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EXPLICIT FORMULAE OF DISTRIBUTIONS AND DENSITIES OF CHARACTERISTICS OF A DYNAMIC ADVERTISING AND PRICING MODEL

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Abstract. We analyze the optimal sales process of a stochastic advertising and pricing model with constant demand elasticities. We derive explicit formulae of the densities of the (optimal) sales times and (optimal) prices when a fixed finite number of units of a product are to be sold during a finite sales period or an infinite one. Furthermore, for any time t the exact distribution of the inventory, i.e. the number of unsold items, at t is determined and will be expressed in terms of elementary functions. Approximations of the densities of sales times by particular beta densities are proposed. Results related to the infinite horizon model are by-products of the finite horizon analysis.

1. Introduction. In this article we shall analyze the optimal sales process determined by the optimal advertising policy and the optimal pricing policy of a marketing model which is characterized by the property that the sales intensity has constant elasticities with respect to price and advertising effort. The model captures situations when a monopolist sells a fixed capacity of N units of a product over a finite or, if discounting is applied, even infinite time interval in a particular market environment. The objective of the seller is to choose an admissible nonnegative advertising policy $w(t,n)$, $n=1,\dots,N$, $0\le t\le T$, T finite or infinite, and an admissible positive pricing policy $p(t,n)$ such that the difference of the expected discounted revenue and the expected discounted advertising expenditures is maximized. The finite horizon problem corresponds to the important case of selling perishable products. Results related to the infinite horizon problem with discounting are by-products of our general analysis but are, of course, of independent interest. The advertising and pricing model is a generalization of the pure pricing model

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with constant elasticity which has been analyzed by McAfee & te Velde ([MV]). The generalized stochastic dynamic model includes as a special case the static and deterministic model which had been proposed by Dorfman & Steiner in the 1950s ([DS]). When unit costs are negligible and no terminal pay-off or terminal penalty term are taken into account the optimal controls of the stochastic advertising and pricing problem have been derived by Helmes and Schlosser, see [HS].

An essential feature of the model under consideration is the property that the sales intensity λ is of the form $\lambda(t,p,w) = a(t)p^{-\varepsilon}w^{\delta}$, $p > 0$, $w \geq 0$, where $\varepsilon > 1$, $0 \leq \delta < 1$, and $a(t)$ is a positive function; the special case $\delta = 0$ corresponds to the pure pricing model. If the optimal policies are applied, then this particular form of the intensity λ together with the objective function described above, and the facts that (i) no positive lower bound is imposed on the values p, (ii) no finite upper bound is imposed on the nonnegative values w, and (iii) no unit costs and no terminal profit terms or cost terms are taken into account imply that for any such finite horizon problem all N items will always be sold, almost surely. Thus, since N is a given integer the optimal sales process is a continuous time death process on a finite state space. In general, any pair of admissible controls (w_t, p_t), see Section 2 for the definition of admissibility, determines—via the intensity $\lambda(t, p(t,n), w(t,n))$—a state-dependent, time inhomogeneous Poisson like jump process with downward jumps of size one. Each such process describes the (random) evolution of the number of units left to be sold. Non-optimal sales processes are not necessarily absorbed at 0 with probability 1. The optimal sales process $(X_t)_t$ with intensity $\lambda^*(t,n) := \lambda(t, p^*(t,n), w^*(t,n))$, $n \geq 1$, and $\lambda^*(t,0) := 0$, where $w^*(t,n)$ and $p^*(t,n)$ are the optimal feedback strategies, see [HS] and formulae (5) below, is characterized by its jump/sales times. Let $\tau_0 := 0$, and let $0 < \tau_1 < \tau_2 < \ldots < \tau_N < T$ denote these random times, i.e. $\tau_k = \inf\{t \,|\, X_t = N-k\}$.

Our analysis starts with the derivation of general expressions of the joint densities of the random vector $(\tau_1, \ldots, \tau_k)$, $1 \leq k \leq N$, and explicit formulae of the 1-dimensional marginal densities of τ_i, $1 \leq i \leq N$, in terms of linear combinations (no mixtures!) of (transformed) power densities. Based on these results we shall derive formulas of the densities of the optimal prices $p_k := p^*(\tau_k, N-k+1)$. Moreover, for each t, analytical formulae of the inventory probabilities $q(t,n) := \mathbb{P}[X_t = n]$, $0 \leq n \leq N$, will be given; in [MV] and [HS] only approximations of q in terms of binomial distributions or recursive integral representations of $q(\cdot, n)$ have been derived. Such integral representations are useful for numerical computations. However, the accuracy of the computed values or of the values based on the binomial approximation do not compare favorably with the values based on analytical calculations. The formula of the joint density of $(\tau_1, \ldots, \tau_N)$ can be also used to compute the distribution function of the optimal revenue and the optimal profit. But any such computation comprises evaluating multiple integrals on the set $\{\vec{t} \in \mathbb{R}^N \,|\, 0 < t_1 < \ldots < t_N\}$. For typical applications of the model in the airline industry [MV], when N ranges between 100 and 400, it is much more efficient to employ simulation techniques to compute histograms of quantities of interest. Specific simulation procedures will be described in Section 5. Throughout, the number N will be fixed. In a future publication we shall analyze the behavior of the system should N become very large.

This paper is organized as follows. A detailed formulation of the control problem and a description of the optimal policies is given in Section 2; for references and literature reviews we refer to [GR], [HS], [HSW] and [MR]. In Section 3, formulas of the densities and distributions of the sales times and the inventory levels are derived. Furthermore, double recursions of the moments $E[\tau_i^m]$, $1 \le i \le N$, $1 \le m$, will be proved. Probabilistic properties of the optimal prices, e.g. marginal densities, properties of the finite moment sequences of the optimal sales prices, and formulas of the conditional, i.e., the inventory is not empty, optimal prices and optimal advertising efforts as functions of time can be found in Section 4. Efficient ways to simulate all aspects of the sales process, in particular, the distribution of the optimal revenue or the optimal profit, are discussed in Section 5. The final section offers approximations of the exact distributions derived in the previous sections in terms of particular beta distributions. These approximations are more accurate than the binomial distributions proposed in [HS] and [MV]. Throughout, the time-homogeneous case with finite or infinite horizon will be used to illustrate the general formulas.

2. Model description and optimal policies. We start by giving a precise formulation of the stochastic control problem which was sketched in Section 1. The model assumes that a finite number N of items shall be sold over a finite time horizon $[0, T]$ or an infinite time horizon $[0, \infty)$ and offers the opportunity to adjust the price $p > 0$ and the advertising rate $w \ge 0$ at every point of time t. To handle the cases of a finite horizon or an infinite one simultaneously, we define the time interval $[0, T\rangle$ by $[0, T\rangle = [0, T]$, if $T < \infty$, and $[0, T\rangle = [0, \infty)$, if $T = \infty$. It is assumed that sales correspond to the jumps of an inhomogeneous Poisson process whose intensity at a time $t \in [0, T\rangle$ depends on the price and advertising rate at time t in the specific way described above, and on an additional time-dependent factor $a(t)$. Let $\varepsilon > 1$ and $\delta \in (0, 1)$ be two fixed parameters, where ε denotes the price elasticity and δ denotes the advertising elasticity of customers. Let $r(t)$—the variable discount rate—be a nonnegative function, and $a(t)$ a positive one, $t \in [0, T\rangle$. Both functions are assumed to be integrable on any bounded subinterval of $\mathbb{R}^+$. For a given price p and advertising rate w the sales intensity at time t is of the form $\lambda(t, p, w) = a(t) p^{-\varepsilon} w^{\delta}$. We consider feedback controls $p : [0, T\rangle \times \{0, 1, \dots, N\} \to (0, \infty)$ and $w : [0, T\rangle \times \{0, 1, \dots, N\} \to [0, \infty)$. Admissible controls $w(t, n)$ and $p(t, n)$—where t denotes the time and n denotes the number of items left to be sold—of the finite horizon problem and admissible controls of the infinite horizon problem are such that all integrals and expectations to be introduced are well defined; see [HS] for more details. Any pair of admissible policies $u = (w, p)$ induces an intensity function $\lambda_u(t, n) := \lambda\big(t, w(t, n), p(t, n)\big)$, $n \ge 1$, $\lambda_u(t, 0) := 0$. The inventory process associated with an (initial) value N and control u is a continuous time process $(N_t^{u,N})_t$ with downside jumps of size one, the intensity λ_u and the initial state $N_0^{u,N} := N$.

We define the discount function $R(t) := \int_0^t r(s)\, ds$. For an admissible policy $u = (w, p)$ the expected discounted profit is the expected discounted revenue minus the expected discounted advertising expenditures; it is given by

$$J(u) = \mathbb{E}\Big[\int_0^T e^{-R(t)} \big(p(t, N_{t-}^{u,N}) \lambda_u(t, N_{t-}^{u,N}) - w(t, N_{t-}^{u,N})\big)\, dt \;\Big|\; N_0^{u,N} = N\Big]. \qquad (1)$$

The objective is to maximize $J(u)$ over all admissible controls. Consider the parameter $\gamma := \frac{\varepsilon-\delta}{1-\delta} > 1$, and the functions

$$A_0(t) := \int_t^T e^{-\gamma R(s)} a(s)^{1/(1-\delta)}\, ds \quad \text{and} \quad A(t) := e^{\gamma R(t)} A_0(t).$$

We call γ the *leveraged price elasticity of demand*, and $A_0(t)$, $A(t)$ resp., the *sales potential*, the *time adjusted sales potential*, resp. It has been shown in [HS] that the optimal controls $w^*(t,n)$ and $p^*(t,n)$ are separable functions of t and n, and can be expressed in terms of $A(t)$ and a particular sequence $(\beta_n)_n$. For a pair $(\varepsilon,\delta) \in (1,\infty)\times[0,1]$ consider the uniquely defined monotone increasing sequence $(\beta_n)_{n\geq 0}$ such that $\beta_0 = 0$ and

$$\beta_n(\beta_n - \beta_{n-1})^{\gamma-1} = \Big(\frac{\gamma-1}{\gamma}\Big)^{\gamma-1}, \quad n \geq 1. \tag{2}$$

For later purposes we introduce the shorthand writing

$$\theta_k := \beta_{N-k+1}^{\gamma/(\gamma-1)}, \quad k = 1,\dots,N. \tag{3}$$

The sequence $(\beta_n)_n$ enjoys several interesting properties, see [HS] and [MV] for details and proofs. The following two properties will be used later on: For $n \geq 2$,

$$\text{(i)} \quad \Big(\frac{\beta_n}{\beta_{n-1}}\Big)^{1/(\gamma-1)} < \big(1+\beta_{n-1}^{-\gamma/(\gamma-1)}\big)^{1/\gamma} \quad \text{and} \quad \text{(ii)} \quad \lim_{n\to\infty}\Big(\frac{\beta_n^{\gamma/(\gamma-1)}}{n}\Big) = 1. \tag{4}$$

The optimal policies derived in [HS] are given by

$$\begin{aligned} w^*(t,n) &= \Big(\frac{\delta}{\varepsilon}\Big)^{\varepsilon/(\varepsilon-\delta)} a(t)^{1/(1-\delta)} A(t)^{-(\gamma-1)/\gamma}\beta_n; \\ p^*(t,n) &= \Big(\frac{\delta}{\varepsilon}\Big)^{\delta/(\varepsilon-\delta)} \Big(\frac{A(t)}{\beta_n^{\gamma/(\gamma-1)}}\Big)^{1/\gamma}. \end{aligned} \tag{5}$$

The corresponding optimal sales intensity λ^* equals

$$\lambda^*(t,n) = \frac{a(t)^{1/(1-\delta)}}{A(t)}\,\beta_n^{\gamma/(\gamma-1)} = -\frac{\dot{A}_0(t)}{A_0(t)}\,\beta_n^{\gamma/(\gamma-1)} = \Big(-\frac{\dot{A}(t)}{A(t)} + \gamma r(t)\Big)\beta_n^{\gamma/(\gamma-1)}. \tag{6}$$

Throughout the paper we will use the following notation:

$$\alpha(t) := a(t)^{1/(1-\delta)}/A(t) = -\dot{A}_0(t)/A_0(t) = -\frac{\partial}{\partial t}\log(A_0(t)). \tag{7}$$

With use of α, the optimal arrival rate is of the separable form

$$\lambda^*(t, N-k+1) = \alpha(t)\cdot\theta_k. \tag{8}$$

Note that the time-homogeneous case, i.e., $r(t) \equiv r \geq 0$ and $a(t) \equiv a > 0$, is special since the time factor of λ^* simplifies and equals

$$\alpha(t) = \frac{a^{1/(1-\delta)}}{e^{r\gamma(T-t)}\int_t^T e^{-r\gamma s} a^{1/(1-\delta)}\, ds} = \begin{cases} \dfrac{1}{T-t} & \text{if } r=0,\ T<\infty \\ \dfrac{r\gamma}{1-e^{-r\gamma(T-t)}} & \text{if } r>0,\ T<\infty \\ r\gamma & \text{if } r>0,\ T=\infty. \end{cases} \tag{9}$$

In [HS], the authors concentrate on solving the stochastic dynamic optimization problem and present analytical expressions of the optimal policies, see above. In this paper we will investigate the optimal inventory process $(X_t)_{t\in[0,T\rangle}$—that is, the inventory process $X_t := N_t^{u^*,N}$ when the optimal policies $u^* = (w^*, p^*)$ stated in (5) are used—in detail. Recall that the optimal inventory process has the intensity (6) and its jump times will be denoted by $(\tau_k)_{1\le k\le N}$, $\tau_0 := 0$. The starting point of our analysis of the optimal inventory process $(X_t)_t$ is the fact that the conditional probabilities of τ_k given τ_{k-1} can be expressed in terms of the function $A_0(t)$.

PROPOSITION 2.1. *For any k, $1 \le k \le N$, and for any $s, t \in [0, T\rangle$ such that $s < t$,*

$$\mathbb{P}[\tau_k \le t \,|\, \tau_{k-1} = s] = 1 - \Big(\frac{A_0(t)}{A_0(s)}\Big)^{\theta_k} = 1 - \exp\Big(-\theta_k \int_s^t \alpha(r)\, dr\Big). \tag{10}$$

Proof. By construction and by (6),

$$\begin{aligned}\mathbb{P}[\tau_k \le t \,|\, \tau_{k-1} = s] &= \int_s^t \exp\Big(-\int_s^\xi \lambda^*(r, N-k+1)\, dr\Big)\lambda^*(\xi, N-k+1)\, d\xi \\ &= 1 - \exp\Big(-\int_s^t \lambda^*(r, N-k+1)\, dr\Big) \\ &= 1 - \exp\Big(\theta_k \int_s^t \frac{\dot{A}_0(r)}{A_0(r)}\, dr\Big)\end{aligned}$$

from which (10) follows. ■

COROLLARY 2.1. *Let $a(t) \equiv a > 0$ and $r(t) \equiv r \ge 0$. Let $k = 1, \dots, N$ and $0 \le s < t < T$. Then*

$$\mathbb{P}[\tau_k \le t \,|\, \tau_{k-1} = s] = \begin{cases} 1 - \Big(\dfrac{1 - e^{-r\gamma(T-t)}}{1 - e^{-r\gamma(T-s)}}\Big)^{\theta_k} & \text{if } r > 0,\ T < \infty \\ 1 - \Big(\dfrac{1 - t/T}{1 - s/T}\Big)^{\theta_k} & \text{if } r = 0,\ T < \infty \\ 1 - e^{-r\gamma\theta_k(t-s)} & \text{if } r > 0,\ T = \infty. \end{cases}$$

3. Sales times and inventory distribution. We continue analyzing the distributions of the optimal sales times τ_k, $1 \le k \le N$. Proposition 2.1 and the form of $A_0(t)$ suggest to take a look at the moment generating functions $E[A_0(\tau_k)^z]$, $z \in \mathbb{R}$. In Section 3.1 we shall show that each such function equals the Laplace transform of a sum of independent exponentially distributed random variables. This result is one of two key observations that underlie the analytical formulas of the densities of τ_k, $1 \le k \le N$, and of the probabilities $q(t, n) := \mathbb{P}[X_t = n]$, $1 \le n \le N$; the latter quantities will be analyzed in Section 3.3. The second key observation is explained below, see Remark 3.1.

3.1. Densities of sales times. Using Proposition 2.1 we easily derive a general expression of the joint density of each random vector $(\tau_1, \dots, \tau_k)$, $1 \le k \le N$. Let $\rho_k(t \,|\, s)$, $0 \le s < t \le T$, denote the conditional density of τ_k given τ_{k-1}. It follows from Proposition 2.1 and (6) that

$$\rho_k(t \,|\, s) = \theta_k \frac{a(t)^{1/(1-\delta)}}{A(t)} \Big(\frac{A_0(t)}{A_0(s)}\Big)^{\theta_k} = \theta_k\Big(-\frac{\dot{A}_0(t)}{A_0(t)}\Big)\Big(\frac{A_0(t)}{A_0(s)}\Big)^{\theta_k}. \tag{11}$$

Next, we concentrate on the individual sales times τ_k and their densities. Obviously, the density of τ_1 equals $\rho_1(\cdot \,|\, 0)$, see (11), and the density f_k of τ_k exists. With use of

Proposition 2.1, the density can be recursively defined by

$$f_k(t) = \int_0^t f_{k-1}(s)\rho_k(t\,|\,s)\,ds, \quad k \geq 2.$$

An alternative to this recursion, which yields analytic formulas, will be given next.

THEOREM 3.1. *For any real number $z \notin \{-\theta_1, \dots, -\theta_k\}$, $1 \leq k \leq N$,*

$$\mathbb{E}\Big[\Big(\frac{A_0(\tau_k)}{A_0(0)}\Big)^z\Big] = \prod_{j=1}^k \frac{\theta_j}{\theta_j + z}\,, \tag{12}$$

and thus, $\big(-\log\big(A_0(\tau_k)/A_0(0)\big)\big)$ is distributed like the sum of k independent random variables, each being exponentially distributed with (intensity) parameter θ_j, $1 \leq j \leq k$. Let $\pi_{1,1} = L_{1,1} := 1$ and, for $1 \leq i \leq k$,

$$\pi_{i,k} := \prod_{\substack{j=1\\ j\neq i}}^k (\theta_j - \theta_i), \qquad L_{i,k} := \prod_{\substack{j=1\\ j\neq i}}^k \frac{\theta_j}{\theta_j - \theta_i} = \pi_{i,k}^{-1} \prod_{\substack{j=1\\ j\neq i}}^k \theta_j \quad \text{and} \quad l_{i,k} := L_{i,k}\cdot\theta_i. \tag{13}$$

The density f_k is given for $0 \leq t < T$ by

$$f_k(t) = \Big(\prod_{i=1}^k \theta_i\Big)\alpha(t) \sum_{i=1}^k \pi_{i,k}^{-1}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i} = \alpha(t)\sum_{i=1}^k l_{i,k}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i} \tag{14}$$

$$= \sum_{i=1}^k L_{i,k}\cdot(\theta_i\cdot\alpha(t))\exp\Big(-\int_0^t \theta_i\cdot\alpha(s)\,ds\Big). \tag{15}$$

Proof. For $z \notin \{-\theta_1, \dots, -\theta_N\}$ we have

$$\begin{aligned}
\mathbb{E}[A_0(\tau_k)^z] &= \mathbb{E}\big[\mathbb{E}[A_0(\tau_k)^z\,|\,\tau_{k-1}]\big] = \int_0^T f_{k-1}(s)\mathbb{E}[A_0(\tau_k)^z\,|\,\tau_{k-1} = s]\,ds \\
&= \int_0^T f_{k-1}(s)\int_s^T \rho_k(t\,|\,s)A_0(t)^z\,dt\,ds \\
&\overset{(11)}{=} \theta_k \int_0^T \frac{f_{k-1}(s)}{A_0(s)^{\theta_k}}\int_s^T -\dot{A}_0(t)A_0(t)^{\theta_k-1+z}\,dt\,ds \qquad (16)\\
&= \theta_k \int_0^T \frac{f_{k-1}(s)}{A_0(s)^{\theta_k}}\Big[-\frac{1}{\theta_k+z}\,A_0(t)^{\theta_k+z}\Big]_s^T\,ds \\
&= \theta_k \int_0^T f_{k-1}(s)\,\frac{1}{\theta_k+z}\,A_0(s)^z\,ds = \frac{\theta_k}{\theta_k+z}\,\mathbb{E}[A_0(\tau_{k-1})^z].
\end{aligned}$$

Repeating the same kind of calculation $(k-1)$ times and taking the identity $\mathbb{E}[A_0(\tau_0)^z] = A_0(0)^z$ into account, we obtain equation (12). Let $Z_1, \dots, Z_N$ be independent random variables with $Z_k \sim Exp(\theta_k)$. We define

$$Y_k := -\log\Big(\frac{A_0(\tau_k)}{A_0(0)}\Big), \quad \text{and} \quad H_k := \sum_{j=1}^k Z_j.$$

Then

$$\mathbb{E}[e^{-zY_k}] = \mathbb{E}[e^{-zH_k}] = \prod_{i=1}^k \frac{\theta_i}{\theta_i+z}\,.$$

Thus, the density of Y_k is given by

$$f_{Y_k}(y) = \Big(\prod_{i=1}^{k} \theta_i\Big) \sum_{i=1}^{k} \Big(\prod_{\substack{j=1\\ j\neq i}}^{k} (\theta_j - \theta_i)\Big)^{-1} \exp(-\theta_i y), \quad y \geq 0,$$

see [JK, Vol. 1, p. 552] (up to a parameter transformation), [MG] or [A]. Since $\dot{A}_0(t) = -e^{-\gamma R(t)} a(t)^{1/(1-\delta)}$, using the chain rule we obtain

$$\begin{aligned} f_k(t) &= e^{-\gamma R(t)} \frac{a(t)^{1/(1-\delta)}}{A_0(t)} f_{Y_k}\Big(-\log\Big(\frac{A_0(t)}{A_0(0)}\Big)\Big) \\ &= \Big(\prod_{i=1}^{k} \theta_i\Big) \alpha(t) \Big[\sum_{i=1}^{k} \pi_{i,k}^{-1} \Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i}\Big] \mathbb{1}_{[0,T)}(t). \ \blacksquare \end{aligned}$$

COROLLARY 3.1.

(i) *If $T < \infty$, $a(t) \equiv a > 0$ and $r(t) \equiv 0$, then, for $0 \leq t < T$,*

$$f_k(t) = \sum_{i=1}^{k} l_{i,k} \frac{(T-t)^{\theta_i - 1}}{T^{\theta_i}}. \tag{17}$$

(ii) *If $T < \infty$, $a(t) \equiv a > 0$ and $r(t) \equiv r > 0$, then, for $0 \leq t < T$,*

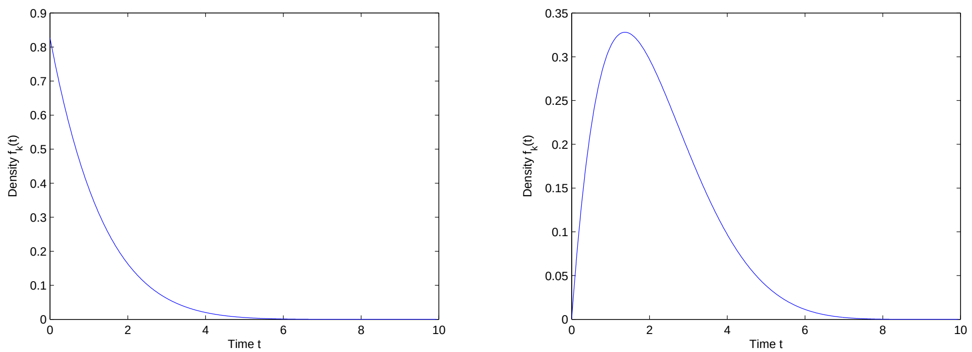
$$f_k(t) = \frac{\gamma r}{1 - e^{-\gamma r(T-t)}} \sum_{i=1}^{k} l_{i,k} \Big(\frac{1 - e^{-\gamma r(T-t)}}{1 - e^{-\gamma r T}}\Big)^{\theta_i}. \tag{18}$$

(iii) *If $T = \infty$, $a(t) \equiv a > 0$ and $r(t) \equiv r > 0$, then τ_k is the sum of k independent random variables Δ_i, $1 \leq i \leq k$, such that $\Delta_i \sim \tau_i - \tau_{i-1} \sim Exp(\gamma r \theta_i)$, and*

$$f_k(t) = \sum_{i=1}^{k} L_{i,k} \cdot (\gamma r \theta_i) e^{-(\gamma r \theta_i) t}, \quad 0 \leq t. \tag{19}$$

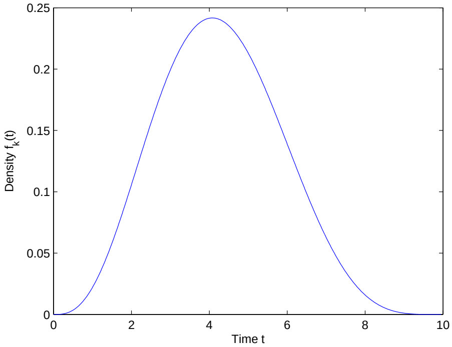
Moreover,

$$\mathbb{E}[\tau_k] = \sum_{i=1}^{k} \mathbb{E}[\Delta_i] = \frac{1}{\gamma r} \sum_{i=1}^{k} \frac{1}{\theta_i}. \tag{20}$$

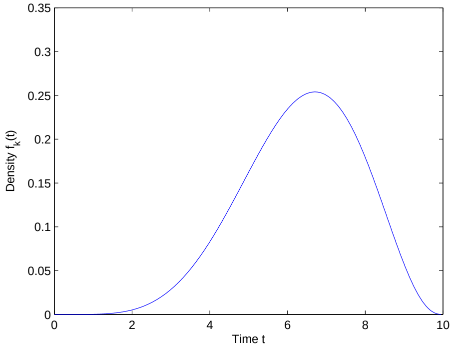


(a) $k = 1$ (b) $k = 2$

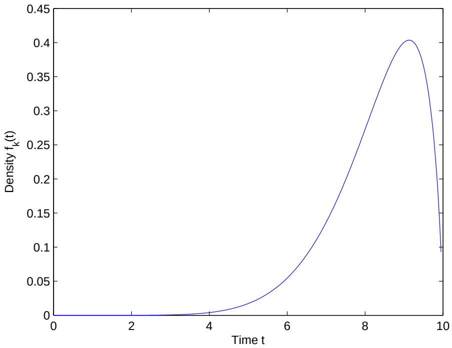
Fig. 1. Plots of density functions f_k on $[0, T)$, $T = 10$; $N = 10$, $\varepsilon = 1.2$, $\delta = 0.5$, $r(t) \equiv 0$; $a(t) \equiv a$; the densities do not depend on the value of the positive constant a.



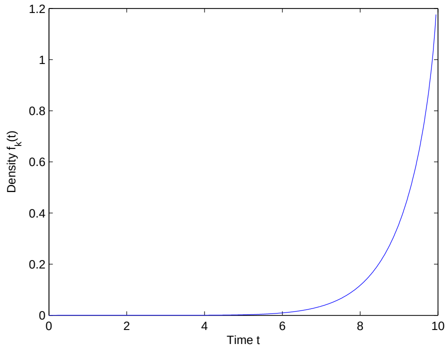
(c) $k = 4$



(d) $k = 6$



(e) $k = 8$



(f) $k = 9$

Fig. 1 (continuation)

Note that for *time homogeneous* models the densities of all sales times are independent of the intensity factor a. Fig. 1 displays density functions $f_k(t)$, $k = 1, 2, 4, 6, 8, 9$, on the time interval $[0, 10]$; notice the different scales on the various y-axes. The probability of when the k-th sale takes place is shifting from being concentrated around zero, the beginning of the sales period, to its end, i.e., $T = 10$, with increasing k. However, while $f_1(0)$ is finite, $\lim_{t\to T} f_{10}(t) = \infty$. Note, the series of graphs shown in Fig. 1 look very much like a gallery of graphs of beta densities, e.g. [JK], p. 320. The approximation of the densities f_k by beta densities will be discussed in detail in Section 6.

Remark 3.1. The coefficients $\pi_{i,k}$, being defined as

$$\pi_{i,k} := \prod_{\substack{j=1\\ j\neq i}}^{k} (\theta_j - \theta_i) = (-1)^{k-i} \Big(\prod_{j=1}^{i-1} (\theta_j - \theta_i) \Big) \Big(\prod_{j=i+1}^{k} (\theta_i - \theta_j) \Big), \tag{21}$$

are the coefficients of the partial fraction decomposition of $1/\big((\theta_1 - z)(\theta_2 - z) \cdot \ldots \cdot (\theta_k - z)\big)$, i.e., for all $z \in \mathbb{R} \setminus \{\theta_1, \ldots, \theta_k\}$,

$$\frac{1}{(\theta_1 - z)(\theta_2 - z) \cdot \ldots \cdot (\theta_k - z)} = \sum_{i=1}^{k} \pi_{i,k}^{-1} (\theta_i - z)^{-1}. \tag{22}$$

These coefficients are alternating in sign, $1 \le i \le k$, i.e. $\operatorname{sign}(\pi_{i,k}) = -\operatorname{sign}(\pi_{i+1,k})$. It follows from (21) that

(i) for $i < k$: $(\theta_i - \theta_k)\pi_{i,k-1} = -\pi_{i,k}$,

(ii) $\sum_{i=1}^{k-1} \pi_{i,k}^{-1} = -\sum_{i=1}^{k-1} \pi_{i,k-1}^{-1}(\theta_i - \theta_k)^{-1} \overset{(22)}{=} -\pi_{k,k}^{-1} \iff \sum_{i=1}^{k} \pi_{i,k}^{-1} = 0,$

(iii) $\sum_{i=1}^{k} l_{i,k} = \Big(\prod_{j=1}^{k} \theta_j\Big) \sum_{i=1}^{k} \pi_{i,k}^{-1} = 0$ and

(iv) $\sum_{i=1}^{k} L_{i,k} = \Big(\prod_{i=1}^{k} \theta_i\Big) \sum_{i=1}^{k} \pi_{i,k}^{-1} \frac{1}{\theta_i - 0} \overset{(22)}{=} 1.$

Note that the coefficients $\pi_{i,k}, L_{i,k}$ and $l_{i,k}$ are not only alternating in sign but also have large absolute values. For example, if $N = 20$, $k = 10$, $\gamma = 2$ and $i = 1, \dots, k$, then $|\pi_{i,k}| \cdot 10^{-5}$, $|L_{i,k}| \cdot 10^{-7}$, $|l_{i,k}| \cdot 10^{-8}$ all range within the interval $[0.008, 4]$. Hence, for large values of N numerical evaluations of formulas like (14)–(15) are problematic. This issue will be addressed next.

3.2. Recursions of moments: The time homogeneous case without discounting. Although we have explicit formulas of the individual sales densities f_k and of the moments too, see below, recursive formulas of the moments $E[\tau_k^m]$, $1 \le m$, are useful to enhance numerical stability. According to Theorem 3.1 the computation of moments requires calculating integrals of the form

$$\int_0^T t^m A_0(t)^{\theta_i - 1}\, dt. \tag{23}$$

Due to Corollary 3.1 such integrals are easy to compute if $r \equiv 0$ and a is a constant.

THEOREM 3.2. *Let $T < \infty$, $a(t) \equiv a > 0$ and let $r(t)$ be identically zero. Then the m-th moment of the k-th selling time τ_k, $1 \le k \le N$, is given by*

$$\mathbb{E}[\tau_k^m] = m!\, T^m \sum_{i=1}^{k} l_{i,k} \Big(\prod_{j=0}^{m} \frac{1}{\theta_i + j}\Big). \tag{24}$$

The array $(E[\tau_k^m])_{1 \le k \le N, 1 \le m}$ satisfies the recursion

$$\mathbb{E}[\tau_k^m] = \frac{1}{\theta_k + m}\big(\theta_k \cdot \mathbb{E}[\tau_{k-1}^m] + mT \cdot \mathbb{E}[\tau_k^{m-1}]\big); \tag{25}$$

if $m = 1$, then

$$\mathbb{E}[\tau_k] = T\Big(1 - \prod_{i=1}^{k} \frac{\theta_i}{\theta_i + 1}\Big). \tag{26}$$

Proof. Since $a(t)$ is constant, $A_0(t) = A(t) = (T-t)a^{1/(1-\delta)}$. To prove (24) it suffices, cf. (17), to evaluate $\int_0^T t^m (T-t)^{\theta_i - 1}\,dt$. Partial integration yields, for $m \geq 1$,

$$\begin{aligned} I_{i,m} &:= \int_0^T t^m (T-t)^{\theta_i-1}\,dt = \frac{m}{\theta_i}\int_0^T t^{m-1}(T-t)^{\theta_i}\,dt \\ &= \frac{m}{\theta_i}\Big(T\int_0^T t^{m-1}(T-t)^{\theta_i-1}\,dt - \int_0^T t^m (T-t)^{\theta_i-1}\,dt\Big) \\ &= \frac{m}{\theta_i}\big(TI_{i,m-1} - I_{i,m}\big). \end{aligned} \tag{27}$$

Since $I_{i,0} = \frac{T^{\theta_i}}{\theta_i}$, equation (27) implies

$$I_{i,1} = \frac{T^{1+\theta_i}}{(1+\theta_i)\theta_i}\,; \tag{28}$$

by induction, we get

$$I_{i,m} = m!\Big(\prod_{j=0}^m \frac{1}{j+\theta_i}\Big)T^{m+\theta_i}.$$

Hence, by (17), we obtain (24):

$$\mathbb{E}[\tau_k^m] = \int_0^T t^m f_k(t)\,dt = \sum_{i=1}^k l_{i,k}\cdot T^{-\theta_i} I_{i,m} = m!\,T^m \sum_{i=1}^k l_{i,k}\Big(\prod_{j=0}^m \frac{1}{j+\theta_i}\Big).$$

To prove (25) we use the properties of $\pi_{i,k}$ referred to in Remark 3.1. By employing formula (24) with m as well as $(m-1)$ the right hand side of (25) can be written as

$$\frac{1}{m+\theta_k}\big(\theta_k\mathbb{E}[\tau_{k-1}^m] + mT\mathbb{E}[\tau_k^{m-1}]\big) = \frac{m!\,T^m}{m+\theta_k}\Big(\prod_{i=1}^k \theta_i\Big)\cdot(*), \tag{29}$$

where

$$(*) = \Big(\sum_{i=1}^{k-1}\pi_{i,k-1}^{-1}\prod_{j=0}^m \frac{1}{j+\theta_i} + \sum_{i=1}^k \pi_{i,k}^{-1}\prod_{j=0}^{m-1}\frac{1}{j+\theta_i}\Big);$$

recall, see Theorem 3.1, that $l_{i,k} = \pi_{i,k}^{-1}\cdot\big(\prod_{j=1}^k \theta_j\big)$. Since $\pi_{i,k-1}^{-1} = -(\theta_i - \theta_k)\pi_{i,k}^{-1}$, see Remark 3.1 (i), we have

$$\sum_{i=1}^{k-1}\pi_{i,k-1}^{-1}\prod_{j=0}^m\frac{1}{j+\theta_i} = -\sum_{i=1}^{k-1}\pi_{i,k}^{-1}(\theta_i-\theta_k)\prod_{j=0}^m\frac{1}{j+\theta_i}\,.$$

Using this equality we express $(*)$ as follows:

$$\begin{aligned} (*) &= \Big(\sum_{i=1}^{k-1}\pi_{i,k}^{-1}\Big(\prod_{j=0}^{m-1}\frac{1}{j+\theta_i}\Big)\Big(1-\frac{\theta_i-\theta_k}{m+\theta_i}\Big) + \pi_{k,k}^{-1}\prod_{j=0}^{m-1}\frac{1}{j+\theta_k}\Big) \\ &= (m+\theta_k)\sum_{i=1}^k \pi_{i,k}^{-1}\prod_{j=0}^m\frac{1}{j+\theta_i}\,. \end{aligned}$$

Together with (24) and (29) this last equation verifies (25). To see (26) we take formula (24) and set $m = 1$. Thus,

$$\mathbb{E}[\tau_k] = T\Big(\prod_{i=1}^{k}\theta_i\Big)\sum_{i=1}^{k}\pi_{i,k}^{-1}\,\frac{1}{\theta_i(1+\theta_i)}\,.$$

Next, define $\tilde{\theta}_i := \theta_i$, $1 \le i \le k$, $\tilde{\theta}_{k+1} := 0$, and $\tilde{\pi}_{i,k}$ accordingly, see (21). Hence, using Remark 3.1 (i), the definition of $\tilde{\theta}_i$, $1 \le i \le k+1$, as well as $\tilde{\pi}_{i,k+1}$ and the facts that $\tilde{\theta}_{k+1} = 0$, $\tilde{\pi}_{i,k} = \pi_{i,k}$, $1 \le i \le k$, we obtain

$$\sum_{i=1}^{k}\pi_{i,k}^{-1}\,\frac{1}{\theta_i(\theta_i+1)} = -\sum_{i=1}^{k}\tilde{\pi}_{i,k+1}^{-1}\,\frac{1}{\theta_i+1} = \frac{\tilde{\pi}_{k+1,k+1}^{-1}}{1+\tilde{\theta}_{k+1}} - \sum_{i=1}^{k+1}\frac{\tilde{\pi}_{i,k+1}^{-1}}{\tilde{\theta}_i-(-1)}$$
$$\overset{(22)}{=} \prod_{i=1}^{k}\frac{1}{\theta_i} - \prod_{i=1}^{k}\frac{1}{1+\theta_i}\,. \tag{30}$$

Multiplying (30) by $T\big(\prod_{i=1}^{k}\theta_i\big)$ yields (26). ∎

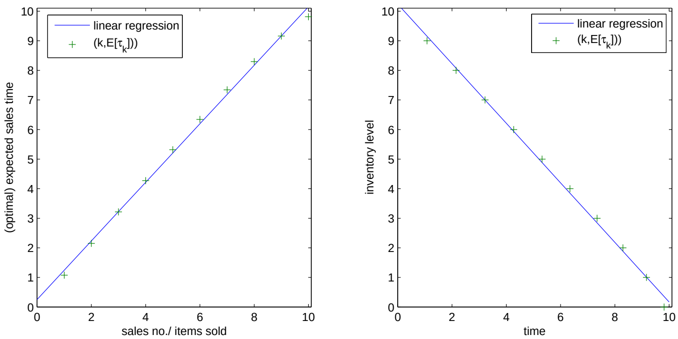


Fig. 2. The sequence of (optimal) expected sales times $(k, E[\tau_k])$, see the panel on the left, and the pairs $(E[\tau_k], N-k)$ on the right, $1 \le k \le N$, together with a joint plot of the linear regression function; $N = 10$, $\varepsilon = 1.2$, $\delta = 0.5$, $r(t) \equiv 0$ and $a(t) \equiv 2$.

Expression (26) looks innocent and not very interesting. However, (26) reveals a remarkable property of the optimal sales process of a time homogeneous model with constant elasticities of demand and no discounting, see Fig. 2. The (optimal) expected sales times evolve (approximately) like an arithmetic sequence! This property is closely related to the question of socially efficient pricing and the issue whether or not a monopolist's pricing policy could coincide with competitive pricing, see [HS] and [MV] for more details. For large values of the initial inventory N and small values of k, $1 \le k \le N$, the (approximate) linear growth of $(E[\tau_k])_k$ is a consequence of the asymptotic property of $(\beta_n)_{n\ge1}$, see (4). For instance,

$$\mathbb{E}[\tau_2] - \mathbb{E}[\tau_1] = \frac{T\cdot\theta_1}{1+\theta_1}\Big(1 - \frac{\theta_2}{1+\theta_2}\Big) \approx \frac{T\cdot N}{N+1}\Big(1 - \frac{N-1}{N}\Big) = \frac{T}{N+1}\,;$$

inductively, we obtain $\mathbb{E}[\tau_k] \approx k\,\frac{T}{N+1}$. A refined approximation will be discussed in Section 6.

Both panels of Fig. 2 display the same information, but in a different format. The panel on the left shows the pairs $(k, \mathbb{E}[\tau_k])$, and the linear regression line reveals the almost arithmetic nature of the sequence; on the right, the graph illustrates how the inventory decreases with time, more precisely, with the expected sales times.

3.3. Exact inventory distribution. To fully describe the behavior of the optimally controlled inventory process we are going to characterize the probability that at a given time $t \in [0, T]$ there are n items left in stock. We denote this probability $\mathbb{P}[X_t = n]$ by $q(t, n)$. It follows from the general theory of continuous time Markov chains that the functions $q(\cdot, n)$ satisfy the Kolmogorov forward equations, $q(t, N+1) := 0$, $0 \le t \le T$, $n = 0, \dots, N$,

$$\dot{q}(t,n) = \lambda^*(t, n+1)q(t, n+1) - \lambda^*(t,n)q(t,n), \tag{31}$$

with initial conditions $q(0, N) = 1$ and $q(0, n) = 0$, $1 \le n < N$. Using the explicit expression of $\lambda^*(t,n)$, see (6), it has been proposed in [HS], see [MV] for the pure pricing model and for the original idea, to replace $\lambda^*(t,n)$ by $\tilde{\lambda}(t,n) := -n(\dot{A}_0(t)/A_0(t))$, etc. and thus, to approximate $q(t,n)$ by binomial probabilities, i.e. $q(t,n) \approx \binom{N}{n}\big(\frac{A(t)}{A(0)}\big)^n \times \big(1 - \frac{A(t)}{A(0)}\big)^{N-n}$. For the time homogeneous case the recursion

$$q(t,n) = \Big(\frac{1 - e^{-\gamma r (T-t)}}{\gamma r}\Big)^{\beta_n^{\gamma/(\gamma-1)}} \int_0^t \beta_{n+1}^{\gamma/(\gamma-1)} \Big(\frac{1 - e^{-\gamma r(T-s)}}{\gamma r}\Big)^{\beta_n^{\gamma/(\gamma-1)}} \cdot q(s, n+1)\, ds,$$

where

$$q(t,N) = \exp\big(-\gamma r \beta_N^{\gamma/(1-\gamma)}\big)\Big(\frac{1 - e^{-\gamma r(T-t)}}{1 - e^{\gamma r T}}\Big)^{\beta_N{}^{\gamma/(1-\gamma)}},$$

can be used to numerically compute the probabilities q, see [HS].

Since $q(t,n) = \mathbb{P}[X_t = n] = \mathbb{P}[\tau_{N-n+1} > t, \tau_{N-n} \le t]$ we can express the inventory probabilities in terms of the density function $f_{N-n+1}(t)$. This representations yields analytic formulas of the probabilities $q(t,n)$.

THEOREM 3.3. *Let $N \ge 1$. For each $n = 1, \dots, N$, let $k_n := N - n + 1 \iff n = N - k_n + 1$, and let f_k denote the density of k-th sales time. For each $t \in [0, T)$,*

$$q(t,n) = \frac{1}{\theta_{k_n}} \sum_{i=1}^{k_n} l_{i,k_N} \cdot \Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i} = \frac{f_{k_n}(t)}{\theta_{k_n} \cdot \alpha(t)} = \frac{f_{N-n+1}(t)}{\lambda^*(t,n)}\,; \tag{32}$$

note that if $n = 0$ then

$$q(t,0) = \Big(\prod_{i=1}^N \theta_i\Big) \sum_{i=1}^{N+1} \pi_{i,N+1}^{-1} \Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i} = 1 - \Big(\prod_{i=1}^N \theta_i\Big) \sum_{i=1}^N \pi_{i,N}^{-1}\theta_i^{-1}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i}.$$

Proof. Set $\tau_{N+1} := T$ and $\theta_{N+1} = 0$. If $0 \le n < N$ and $k = N - n + 1$, then it follows from Proposition 2.1 and Remark 3.1 (ii) that

$$\begin{aligned} q(t,n) &= \mathbb{P}[\tau_{N-n+1} > t,\, \tau_{N-n} \le t] \\ &= \int_0^t \mathbb{P}[\tau_k > t \,|\, \tau_{k-1} = s] f_{k-1}(s)\, ds \\ &= \int_0^t \frac{A_0(t)^{\theta_k}}{A_0(s)^{\theta_k}} \Big(\prod_{i=1}^{k-1} \theta_i\Big) \alpha(s) \Big(\sum_{i=1}^{k-1} \pi_{i,k-1}^{-1} \Big(\frac{A_0(s)}{A_0(0)}\Big)^{\theta_i}\Big)\, ds \end{aligned}$$

$$= A_0(t)^{\theta_k}\Big(\prod_{i=1}^{k-1}\theta_i\Big)\sum_{i=1}^{k-1}\pi_{i,k-1}^{-1}A_0(0)^{-\theta_i}\int_0^t -\dot{A}_0(s)\cdot A_0(s)^{\theta_i-\theta_k-1}\,ds$$

$$= \Big(\prod_{i=1}^{k-1}\theta_i\Big)\sum_{i=1}^{k-1}\pi_{i,k}^{-1}\Big(\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i} - \Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_k}\Big)$$

$$= \Big(\prod_{i=1}^{k-1}\theta_i\Big)\sum_{i=1}^{k}\pi_{i,k}^{-1}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i}, \quad \text{since } \sum_{i=1}^{k}\pi_{i,k}^{-1}=0,$$

$$\overset{n\geq 0}{=} \frac{f_k(t)}{\alpha(t)\theta_k}.$$

If $n = N$, then $q(t,N) = 1 - \mathbb{P}[\tau_1 \leq t] = \big(\frac{A_0(t)}{A_0(0)}\big)^{\theta_1}$. ■

REMARK 3.2. Since $\pi_{1,1} = 1$ and $\sum_{i=1}^{k}\pi_{i,k}^{-1} = 0$ for all $k = 2,\ldots,N$, the general expressions of Theorem 3.3 immediately yield $q(0,N) = 1$, $q(0,n) = 0$, $n = 1,\ldots,N-1$, and $\lim_{t\to T} q(t,0) = 1$ as well as $\lim_{t\to T} q(t,n) = 0$, $1 \leq n \leq N$, $T \leq \infty$. It is an elementary exercise to check that the functions $q(\cdot,n)$ satisfy the forward equation (31), and the recursion formulas of the time homogeneous case with discounting. The (most) right hand side expression of (32) is most helpful. The product of $q(t,n)$ and $h\cdot\lambda^*(t,n)$, h "small", is approximately equal to the probability of the event that at time t the inventory level is n and a jump occurs in the small interval $(t,t+h)$. Of course, this expression corresponds to the probability that the $(N-n+1)$-th sale takes place in this interval.

4. Optimal sales prices and revenue characteristics

4.1. Optimal prices. We shall abbreviate the optimal k-th sales price $p^*(\tau_k, X_{\tau_k-})$ by p_k. Since the optimal discounted sales prices $\hat{p}_k := e^{-R(\tau_k)}p_k$ are transformed values of τ_k, $1 \leq k \leq N$, knowing formulas for f_k readily yields density formulas of $\hat{p}_k$. The optimal pricing policy, see Section 2, implies the formula

$$\hat{p}_k = e^{-R(\tau_k)}p_k = e^{-R(\tau_k)}\underbrace{\beta_{N-k+1}^{-1/(\gamma-1)}}_{=:\phi_k}\underbrace{\Big(\frac{\delta}{\varepsilon}\Big)^{\delta/(\varepsilon-\delta)}}_{=:\kappa}A(\tau_k)^{1/\gamma} = \phi_k\cdot\kappa\cdot A_0(\tau_k)^{1/\gamma}. \qquad (33)$$

By definition, $\phi_k^{\gamma} = \theta_k^{-1}$ and $\kappa^{-\gamma} = \big(\frac{\varepsilon}{\delta}\big)^{\delta/(1-\delta)}$. Since $a(t)$ is assumed to be a positive function the mapping $A_0 : [0,T] \to [0, A_0(0)]$, $A_0 : [0,\infty) \to (0, A_0(0)]$ resp., see Section 2, is strictly monotone decreasing and it is a continuous bijection with inverse $A_0^{-1} : (0, A_0(0)] \to [0,T)$. Let $h_k(x) := (x/(\kappa\phi_k))^{\gamma}$, $x \geq 0$, $t_k(x) := A_0^{-1}(h_k(x))$ and $I_{\hat{p}_k} := \big(0, \kappa(A_0(0)/\theta_k)^{1/\gamma}\big)$. Note that the right boundary of $I_{\hat{p}_k}$ equals $p^*(N-k+1,0)$, cf. (5), since $A(0) = A_0(0)$. If $r \equiv 0$, i.e. in cases without discounting, $p_k = \hat{p}_k$.

THEOREM 4.1. *The support of the density $f_{\hat{p}_k}$ of the discounted k-th sales price $\hat{p}_k$ is the interval $I_{\hat{p}_k}$. For each $x \in I_{\hat{p}_k}$, $1 \leq k \leq N$,*

$$f_{\hat{p}_k}(x) = \frac{\gamma}{x}\Big(\sum_{i=1}^{k} l_{i,k}\Big(\frac{h_k(x)}{A_0(0)}\Big)^{\theta_i}\Big); \qquad (34)$$

outside of $I_{\hat{p}_k}$ the density is zero.

Proof. For any $x \in I_{\hat{p}_k}$ we have

$$\mathbb{P}[\hat{p}_k \leq x] = \mathbb{P}\Big[A_0(\tau_k) \leq \Big(\frac{x}{\kappa\phi_k}\Big)^{\gamma}\Big] = 1 - \mathbb{P}\big[\tau_k \leq A_0^{-1}\big(\theta_k \kappa^{-\gamma} x^{\gamma}\big)\big].$$

Differentiating and using the notation introduced above, we get

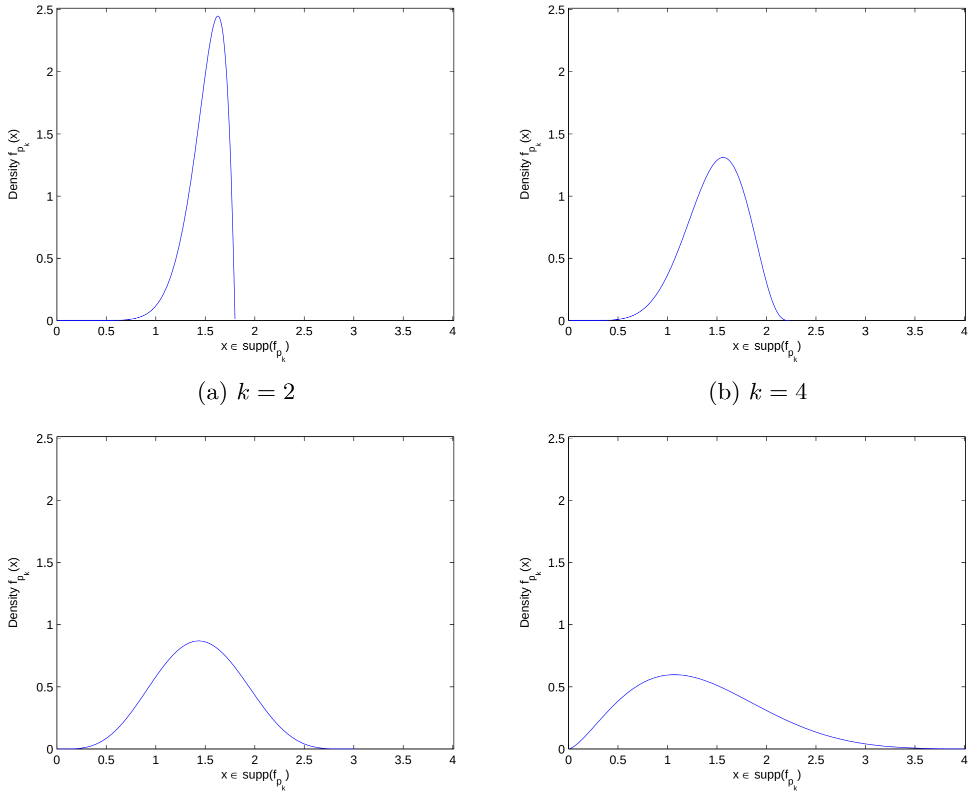
$$f_{\hat{p}_k}(x) = -f_k\big(A_0^{-1}(\theta_k(x/\kappa)^{\gamma})\big) \cdot (A_0^{-1})'(\theta_k(x/\kappa)^{\gamma}) \cdot \gamma\theta_k\kappa^{-\gamma}x^{\gamma-1} \tag{35}$$

$$= \frac{\gamma}{x} \cdot h_k(x) \cdot f_k(t_k(x)) \cdot (-1) \cdot \big(A_0^{-1}\big)'\,(h_k(x)). \tag{36}$$

Since $A_0'(t) = -e^{-\gamma R(t)} a(t)^{1/(1-\delta)}$, and

$$(A_0^{-1})'(h_k(x)) = \frac{1}{A_0'(A_0^{-1}(h_k(x)))} = \frac{1}{A_0'(t_k(x))} = \frac{1}{\alpha(t_k(x))A_0(t_k(x))}\,,$$

the result follows from (14) and the identity $A_0(t_k(x)) = h_k(x)$. ■



(a) $k = 2$ (b) $k = 4$

(c) $k = 6$ (d) $k = 8$

Fig. 3. Plots of the price densities f_{p_k}; $N = 10$, $T = 10$, $\varepsilon = 1.2$, $\delta = 0.5$, $r = 0$ and $a(t) \equiv 2$.

Fig. 3 shows plots of price densities f_{p_k} of a finite horizon time homogeneous model without discounting, $k = 2$, 4, 6 and 8, and $N = 10$. The first plot, Fig. 3 a, illustrates the fact that at the beginning of the sales period optimal sales prices p_k are highly concentrated around $p^*(0, N) = 1.6489$ and do not exceed the initial price $p^*(0, N)$ by much, e.g. $p^*(0, N-1) - p^*(0, N) = 0.1514$, when $T = 10$, $\varepsilon = 1.2$, $\delta = 0.5$, and $a = 2$.

At the other end of the spectrum, Fig. 3 d illustrates the common knowledge that very often the last few items of a batch may pass over the counter either at low or, just the opposite, high prices. The fact that for the numerical example considered the 95% percentile of $f_{\hat{p}_8}$ is more than twice as large as the mean of the price p_2 could explain why risk averse and/or low budget customers usually purchase (airline) tickets long before the end of the sales period.

For the general model, Theorem 3.1 yields an elegant formula of the m-th moment of the discounted price $\hat{p}_k$ which reveals a surprising property of optimal expected discounted sales prices, see below.

PROPOSITION 4.1. *Let $a(t)$ be a positive function and $r(t) \geq 0$. For any $m \in \mathbb{N}$, $1 \leq k \leq N$, the m-th moment of the discounted k-th sales price is given by the formula*

$$\mathbb{E}[\hat{p}_k^m] = \mathbb{E}\big[(e^{-R(\tau_k)}p_k)^m\big] = \big(\kappa^\gamma A_0(0)/\theta_k\big)^{m/\gamma} \prod_{i=1}^{k} \frac{\theta_i}{\theta_i + m/\gamma}\,. \tag{37}$$

Proof. The result follows directly from the definition of $\hat{p}_k$, cf. (33), and Theorem 3.1, by exploiting the identity $\phi_k^\gamma = \theta_k^{-1}$. ∎

The next result is a generalization of Proposition 5.1 in [HS]. The proof relies on the analytical formula of $f_{\hat{p}_k}$, see Theorem 4.1.

PROPOSITION 4.2. *If $m/\gamma \leq 1$, $r(t) \geq 0$, then the (finite) sequence*

$$\Big(\mathbb{E}\big[\big(e^{-R(\tau_k)}p_k\big)^m\big]\Big)_{1\leq k\leq N}$$

is strictly monotone decreasing (in k).

Proof. For any $k = 1, \dots, N-1$, and $m \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}\big[\big(e^{-R(\tau_{k+1})}p_{k+1}\big)^m\big] &\overset{(37)}{=} \big(\kappa^\gamma A_0(0)/\theta_{k+1}\big)^{m/\gamma} \prod_{i=1}^{k+1} \frac{\theta_i}{\theta_i + m/\gamma} \\ &\overset{(37)}{=} \mathbb{E}\big[\big(e^{-R(\tau_k)}p_k\big)^m\big]\Big(\frac{\theta_k}{\theta_{k+1}}\Big)^{m/\gamma}\Big(\frac{\theta_{k+1}}{\theta_{k+1} + m/\gamma}\Big).\end{aligned}$$

If $m/\gamma < 1$, then employing Bernoulli's inequality with the exponent $m/\gamma < 1$ and using (4) (ii) yields

$$\begin{aligned}\Big(\frac{\theta_k}{\theta_{k+1}}\Big)^{m/\gamma}\Big(\frac{\theta_{k+1}}{\theta_{k+1} + m/\gamma}\Big) &= \Big(\frac{\beta_n^{\gamma/(\gamma-1)}}{\beta_{n-1}^{\gamma/(\gamma-1)}}\Big)^{m/\gamma} \frac{\beta_{n-1}^{\gamma/(\gamma-1)}}{\beta_{n-1}^{\gamma/(\gamma-1)} + m/\gamma} \\ &= \Big(\frac{\beta_n}{\beta_{n-1}}\Big)^{1/(\gamma-1)}\Big(\frac{1}{1 + (m/\gamma)\beta_{n-1}^{-\gamma/(\gamma-1)}}\Big) \\ &\leq \Big(\Big(\frac{\beta_n}{\beta_{n-1}}\Big)^{1/(\gamma-1)}\Big(\frac{1}{1 + \beta_{n-1}^{-\gamma/(\gamma-1)}}\Big)^{1/\gamma}\Big)^m < 1.\end{aligned}$$

If $m/\gamma = 1$ the strict inequality follows directly from (4) (ii). Hence, the expected discounted prices are strictly monotone decreasing in k. ∎

Fig. 4 illustrates the behavior of the moments of the non-discounted prices for a particular choice of parameters, viz. $\varepsilon = 2$ and $\delta = 0.5$; these parameter values yield

$\gamma = 3$. Thus, the first three moments are decreasing (in k). The second window shows that the condition $m/\gamma \le 1$ is sharp.

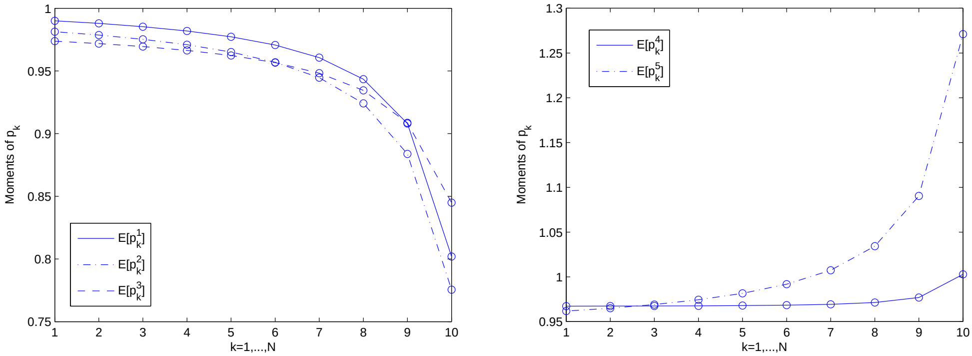


Fig. 4. The graphs are an illustration of Proposition 4.2; note, $\gamma = 3$. The window on the left shows the decreasing sequences of the first, second and third moments of the optimal sales prices for a time-homogeneous model without discounting; the window on the right shows the increasing sequences of the 4th and 5th moments.

4.2. A formula of the optimal expected revenue: Infinite time horizon. The optimal expected revenue, $\mathbb{E}[U_N] := \mathbb{E}\big[\sum_{k=1}^{N} p_k e^{-r\tau_k}\big]$, can be easily computed for any $N \in \mathbb{N}$, using Corollary 3.1, see also (37), when $a(t) \equiv a > 0$, $r(t) \equiv r > 0$ and $T = \infty$. Since

$$p_k := \left(\frac{\delta}{\varepsilon}\right)^{\delta/(\varepsilon-\delta)} \left(a^{1/(1-\delta)}/r\gamma\right)^{1/\gamma} \theta_k^{-1/\gamma} = \text{const} \cdot \theta_k^{-1/\gamma}$$

we get

$$\mathbb{E}[U_N] = \sum_{k=1}^{N} p_k \prod_{i=1}^{k} \frac{\theta_i}{\theta_i + 1/\gamma} = \text{const} \sum_{k=1}^{N} \theta_k^{-1/\gamma} \prod_{i=1}^{k} \frac{\theta_i}{\theta_i + 1/\gamma}. \tag{38}$$

The following lemma shows that the second sum in (38) equals β_N. This nontrivial formula has been verified in [HS] by exploiting the explicit expression of the value function of the infinite horizon control problem with discounting together with a dynamic Dorfman–Steiner identity. Here, we will provide an elementary proof using an induction argument.

Lemma 4.1. *For all $N \in \mathbb{N}$, $\gamma > 1$ and $(\beta_n)_n$ defined by* (2),

$$\beta_N = \sum_{k=1}^{N} \beta_{N-k+1}^{-1/(\gamma-1)} \left(\prod_{j=1}^{k} \frac{\beta_{N-j+1}^{\gamma/(\gamma-1)}}{\beta_{N-j+1}^{\gamma/(\gamma-1)} + 1/\gamma} \right). \tag{39}$$

Proof. We prove this lemma by induction. Let the right-hand side of (39) be abbreviated by B_N. Since $\beta_1 = \left(\frac{\gamma-1}{\gamma}\right)^{(\gamma-1)/\gamma}$, we get

$$B_1 = \beta_1^{-1/(\gamma-1)} \left(\frac{\beta_1^{\gamma/(\gamma-1)}}{\beta_1^{\gamma/(\gamma-1)} + 1/\gamma} \right) = \beta_1 \left(\beta_1^{\gamma/(\gamma-1)} + \frac{1}{\gamma} \right)^{-1} = \beta_1.$$

$\underline{N \to N+1}$: Assume $B_N = \beta_N$, then

$$
\begin{aligned}
B_{N+1} &= \sum_{k=1}^{N+1} \beta_{(N+1)-k+1}^{-1/(\gamma-1)} \left[\prod_{j=1}^{k} \frac{\beta_{(N+1)-j+1}^{\gamma/(\gamma-1)}}{\beta_{(N+1)-j+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \right] \\
&= \beta_{N+1}^{-1/(\gamma-1)} \left[\frac{\beta_{N+1}^{\gamma/(\gamma-1)}}{\beta_{N+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \right] + \frac{\beta_{N+1}^{\gamma/(\gamma-1)}}{\beta_{N+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \sum_{k=2}^{N+1} \beta_{(N+1)-k+1}^{-1/(\gamma-1)} \left[\prod_{j=2}^{k} \frac{\beta_{(N+1)-j+1}^{\gamma/(\gamma-1)}}{\beta_{(N+1)-j+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \right] \\
&\overset{B_N = \beta_N}{=} \beta_{N+1}^{-1/(\gamma-1)} \left[\frac{\beta_{N+1}^{\gamma/(\gamma-1)}}{\beta_{N+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \right] + \frac{\beta_{N+1}^{\gamma/(\gamma-1)}}{\beta_{N+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}} \beta_N \\
&= \left(\beta_{N+1}^{\gamma/(\gamma-1)} + \frac{1}{\gamma}\right)^{-1} \left[\beta_{N+1} + \beta_N \beta_{N+1}^{\gamma/(\gamma-1)}\right] = \frac{1 + \beta_N \beta_{N+1}^{1/(\gamma-1)}}{\beta_{N+1}^{1/(\gamma-1)} + (\gamma\beta_{N+1})^{-1}} .
\end{aligned}
$$

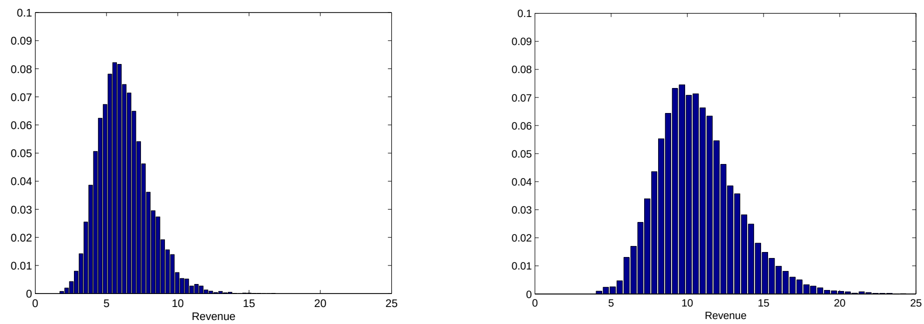
By the definition of the β-sequence we know that $\beta_{n+1}^{1/(\gamma-1)} = \frac{\gamma-1}{\gamma}(\beta_{n+1} - \beta_n)^{-1}$; thus,

$$
B_{N+1} = \frac{\gamma\beta_{N+1} - \beta_N}{\gamma - 1 + (\beta_{N+1} - \beta_N)/\beta_{N+1}} = \beta_{N+1} \frac{\gamma\beta_{N+1} - \beta_N}{\gamma\beta_{N+1} - \beta_N} = \beta_{N+1}. \quad \blacksquare
$$

Using Lemma 4.1 and formula (38) we obtain the following formula for the expected optimal revenue.

Proposition 4.3. *Let $a(t) \equiv a > 0$, $r(t) \equiv r > 0$, $T = \infty$ and $N \in \mathbb{N}$. The optimal revenue $U_N = \sum_{k=1}^{N} p_k \cdot e^{-r\tau_k}$ has an expected value of*

$$
\mathbb{E}[U_N] = \left(\frac{\delta}{\varepsilon}\right)^{\delta/(\varepsilon-\delta)} \left(\frac{a^{1/(1-\delta)}}{r\gamma}\right)^{1/\gamma} \cdot \beta_N. \tag{40}
$$



(a) (b)

Fig. 5. Histograms of U_N of two cases based on 10.000 simulations of optimal sales times sequences $(\tau_1, \ldots, \tau_N)$. (a) illustrates the distribution of the optimal revenue of a finite horizon model with discounting, $T = 10$, while (b) shows the histogram of the infinite horizon case, $T = \infty$; $N = 10$, $\varepsilon = 1.2$, $\delta = 0.5$, $r \equiv 0.1$ and $a \equiv 2$.

Fig. 5 shows the histograms of the optimal revenue of a finite horizon model and the corresponding infinite horizon one based on 10,000 simulated sales times sequences; see the next section for details of the simulation procedure being used. The empirical mean equals the exact value 10.73, see formula (38). Notice the heavier tail of the distribution of U_N if $T = \infty$ and, of course, the expected value which is larger compared to the expected value in the finite horizon case. If $T = 10$, the mean of U is 7.56.

5. Simulation procedures. The results of Section 3 could be used to simulate optimal sales times and optimal prices, etc. However, it is more efficient to implement an idea proposed by McAfee and te Velde [MV], see also [HS]: The random variables $\big(A_0(\tau_k)/A_0(\tau_{k-1})\big)^{\theta_k}$ are (conditionally) independent and uniformly distributed on the unit interval, cf. Proposition 2.1. Hence, realizations of the optimal sales times $\tau_1, \ldots, \tau_N$ can be constructed as follows: Take iid random variables $u_1, \ldots, u_N$, $u_k \sim \mathcal{U}[0,1]$, and use the fact

$$\big(A_0(\tau_k)\,|\,\tau_{k-1}\big) \sim u_k^{1/\theta_k} A_0(\tau_{k-1}).$$

Define the sequence of realizations of sales times, $0 = t_0 < t_1 < \ldots < t_n < T$, by

$$t_k := A_0^{-1}\big(u_k^{1/\theta_k} A_0(t_{k-1})\big), \quad 1 \le k \le N. \tag{41}$$

If $a(t)$ and $r(t)$ are independent of t, then

$$A_0(t) = \frac{a^{1/(1-\delta)}}{\gamma r}\big(e^{-\gamma r t} - e^{-\gamma r T}\big),$$

and (41) simplifies to

$$t_k = -\frac{1}{\gamma r}\ln\Big(u_k^{1/\theta_k} e^{-\gamma r t_{k-1}} + (1 - u_k^{1/\theta_k})e^{-\gamma r T}\Big).$$

If, in addition, $r(t) \equiv 0$, then, see Section 3, the following efficient implementation is recommended: Choose iid random variables $(u_k)_{1\le k\le N}$, and define

$$\bar{u}_k := u_1^{1/\theta_1} \cdot \ldots \cdot u_k^{1/\theta_k}.$$

Thus,

$$\bar{u}_k \sim \frac{T-\tau_1}{T}\,\frac{T-\tau_2}{T-\tau_1}\cdots\frac{T-\tau_k}{T-\tau_{k-1}} = \frac{T-\tau_k}{T}.$$

Hence, with realized values of $\bar{u}_k$ define

$$t_k := T(1-\bar{u}_k).$$

The case with constant discount rate and $T = \infty$ is even easier to handle, cf. Corollary 3.1. Just generate realizations Δ_i of independent exponential random variables with intensity $\gamma r \theta_i$ and set

$$t_k := \sum_{i=1}^{k} \Delta_i.$$

6. Approximation results. Besides the limit result $\beta_n^{\gamma/(\gamma-1)} \approx n$, if n is large, McAfee and te Velde [MV] have proposed the following refined approximation of β_n:

$$\beta_n^{\gamma/(\gamma-1)} \approx n - \frac{1.5}{\gamma}, \quad n \ge 2. \tag{42}$$

This approximation is remarkably accurate for all $\gamma \ge 1.1$ and $n \ge 2$. In this section, we shall use the approximation (42) and replace θ_k by $(N-k+1-\frac{1.5}{\gamma})$, $1 \le k \le N$, in Theorems 3.1 and 3.3. This way the formulas of f_k, etc. will simplify, and we shall get accurate density approximations by beta densities, see below. To this end, using (42) we obtain simple approximations of the expressions $\pi_{i,k}$ in terms of products of factorials:

$$\pi_{i,k} = \prod_{\substack{j=1\\ j\neq i}}^{k} (\theta_j - \theta_i) \approx \prod_{\substack{j=1\\ j\neq i}}^{k} (i-j) = \Big(\prod_{j=1}^{i-1}(i-j)\Big)\Big(\prod_{j=i+1}^{k}(i-j)\Big) = (-1)^{k-i}(i-1)!\,(k-i)!\,.$$

6.1. Approximation of the sales times densities f_k

THEOREM 6.1. *Assume the conditions of Theorem* 3.1 *to hold. For* k, $1 \le k < N$, *let* $c_k := k$ *and* $d_k := N-(k-1)-\frac{1.5}{\gamma}$. *For* t, $0 \le t \le T$, *let* $x := 1 - A_0(t)/A_0(0) \in [0,1]$, *and define* $b_k(x) := (1/B(c_k,d_k))x^{c_k-1}(1-x)^{d_k-1}$, *where* $B(c_k,d_k) := \Gamma(c)\Gamma(d)/\Gamma(c+d)$, *i.e.* b_k *denotes the beta density with nonnegative parameters* c_k, d_k. *Let for* $t \in [0,T)$,

$$\tilde{f}_k(t) := \frac{a(t)^{1/(1-\delta)}}{A(t)}\,\frac{A_0(t)}{A_0(0)}\,\frac{1}{B(c_k,d_k)}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{d_k-1}\Big(1-\frac{A_0(t)}{A_0(0)}\Big)^{c_k-1}.$$

Then,

$$f_k(t)\,dt \approx \tilde{f}_k(t)\,dt = b_k(x)\,dx.$$

If $a(t) \equiv a > 0$ *and* $r(t) \equiv 0$, *then*

$$\mathbb{E}[\tau_k] \approx \int_0^T t\tilde{f}_k(t)\,dt = \frac{k}{N+1-1.5/\gamma}\cdot T. \tag{43}$$

Proof.

$$\begin{aligned}
f_k(t) &= e^{-\gamma R(t)}\Big(\prod_{i=1}^{k}\theta_i\Big)\frac{a(t)^{1/(1-\delta)}}{A_0(t)}\sum_{i=1}^{k}\pi_{i,k}^{-1}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{\theta_i}\\
&\approx e^{-\gamma R(t)}\Big(\prod_{i=1}^{k}\Big(N-i+1-\frac{1.5}{\gamma}\Big)\Big)\frac{a(t)^{1/(1-\delta)}}{A_0(t)}\\
&\quad\cdot\sum_{i=1}^{k}(-1)^{k-i}\,\frac{1}{(i-1)!\,(k-i)!}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{N-(i-1)-1.5/\gamma}\\
&= e^{-\gamma R(t)}\frac{\prod_{i=1}^{k}\big(N-i+1-\frac{1.5}{\gamma}\big)}{(k-1)!}\,\frac{a(t)^{1/(1-\delta)}}{A_0(t)}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{N-1.5/\gamma}\\
&\quad\cdot\sum_{i=0}^{k-1}(-1)^{(k-1)-i}\binom{k-1}{i}\Big(\frac{A_0(0)}{A_0(t)}\Big)^{i}\\
&= e^{-\gamma R(t)}\,\frac{\prod_{i=1}^{k}\big(N-i+1-\frac{1.5}{\gamma}\big)}{(k-1)!}\,\frac{a(t)^{1/(1-\delta)}}{A_0(t)}\\
&\quad\cdot\Big(\frac{A_0(t)}{A_0(0)}\Big)^{N-1.5/\gamma}\Big[\frac{A_0(0)}{A_0(t)}-1\Big]^{k-1}\\
&= e^{-\gamma R(t)}\,\frac{\Gamma(c_k+d_k)}{\Gamma(c_k)\Gamma(d_k)}\,\frac{a(t)^{1/(1-\delta)}}{A_0(0)}\Big(1-\frac{A_0(t)}{A_0(0)}\Big)^{k-1}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{N-1.5/\gamma-k}\\
&= e^{-\gamma R(t)}\,\frac{a(t)^{1/(1-\delta)}}{A_0(0)}\,\frac{\Gamma(c_k+d_k)}{\Gamma(c_k)\Gamma(d_k)}\Big(\frac{A_0(t)}{A_0(0)}\Big)^{d_k-1}\Big(1-\frac{A_0(t)}{A_0(0)}\Big)^{c_k-1} =: \tilde{f}_k(t).
\end{aligned} \tag{44}$$

If we define $x := 1 - A_0(t)/A_0(0)$ we get

$$\frac{dx}{dt} = -\frac{\dot{A}_0(t)}{A_0(0)} = \frac{e^{-\gamma R(t)} a(t)^{1/(1-\delta)}}{A_0(0)},$$

and

$$\begin{aligned} f_k(t)\,dt \approx \tilde{f}_k(t)\,dt &= \tilde{f}_k(t) \cdot \Big(\frac{A_0(0)e^{\gamma R(t)}}{a(t)^{1/(1-\delta)}}\Big)\,dx \\ &= \underbrace{\frac{1}{B(c_k,d_k)} x^{c_k-1}(1-x)^{d_k-1}}_{=b_k(x)}\,dx. \end{aligned} \tag{45}$$

Moreover,

$$\begin{aligned} \mathbb{E}[\tau_k] \approx \int_0^T t \cdot \tilde{f}_k(t)\,dt &= \int_0^1 T \cdot x b_k(x)\,dx = T \cdot \mathbb{E}_{\beta(c_k,d_k)}(X) \\ = T \cdot \frac{c_k}{c_k+d_k} &= \frac{k}{N+1-1.5/\gamma} \cdot T. \ \blacksquare \end{aligned} \tag{46}$$

Remark 6.1. Note that while $\tilde{f}_k$ involves the term $(A_0(t)/A_0(0))$ with exponent (d_k-1), the density $b_k(x)$ actually involves the term x^{c_k-1}. The approximation (43) is valid if $k < N$. Since $\gamma > 1$, the value $(d_k - 1)$ is positive iff $k < N$.

Proposition 6.1. *A good approximation of* $\mathbb{E}[\tau_N]$ *which is an improvement over* $T \cdot N/(N+1)$, *see Subsection* 3.2, *is given by*

$$\mathbb{E}[\tau_N] \approx T\Big(1 - \frac{\gamma-1}{2\gamma-1}\,\frac{2-1.5/\gamma}{N+1-1.5/\gamma}\Big). \tag{47}$$

Proof. It follows by the definition of β_1, see Section 2, that $\theta_N = (\gamma-1)/\gamma \iff \theta_N/(1+\theta_N) = (\gamma-1)/(2\gamma-1)$. Thus,

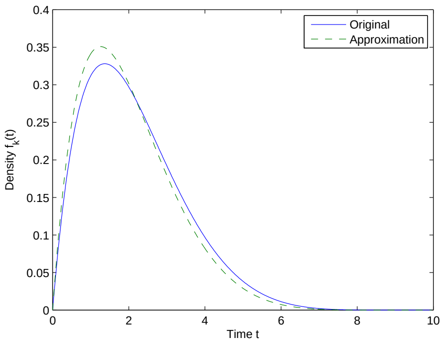
$$\mathbb{E}[\tau_N] = T\Big(1 - \prod_{k=1}^{N} \frac{\theta_k}{\theta_k+1}\Big) = T\Big(1 - \frac{\theta_N}{\theta_N+1} \prod_{k=1}^{N-1} \frac{\theta_k}{\theta_k+1}\Big),$$

and

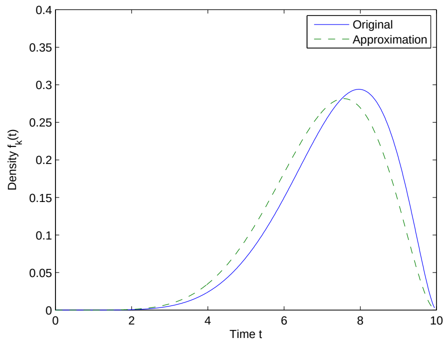
$$\prod_{k=1}^{N-1} \frac{\theta_k}{\theta_k+1} \approx \prod_{k=1}^{N-1} \frac{N+1-\frac{1.5}{\gamma}-k}{N+2-\frac{1.5}{\gamma}-k} = \frac{\Gamma\big(N+1-\frac{1.5}{\gamma}\big)\Gamma\big(3-\frac{1.5}{\gamma}\big)}{\Gamma\big(2-\frac{1.5}{\gamma}\big)\Gamma\big(N+2-\frac{1.5}{\gamma}\big)} = \frac{2-\frac{1.5}{\gamma}}{N+1-\frac{1.5}{\gamma}}.$$

This implies (47). $\blacksquare$

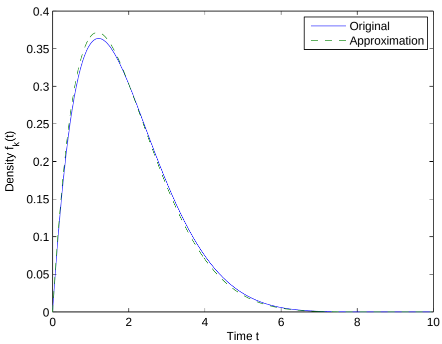
Fig. 6 illustrates the quality of the approximation of the densities f_k by the beta densities $\tilde{f}_k$. If ε becomes larger, i.e. γ increases, see Fig. 6 c and Fig. 6 d, the approximation by beta densities becomes even better. Fig. 7 shows the approximation of optimal mean sales times according to formulas (46) and (47). Fig. 7 b nicely illustrates the improvement of the beta-approximation over the binomial-approximation by looking at the average inventory level over time.



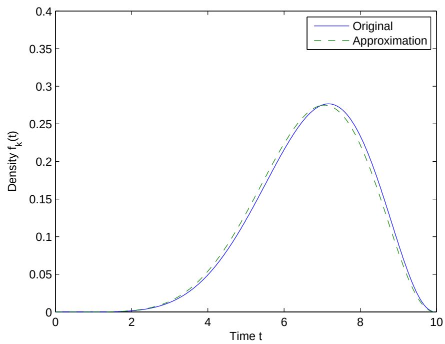
(a) $k = 2,\ \varepsilon = 1.2$



(b) $k = 7,\ \varepsilon = 1.2$

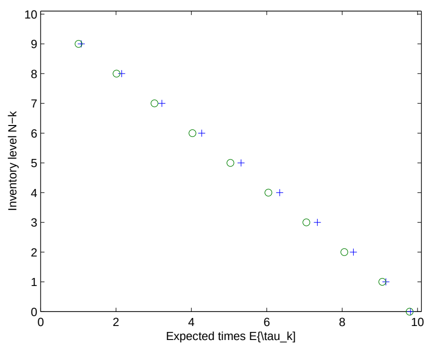


(c) $k = 2,\ \varepsilon = 2$

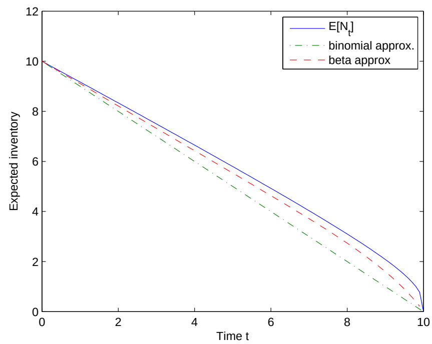


(d) $k = 7,\ \varepsilon = 2$

Fig. 6. A comparison of the densities f_k and $\tilde{f}_k$, see Theorem 6.1; $N = 10, T = 10, \delta = 0.5$, $a(t) \equiv a = 2$ and $r(t) \equiv 0$. (a) and (b) illustrate the approximation when $\varepsilon = 1.2$, while (c) and (d) illustrate the situation when ε gets larger, e.g. $\varepsilon = 2$.



(a)



(b)

Fig. 7. The inventory level $N - k$ (marked on the y-axis) versus the exact expected sales times $E[\tau_k]$, (x-values), and their approximations based on (46) and (47). The exact $E[\tau_k]$ are marked by (+) while the approximated values are marked by (o), see (a). (b) shows plots of the average inventory level over time; $T = N = 10$, $\varepsilon = 1.3$, $\delta = 0.5$, $a = 2$.

6.2. Approximations of sales prices. To approximate the density of the discounted sales prices $\hat{p}_k = e^{-R(\tau_k)} p_k$ we take up the ideas that have already been applied in the previous subsection.

THEOREM 6.2. *Assume all conditions of Theorem* 4.1 *to hold and adopt the notation introduced in Subsection* 4.1. *Let* $c_k = k$ *and* $d_k = N - (k-1) - 1.5/\gamma$. *Let* $b_k^{(p)}(y) := (B(d_k, c_k))^{-1} y^{d_k - 1}(1-y)^{c_k - 1}$, $0 \le y \le 1$, *and*

$$\tilde{f}_{\hat{p}_k}(x) := \frac{\gamma}{A_0(0)} \frac{\Gamma(c_k + d_k)}{\Gamma(c_k)\Gamma(d_k)} \frac{h_k(x)}{x} \Big(\frac{h_k(x)}{A_0(0)}\Big)^{d_k - 1} \Big(\frac{h_k(x)}{A_0(0)} - 1\Big)^{c_k - 1},$$

for $h_k(x) = \theta_k (x/\kappa)^\gamma$ *as defined in Subsection* 4.1. *Then, for* $0 \le x \le \kappa \phi_k A_0(0)^{1/\gamma}$,

$$f_{\hat{p}_k}(x)\, dx \approx \tilde{f}_{\hat{p}_k}(x)\, dx = b_k^{(p)}(y)\, dy. \tag{48}$$

Proof.

$$\begin{aligned} f_{\hat{p}_k}(x) &= \frac{\gamma}{x}\Big(\prod_{i=1}^k \theta_i\Big) \sum_{i=1}^k \pi_{i,k}^{-1} \Big(\frac{h_k(x)}{A_0(0)}\Big)^{\theta_i} \\ &\approx \frac{\gamma}{x}\Big(\prod_{i=1}^k \Big(N - i + 1 - \frac{1.5}{\gamma}\Big)\Big) \sum_{i=1}^k (-1)^{k-i} \frac{1}{(i-1)!\,(k-i)!} \Big(\frac{h_k(x)}{A_0(0)}\Big)^{d_i} \\ &= \frac{\gamma}{x} \frac{\Gamma(c_k + d_k)}{\Gamma(c_k)\Gamma(d_k)} \Big(\frac{h_k(x)}{A_0(0)}\Big)^{N - 1.5/\gamma} \Big[\frac{A_0(0)}{h_k(x)} - 1\Big]^{k-1} \\ &= \frac{\gamma}{x} \frac{h_k(x)}{A_0(0)} \frac{\Gamma(c_k + d_k)}{\Gamma(c_k)\Gamma(d_k)} \Big(\frac{h_k(x)}{A_0(0)}\Big)^{d_k - 1} \Big[1 - \frac{h_k(x)}{A_0(0)}\Big]^{c_k - 1}, \end{aligned}$$

which equals $\tilde{f}_{\hat{p}_k}(x)$. Moreover, let $y(x) := h_k(x)/A_0(0)$. This transformation is obviously continuous, $y(0) = 0$, and $y(\kappa\phi_k A(0)^{1/\gamma}) = 1$. Hence, y maps the support of f_{p_k} onto the interval $(0,1)$ and (48) follows,

$$\begin{aligned} f_{\hat{p}_k}(x)\, dx \approx \tilde{f}_{\hat{p}_k}(x)\, dx &= \tilde{f}_k(x) \frac{x}{\gamma \cdot y}\, dy \\ &= \frac{1}{B(d_k, c_k)} y^{d_k - 1}(1-y)^{c_k - 1}\, dy = b_k^{(p)}(y)\, dy. \ \blacksquare \end{aligned}$$

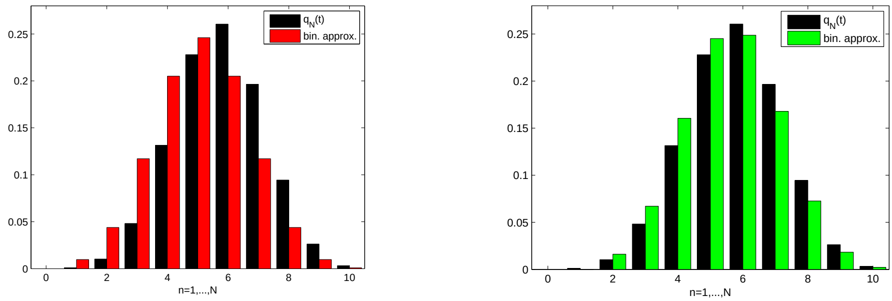
COROLLARY 6.1. *For* $1 \le k < N$, $d_k = N - (k-1) - 1.5/\gamma$,

$$\begin{aligned} \mathbb{E}[e^{-R(\tau_k)} p_k] &= \kappa\phi_k A_0(0)^{1/\gamma} \prod_{i=1}^k \frac{\theta_i}{\theta_i + 1/\gamma} \\ &\approx \kappa A_0(0)^{1/\gamma} d_k^{-(1+\gamma)/\gamma} \frac{\Gamma(N + 1 - \frac{1.5}{\gamma})\Gamma(N - (k-1) - \frac{0.5}{\gamma})}{\Gamma(N + 1 - \frac{0.5}{\gamma})}; \end{aligned}$$

moreover,

$$\mathbb{E}[p_N] \approx \kappa\Big(\frac{\gamma - 1}{\gamma}\Big)^{-(\gamma-1)/\gamma} A_0(0)^{1/\gamma} \frac{\Gamma(N + 1 - \frac{1.5}{\gamma})\Gamma(2 - \frac{0.5}{\gamma})}{\Gamma(N + 1 - \frac{0.5}{\gamma})\Gamma(2 - \frac{1.5}{\gamma})}.$$

6.3. An approximation of the expected inventory values. For the pure pricing model with $r(t) \equiv 0$ and $T < \infty$, McAfee and te Velde have shown that $q(t,n)$ can be approximated by binomial probabilities $\tilde{q}(t,n) := \binom{N}{n}\big(\frac{A(t)}{A(0)}\big)^n \big(1 - \frac{A(t)}{A(0)}\big)^{N-n}$. Following the ideas which were put forward by McAfee and te Velde, this approximation was extended to the advertising and pricing model in [HS]. In this subsection we propose a second approximation $\bar{q}(t,n)$ based on beta distributions. For $t = T/2$, Fig. 8 shows $q(t,n)$ (black bars), and approximations $\tilde{q}(t,n)$ (dark gray bars, Fig. 8 a), and $\bar{q}(t,n)$ (light gray bars, Fig. 8 b) next to each other, $n = 0,\ldots,N$. At half time, i.e. $t = T/2$, the graph illustrates that both approximations are shifted towards lower inventory levels, i.e. $\tilde{q}(t,n) > q(t,n)$, $n = 0,1,\ldots,5$, while $\tilde{q}(t,n) < q_n(t)$, $n = 6,\ldots,10$. Furthermore, the two figures support the fact that the approximation by beta distributions is better than the binomial distribution. The graphs are representative for all $0 \le t < T$, i.e. the shift of the approximations towards lower inventory values persists over the whole sales period.



(a) binomial approximation (b) approximation based on beta distribution

Fig. 8. The exact inventory probabilities $q_n(t)$ (black bars), and approximated probabilities $\tilde{q}(t,n)$, and $\bar{q}(t,n)$ (gray bars) for the time homogeneous model without discounting; $t = T/2$, $T = 10$, $N = 10$, $\varepsilon = 1.2$ and $\delta = 0.3$.

For any $t \in [0,T)$ the expected inventory $\sum_{n=1}^{N} nq(t,n)$ can be computed using the exact values of $q(t,n)$, see Theorem 3.3. As has been pointed out in [HS], cf. Section 3, an approximation of the expected inventory is given by $NA(t)/A(0)$. A refined approximation can be given using the last two theorems. Let

$$\hat{\theta}_k := \begin{cases} N + 1 - k - \dfrac{1.5}{\gamma}, & k = 1,\ldots,N-1 \\ \dfrac{\gamma - 1}{\gamma}, & k = N, \end{cases}$$

and define, for $1 \le n \le N$,

$$\bar{q}(t,n) = \begin{cases} \dfrac{A(t)}{a(t)^{1/(1-\delta)}} \, \dfrac{\tilde{f}_{N-n+1}(t)}{\hat{\theta}_{N-n+1}}, & 0 < n < N \\ \left(\dfrac{A_0(t)}{A_0(0)}\right)^{N-1.5/\gamma}, & n = N \\ \left(\displaystyle\prod_{i=1}^{N-1} \hat{\theta}_i\right) \displaystyle\sum_{i=1}^{N} \hat{\pi}_{i,N}^{-1} \left(\dfrac{A_0(t)}{A_0(0)}\right)^{\hat{\theta}_i}, & n = 0. \end{cases}$$

It then follows from Theorem 6.1 and 6.2 that

$$\mathbb{E}[X_t] = \sum_{n=1}^{N} nq(t,n) \approx \sum_{n=1}^{N} n\bar{q}(t,n). \tag{49}$$

Fig. 7 b illustrates the approximation of $\mathbb{E}[X_t]$ based on (49), i.e. the beta approximation, and on the binomial approximation, cf. Section 3.

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