le \mathbb{E}\Big|\int_t^T \Psi(s-t)P\,dB_H(s)\Big|^2 ds \\ &\le |P|^2_{\mathcal{L}(V)} \int_t^T \int_t^T |\Psi(s-t)|_{\mathcal{L}(V)}|\Psi(r-t)|_{\mathcal{L}(V)}\phi_H(r-s)\,dr\,ds \\ &\le c_1 \int_t^T \int_t^T e^{(-\tilde\omega(s-t))}e^{(-\tilde\omega(r-t))}\phi_H(r-s)\,dr\,ds \\ &\le c_1 \int_0^\infty \int_0^\infty e^{-\tilde\omega s}e^{-\tilde\omega r}\phi_H(r-s)\,dr\,ds < \infty.\end{aligned} \tag{31}$$

From the Gaussian property of V_T and the stochastic integral this inequality verifies (27) for any $q > 0$. ■

Proof of Theorem 3.2. Let $u \in \mathcal{U}$ be such that (15) is satisfied. Dividing both sides of (19) by T and letting $T \to \infty$ implies the inequality

$$\begin{aligned}\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}J(x,u) \ge \limsup_{T\to\infty}\Big(\frac{-1}{2T}\,\mathbb{E}\int_0^T |R^{-1/2}B^*V_T(s)|_U^2\,ds \\ + \frac{1}{T}\int_0^T\int_0^s \operatorname{Tr}(\widetilde{Q}P\Phi(s-r))\phi_H(s-r)\,dr\,ds\Big).\end{aligned} \tag{32}$$

From (10)

$$\big|\operatorname{Tr}(\widetilde{Q}P\Phi(s-r))\big|\phi_H(s-r) \le |P|_{\mathcal{L}(V)} \operatorname{Tr}(\widetilde{Q}) M_P e^{(-\tilde\omega(s-r))}\phi_H(s-r) \tag{33}$$

and therefore

$$\lim_{s\to\infty}\int_0^s \operatorname{Tr}(\widetilde{Q}P\Phi(s-r))\phi_H(s-r)\,dr = \int_0^\infty \operatorname{Tr}(\widetilde{Q}P\Phi(r))\phi_H(r)\,dr < \infty \tag{34}$$

which implies

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^s \operatorname{Tr}(\widetilde{Q}P\Phi(s-r))\phi_H(s-r)\,dr\,ds = \int_0^\infty \operatorname{Tr}(\widetilde{Q}P\Phi(r))\phi_H(r)\,dr. \tag{35}$$

The following inequality is satisfied.

$$\begin{aligned}&\frac{1}{T}\,\mathbb{E}\int_0^T \big(|R^{-1/2}B^*V_T(s)|_U^2 - |R^{-1/2}B^*W(s)|_U^2\big)\,ds \\ &\le \frac{1}{2T}\,\mathbb{E}\int_0^T \big(|R^{-1/2}B^*V_T(s)|_U + |R^{-1/2}B^*W(s)|_U\big) \\ &\qquad \times |R^{-1/2}B^*|_{\mathcal{L}(V,U)}|V_T(s)-W(s)|\,ds \\ &\le \frac{c}{T}\int_0^T \big(\mathbb{E}(|V_T(s)|^2 + |W(s)|^2)\big)^{1/2}(\mathbb{E}|V_T(s)-W(s)|^2)^{1/2}\,ds\end{aligned} \tag{36}$$

for some constant $c > 0$, and by Lemma 3.4 it follows that

$$\lim_{T\to\infty}\frac{1}{T}\,\mathbb{E}\int_0^T \big(|R^{-1/2}B^*V_T(s)|^2 - |R^{-1/2}B^*W(s)|^2\big)\,ds = 0 \tag{37}$$

Now the limits in (34) and (37) verify the first statement of the theorem. It remains to prove that the feedback control $\hat{u}$ is admissible and (18) is satisfied. The closed loop stochastic equation with $\hat{u}$ is

$$dX^{\hat{u}}(t) = (A - BR^{-1}B^*P)X^{\hat{u}}(t)\,dt - BR^{-1}B^*W(t)\,dt + dB_H(t) \tag{38}$$

$$X^{\hat{u}}(0) = x. \tag{39}$$

The mild solution can be written in terms of the semigroup $(\Phi(t), t \geq 0)$ corresponding to the generator A_P (cf. Proposition 3.1)

$$X^{\hat{u}}(t) = \Phi(t)x + \int_0^t \Phi(t-s)BR^{-1}B^*W(s)\,ds + Z(t) \tag{40}$$

where

$$Z(t) = \int_0^t \Phi(t-s)\,dB_H(s). \tag{41}$$

Recall that by Proposition 3.1 it follows that

$$|\Phi(t)|_{\mathcal{L}(V)} \leq M_P e^{-\tilde{\omega}t}. \tag{42}$$

Now it is verified that for each $m > 0$ there is a constant C_m such that

$$\mathbb{E}|X^{\hat{u}}(t)|^{2m} \leq C_m \tag{43}$$

for each $t > 0$. Since $(X^{\hat{u}}(t), t \geq 0)$ is a Gaussian process it is sufficient to verify the inequality (43) for $m = 1$. Initially for the process $(Z(t), t \geq 0)$ there is the following inequality

$$\begin{aligned}\mathbb{E}|Z(t)|^2 &\leq \int_0^t \int_0^t |\Phi(t-s)|_{\mathcal{L}(V)}|\Phi(t-s)|_{\mathcal{L}(V)}\phi_H(s-r)\,ds\,dr \\ &\leq M_P^2 \int_0^t \int_0^t e^{-\tilde{\omega}(t-s)}e^{-\tilde{\omega}(t-r)}\phi_H(s-r)\,ds\,dr \\ &\leq M_P^2 \int_0^\infty \int_0^\infty e^{-\tilde{\omega}s}e^{-\tilde{\omega}r}\phi_H(s-r)\,ds\,dr = M_Z < \infty.\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}|X^{\hat{u}}(t)|^2 &\leq 3M_P^2 e^{-2\tilde{\omega}t}|x|^2 + 3M_Z^2 \\ &\quad + 3\mathbb{E}\Big(\int_0^t |\Phi(t-s)_{\mathcal{L}(V)}|BR^{-1}|_{\mathcal{L}(U,V)}|B^*|_{\mathcal{L}(V,U)}|W(s)|\,ds\Big)^2 \\ &\leq c\bigg(1 + \mathbb{E}\Big(\int_0^t e^{-\tilde{\omega}(t-s)}|W(s)|\,ds\Big)^2\bigg) = c(1 + I(t)).\end{aligned} \tag{44}$$

where $c > 0$ is a constant independent of $t > 0$. In what follows $c > 0$ is a generic constant.

By the Cauchy–Schwarz inequality

$$\begin{aligned}I(t) &\leq \mathbb{E}\Big(\int_0^t e^{-\tilde{\omega}(t-s)/2}e^{-\tilde{\omega}(t-s)/2}|W(s)|\,ds\Big)^2 \\ &\leq \int_0^\infty e^{-\tilde{\omega}s}\,ds \int_0^t e^{-\tilde{\omega}(t-s)}\mathbb{E}|W(s)|^2\,ds.\end{aligned} \tag{45}$$

By the convergence $\mathbb{E}|V_T(s) - W(s)|^2 \to 0$ as $T \to \infty$ proved in (29) and by (27) it follows that

$$\mathbb{E}|W(s)|^2 \leq C_1 \tag{46}$$

where C_1 is the constant from (27) and it follows that $(I(t), t \geq 0)$ is bounded. Therefore, (44) implies that (43) is satisfied and consequently $(X^{\hat{u}}(t), t \geq 0)$ satisfies the stability condition (15) and $\widetilde{u} \in \mathcal{U}$. It remains to prove (18). Evaluate (19) with $u = \hat{u}$ and by (35) and (37) it follows that

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}|J_T(x,\hat{u}) - J_\infty| \\
&= \limsup_{T\to\infty} \frac{1}{T}\,\mathbb{E}\int_0^T |R^{1/2}(-R^{-1}B^*PX^{\hat{u}}(s) - R^{-1}B^*W(s) \\
&\qquad + R^{-1}B^*PX^{\hat{u}}(s) + R^{-1}B^*V_T(s))|_U^2\, ds \\
&\leq \limsup_{T\to\infty} \frac{1}{T}\int_0^T |R^{-1/2}B^*|^2_{\mathcal{L}(V,U)}\mathbb{E}|V_T(s) - W(s)|^2\, ds.
\end{aligned} \tag{47}$$

The right hand side is zero by Lemma 3.4 which concludes the proof of the theorem. ∎

4. Examples

EXAMPLE 4.1 (finite dimensional equation). Let $V = \mathbb{R}^m$, $U = \mathbb{R}^k$ and consider the system (1)–(2) with the ergodic cost (7), that is, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times k}$, $L \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{k\times k}$ are matrices, R is symmetric and regular, and $(B_H(t), t > 0)$ is a fractional Brownian motion in $\mathbb{R}^m$ (with no loss of generality let $\widetilde{Q} = I$). In this case the conditions verifying (A1) and (A2) are well known (see e.g. Zabczyk [Za], Theorem 2.9). The exponential stabilizability condition (A2) follows from controllability of the pair (A, B), which is equivalent to the rank condition

$$\operatorname{rank}(A\,|\,B) = m \tag{48}$$

where

$$(A\,|\,B) = (B, AB, \ldots, A^{m-1}B) \in \mathbb{R}^{m\times mk}. \tag{49}$$

Similarly, detectability of the pair (A, L) (condition (A1)) is implied by its observability that is equivalent to the rank condition

$$\operatorname{rank}(A^*\,|\,L^*) = m. \tag{50}$$

It may be concluded that if (48), (50) are satisfied (in particular if A and L are an observable pair of matrices) the assumptions (A1), (A2) of Theorem 3.2 are fulfilled.

EXAMPLE 4.2 (stochastic parabolic equation). Consider the $2m$-th order controlled stochastic parabolic equation

$$\frac{\partial y}{\partial t}(t,\xi) = (L_{2m}y)(t,\xi) + (Bu_t)(\xi) + \eta^H(t,\xi) \tag{51}$$

for $(t,\xi) \in \mathbb{R}_+ \times D$ with the initial condition

$$y(0,\xi) = x(\xi) \tag{52}$$

for $\xi \in D$ and the Dirichlet boundary conditions

$$\frac{\partial^k}{\partial \nu^k} y(t,\xi) = 0 \tag{53}$$

for $(t,\xi) \in \mathbb{R}_+ \times \partial D$, $k = 0, 1, \ldots, m-1$, where $D \subset \mathbb{R}^d$ is a bounded domain with a smooth boundary, $\frac{\partial}{\partial \nu}$ stands for conormal derivative, $x \in L^2(D)$ and L_{2m} is a $2m$-th order uniformly elliptic operator of the form

$$L_{2m} = \sum_{|\alpha| \le 2m} a_\alpha(\xi) D^\alpha$$

with $a_\alpha \in C_b^\infty(D)$. Furthermore, $u \in \mathcal{U}$ where the set $\mathcal{U}$ of admissible controls is defined in Section 2 with an arbitrary Hilbert space U, and $B \in \mathcal{L}(U, L^2(D))$. Finally, η^H formally describes a space-dependent fractional noise with the Hurst parameter $H > \frac{1}{2}$. As usual, the formal system (51)–(53) is rewritten in the form (1)–(2) where $V = L^2(D)$, $A = L_{2m}$,

$$\mathrm{Dom}(A) = \Big\{\varphi \in H^{2m}(D) : \frac{\partial^k}{\partial \nu^k}\varphi = 0 \text{ on } \partial D \text{ for } k = 0, 1, \dots, m-1\Big\} \tag{54}$$

and $(B_H(t), t > 0)$ is an $L^2(D)$-valued fractional Brownian motion with a trace class incremental covariance $\widetilde{Q}$ (formally $\eta^H(t, \cdot) = \frac{\partial}{\partial t} B_H(t)$) and $H \in (\frac{1}{2}, 1)$. It is well known that the operator A generates a strongly continuous (and analytic) semigroup $(S(t), t \ge 0)$. The cost functional takes the form (6) where $L \in \mathcal{L}(L^2(D))$ and $R \in \mathcal{L}(U)$ is self-adjoint and invertible.

In order to verify applicability of Theorem 3.2 to the above formulated control problem it is necessary to examine the detectability and the stabilizability conditions (A1) and (A2). By [Za, Theorem 3.3] (A1) and (A2) follow from exact null controllability of the pairs (A^*, L^*) and (A, B), respectively, that is, the following conditions must be satisfied

$$\mathrm{Range}(S^*(T)) \subset \mathrm{Range}\bigg(\Big(\int_0^T S^*(r) L^* L S(r)\, dr\Big)^{1/2}\bigg) \tag{55}$$

and

$$\mathrm{Range}(S(T)) \subset \mathrm{Range}\bigg(\Big(\int_0^T S(r) B B^* S^*(r)\, dr\Big)^{1/2}\bigg) \tag{56}$$

for some $T > 0$. Conditions of the type (55), (56) have been widely studied. For example, they are satisfied if $L^*L \ge \alpha I$, $BB^* \ge \alpha I$ for some $\alpha > 0$ (see e.g. [DZ1], Remark B.9). Furthermore (56) is equivalent to the strong Feller property of solutions to the auxiliary stochastic linear equation

$$dY(t) = AY(t)\, dt + B\, dW(t), \quad t > 0, \tag{57}$$

where $(W(t), t \ge 0)$ is an arbitrary cylindrical Wiener process on U (and similarly for (55)). The strong Feller property of processes defined by (57) has also been extensively studied (see e.g. [MaNe] for a general result of this type).

Example 4.3 (stochastic wave equation). General results of Section 2 may also be applied to stochastic controlled hyperbolic equations. Only a simple example following Example 6.14 in [CZ] is given. Consider the controlled stochastic wave equation

$$\frac{\partial^2 w}{\partial t^2}(t, \xi) = \frac{\partial^2 w}{\partial \xi^2}(t, \xi) + u_t(\xi) + \eta^H(t, \xi) \tag{58}$$

for $(t, \xi) \in \mathbb{R}_+ \times (0, 1)$ with the boundary condition

$$w(t, 0) = w(t, 1) = 0, \quad t > 0, \tag{59}$$

and initial condition

$$w(0,\xi) = x_1(\xi),\ \frac{\partial w}{\partial t}(0,\xi) = x_2(\xi),\ \xi \in (0,1),$$

where $u_t \in L^2(0,1)$ and η^H is a fractional noise on $L^2(0,1)$. The above system is rewritten in the abstract form (1)–(2) in the space $V = \mathrm{Dom}(A_0^{1/2}) \times L^2(0,1)$ with

$$A_0 = \frac{\partial^2}{\partial \xi^2},$$

$$\mathrm{Dom}(A_0) = \big\{h \in L^2(0,1) : h, h' \text{ absolutely continuous},\ h'' \in L^2(0,1),\ h(0) = h(1) = 0\big\}.$$

Then the operator

$$A := \begin{pmatrix} 0 & I \\ A_0 & 0 \end{pmatrix}, \quad \mathrm{Dom}(A) = \mathrm{Dom}(A_0) \times \mathrm{Dom}(A_0^{1/2})$$

generates a strongly continuous semigroup in V and we define

$$B = \begin{pmatrix} 0 \\ I \end{pmatrix} \in \mathcal{L}(L^2(0,1), V), \quad U = L^2(0,1).$$

For the noise term choose an arbitrary fractional Brownian motion $(B_H(t), t \geq 0)$ in V with $H \in (\frac{1}{2}, 1)$ and an incremental covariance of the form

$$\widetilde{Q} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{Q}_1 \end{pmatrix} \in \mathcal{L}(V)$$

where $\widetilde{Q}_1$ is a nonnegative, self-adjoint and trace class operator on $L^2(0,1)$. The cost functional takes the form (6) where

$$J_T(x,u) = \int_0^T \int_0^1 |X_2(t)|^2 + |u_t(\xi)|^2 \, d\xi \, dt,$$
$$x = (x_1, x_2), \quad X(t) = (X_1(t), X_2(t)).$$

In this case the detectability and stabilizability conditions (A1), (A2) are satisfied ([CZ], p. 310) and Theorem 3.2 is applicable.

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