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ROBUST ESTIMATES OF CERTAIN LARGE DEVIATION PROBABILITIES FOR CONTROLLED SEMI-MARTINGALES

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Abstract. Motivated by downside risk minimization on the wealth process in an incomplete market model, we have studied in the recent work the asymptotic behavior as time horizon $T \to \infty$ of the minimizing probability that the empirical mean of a controlled semi-martingale falls below a certain level on the time horizon T. This asymptotic behavior relates to a risk-sensitive stochastic control problem in the risk-averse case. Indeed, we obtained an expression of the decay rate of the probability by the Legendre transform of the limit value of the value function of the stochastic control problem, which is characterized as the solution to the H–J–B equation of ergodic type. In the current work we present the results on its robust version, admitting model uncertainty.

1. Introduction. Let X_t be a solution to the following stochastic differential equation:

$$dX_t = \lambda(X_t)\, dW_t + \beta(X_t)\, dt, \quad X_0 = x \in R^N, \tag{1.1}$$

with $\lambda(x) \in C_{\mathrm{Lip}}(R^N; R^N \otimes R^M)$, $\beta(x) \in C_{\mathrm{Lip}}(R^N; R^N)$ and consider the following asymptotic behavior of the probability for the given controlled semi-martingale $F_T(X, h)$ of falling below the target growth rate κ:

$$J_0(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \log P\Big(\frac{1}{T} F_T(X_\cdot, h_\cdot) \le \kappa\Big), \tag{1.2}$$

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where

$$F_T(X_\cdot, h_\cdot) = F_0 + \int_0^T f(X_s, h_s)\, ds + \int_0^T \varphi(X_s, h_s)^*\, dW_s,$$

W_t is an M-dimensional Brownian motion process defined on a probability space $(\Omega, \mathcal{F}, P)$ with filtration $\mathcal{F}_t$, F_0 is an $\mathcal{F}_0$-measurable random variable, and $h_t = h(t, X_t)$ is an $\mathcal{F}_t$-progressively measurable R^m-valued process with $m, N \leq M$. In particular, we consider the case where

$$f(x,h) := -\frac{1}{2}\, h^* S(x) h + h^* g(x) + U(x), \quad \varphi(x,h) = \delta(x) h,$$

with

$$S(x) \in C(R^N; R^m \otimes R^m), \quad g(x) \in C(R^N; R^m), \quad \delta(x) \in C(R^N; R^M \otimes R^m),$$

and $U(x) \in C(R^N; R^1)$. The problem is motivated by certain problems arising from mathematical finance called "downside risk minimization", which have been studied in [10], [18], [19], [20], [28] in various kinds of situation (cf. also references [9], [11], [23], [24], [26], [25] on "upside chance maximization"). See also [13] about robust downside risk minimization for a one factor model with 1 risky asset. In the recent article [22], as a generalization of "downside risk minimization", we have discussed such kinds of asymptotic problem as (1.2) apart from the problems in mathematical finance. In a similar manner to those works, we relate (1.2) to the limit value:

$$\chi_0(\theta) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \log E\big[e^{\theta F_T(X_\cdot, h_\cdot)}\big], \quad \theta < 0, \tag{1.3}$$

which is interpreted as the averaging limit of the value function of the risk-sensitive stochastic control problem and characterized by the solution to the relevant H–J–B equation of ergodic type. Indeed, we establish the duality relationship between (1.2) and (1.3) as

$$J_0(\kappa) = -\inf_{k\in(-\infty,\kappa]} \sup_{\theta<0} \{\theta k - \chi_0(\theta)\}. \tag{1.4}$$

Now, in the current paper, we shall further study the problem of its robust version, admitting model uncertainty. Namely, suppose that the drift coefficient $\beta(x)$ is uncertain and we take up the solution to

$$dX_t = \lambda(X_t)\, dW_t^\zeta + \{\beta(X_t) + \lambda(X_t)\zeta_t\}\, dt, \quad X_0 = x \in R^N, \tag{1.5}$$

instead of (1.1), introducing an uncertainty parameter process ζ_t. Here, W_t^ζ is an M-dimensional standard Brownian motion process under the probability measure P^ζ defined by

$$\left.\frac{dP^\zeta}{dP}\right|_{\mathcal{F}_T} := \exp\Big(\int_0^T \zeta_s\, dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2\, ds\Big).$$

We consider the worst case probability that the controlled semi-martingale F_T added by the correction term $\frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds$ falls below the target growth rate κ among the set of probability measures P^ζ, parameterized by uncertainty parameter ζ, each of which is absolutely continuous with respect to the original probability measure P. Then we investigate the asymptotic behavior of the infimum of the worst case probability:

$$J(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \sup_{\zeta_\cdot} \log P^{\zeta}\Big(\frac{1}{T}\Big\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds\Big\} \le \kappa\Big) \tag{1.6}$$

over the set of admissible strategy $h_\cdot$. To study (1.6), we relate it to the following averaging limit:

$$\chi(\theta) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot} \sup_{\zeta_\cdot} \log E^{\zeta}\Big[\exp\Big(\theta(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds)\Big)\Big] \tag{1.7}$$

and show the duality relationship:

$$J(\kappa) = -\inf_{k\in(-\infty,\kappa]} \sup_{\theta<0}\{\theta k - \chi(\theta)\} \tag{1.8}$$

for each given constant $\chi'(-\infty) < \kappa < \chi'(-0)$, where $\chi'(\theta)$ is the derivative of $\chi(\theta)$ with respect to θ and $\chi'(-0) = \lim_{\theta\uparrow 0}\chi'(\theta)$.

Taking the supremum with respect to $\zeta_\cdot$ in (1.6), we consider the worst case probability and are concerned with the robustness of the estimate. Analyzing this problem results in providing analysis of the H–J–B equations of the stochastic control problem with a new aspect. Indeed, we here come to deal with the H–J–B–Isaacs equations relevant to $\chi(\theta)$. This robust version of the asymptotic analysis is motivated by the modeling issues and we here mention it. Suppose that the log prices of the securities in a financial market are governed by stochastic differential equation (1.1). In that case we can obtain the enormous market data about X_t, which is considered to be the solution to (1.1). By using such data, we need to estimate the coefficients consisting of the volatilities λ and the drift β based on the theory of statistical inference. However, to have accurate estimates of β is considered almost impossible since for that one needs to utilize the data on an extremely long term, which may be not accessible. To obtain the good estimates of volatilities λ, one needs the high frequency data in a relatively shorter period, which may be accessible. Thus, robust approach admitting the drift uncertainty may be meaningful. Further, we mention the meaning of the constant μ appearing in the correction term $\frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds$. It reflects the certainty level of the given drift coefficient β (cf. Remark 6.1) since we have the estimate $\|\tilde{\zeta}(x)\|_{L^\infty_{\mathrm{loc}}} \le O(\frac{1}{\mu})$ for the worst case uncertainty $\tilde{\zeta}(x)$. Namely, it goes to 0 as μ tends to ∞, which means the certainty level of β tends to ∞.

From the practical view points, numerical implementation may be also important. In the last section, we illustrate the case of linear Gaussian model, where the solution to the H–J–B equation has the explicit form expressed by using the solution to the Riccati equation. In that case, the relevant quantities are given in the concrete forms (cf. (7.1), (7.3) and (7.4)), which may be helpful in numerical implementation.

2. Studies of the problems on a finite time horizon. In the following we always assume that $F_0 = 0$ for simplicity and so

$$F_T(X_\cdot, h_\cdot) = \int_0^T f(X_s, h_s)\, ds + \int_0^T \varphi(X_s, h_s)^*\, dW_s$$

is considered. Let us make the following assumptions.

$$\begin{cases} \lambda,\ \beta,\ S,\ g,\ \delta \text{ are smooth and globally Lipschitz,} \\ U \text{ is a smooth function bounded below such that} \\ \qquad |U(x)|,\ |DU(x)| \le M_1|x|^2 + M_2 \end{cases} \tag{2.1}$$

$$c_0\delta^*\delta(x) \le S(x) \le c_1\delta^*\delta(x), \quad x \in R^N, \quad c_0, c_1 > 0 \tag{2.2}$$

$$c_\delta I_m \le \delta^*\delta(x) \le c'_\delta I_m, \quad c_\delta, c'_\delta > 0 \tag{2.3}$$

$$c_2|\xi|^2 \le \xi^*\lambda\lambda^*(x)\xi \le c_3|\xi|^2, \quad c_2, c_3 > 0, \quad \xi \in R^n, \tag{2.4}$$

where $DU(x)$ is the gradient vector of the function $U(x)$ with respect to the variable $x \in R^n$, δ^* denotes the transposed matrix of δ and I_m stands for the $m \times m$ unit matrix. Note that, under these assumptions, Q_θ defined by

$$Q_\theta = S - \theta\delta^*\delta$$

satisfies

$$(c_0 - \theta)\delta^*\delta(x) \le Q_\theta(x) \le (c_1 - \theta)\delta^*\delta(x) \tag{2.5}$$

and thus

$$\frac{1}{c_1 - \theta}\,(\delta^*\delta(x))^{-1} \le Q_\theta^{-1}(x) \le \frac{1}{c_0 - \theta}\,(\delta^*\delta(x))^{-1}. \tag{2.6}$$

Moreover,

$$\frac{c_0}{c_0 - \theta}\,I_M \le I_M + \theta\delta Q_\theta^{-1}\delta^* \le I_M. \tag{2.7}$$

Indeed, the left hand side of (2.7) is seen since

$$\frac{\theta}{c_0 - \theta}\,I_M \le \frac{\theta}{c_0 - \theta}\,\delta(\delta^*\delta)^{-1}\delta^* \le \theta\delta Q_\theta^{-1}\delta^*,$$

which follows from (2.6). The right hand side of (2.7) is obvious.

Let $\mathcal{Z}_0$ be the totality of R^M-valued Borel functions $\zeta(t,x)$ such that $|\zeta(t,x)| \le C(1 + |x|)$ for some $C > 0$ and set

$$\mathcal{Z} = \{\zeta_t : \zeta_t = \zeta(t, X_t) \text{ is progressively measurable, } \zeta(t,x) \in \mathcal{Z}_0\}.$$

Then for given $\zeta_\cdot \in \mathcal{Z}$, we can define a probability measure P^ζ by

$$\left.\frac{dP^\zeta}{dP}\right|_{\mathcal{F}_T} := \exp\Big(\int_0^T \zeta_s^*\,dW_s - \frac{1}{2}\int_0^T |\zeta_s|^2\,ds\Big)$$

since we see that $\exp\big(\int_0^t \zeta_s^*\,dW_s - \frac{1}{2}\int_0^t |\zeta_s|^2\,ds\big)$, $0 \le t \le T$, is a martingale for each T. Indeed, we can prove it in a similar manner to Lemma 4.1.1 in [1] (cf. also Lemma 2.2 below). Then $W^\zeta = W_t - \int_0^t \zeta_s\,ds$ becomes a standard Brownian motion process and X_t satisfies the stochastic differential equation (1.5). We are going to investigate the asymptotic behavior

$$J(\kappa) := \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\zeta_\cdot \in \mathcal{Z}} \log P^\zeta\Big(\frac{1}{T}\,\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\} \le \kappa\Big), \tag{2.8}$$

where

$$\Delta_{\mathcal{H}} = \{h_t; h_t = h(t, X_t, \zeta_t) \ \text{ is progressively measurable, } \ h(t,x,\zeta) \in \mathbf{H}\}.$$

Here, $\mathbf{H}$ denotes the totality of Borel functions $h(t,x,\zeta) : [0,T] \times R^N \times R^M \to R^m$ such that $|h(t,x,\zeta)| \le C(1+|x|+|\zeta|)$. To study (2.8) we are going to analyze

$$\chi(\theta) = \varliminf_{T\to\infty} \frac{1}{T} \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\zeta_\cdot \in \mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds\Big\}\Big)\Big]. \tag{2.9}$$

For that we first set the lower value function for the differential game on a finite time horizon by

$$u_*(0,x;T) := \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\zeta_\cdot \in \mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds\Big\}\Big)\Big]. \tag{2.10}$$

Available strategies in (2.10) would be considered to be a Markovian counterpart of Elliott–Kalton strategies and confer [6] and [7] for Elliott–Kalton strategies concerning differential games and their stochastic counterpart respectively. When introducing another probability measure $P^{\zeta,h}$ defined by

$$\frac{dP^{\zeta,h}}{dP^{\zeta}}\bigg|_{\mathcal{F}_T} = \exp\Big(\theta\int_0^T \varphi(X_s,h_s)^*\, dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^T |\varphi(X_s,h_s)|^2\, ds\Big)$$

the lower value function is written as

$$\begin{aligned} u_*(0,x;T) &= \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\zeta_\cdot \in \mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta|^2\, ds\Big\}\Big)\Big] \\ &= \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\zeta_\cdot \in \mathcal{Z}} \log E^{\zeta,h}\Big[\exp\Big(\theta\int_0^T \eta(X_s,h_s,\zeta_s)\, ds\Big)\Big], \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \eta(x,h,\zeta) &= f(x,h) + h^*\delta(x)^*\zeta + \frac{\mu}{2}|\zeta|^2 + \frac{\theta}{2}|\delta(x)h|^2 \\ &= -\frac{1}{2} h^*Q_\theta(x)h + h^*(\delta(x)^*\zeta + g(x)) + U(x) + \frac{\mu}{2}|\zeta|^2. \end{aligned}$$

Note that it is also seen by Lemma 2.2 below that

$$\exp\Big(\theta\int_0^t \varphi(X_s,h_s)^*\, dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^t |\varphi(X_s,h_s)|^2\, ds\Big), \quad 0 \le t \le T,$$

is a martingale under $(\Omega, \mathcal{F}_t, P^{\zeta})$ and $P^{\zeta,h}$ is well defined. Then

$$W^{\zeta,h} = W^{\zeta} - \theta\int_0^t \varphi(X_s,h_s)\, ds$$

is a Brownian motion process under $P^{\zeta,h}$ and X_t turns out to satisfy

$$dX_t = \lambda(X_t)\, dW_t^{\zeta,h} + (\beta(X_t) + \lambda(X_t)\zeta_t + \theta\lambda\delta(X_t)h_t)\, dt. \tag{2.12}$$

Then we see that to the lower value there corresponds the lower Isaacs H–J–B equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \dfrac{1}{2}(Du)^*\lambda\lambda^* Du + H_-(x,Du) = 0 \\ u(T,x) = 0, \end{cases} \tag{2.13}$$

where

$$\begin{aligned}
H_-(x,p) &= \sup_{\zeta\in R^M}\inf_{h\in R^m}\Lambda(x,p,\zeta,h)\\
&\equiv \sup_{\zeta\in R^M}\inf_{h\in R^m}\left[\{\zeta+\theta\delta(x)h\}^*\lambda(x)^*p+\theta\eta(x,\zeta,h)\right]\\
&= -\frac{1}{2\theta\mu}\,(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)^*R_{\theta,\mu}^{-1}(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)\\
&\quad +\frac{\theta}{2}\,(g+\delta^*\lambda^*p)^*Q_\theta^{-1}(g+\delta^*\lambda^*p)+\theta U,
\end{aligned}$$

and

$$N_\theta = I_M+\theta\delta Q_\theta^{-1}\delta^*,\quad R_{\theta,\mu}=I_M+\frac{1}{\mu}\,\delta Q_\theta^{-1}\delta^*, \tag{2.14}$$

because

$$\begin{aligned}
\Lambda(x,p,\zeta,h) = \Bigg[&-\frac{\theta}{2}\,\{h-Q_\theta^{-1}(g+\delta^*\zeta+\delta^*\lambda^*p)\}^*Q_\theta\{h-Q_\theta^{-1}(g+\delta^*\zeta+\delta^*\lambda^*p)\}\\
&+\frac{\theta}{2}\,(g+\delta^*\lambda^*p)^*Q_\theta^{-1}(g+\delta^*\lambda^*p)+\theta U\\
&+\frac{\theta\mu}{2}\left\{\zeta+\frac{1}{\theta\mu}\,R_{\theta,\mu}^{-1}(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)\right\}^*R_{\theta,\mu}\left\{\zeta+\frac{1}{\theta\mu}\,R_{\theta,\mu}^{-1}(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)\right\}\\
&-\frac{1}{2\theta\mu}\,(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)^*R_{\theta,\mu}^{-1}(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g)\Bigg].
\end{aligned} \tag{2.15}$$

Indeed, set

$$\hat\zeta(x,p) = -\frac{1}{\theta\mu}\,R_{\theta,\mu}^{-1}(N_\theta\lambda^*p+\theta\delta Q_\theta^{-1}g) \tag{2.16}$$

$$\hat h(x,\zeta,p) = Q_\theta^{-1}(g+\delta^*\zeta+\delta^*\lambda^*p). \tag{2.17}$$

Then, for a given solution u to lower Isaacs equation (2.13), $\bar\zeta(t,x)$ and $\bar h(t,x,\zeta)$ defined by

$$\bar\zeta(t,x):=\hat\zeta(x,Du(t,x)),\quad \bar h(t,x,\zeta):=\hat h(x,\zeta,Du(t,x))$$

satisfy

$$\bar h(t,x,\zeta)=\arg\min_{h\in R^m}\Lambda(x,Du(t,x),\zeta,h)$$

$$\bar\zeta(t,x)=\arg\max_{\zeta\in R^M}\Lambda(x,Du(t,x),\zeta,\bar h(t,x,\zeta))$$

and

$$\begin{aligned}
H_-(x,Du) &= \sup_{\zeta\in R^M}\inf_{h\in R^m}\Lambda(x,Du(t,x),\zeta,h)\\
&= \Lambda(x,Du(t,x),\bar\zeta(t,x),\bar h(t,x,\bar\zeta(t,x))).
\end{aligned}$$

Let us show that the Isaacs condition holds in the Isaacs equation (2.13). To see that we notice that

$$
\begin{aligned}
H_+(x,p) &:= \inf_{h\in R^m}\sup_{\zeta\in R^M}\Lambda(x,p,\zeta,h)\\
&\equiv \inf_{h\in R^m}\sup_{\zeta\in R^M}\big[\{\zeta+\theta\delta(x)h\}^*\lambda(x)^*p+\theta\eta(x,\zeta,h)\big]\\
&= \frac{1}{2\theta\mu^2}\{(1-\theta\mu)\delta^*\lambda^*(x)p-\theta\mu g(x)\}^* Q^{-1}_{\theta-1/\mu}(x)\{(1-\theta\mu)\delta^*\lambda^*(x)p-\theta\mu g(x)\}\\
&\qquad -\frac{1}{2\theta\mu}\,p^*\lambda\lambda^*(x)p+\theta U(x)
\end{aligned}
$$

holds, since

$$
\begin{aligned}
\Lambda(x,p,\zeta,h) &= \frac{\theta\mu}{2}\Big[\zeta+\frac{1}{\theta\mu}\{\lambda^*p+\theta\delta h\}\Big]^*\Big[\zeta+\frac{1}{\theta\mu}\{\lambda^*p+\theta\delta h\}\Big]\\
&-\frac{\theta}{2}\Big\{h+\frac{1}{\theta\mu}Q^{-1}_{\theta-1/\mu}((1-\theta\mu)\delta^*\lambda^*p-\theta\mu g)\Big\}^{-1}Q_{\theta-1/\mu}\\
&\qquad\times\Big\{h+\frac{1}{\theta\mu}Q^{-1}_{\theta-1/\mu}((1-\theta\mu)\delta^*\lambda^*p-\theta\mu g)\Big\}\\
&+\frac{1}{2\theta\mu^2}\{(1-\theta\mu)\delta^*\lambda^*p-\theta\mu g\}^*Q^{-1}_{\theta-1/\mu}\{(1-\theta\mu)\delta^*\lambda^*p-\theta\mu g\}-\frac{1}{2\theta\mu}\,p^*\lambda\lambda^*p+\theta U.
\end{aligned}
$$

The following lemma says that the Isaacs condition holds in the current case. Instead of appealing to the general theory of convex analysis, we give a direct proof of the lemma in which we have useful formulae for the succeeding arguments.

Lemma 2.1. *The Isaacs condition holds for* $\Lambda(x,p;\zeta,h)$:

$$H_+(x,p)=H_-(x,p).$$

Proof. It suffices to prove that

$$-\frac{1}{\mu}Q_\theta^{-1}\delta^*R^{-1}_{\theta,\mu}\delta Q_\theta^{-1}+Q_\theta^{-1}=Q^{-1}_{\theta-1/\mu}, \tag{2.18}$$

$$-\frac{1}{\theta\mu}N_\theta R^{-1}_{\theta,\mu}N_\theta+\theta\delta Q_\theta^{-1}\delta^*=\frac{(1-\theta\mu)^2}{\theta\mu^2}\delta Q^{-1}_{\theta-1/\mu}\delta^*-\frac{1}{\theta\mu}I_M \tag{2.19}$$

and

$$-\frac{1}{\mu}Q_\theta^{-1}\delta^*R^{-1}_{\theta,\mu}N_\theta+\theta Q_\theta^{-1}\delta^*=-\frac{1-\theta\mu}{\mu}Q^{-1}_{\theta-1/\mu}\delta^*. \tag{2.20}$$

We first show the following useful formula:

$$Q^{-1}_{\theta-1/\mu}=Q_\theta^{-1}\delta^*R^{-1}_{\theta,\mu}\delta(\delta^*\delta)^{-1}. \tag{2.21}$$

Since $Q_{\theta-1/\mu}=Q_\theta+\frac{1}{\mu}\delta^*\delta$ we have

$$
\begin{aligned}
Q_{\theta-1/\mu}(Q_\theta^{-1}\delta^*R^{-1}_{\theta,\mu}\delta(\delta^*\delta)^{-1}) &= \delta^*R^{-1}_{\theta,\mu}\delta(\delta^*\delta)^{-1}+\frac{1}{\mu}\delta^*\delta Q_\theta^{-1}\delta^*R^{-1}_{\theta,\mu}\delta(\delta^*\delta)^{-1}\\
&= \delta^*R^{-1}_{\theta,\mu}\delta(\delta^*\delta)^{-1}+\frac{1}{\mu}\delta^*R^{-1}_{\theta,\mu}\delta Q_\theta^{-1}\delta^*\delta(\delta^*\delta)^{-1}\\
&= \delta^*R^{-1}_{\theta,\mu}\Big(I_M+\frac{1}{\mu}\delta Q_\theta^{-1}\delta^*\Big)\delta(\delta^*\delta)^{-1}=I_m
\end{aligned}
$$

because from the definition of $R_{\theta,\mu}$ we can see that $\delta Q_\theta^{-1}\delta^*$ and $R_{\theta,\mu}^{-1}$ are commutative. Thus, (2.21) holds. We further obtain the formula

$$N_{\theta-1/\mu} = R_{\theta,\mu}^{-1} N_\theta \tag{2.22}$$

from (2.21). Indeed, we obtain

$$\begin{aligned} N_{\theta-1/\mu} &= I_M + \Big(\theta - \frac{1}{\mu}\Big)\delta Q_{\theta-1/\mu}^{-1}\delta^* = I_M + \Big(\theta - \frac{1}{\mu}\Big)\delta Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1}\delta^* \\ &= I_M + \Big(\theta - \frac{1}{\mu}\Big)\mu(I_M - R_{\theta,\mu}^{-1}) = \mu\theta I_M + (1-\theta\mu)R_{\theta,\mu}^{-1} = R_{\theta,\mu}^{-1}N_\theta \end{aligned}$$

since $I_M = R_{\theta,\mu}^{-1} + \frac{1}{\mu}\, R_{\theta,\mu}^{-1}\delta Q_\theta^{-1}\delta^*$ and $N_\theta = (1-\theta\mu)I_M + \theta\mu R_{\theta.\mu}$.

Now let us show (2.18).

$$\begin{aligned} -\frac{1}{\mu}\, Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta Q_\theta^{-1} + Q_\theta^{-1} &= -\frac{1}{\mu}\,(\delta^*\delta)^{-1}\delta^*\delta Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta Q_\theta^{-1} + Q_\theta^{-1} \\ &= -(\delta^*\delta)^{-1}\delta^*(I_M - R_{\theta,\mu}^{-1})\delta Q_\theta^{-1} + Q_\theta^{-1} = (\delta^*\delta)^{-1}\delta^* R_{\theta,\mu}^{-1}\delta Q_\theta^{-1} \\ &= Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1} = Q_{\theta-1/\mu}^{-1}. \end{aligned}$$

Next we shall see (2.19).

$$-\frac{1}{\theta\mu}\, N_\theta R_{\theta,\mu}^{-1} N_\theta + \theta\delta Q_\theta^{-1}\delta^* = -\frac{1}{\theta\mu}\, N_\theta R_{\theta,\mu}^{-1} N_\theta + N_\theta - I_M = \frac{\theta\mu - 1}{\theta\mu}\, N_\theta R_{\theta,\mu}^{-1} - I_M$$

since $R_{\theta,\mu}^{-1}N_\theta = (1-\theta\mu)R_{\theta,\mu}^{-1} + \theta\mu I_M$. On the other hand

$$\begin{aligned} \frac{(1-\theta\mu)^2}{\theta\mu^2}\,\delta Q_{\theta-1/\mu}^{-1}\delta^* - \frac{1}{\theta\mu}\, I_M + I_M &= \frac{(1-\theta\mu)^2}{\theta\mu}\,\delta Q_{\theta-1/\mu}^{-1}\delta^* + \frac{\theta\mu-1}{\theta\mu}\, I_M \\ &= \frac{\theta\mu-1}{\theta\mu}\Big\{I_M + \Big(\theta - \frac{1}{\mu}\Big)\delta Q_{\theta-1/\mu}^{-1}\delta^*\Big\} = \frac{\theta\mu-1}{\theta\mu}\, N_{\theta-1/\mu} \end{aligned}$$

and thus, (2.22) implies (2.19).

Finally we shall see (2.20).

$$-\frac{1}{\mu}\, Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1} N_\theta + \theta Q_\theta^{-1}\delta^* = -\frac{1}{\mu}\, Q_\theta^{-1}\delta^*(R_{\theta,\mu}^{-1}N_\theta - \theta\mu I_M) = -\frac{1-\theta\mu}{\mu}\, Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}$$

while

$$Q_{\theta-1/\mu}^{-1}\delta^* = Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1}\delta^* = Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}$$

since $\delta(\delta^*\delta)^{-1}\delta^*$ and $R_{\theta,\mu}$ are commutative, and so are $\delta(\delta^*\delta)^{-1}\delta^*$ and $R_{\theta,\mu}^{-1}$. Hence, (2.20) is seen to hold. ∎

Owing to Lemma 2.1, $H_-(x,p) = H_+(x,p)$ and the Isaacs equation (2.13) can be written as

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2 u] + \Big\{\beta + \Big(\theta - 1/\mu\Big)\lambda\delta Q_{\theta-1/\mu}^{-1} g\Big\}^* Du \\ &\qquad + \frac{\theta\mu-1}{2\theta\mu}\,(Du)^*\lambda\Big(I + \Big(\theta - \frac{1}{\mu}\Big)\delta Q_{\theta-1/\mu}^{-1}\delta^*\Big)\lambda^* Du + \frac{\theta}{2}\, g^* Q_{\theta-1/\mu}^{-1} g + \theta U = 0 \\ &u(T,x) = 0. \end{aligned} \tag{2.23}$$

Thus, we see that for a given solution u to lower Isaacs equation (2.13), $\check{\zeta}(t,x)$ and $\check{h}(t,x,\zeta)$ defined by

$$\check{\zeta}(t,x,h) = -\frac{1}{\theta\mu}\,(\lambda^* Du + \theta\delta h)$$
$$\check{h}(t,x) = -\frac{1}{\theta\mu}\,Q_{\theta-1/\mu}^{-1}((1-\theta\mu)\delta^*\lambda^* Du - \theta\mu g)$$

satisfy

$$\check{h}(t,x) = \arg\min_{h\in R^m} \Lambda(x, Du(t,x), \check{\zeta}(t,x,h), h)$$
$$\check{\zeta}(t,x,h) = \arg\max_{\zeta\in R^M} \Lambda(x, Du(t,x), \zeta, h)$$

and

$$H_+(x,Du) = \inf_{h\in R^m}\sup_{\zeta\in R^M} \Lambda(x, Du(t,x), \zeta, h)$$
$$= \Lambda\bigl(x, Du(t,x), \check{\zeta}(t,x,\check{h}(t,x)), \check{h}(t,x)\bigr).$$

Then we can see that equation (2.23) has a sufficiently smooth solution satisfying nice gradient estimates, similarly to Theorem 2.1 in [19]. Indeed, we have the following proposition (cf. [3], [17]).

Proposition 2.1. *Under the assumptions* (2.1)–(2.4) *the Isaacs equation* (2.23) *has a solution such that*

$$v(t,x) \le K_0,$$
$$v,\ \frac{\partial v}{\partial t},\ \frac{\partial v}{\partial x_i},\ \frac{\partial^2 v}{\partial x_i\partial x_j} \in L^p(0,T;L^p_{\mathrm{loc}}(R^n)) \quad \forall p>1,$$
$$\frac{\partial v}{\partial t} \ge -C,$$
$$\frac{\partial^2 v}{\partial^2 t},\ \frac{\partial^2 v}{\partial x_i\partial t},\ \frac{\partial^3 v}{\partial x_i\partial x_j\partial x_k},\ \frac{\partial^3 v}{\partial x_i\partial x_j\partial t} \in L^p(0,T;L^p_{\mathrm{loc}}(R^n)) \quad \forall p>1,$$
$$|Dv|^2 + \frac{(c_0-\theta)(1+c)}{c_0c_2}\Bigl(\frac{\partial v}{\partial t}+C\Bigr) \le c'\bigl(|DN_{\theta,\mu}|^2_{2r} + |N_{\theta,\mu}|^2_{2r} + |D(\lambda\lambda^*)|^2_{2r}$$
$$+ |D\beta_{\theta,\mu}|_{2r} + |\beta_{\theta,\mu}|^2_{2r} + |\theta U|_{2r} + |\theta DU|_{2r} + |g^*Q_{\theta,\mu}g|_{2r} + |D(g^*Q_{\theta,\mu}g)|_{2r} + 1\bigr),$$
$$x\in B_r,\quad t\in[0,T),$$

where

$$\beta_{\theta,\mu} = \beta + \Bigl(\theta - \frac{1}{\mu}\Bigr)\lambda\delta Q_{\theta-1/\mu}^{-1}g,\quad N_{\theta,\mu} = \frac{\theta\mu-1}{2\theta\mu}\,N_{\theta-1/\mu},\quad Q_{\theta,\mu} = \frac{\theta}{2}\,Q_{\theta-1/\mu}^{-1},$$

$B_r = \{x : |x| < r\}$ *and* $c>0$ *is an arbitrary positive constant,* c' *is a positive constant depending on* c_0, c_2, c, C, θ *and* n *but not on* r, *and* $-C$ *is the lower bound of* U.

Here, we note that $\frac{\partial v}{\partial t} \ge -C$ because of assumption (2.1) and the gradient estimates have a minor difference from Theorem 2.1 in [20].

Let us set

$$J(\zeta, h(\zeta)) := \log E^{\zeta,h(\zeta)}\Bigl[\exp\Bigl(\theta\int_0^T \eta(X_s, h(s,X_s,\zeta_s),\zeta_s)\,ds\Bigr)\Bigr],$$

where

$$h(\zeta) = h(t,x,\zeta), \quad \zeta \in \mathcal{Z}.$$

Then we can see that for the solution $u(t,x;T)$ to (2.23),

$$u(0,x;T) = J(\bar{\zeta}, \bar{h}(\bar{\zeta})),$$

where

$$\bar{\zeta} = \bar{\zeta}(t,X_t), \quad \bar{h}(\bar{\zeta}) = \bar{h}(t,X_t,\bar{\zeta}(t,X_t)). \tag{2.24}$$

Further, $(\bar{\zeta}, \bar{h}(\bar{\zeta}))$ turns out to be a saddle point of the game and attain the value of the game. Indeed, we have the following proposition.

Proposition 2.2. *Let $u(t,x;T)$ be a solution to* (2.23). *Then, under the assumptions* (2.1)–(2.4), *the pair of the strategies $\bar{\zeta} \in \mathcal{Z}$ and $\bar{h}(\bar{\zeta}) \in \Delta_{\mathcal{H}}$ defined by* (2.24) *is the saddle point, namely,*

$$J(\zeta, \bar{h}(\zeta)) \le J(\bar{\zeta}, \bar{h}(\bar{\zeta})) \le J(\bar{\zeta}, h(\bar{\zeta})), \tag{2.25}$$

and attains the value of the game:

$$u(0,x;T) = J(\bar{\zeta}, \bar{h}(\bar{\zeta})) = u_*(0,x;T).$$

To prove this proposition, the following lemma is useful.

Lemma 2.2. *Let Y_t be a solution to the stochastic differential equation*

$$dY_t = \sigma(t,Y_t)\,dW_t + \mu(t,Y_t)\,dt, \quad Y_0 = x,$$

where σ, μ are locally Lipschitz continuous functions such that

$$|r(t,x)| \le K(1+|x|) \tag{2.26}$$

for $r = \sigma, \mu$ and $\psi(t,x)$ be a continuous function satisfying (2.26). *We further assume that σ is bounded. Then, setting*

$$\rho_t := \exp\Big(\int_0^t \psi(s,Y_s)^*\,dW_s - \frac{1}{2}\int_0^t |\psi(s,X_s)|^2\,ds\Big)$$

we have

$$E[\rho_t] = 1, \quad \forall t.$$

The proof of this lemma is similar to that of Lemma 4.1.1 in [1].

Proof of Proposition 2.2. Let us first show that $\bar{\zeta}_t := \bar{\zeta}(t,X_t) \in \mathcal{Z}$ and that $\bar{h}_t := \bar{h}(t,X_t,\bar{\zeta}_t) \in \Delta_{\mathcal{H}}$. For that we shall show that (2.26) holds for each of $r(t,x) = \bar{\zeta}(t,x)$, $\bar{h}(t,x,\bar{\zeta}(t,x))$ in light of the gradient estimates given in Proposition 2.1. Indeed, it follows that

$$-\frac{c_0}{\theta\mu(c_0-\mu)}\,I_M \le -\frac{1}{\theta\mu}\,R_{\theta,\mu}^{-1}N_\theta \le -\frac{1}{\theta\mu}\,I_M$$

from (2.7) and (2.22), and that

$$\frac{1}{c_1-\theta+1/\mu}\,(\delta^*\delta)^{-1} \le Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1} \le \frac{1}{c_0-\theta+1/\mu}\,(\delta^*\delta)^{-1}$$

from (2.6) and (2.21). Therefore, since

$$R_{\theta,\mu}^{-1}\delta Q_\theta^{-1} = R_{\theta,\mu}^{-1}\delta Q_\theta^{-1}\delta^*\delta(\delta^*\delta)^{-1} = \delta Q_\theta^{-1}\delta^* R_{\theta,\mu}^{-1}\delta(\delta^*\delta)^{-1},$$

$\bar\zeta(t,x)$ is seen to satisfy (2.26). Similarly, we can see that $|\bar h(t,x,\zeta)| \le C(1+|x|+|\zeta|)$ and thus (2.26) holds for $\bar h(t,x,\bar\zeta(t,x))$. Therefore, owing to Lemma 2.2 we see that $\bar\zeta_t = \bar\zeta(t,X_t) \in \mathcal{Z}$ and that $\bar h_t = \bar h(t,X_t,\bar\zeta_t) \in \Delta_{\mathcal{H}}$.

Next, we shall show that

$$J(\bar\zeta,\bar h) = \inf_{h\in\Delta_{\mathcal{H}}} J(\bar\zeta, h(\bar\zeta)). \tag{2.27}$$

Let X_t be a solution to

$$dX_t = \left\{\beta(X_t) + \lambda(X_t)\bar\zeta(t,X_t) + \theta\delta(X_t)h(\bar\zeta)_t\right\}dt + \lambda\, dW_t^{\bar\zeta,h}, \quad X_0 = x,$$

for a given $h\in\Delta_{\mathcal{H}}$, where $h(\bar\zeta)_t = h(t,X_t,\bar\zeta(t,X_t))$. Noting that the H–J–B equation for the control problem (2.27) turns out to be

$$\begin{aligned}
&\frac{\partial u}{\partial t} + \frac12\operatorname{tr}[\lambda\lambda^* D^2u] + \beta^* Du + \frac12 (Du)^*\lambda\lambda^* Du\\
&\qquad + \inf_{h\in R^m}\left[\{\bar\zeta(t,x)\lambda^*(x) + \theta h^*\delta^*(x)\}Du + \theta\eta(x,h,\bar\zeta)\right] = 0\\
&u(T,x) = 0.
\end{aligned}$$

Thus, through the standard argument using Itô's formula, we see that

$$\begin{aligned}
&E^{\bar\zeta,\bar h}\Big[\exp\Big(\theta\int_0^T \eta(X_s,\bar h_s,\bar\zeta_s)\,ds\Big)\Big]\\
&\quad = E^{\bar\zeta}\Big[\exp\Big(\theta\int_0^T \eta(X_s,\bar h_s,\bar\zeta_s)\,ds + \theta\int_0^T [\delta\bar h_s]^*\,dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^T |\delta\bar h_s|^2\,ds\Big)\Big]\\
&\quad = e^{u(0,x)}.
\end{aligned}\tag{2.28}$$

Indeed, by Itô's formula,

$$\begin{aligned}
u(T,X_T) - u(0,X_0) &= \int_0^T \Big(\frac{\partial u}{\partial t} + L^{\bar\zeta}u\Big)(s,X_s)\,ds + \int_0^T (Du)^*(s,X_s)\lambda(X_s)\,dW_s^{\bar\zeta}\\
&= -\theta\int_0^T [\lambda\delta\bar h_s]^* Du(s,X_s)\,ds - \frac12\int_0^T (Du)^*\lambda\lambda^* Du(s,X_s)\,ds\\
&\qquad - \int_0^T \eta(X_s,\bar h_s,\bar\zeta_s)\,ds + \int_0^T (Du)^*\lambda(s,X_s)\,dW_s^{\bar\zeta}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\exp\Big(\theta\int_0^T \eta(X_s,\bar h_s,\zeta_s)\,ds + \theta\int_0^T [\delta\bar h_s]^*\,dW_s^{\zeta} - \frac{\theta^2}{2}\int_0^T |\delta\bar h_s|^2\,ds\Big)\\
&\quad = \exp\Big(u(0,x) + \int_0^T [\theta\delta\bar h_s + \lambda^* Du(s,X_s)]^*\,dW_s^{\bar\zeta} - \frac12\int_0^T |\theta\delta\bar h_s + \lambda^* Du(s,X_s)|^2\,ds\Big).
\end{aligned}$$

and

$$\begin{aligned}
&E^{\bar\zeta}\Big[\exp\Big(\int_0^T [\theta\delta\bar h_s + \lambda^* Du(s,X_s)]^*\,dW_s^{\bar\zeta} - \frac12\int_0^T \big|\theta\delta\bar h_s + \lambda^* Du(s,X_s)\big|^2\,ds\Big)\Big]\\
&= E\Big[\exp\Big(\int_0^T [\bar\zeta_s + \theta\delta\bar h_s + \lambda^* Du(s,X_s)]^*\,dW_s - \frac12\int_0^T \big|\bar\zeta_s + \theta\delta\bar h_s + \lambda^* Du(s,X_s)\big|^2\,ds\Big)\Big] = 1
\end{aligned}$$

follows from the above lemma. Hence we see that (2.28) holds.

Next, we shall prove that

$$e^{u(0,x)} \le E^{\bar\zeta,h(\bar\zeta)}\Big[\exp\Big(\theta\int_0^T \eta(X_s,h(\bar\zeta)_s,\bar\zeta_s)\,ds\Big)\Big] \tag{2.29}$$

for each $h\in\Delta_{\mathcal H}$. In a similar manner to the above we have

$$\begin{aligned}
u(T,X_T)-u(0,X_0) &= \int_0^T\Big(\frac{\partial u}{\partial t}+L^{\bar\zeta}u\Big)(s,X_s)\,ds+\int_0^T (Du)^*(s,X_s)\,dW_s^{\bar\zeta}\\
&\ge -\theta\int_0^T[\lambda\delta h(\bar\zeta)_s]^*Du(s,X_s)\,ds-\frac12\int_0^T (Du)^*\lambda\lambda^*Du(s,X_s)\,ds\\
&\qquad -\int_0^T\theta\eta(X_s,h(\bar\zeta)_s,\bar\zeta_s)\,ds+\int_0^T (Du)^*\lambda(s,X_s)\,dW_s^{\bar\zeta},
\end{aligned}$$

where $h(\bar\zeta)_s=h(s,X_s,\bar\zeta_s)$. Thus,

$$\begin{aligned}
&\exp\Big(\theta\int_0^T\eta(X_s,h(\bar\zeta)_s,\bar\zeta_s)\,ds+\theta\int_0^T[\delta\bar h_s]^*\,dW_s^{\bar\zeta}-\frac{\theta^2}{2}\int_0^T|\delta h(\bar\zeta)_s|^2\,ds\Big)\\
&\ge\exp\Big(u(0,x)+\int_0^T[\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)]^*\,dW_s^{\bar\zeta}-\frac12\int_0^T\big|\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)\big|^2\,ds\Big)
\end{aligned}$$

Since we have

$$\begin{aligned}
&E^{\bar\zeta}\Big[\exp\Big(\int_0^T[\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)]^*\,dW_s^{\bar\zeta}-\frac12\int_0^T\big|\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)\big|^2\,ds\Big)\Big]\\
&\qquad=E\Big[\exp\Big(\int_0^T\big[\bar\zeta_s+\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)\big]^*\,dW_s\\
&\qquad\qquad-\frac12\int_0^T\big|\bar\zeta_s+\theta\delta h(\bar\zeta)_s+\lambda^*Du(s,X_s)\big|^2\,ds\Big)\Big]=1
\end{aligned}$$

because of the above lemma, we obtain (2.29).

Finally we prove that

$$J(\bar\zeta,\bar h(\bar\zeta))\ge J(\zeta,\bar h(\zeta)),\quad \forall\zeta\in\mathcal Z. \tag{2.30}$$

For that it suffices to show that

$$E^{\zeta,\bar h(\zeta)}\Big[\exp\Big(\theta\int_0^T\eta(X_s,\bar h(\zeta)_s,\zeta_s)\,ds\Big)\Big]\le e^{u(0,x)} \tag{2.31}$$

for each $\zeta\in\mathcal Z$, where X_t is a solution to

$$dX_t=\big\{\beta(X_t)+\lambda(X_t)\zeta_t+\theta\delta(X_t)\bar h(t,X_t,\zeta_t)\big\}\,dt+\lambda\,dW_t^{\zeta,\bar h},\quad X_0=x.$$

By looking at (2.15) we see that

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac12\operatorname{tr}[\lambda\lambda^*D^2u]+\beta^*Du+\frac12(Du)^*\lambda\lambda^*Du\\
&\qquad+\big\{\zeta\lambda^*(x)+\theta\bar h(t,x,\zeta)^*\delta^*(x)\big\}Du+\eta(x,\bar h(t,x,\zeta),\zeta)\le0,\quad\forall\zeta\in R^M,\\
&u(T,x)=0,
\end{aligned}$$

and hence we obtain (2.31) similarly to the above. ∎

We further have the following proposition.

PROPOSITION 2.3. $\bar{h}(t, X_t, \bar{\zeta}(t, X_t)) = \check{h}(t, X_t)$.

To see the convexity of the value of the game with respect to θ, rewriting the Isaacs equation as follows is useful.

$$\begin{aligned}&\frac{\partial u}{\partial t} + \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2 u] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* Du \\ &\quad - \frac{1-\theta\mu}{2\theta\mu}\Big\{\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\,\delta(\delta^*\delta)^{-1}g\Big\}^* N_{\theta-1/\mu}\Big\{\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\,\delta(\delta^*\delta)^{-1}g\Big\} \\ &\quad + \frac{\theta\mu}{2(1-\theta\mu)}\, g^*(\delta^*\delta)^{-1}g + \theta U = 0 \\ &u(T, x) = 0,\end{aligned}$$

which can be regarded as

$$\begin{aligned}&\frac{\partial u}{\partial t} + \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2 u] + (\beta - \lambda\delta(\delta^*\delta)^{-1}g)^* Du \\ &\quad + \sup_{z\in R^M}\Big\{\frac{\theta\mu}{2(1-\theta\mu)}\, z^* N_{\theta-1/\mu}^{-1} z + z^*(\lambda^* Du - \frac{\theta\mu}{1-\theta\mu}\,\delta(\delta^*\delta)^{-1}g)\Big\} \\ &\quad + \frac{\theta\mu}{2(1-\theta\mu)}\, g^*(\delta^*\delta)^{-1}g + \theta U = 0 \\ &u(T, x) = 0.\end{aligned} \tag{2.32}$$

Equation (2.32) is seen to be the H–J–B equation of the stochastic control problem

$$\tilde{u}_*(0, x; T) = \sup_{Z_\cdot \in \mathbf{Z}} E\Big[\int_0^T \Phi(X_s, Z_s)\, ds\Big], \tag{2.33}$$

$$\begin{aligned}\Phi(x, z; \theta) &= \frac{\theta\mu}{2(1-\theta)}\, z^* N_{\theta-1/\mu}^{-1} z - \frac{\theta\mu}{1-\theta\mu}\, z^*\delta(\delta^*\delta)^{-1}g \\ &\quad + \frac{\theta\mu}{2(1-\theta\mu)}\, g^*(\delta^*\delta)^{-1}g + \theta U,\end{aligned}$$

with the controlled process X_t governed by the stochastic differential equation

$$dX_t = \lambda(X_t)\, dW_t + \{G(X_t) + \lambda(X_t)Z_t\}\, dt, \quad X_0 = x, \tag{2.34}$$

where

$$G(x) = \beta - \lambda\delta(\delta^*\delta)^{-1}g.$$

Here, $\mathbf{Z}$ is the totality of R^M-valued, progressively measurable processes Z_t such that $E[\int_0^T |Z_s|^2\, ds] < \infty$.

LEMMA 2.3. *The above defined $\Phi(x, z) = \Phi(x, z; \theta)$ is a convex function of θ.*

Proof. Noting that

$$N_{\theta-1/\mu}^{-1} = R_{\theta,\mu} N_\theta^{-1} = \frac{I_M - (1-\theta\mu)N_\theta^{-1}}{\theta\mu}\,,$$

we see that

$$\begin{aligned}\Phi(x,z) &= \frac{1}{2(1-\theta\mu)}\, z^*z - \frac{1}{2}\, z^*N_\theta^{-1}(x)z - \frac{\theta\mu}{1-\theta\mu}\, z^*\delta(\delta^*\delta)^{-1}g(x)\\ &\quad + \frac{\theta\mu}{2(1-\theta\mu)}\, g^*(\delta^*\delta)^{-1}g(x) + \theta U(x)\\ &= \frac{1}{2(1-\theta\mu)}\,(z-\theta\mu\delta(\delta^*\delta)^{-1}g(x))^*(z-\theta\mu\delta(\delta^*\delta)^{-1}g(x))\\ &\quad - \frac{1}{2}\, z^*N_\theta^{-1}(x)z + \frac{\theta\mu}{2}\, g^*(\delta^*\delta)^{-1}g(x) + \theta U(x).\end{aligned} \tag{2.35}$$

Set

$$\varphi(x,z;\theta) := \frac{1}{2(1-\theta\mu)}\,(z-\theta\mu\delta(\delta^*\delta)^{-1}g)^*(z-\theta\mu\delta(\delta^*\delta)^{-1}g).$$

Then we can see by calculation that

$$\frac{\partial^2\varphi}{\partial\theta^2} = \frac{\mu^2}{(1-\theta\mu)^3}\,(z-\delta(\delta^*\delta)^{-1}g)^*(z-\delta(\delta^*\delta)^{-1}g) \geq 0. \tag{2.36}$$

Further, we have

$$\frac{\partial^2 N_\theta^{-1}}{\partial\theta^2} = 0. \tag{2.37}$$

Indeed, since $\frac{\partial Q_\theta}{\partial\theta} = -\delta^*\delta$ we have

$$\frac{\partial Q_\theta^{-1}}{\partial\theta} = -Q_\theta^{-1}\,\frac{\partial Q_\theta}{\partial\theta}\,Q_\theta^{-1} = Q_\theta^{-1}\delta^*\delta Q_\theta^{-1} \tag{2.38}$$

and thus,

$$\frac{\partial N_\theta}{\partial\theta} = \delta Q_\theta^{-1}\delta^* + \theta\delta Q_\theta^{-1}\delta^*\delta Q_\theta^{-1}\delta^* = \delta Q_\theta^{-1}\delta^* N_\theta = N_\theta\delta Q_\theta^{-1}\delta^*. \tag{2.39}$$

Therefore,

$$\frac{\partial^2 N_\theta}{\partial\theta^2} = \frac{\partial N_\theta}{\partial\theta}\,\delta Q_\theta^{-1}\delta^* + N_\theta\delta\,\frac{\partial Q_\theta^{-1}}{\partial\theta}\,\delta^* = 2N_\theta(\delta Q_\theta^{-1}\delta^*)^2.$$

Then

$$\frac{\partial N_\theta^{-1}}{\partial\theta} = -N_\theta^{-1}\,\frac{\partial N_\theta}{\partial\theta}\,N_\theta^{-1} = -\delta Q_\theta^{-1}\delta^* N_\theta^{-1} = -N_\theta^{-1}\delta Q_\theta^{-1}\delta^*$$

and so we see (2.37) as follows.

$$\begin{aligned}\frac{\partial^2 N_\theta^{-1}}{\partial\theta^2} &= -\frac{\partial N_\theta^{-1}}{\partial\theta}\,\frac{\partial N_\theta}{\partial\theta}\,N_\theta^{-1} - N_\theta^{-1}\,\frac{\partial^2 N_\theta}{\partial\theta^2}\,N_\theta^{-1} - N_\theta^{-1}\,\frac{\partial N_\theta}{\partial\theta}\,\frac{\partial N_\theta^{-1}}{\partial\theta}\\ &= N_\theta^{-1}(\delta Q_\theta^{-1}\delta^*)^2 - 2N_\theta^{-1}(\delta Q_\theta^{-1}\delta^*)^2 + N_\theta^{-1}(\delta Q_\theta^{-1}\delta^*)^2 = 0.\end{aligned}$$

Hence, convexity of Φ follows from (2.35), (2.36) and (2.37). ■

LEMMA 2.4. *Let assumptions* (2.1)–(2.4) *be satisfied and let* $u(t,x;T)$ *be a solution to H–J–B equation* (2.32). *Then* $u(0,x;T)$ *is a convex function of* θ.

Proof. Set

$$\hat{Z}(t,x) := \frac{1-\theta\mu}{\theta\mu}\,N_{\theta-1/\mu}(x)\Big(\lambda^*(x)Du(t,x) - \frac{\theta\mu}{1-\theta\mu}\,\delta(\delta^*\delta)^{-1}g(x)\Big),$$

and let $\hat{X}_t$ be a solution to (2.34) for $Z_t = \hat{Z}_t \equiv \hat{Z}(t, X_t)$. Then the verification theorem for problem (2.33) holds:

$$\tilde{u}_*(0,x;T) = E\Big[\int_0^T \Phi(\hat{X}_t, \hat{Z}_t)\,dt\Big] = u(0,x;T).$$

Therefore, since $\Phi(x,z)$ is a convex function of θ by Lemma 2.1 convexity of $u(0,x;T)$ follows. ∎

3. H–J–B equations of ergodic type.

Set $\bar{u} = -u$. Then

$$\begin{aligned}&\frac{\partial \bar{u}}{\partial t} + \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2\bar{u}] + \beta^*_{\theta,\mu} D\bar{u} - \frac{1}{2}\,(D\bar{u})^*\lambda N_{\theta,\mu}\lambda^* D\bar{u} - \frac{1}{2}\,g^* Q_{\theta,\mu} g - \theta U = 0,\\ &\bar{u}(T,x) = 0.\end{aligned} \tag{3.1}$$

Now let us consider the infinite horizon counterpart of H–J–B equation (3.1) (or (2.23)), which is called H–J–B equation of ergodic type. We have the following according to [2] and Proposition 3.2 in [20].

PROPOSITION 3.1.

i) *Assume that*

$$\lim_{r\to\infty} \inf_{|x|\ge r} \{g^*(\delta^*\delta)^{-1} g(x) + U(x)\} = \infty \tag{3.2}$$

besides assumptions (2.1)–(2.4). *Then we have a solution* $(\bar{\chi}(\theta), \bar{w})$ *of*

$$\bar{\chi}(\theta) = \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2\bar{w}] + \beta^*_{\theta,\mu} D\bar{w} - \frac{1}{2}\,(D\bar{w})^*\lambda N_{\theta,\mu}\lambda^* D\bar{w} - \frac{1}{2}\,g^* Q^{-1}_{\theta,\mu} g - \theta U, \tag{3.3}$$

such that $\bar{w}(x)$ *is bounded below. Moreover, such a solution* $(\bar{\chi}, \bar{w})$ *is unique up to additive constants with respect to* $\bar{w}$ *and* $\bar{w}(x) \to \infty$ *as* $|x| \to \infty$. *Further, the solution satisfies the following estimate*

$$|D\bar{w}(x)|^2 \le C_w|x|^2 + C'_w. \tag{3.4}$$

If we assume the stronger assumption than (3.2),

$$c_4|x|^2 - c_5 \le \frac{1}{c_1 - \theta}\, g^*(\delta^*\delta)^{-1} g(x) + U(x), \tag{3.5}$$

then we have

$$c_w|x|^2 - c'_w \le \bar{w}(x), \quad c_w,\ c'_w > 0. \tag{3.6}$$

ii) *Assume that*

$$\beta(x)^* x \le -c_\beta|x|^2 + c'_\beta, \quad c_\beta > 0,\ c'_\beta > 0 \tag{3.7}$$

besides assumptions (2.1)–(2.4). *Then there exists a positive constant* $b_* > 0$ *such that* $\psi_{b^*}(x) := -b_*|x|^2$ *satisfies*

$$F(\psi_{b^*})(x) \to \infty, \ \text{as } |x| \to \infty,$$

where

$$F(\psi) = \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2\psi] + \beta^*_{\theta,\mu} D\psi - \frac{1}{2}\,(D\psi)^*\lambda N_{\theta,\mu}\lambda^* D\psi - \frac{1}{2}\,g^* Q^{-1}_{\theta,\mu} g - \theta U.$$

Moreover, we have a solution $(\bar{\chi}(\theta), \bar{w})$ *to* (3.3) *such that* $\bar{w} - \psi_b(x)$ *with* $0 < b < b_*$ *is bounded below.*

Here we note that c_w in (3.6) is a positive constant such that

$$2c_3c_w^2 + C_\beta c_w < \frac{-c_4\theta}{4}$$

and C_β is the one such that $|\beta_{\theta,\mu}(x)| \le C_\beta|x| + C$ (3.4) and (3.6) can be seen in [20]. Existence and uniqueness of the solution to (3.3) are due to [2].

In the statement ii), that $F(\psi_{b^*})(x) \to \infty$, $|x| \to \infty$, can be seen by looking at

$$\begin{aligned}
&\Big[\Big(\theta - \frac{1}{\mu}\Big)\lambda\delta Q_{\theta-1/\mu}^{-1} g\Big]^* D\psi_b - \frac{\theta}{2}\, g^* Q_{\theta-1/\mu}^{-1} g \\
&= -\frac{\theta}{2}\Big\{g - \Big(1 - \frac{1}{\theta\mu}\Big)\delta^*\lambda^* D\psi_b\Big\}^* Q_{\theta-1/\mu}^{-1}\Big\{g - \Big(1 - \frac{1}{\theta\mu}\Big)\delta^*\lambda^* D\psi_b\Big\} \\
&\quad + \frac{\theta}{2}\Big(1 - \frac{1}{\theta\mu}\Big)^2 (D\psi_b)^*\lambda\delta Q_{\theta-1/\mu}^{-1}\delta^*\lambda^* D\psi_b
\end{aligned}$$

since $\beta_{\theta,\mu} = \beta + (\theta - \frac{1}{\mu})\lambda\delta Q_{\theta-1/\mu}^{-1} g$ and $Q_{\theta,\mu} = \frac{\theta}{2}\, Q_{\theta-1/\mu}^{-1}$. Indeed, we see that

$$\exists c_*, c'_* > 0 \;\; F(\psi_{b^*}) \ge c_*|x|^2 - c'_*,$$

by taking sufficiently small b^*. Thus, when considering $\bar{w} - \psi_b$ in place of $\bar{w}$ the equation (3.3) can be written as

$$\begin{aligned}
\bar{\chi}(\theta) = \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2(\bar{w} - \psi_b)] + (\beta_{\theta,\mu} - \lambda N_{\theta,\mu}\lambda^* D\psi_b)^* D(\bar{w} - \psi_b) \\
- \frac{1}{2}\, D(\bar{w} - \psi_b)^*\lambda N_{\theta,\mu}\lambda^* D(\bar{w} - \psi_b) + F(\psi_b),
\end{aligned}$$

and then the case of ii) is reduced to i). Indeed, this equation has a solution $(\bar{\chi}(\theta), \bar{w} - \psi_b)$ because of Proposition 3.1 i), while it means that $(\bar{\chi}, \bar{w})$ is the solution to (3.3).

In what follows we shall proceed assuming the assumptions of Proposition 3.1 i) with (3.5). We can develop parallel arguments as well in the case of ii) of the proposition.

Let us write H–J–B equation (3.1) as

$$\begin{aligned}
&\frac{\partial \bar{u}}{\partial t} + \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2\bar{u}] + H(x, D\bar{u}) = 0, \\
&\bar{u}(T, x) = 0,
\end{aligned} \tag{3.1$'$}$$

by introducing

$$H(x,p) = \beta_{\theta,\mu}^*(x)p - \frac{1}{2}\, p^*\lambda N_\theta\lambda^*(x)p - \frac{1}{2}\, g^* Q_{\theta,\mu} g(x) - \theta U(x),$$

which is concave with respect to p. Here we note that the optimal diffusion process for stochastic control problem (2.25) turns out to be governed by stochastic differential equation

$$d\bar{Y}_t = \lambda(\bar{Y}_t)\, dW_t + D_p H(\bar{Y}_t, \bar{u}(t, \bar{Y}_t))\, dt, \quad \bar{Y}_0 = x,$$

where

$$D_p H(x,p) = \frac{\partial H(x,p)}{\partial p} = \beta_{\theta,\mu} + \lambda N_{\theta,\mu}\lambda^* p.$$

Now let us extend the solution to (3.1$'$) to $(-\infty, 0]$. It is indeed possible because the coefficients of the H–J–B equation do not depend on t and for each $-s < 0$, the solution $\bar{u}(-s, x; T)$ is identical to $\bar{u}(0, x; T + s)$, which is the solution $\bar{u}(\cdot, \cdot; T + s)$ of (3.1$'$) with the terminal condition $\bar{u}(T+s, x; T+s) = 0$. Thus, we have $\bar{u}(-s, x; T)$ defined on $-T \le s < \infty$. In what follows we specify the solution $\bar{w}(x)$ of H–J–B equation (3.3) such that $\bar{w}(x) \ge 0$ by suitably taking an additive constant. Then from the maximum principle it follows that

$$\bar{u}(-s, x; T) \le \bar{w}(x) + \bar{\chi}(T+s), \quad -T \le s < \infty, \tag{3.8}$$

since $\bar{w}(x) + \bar{\chi}(T+s)$ is a super-solution to (3.1). Now we have the following proposition.

Proposition 3.2. *Under the assumptions of Proposition* 3.1 i) *with* (3.5)*,*

$$\bar{u}(-s, x; T) - (\bar{w}(x) + \bar{\chi}(T+s)) \ge -c_E|x|^2 - c'_E \tag{3.9}$$

where c_E, and c'_E are positive constants independent of s.

Proof. For given $-s \le 0$, let $\bar{Y}_t$ be a solution to

$$d\bar{Y}_t = \lambda(\bar{Y}_t)\, dW_t + D_pH(\bar{Y}_t, \bar{u}(-s+t, \bar{Y}_t))\, dt, \quad \bar{Y}_0 = x. \tag{3.10}$$

Then we see that

$$\bar{u}(-s, x; T) = \inf_{z_\cdot} E\Big[\int_0^{T+s} -\Phi(Y_s, z_s)\, ds\Big] = E\Big[\int_0^{T+s} -\Phi(\bar{Y}_t, \hat{z}(-s+t, \bar{Y}_t))\, dt\Big]$$

and also

$$\bar{w}(x) + \bar{\chi}(T+s) \le E\Big[\int_0^{T+s} -\Phi(Y_t, z_t)\, dt + \bar{w}(Y_{T+s})\Big]$$

for each control z_t. Therefore,

$$\bar{u}(-s, x; T) - (\bar{w}(x) + \bar{\chi}(T+s)) \ge -E[\bar{w}(\bar{Y}_{T+s})] \ge -c_w E[|\bar{Y}_{T+s}|^2] + c'_w.$$

We conclude the present proposition since we see that $E[|\bar{Y}_{T+s}|^2] \le C|x|^2 + C'$ for some positive constants C and C' independent of s from the following lemma and the estimate $|\bar{w}(x)|^2 \le C|x|^2 + C'$ obtained from (3.4).

Lemma 3.1. *Let the assumptions of Proposition* 3.2 *be satisfied and set $\psi(x) = -c_*|x|^2$ with a positive constant c_* such that*

$$c_\beta c_* + \frac{c_3}{2}\, c_*^2 \le -c_4\theta. \tag{3.11}$$

Then

$$E[c_*|\bar{Y}_{T+s}|^2] \le e^{-\alpha(T+s)}(\bar{w}(x) - \psi(x)) + \frac{C}{\alpha}\,(1 - e^{-\alpha(T+s)}), \quad -T \le s < \infty, \tag{3.12}$$

for sufficiently small $\alpha > 0$ and a positive constant C.

Proof. First note that there exist positive constants c_ψ and c'_ψ such that

$$\frac{1}{2}\,\mathrm{tr}[\lambda\lambda^* D^2\psi] + H(x, D\psi) \ge c_\psi|x|^2 - c'_\psi \tag{3.13}$$

because of the estimates $|\beta^*_{\theta,\mu} D\psi| \le c_\beta c_*|x|^2 + C$, $\frac{1}{2}\,(D\psi)^*\lambda N_{\theta,\mu}\lambda^* D\psi \le \frac{c_3}{2}\, c_*^2$, and assumption (3.5). Further, since $H(x,p)$ is concave in p we have

$$H(x, D\psi) - H(x, D\bar{u}) \le D_pH(x, D\bar{u}) \cdot (D\psi - D\bar{u}).$$

Therefore, by setting

$$\hat{u}(-s+t, x; T) = \bar{u}(-s+t, x; T) - \{\psi(x) + \bar{\chi}(T-t+s)\}, \quad 0 \le t \le T+s,$$

and applying Itô's formula to $e^{\alpha t}\hat{u}(-s+t,x;T)$, we see that

$$
\begin{aligned}
&e^{\alpha t}\hat{u}(-s+t,\bar{Y}_t,;T)-\hat{u}(-s,x;T)\\
&\quad=\int_0^t e^{\alpha r}\{L\hat{u}(-s+r,\bar{Y}_r)+D_pH(\bar{Y}_r,D\bar{u}(-s+r,\bar{Y}_r))^*D\hat{u}(-s+r,\bar{Y}_r)\}\,dr\\
&\qquad\qquad\qquad\qquad+\alpha\int_0^t e^{\alpha r}\hat{u}(-s+r,\bar{Y}_r)\,dr+\text{local mart.},
\end{aligned}
$$

where $Lu:=\frac{\partial u}{\partial t}+\frac{1}{2}\operatorname{tr}[\lambda\lambda^*D^2u]$. Thus, by using the stopping time arguments, we obtain

$$
\begin{aligned}
&e^{\alpha(T+s)}E[\hat{u}(T,X_{T+s};T)]-\hat{u}(-s,x;T)\\
&\quad\le\int_0^{T+s}e^{\alpha r}E\big[L\bar{u}(-s+r,\bar{Y}_r)+H(\bar{Y}_r,D\bar{u}(-s+r,\bar{Y}_r))\big]\,dr\\
&\qquad-\int_0^{T+s}e^{\alpha r}E\big[\tfrac{1}{2}\operatorname{tr}[\lambda\lambda^*D^2\psi(\bar{Y}_r)]+H(\bar{Y}_r,D\psi(\bar{Y}_r))\big]\,dr\\
&\qquad+\alpha\int_0^{T+s}e^{\alpha r}E[\hat{u}(-s+r,\bar{Y}_r)]\,dr\\
&\quad\le\int_0^{T+s}e^{\alpha r}E\big[\alpha(\bar{w}-\psi)(\bar{Y}_r)-c_\psi|\bar{Y}_r|^2+c'_\psi\big]\,dr,
\end{aligned}
$$

from (3.13) and (3.8). Since $\alpha(\bar{w}-\psi)(x)\le c_\psi|x|^2+C'$ for sufficiently small $\alpha>0$, we have (3.12) from (3.8). ∎

REMARK 3.1. Let $\bar{X}_t$ be the solution to

$$
d\bar{X}_t=\lambda(\bar{X}_t)\,dW_t+D_pH(\bar{X}_t,D\bar{w}(\bar{X}_t))\,dt,\quad \bar{X}_0=x,
$$

and set

$$
\begin{aligned}
L^{\bar{w}}u&:=\frac{1}{2}\operatorname{tr}[\lambda\lambda^*D^2u]+D_pH(x,D\bar{w})^*Du\\
&=\frac{1}{2}\operatorname{tr}[\lambda\lambda^*D^2u]+\beta^*_{\theta,\mu}Du-(D\bar{w})^*\lambda N_{\theta,\mu}\lambda^*Du.
\end{aligned}\tag{3.14}
$$

Then the diffusion process $\bar{X}_t$ is ergodic under the assumptions of Proposition 3.1 i). Indeed, we see that $\bar{w}(x)$, and also $K(x;\bar{w})$ defined by

$$
K(x;\bar{w}):=-L^{\bar{w}}\bar{w}(x)=\frac{1}{2}(D\bar{w})^*\lambda N_{\theta,\mu}\lambda^*D\bar{w}(x)-\frac{1}{2}g^*Q_{\theta,\mu}g(x)-\theta U(x)-\chi(\theta)\tag{3.15}
$$

goes to ∞ as $|x|\to\infty$ and thus the Hasminskii's conditions are satisfied.

On the other hand, we assume the conditions in Proposition 3.1 ii) and take up $\bar{w}-\psi_b$ in place of $\bar{w}$ and operate $L^{\bar{w}}$ to $\bar{w}-\psi_b$. Then we can see that

$$
\chi(\theta)=L^{\bar{w}}(\bar{w}-\psi_b)+\frac{1}{2}(D\bar{w}-D\psi_b)^*\lambda N^*_{\theta,\mu}D(\bar{w}-\psi_b)+F(\psi_b)
$$

and thus

$$
K(x;\bar{w}-\psi_b):=-L^{\bar{w}}(\bar{w}-\psi_b)=\frac{1}{2}(D\bar{w}-\psi_b)^*\lambda N_{\theta,\mu}\lambda^*D(\bar{w}-\psi_b)+F(\psi_b)-\chi(\theta)
$$

goes to ∞ as $|x|\to\infty$. Hence, we can see that $L^{\bar{w}}$ is ergodic under the assumptions in Proposition 3.1 ii).

Now set

$$\bar{v}(0,x;T) := \bar{u}(0,x;T) - \{\bar{w}(x) + \bar{\chi}T\}$$

for a specified solution $\bar{w}(x)$ to H–J–B equation (3.3) such that $\bar{w}(x) \geq 0$. Then we have the following theorem.

THEOREM 3.1. *Under the assumptions of Proposition* 3.2*, as* $T \to \infty$*,* $\bar{v}(0,x;T)$ *converges to a constant* $c_\infty \in R$ *uniformly on each compact set.*

Proof. Let us first note that $\bar{v}(0,x;T)$ is nonpositive and bounded below by $-c'_E - c_E|x|^2$ owing to (3.8) and (3.9). Thus, its bounds do not depend on T. Moreover, $|D\bar{v}(0,x;T)|^2 \leq C_R$, $x \in B_R$, where C_R is a positive constant depending only on R. Thus, we see that $\{\bar{u}(0,x;T)\}_{T\geq 1}$ is relatively compact in $C(R^N)$ and so, there exists a sequence T_j such that $\bar{v}(0,x;T_j)$ converges to a function $v_\infty \in C(R^N)$ as $T_j \to \infty$. Set

$$\mathcal{V} := \{v_\infty \in C(R^N) : \lim_{T_j\to\infty} \bar{v}(0,x;T_j) = v_\infty \text{ in } C(R^N), \text{ for some } \{T_j\}\}.$$

Then proving that $\mathcal{V} = \{c_\infty\}$ for some constant $c_\infty \in R$ makes the proof of the theorem complete. Apply Itô's formula to $\bar{v}(-s+t,x;T)$, $0 \leq t \leq T+s$, and we have

$$\begin{aligned}
&\bar{v}(-s+t,\bar{X}_t;T) - \bar{v}(-s,X_0;T) \\
&\quad = \int_0^t \{L\bar{v} + D_pH(\cdot,D\bar{w})^*D\bar{u}\}(-s+r,\bar{X}_r)\,dr + \int_0^t D\bar{v}(-s+r,\bar{X}_r)^*\lambda(\bar{X}_r)\,dW_r \\
&\quad = \int_0^t \frac{1}{2}\,(D\bar{v})^*\lambda N_{\theta,\mu}\lambda^* D\bar{v}(-s+r,\bar{X}_r)\,dr + \int_0^t D\bar{v}(-s+r,\bar{X}_r)^*\lambda(\bar{X}_r)\,dW_r \\
&\quad \equiv \int_0^t Q(\bar{X}_r, D\bar{v}(-s+r,\bar{X}_r))\,dr + \int_0^t D\bar{v}(-s+r,\bar{X}_r)^*\lambda(\bar{X}_r)\,dW_r
\end{aligned}$$

when noting that

$$H(x,D\bar{w}+p) - H(x,D\bar{w}) = D_pH(x,D\bar{w})^*p - \frac{1}{2}\,p^*\lambda N_{\theta,\mu}\lambda^*p.$$

Thus, setting $\gamma := \frac{-c_0}{\theta(1+\mu c_0-\mu\theta)}$,

$$\begin{aligned}
&\exp\Big(-\gamma\bar{v}(-s,x;T) - \gamma\int_0^t D\bar{v}(-s+r,\bar{X}_r)^*\lambda(\bar{X}_r)\,dW_r \\
&\qquad - \frac{\gamma^2}{2}\int_0^t (D\bar{v})^*\lambda\lambda^* D\bar{v}(-s+r,\bar{X}_r)\,dr\Big) \\
&\quad = \exp\Big(-\gamma\bar{v}(-s+t,X_t;T) + \gamma\int_0^t Q(\bar{X}_r, D\bar{v}(-s+r,\bar{X}_r))\,dr \\
&\qquad - \frac{\gamma^2}{2}\int_0^t (D\bar{v})^*\lambda\lambda^* D\bar{v}(-s+r,\bar{X}_r)\,dr\Big) \geq e^{-\gamma\bar{v}(-s+t,X_t;T)}
\end{aligned}$$

since $Q(x,p) \geq \frac{\gamma}{2}\,|p|^2$. Therefore we have

$$E\big[e^{-\gamma\bar{v}(-s+t,X_t;T)}\big] \leq e^{-\gamma\bar{v}(-s,x;T)}, \quad 0 \leq t \leq T+s.$$

Choose a sequence T_j such that $\{\bar{v}(T-T_j,x;T)\} = \{\bar{v}(0,x;T_j)\}$ converges to some $u_\infty \in C(R^N)$ as $T_j \to \infty$ and set $s = t = T_j - T$. Then we obtain

$$E[e^{-\gamma\bar{v}(0,X_{T_j-T};T)}] \leq e^{-\gamma\bar{v}(T-T_j,x;T)}$$

which implies that

$$\int e^{-\gamma\bar{v}(0,x;T)}\, m_\theta(dx) \le e^{-\gamma u_\infty(x)},$$

where $m_\theta(dx)$ is the invariant measure of the diffusion process with the generator $L^{\bar{w}}$. Taking $T = T_j$ sending it to ∞ again, we have

$$\int e^{-\gamma u_\infty(y)}\, m_\theta(dy) \le \varliminf_{T_j\to\infty} \int e^{-\gamma\bar{v}(0,y;T_j)}\, m_\theta(dy) \le e^{-\gamma u_\infty(x)}$$

for each $x \in R^N$. By taking the infimum of both sides with respect to x we obtain

$$\int \big(e^{-\gamma u_\infty(y)} - \inf_{x\in R^n} e^{-\gamma u_\infty(x)}\big)\, m_\theta(dx) \le 0.$$

Since $\operatorname{supp} m_\theta = R^N$ we see that u_∞ is a constant function.

Suppose that $\bar{v}(0,x;T_j)$ (respectively $\bar{v}(0,x;T_j'))$ converges to a constant c_∞ (resp. c_∞'), as T_j and $T_j' \to \infty$. Then we see that

$$\int e^{-\gamma\bar{v}(0;y;T)}\, m_\theta(dy) \le e^{-\gamma c_\infty}.$$

By setting $T = T_j'$ and sending T_j' to ∞ we have

$$e^{-\gamma c_\infty'} \le e^{-\gamma c_\infty}$$

and thus we obtain $c_\infty' \ge c_\infty$. By replacing the roles of the both we also see that $c_\infty \ge c_\infty'$ and hence $c_\infty = c_\infty'$. ■

Now the following is the direct consequence of Theorem 3.1 and Lemma 2.4.

Corollary 3.1. *Under the assumptions of Theorem* 3.1, *for some constant* c_∞,

$$\lim_{T\to\infty} \big\{u(0,x;T) - \chi(\theta)T - c_\infty - w(x)\big\} = 0,$$

uniformly on each compact set, which implies that

$$\lim_{T\to\infty} \frac{u(0,x;T)}{T} = \chi(\theta),$$

where $(\chi(\theta), w(x))$ *is the solution to H–J–B equation of ergodic type*:

$$\chi(\theta) = \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2 w] + \beta_{\theta,\mu}^* Dw + \frac{1}{2}\,(Dw)^*\lambda N_{\theta,\mu}\lambda^* Dw + \frac{1}{2}\, g^* Q_{\theta,\mu} g + \theta U. \tag{3.16}$$

Further, $\chi(\theta)$ *is convex.*

Moreover, similarly to Lemma 4.2 in [20], we have the following useful lemma.

Lemma 3.2. *Under the assumptions of Theorem* 3.1, *for each* $\theta_1 \le \theta \le \theta_0$ *there exist positive constants* δ *and* C *independent of* T *and* $\theta \in [\theta_1, \theta_0]$ *such that*

$$E[e^{\delta\bar{w}(\bar{X}_T)}] \le C. \tag{3.17}$$

4. Differentiability of H–J–B equation. Let us take a solution $(\bar\chi(\theta), \bar w)$ to (3.3) such that $\bar w(x) > 0$. Then consider the equation formally obtained by differentiating equation (3.3) with respect to θ.

$$\begin{aligned}\bar\chi'(\theta) &= \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2\bar w'] + \beta^*_{\theta,\mu} D\bar w' - (D\bar w)^*\lambda N_{\theta,\mu}\lambda^* D\bar w' \\ &\quad + \Big(\frac{\partial\beta_{\theta,\mu}}{\partial\theta}\Big)^* D\bar w - \frac{1}{2}(D\bar w)^*\lambda\frac{\partial N_{\theta,\mu}}{\partial\theta}\lambda^* D\bar w - \frac{1}{2}g^*\frac{\partial Q_{\theta,\mu}}{\partial\theta}g - U,\end{aligned}$$

namely,

$$\bar\chi'(\theta) = L^{\bar w}\bar w' + \Big(\frac{\partial\beta_{\theta,\mu}}{\partial\theta}\Big)^* D\bar w - \frac{1}{2}(D\bar w)^*\lambda\frac{\partial N_{\theta,\mu}}{\partial\theta}\lambda^* D\bar w - \frac{1}{2}g^*\frac{\partial Q_{\theta,\mu}}{\partial\theta}g - U, \tag{4.1}$$

where $w' = \frac{\partial w}{\partial\theta}$. Thus, we are led to consider a solution $(\rho(\theta), v)$ to the linear equation

$$\rho(\theta) = L^{\bar w}v(x) + f^{(\theta)}(x), \tag{4.2}$$

where

$$f^{(\theta)}(x) = \Big(\frac{\partial\beta_{\theta,\mu}}{\partial\theta}\Big)^* D\bar w(x) - \frac{1}{2}(D\bar w)^*\lambda\frac{\partial N_{\theta,\mu}}{\partial\theta}\lambda^* D\bar w(x) - \frac{1}{2}g^*\frac{\partial Q_{\theta,\mu}}{\partial\theta}g(x) - U(x). \tag{4.3}$$

Note that

$$\begin{aligned}f^{(\theta)}(x) &= (\lambda N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}g)^* D\bar w(x) - \frac{1}{2\theta^2\mu}(D\bar w)^*\lambda N_{\theta-1/\mu}\lambda^* D\bar w(x) - U(x) \\ &\quad - \frac{\theta\mu-1}{2\theta\mu}(D\bar w)^*\lambda N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}\delta^*\lambda^* D\bar w(x) \\ &\quad + \frac{1}{2}g^*(Q^{-1}_{\theta-1/\mu} + \theta Q^{-1}_{\theta-1/\mu}\delta^*\delta Q^{-1}_{\theta-1/\mu})g(x) \\ &= -\frac{1}{2}(\delta(\delta^*\delta)^{-1}g - \lambda^* D\bar w)^* N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}\delta^*(\delta(\delta^*\delta)^{-1}g - \lambda^* D\bar w)(x) - U(x) \\ &\quad - \frac{1}{2\theta^2\mu}(D\bar w)^*\lambda N_{\theta-1/\mu}\lambda^* D\bar w(x) + \frac{1}{2\theta\mu}(D\bar w)^*\lambda N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}\delta^*\lambda^* D\bar w(x) \\ &\quad - \frac{1}{2\mu}g^* Q^{-1}_{\theta-1/\mu}\delta^*\delta Q^{-1}_{\theta-1/\mu}g(x)\end{aligned} \tag{4.4}$$

holds since

$$\frac{\partial\beta_{\theta,\mu}}{\partial\theta} = \lambda\frac{\partial N_{\theta-1/\mu}}{\partial\theta}\delta(\delta^*\delta)^{-1}g, \tag{4.5}$$

$$\frac{\partial N_{\theta-1/\mu}}{\partial\theta} = \delta Q^{-1}_{\theta-1/\mu}\delta^* + \Big(\theta - \frac{1}{\mu}\Big)\delta Q^{-1}_{\theta-1/\mu}\delta^*\delta Q^{-1}_{\theta-1/\mu}\delta^* = N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}\delta^* \tag{4.6}$$

$$\frac{\partial N_{\theta,\mu}}{\partial\theta} = \frac{1}{\theta^2\mu}N_{\theta-1/\mu} + \Big(1 - \frac{1}{\theta\mu}\Big)N_{\theta-1/\mu}\delta Q^{-1}_{\theta-1/\mu}\delta^* \tag{4.7}$$

$$\begin{aligned}\frac{\partial Q_{\theta,\mu}}{\partial\theta} &= Q_{\theta-1/\mu}^{-1} + \theta Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}\\ &= Q_{\theta-1/\mu}^{-1} + \Big(\theta-\frac{1}{\mu}\Big)Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1} + \frac{1}{\mu}\,Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}\\ &= (\delta^*\delta)^{-1}\delta^*\Big(I+\Big(\theta-\frac{1}{\mu}\Big)\delta Q_{\theta-1/\mu}^{-1}\delta^*\Big)\delta Q_{\theta-1/\mu}^{-1} + \frac{1}{\mu}\,Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}\\ &= (\delta^*\delta)^{-1}\delta^* N_{\theta-1/\mu}\delta Q_{\theta-1/\mu}^{-1} + \frac{1}{\mu}\,Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}.\end{aligned} \tag{4.8}$$

Noting that $\frac{-c_0}{\theta(1+\mu c_0-\mu\theta)} I \le N_{\theta,\mu}$, we see that for sufficiently large $R_0>0$

$$K(x;\bar w) \equiv -L^{\bar w}\bar w(x) > \frac{-c_0}{\theta(1+\mu c_0-\mu\theta)\bar w}\,(D\bar w)^*\lambda\lambda^* D\bar w(x), \quad x\in B_{R_0}^c := \{x; |x|\ge R_0\}.$$

and condition (A.3) of Appendix in [20] is satisfied for $L^{\bar w}$ and $\bar w$. The other conditions are also satisfied. Thus, setting

$$F_{\bar w} = \Big\{u\in W^{2,p}_{\rm loc} : \operatorname*{ess\,sup}_{x\in B^c_{R_0}} \frac{|u(x)|}{\bar w(x)} < \infty\Big\}$$

and

$$F_K = \Big\{f\in L^\infty_{\rm loc} : \operatorname*{ess\,sup}_{x\in B^c_{R_0}} \frac{|f(x)|}{K(x;\bar w)} < \infty\Big\},$$

we see that equation (4.2) has a solution $v\in F_{\bar w}$ such that

$$\rho(\theta) = \int f^{(\theta)}(x)\, m_\theta(dx) \tag{4.9}$$

(cf. Corollary 5.1 in [20]). Indeed, we can see that

$$K(x,\bar w) \ge C|x|^2 - C, \quad x\in B_R^c,$$

for sufficiently large $R>0$ because of (3.3) and (3.5), while

$$|f^{(\theta)}(x)|^2 \le M_1|x|^2 + M_2$$

holds for some positive constants M_1 and M_2 in light of (4.4) and (3.4).

5. Interpretation in terms of a stochastic differential game. Let us introduce a stochastic differential game

$$\begin{aligned}\bar J(0,x;T) = \inf_{h_\cdot\in\Delta_{\mathcal H}} \sup_{\zeta_\cdot,\nu_\cdot\in\mathcal Z} E^{\zeta,h,\nu}\Big[\theta\Big\{\int_0^T f(X_s,h_s)\,ds + \int_0^T \varphi(X_s,h_s)^*\,dW_s\Big\}\\ + \frac{\theta\mu}{2}\int_0^T|\zeta_s|^2\,ds - \frac{1}{2}\int_0^T \big|\nu_s+\theta\delta(X_s)h_s\big|^2\,ds\Big],\end{aligned} \tag{5.1}$$

where X_t is a solution to the controlled stochastic differential equation

$$dX_t = \{\beta(X_t)+\lambda(X_t)(\zeta_t+\nu_t+\theta\delta(X_t)h_t)\}\,dt + \lambda(X_t)\,d\tilde W_t, \quad X_0=x,$$

ν_t is a progressively measurable process such that $\nu_t=\nu(t,X_t)$ with $|\nu(t,x)|\le C(1+|x|)$ for some positive constant C by the definition of $\mathcal Z$ similarly to ζ_t, $P^{\zeta,h,\nu}$ is a probability measure defined by

$$P^{\zeta,h,\nu}(A) = E^{\zeta}\Big[\exp\Big(\int_0^T (\nu_s + \theta\delta(X_s)h_s)^* \, dW_s^{\zeta} - \frac{1}{2}\int_0^T |\nu_s + \theta\delta(X_s)h_s|^2 \, ds\Big); A\Big]$$

and

$$\tilde{W}_t = W_t^{\zeta} - \int_0^t (\nu_s + \theta\delta(X_s)h_s) \, ds = W_t - \int_0^t \{\zeta_s + \nu_s + \theta\delta(X_s)h_s\} \, ds.$$

Note that

$$\begin{aligned} E^{\zeta,h,\nu}\Big[\theta\Big\{&\int_0^T f(X_s,h_s)\, ds + \int_0^T \varphi(X_s,h_s)^* \, dW_s + \frac{\mu}{2}\int_0^T |\zeta_s|^2 \, ds\Big\} \\ &- \frac{1}{2}\int_0^T |\nu_s + \theta\delta(X_s)h_s|^2 \, ds\Big] \\ = E^{\zeta,h,\nu}\Big[&\int_0^T \Big\{-\frac{\theta}{2}\, h_s^* Q_\theta(X_s)h_s + \theta h_s^*(g(X_s) + \delta(X_s)\zeta_s) + \theta U(X_s) + \frac{\theta\mu}{2}\,|\zeta_s|^2\Big\}\, ds \\ &- \frac{1}{2}\int_0^T |\nu_s|^2 \, ds\Big] \end{aligned}$$

and set

$$\Xi_1(x,h,\nu;\theta) = -\frac{\theta}{2}\, h^* Q_\theta(x)h + \theta h^*(g(x) + \delta(x)\zeta) + \theta U(x) + \frac{\theta\mu}{2}\,|\zeta|^2 - \frac{1}{2}\,|\nu|^2.$$

Then (5.1) is written as

$$\bar{J}(0,x;T) = \inf_{h_\cdot \in \Delta_{\mathcal{H}}} \sup_{\nu_\cdot,\zeta_\cdot \in \mathcal{Z}} E^{\zeta,h,\nu}\Big[\int_0^T \Xi_1(X_s,h_s,\zeta_s,\nu_s;\theta)\, ds\Big].$$

In this fictitious game the player controlling h_t wants to minimize the criterion taking into account all possibility of uncertainty processes ζ_t and the other player (considered to be the nature) controlling ν_t tries to maximize the criterion. Control parameters h_t and ν_t separately act in the above defined controlled dynamics and do not correlate in the cost functional Ξ_1. Therefore, we can change the order of the infimum over h and the supremum over ν. Thus, its corresponding lower Isaacs equation is seen to be

$$\frac{\partial u}{\partial t} + \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2 u] + \sup_{\zeta,\nu}\inf_h \big[\{\beta + \lambda(\nu + \zeta + \theta\delta h)\}^* Du + \Xi_1(x,h,\zeta,\nu;\theta)\big] = 0.$$

It is also written as

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2 u] + \beta^* Du + \sup_{\nu\in R^M}\{\nu^*\lambda^* Du - \tfrac{1}{2}\,|\nu|^2\} + \sup_{\zeta}\inf_h \Lambda(x,Du,\zeta,h) = 0$$

and thus seen to be the same equation as (2.13). Thus, we have an interpretation of our problem (2.10) in terms of a stochastic differential game of classical type. We do not need the proof of the verification theorem which shows the solution u to (2.13) is identical to the value, namely $\bar{J}(0,x;T) = u(0,x;T)$. In the proof of our duality theorem to be seen later, we only need the expression of the H–J–B equation (3.3) of ergodic type as the one of the stochastic differential game which is the infinite horizon counterpart of (5.1).

When considering the problem on infinite time horizon with the averaging cost criterion, we obtain the ergodic type equation

$$\chi(\theta) = \frac{1}{2}\operatorname{tr}[\lambda\lambda^* D^2 w] + \sup_{\nu\in R^n,\zeta\in R^M}\inf_{h\in R^m}[\{\beta+\lambda\zeta+\lambda(\nu+\theta\delta h)\}^* Dw + \Xi_1(x,h,\zeta,\nu)],$$

which can be written as

$$\chi(\theta) = L^{\bar{w}} w + \Xi_1(x,\tilde{h},\tilde{\zeta},\tilde{\nu}), \tag{5.2}$$

where

$$\begin{aligned}\tilde{h} &= Q_\theta^{-1}(g+\delta^*\tilde{\zeta}+\delta^*\lambda^* Dw)\\ \tilde{\zeta} &= -\frac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta\lambda^* Dw+\theta\delta Q_\theta^{-1} g)\\ \tilde{\nu} &= \lambda^* Dw.\end{aligned}$$

It is same as

$$\chi(\theta) = L^{\bar{w}} w - \frac{1}{2}(Dw)^*\lambda N_{\theta,\mu}\lambda^* Dw + \frac{1}{2} g^* Q_{\theta,\mu} g + \theta U, \tag{5.3}$$

whose derivative with respect to θ is

$$\chi'(\theta) = L^{\bar{w}} w' + \Big(\frac{\partial\beta_{\theta,\mu}}{\partial\theta}\Big)^* Dw + \frac{1}{2}(Dw)^*\lambda\,\frac{\partial N_{\theta,\mu}}{\partial\theta}\,\lambda^* Dw + \frac{1}{2} g^*\,\frac{\partial Q_{\theta,\mu}}{\partial\theta}\, g + U. \tag{5.4}$$

From (5.3) and (5.4) we obtain

$$\begin{aligned}\chi(\theta)-\theta\chi'(\theta) &= L_1^w(w-\theta w')\\ &\quad-\frac{1}{2}\{N_{\theta-1/\mu}\lambda^* Dw+\theta\delta Q_{\theta-1/\mu}^{-1} g\}^*\{N_{\theta-1/\mu}\lambda^* Dw+\theta\delta Q_{\theta-1/\mu}^{-1} g\},\end{aligned} \tag{5.5}$$

which is seen to be

$$\chi(\theta)-\theta\chi'(\theta) = L_1^w(w-\theta w') - \frac{1}{2}|\tilde{\nu}+\theta\delta\tilde{h}|^2. \tag{5.6}$$

Indeed, from (4.6) it follows that

$$\begin{aligned}&-\frac{1}{2} N_{\theta,\mu} - \frac{\theta}{2}\frac{\partial N_{\theta,\mu}}{\partial\theta}\\ &= -\frac{1}{2}\Big(1-\frac{1}{\theta\mu}\Big)N_{\theta-1/\mu} - \frac{\theta}{2}\Big\{\frac{1}{\theta^2\mu} N_{\theta-1/\mu} + \Big(1-\frac{1}{\theta\mu}\Big)N_{\theta-1/\mu}\delta Q_{\theta-1/\mu}^{-1}\delta^*\Big\}\\ &= -\frac{1}{2} N_{\theta-1/\mu} - \frac{1}{2}\Big(\theta-\frac{1}{\mu}\Big)N_{\theta-1/\mu}\delta Q_{\theta-1/\mu}^{-1}\delta^* = -\frac{1}{2} N_{\theta-1/\mu}N_{\theta-1/\mu}.\end{aligned}$$

On the other hand, from (4.8) we obtain

$$\begin{aligned}-\frac{1}{2} Q_{\theta,\mu} - \frac{\theta}{2}\frac{\partial Q_{\theta,\mu}}{\partial\theta} &= \frac{\theta}{2} Q_{\theta-1/\mu}^{-1} - \frac{\theta}{2} Q_{\theta-1/\mu}^{-1} - \frac{\theta^2}{2} Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}\\ &= -\frac{\theta^2}{2} Q_{\theta-1/\mu}^{-1}\delta^*\delta Q_{\theta-1/\mu}^{-1}.\end{aligned}$$

Further,

$$\begin{aligned}-\theta\,\frac{\partial\beta_{\theta,\mu}}{\partial\theta} &= -\theta g^*(\delta^*\delta)^{-1}\delta^*\,\frac{\partial N_{\theta-1/\mu}}{\partial\theta}\,\lambda^* Dw\\ &= -\theta g^*(\delta^*\delta)^{-1}\delta^* N_{\theta-1/\mu}\delta Q_{\theta-1/\mu}^{-1}\delta^*\lambda^* Dw = -\theta g^* Q_{\theta-1/\mu}^{-1}\delta^* N_{\theta-1/\mu}\delta\lambda^* Dw.\end{aligned}$$

Thus, we see that (5.5) holds, while

$$\begin{aligned}\tilde{\nu} + \theta\delta\tilde{h} &= \lambda^* Dw - \frac{1}{\mu}\,\delta Q_{\theta-1/\mu}^{-1}\big((1-\theta\mu)\delta^*\lambda^* Dw - \theta\mu g\big)\\ &= \Big(I + \Big(\theta - \frac{1}{\mu}\Big)\delta Q_{\theta-1/\mu}^{-1}\delta^*\Big)\lambda^* Dw + \theta\delta Q_{\theta-1/\mu}^{-1} g\\ &= N_{\theta-1/\mu}\lambda^* Dw + \theta\delta Q_{\theta-1/\mu}^{-1} g,\end{aligned}$$

from which we obtain (5.6).

Formula (5.6) is useful in the proof of our duality theorem.

6. Duality theorem. In this section we shall present the robust estimates of the large deviation probability for the controlled semi-martingales given in Section 1. We have the following duality theorem.

THEOREM 6.1. *For $\kappa \in (\chi'(-\infty), \chi'(0-))$, we have*

$$\varliminf_{T\to\infty} \frac{1}{T} \inf_{h\in\Delta_{\mathcal{H}}} \sup_{\zeta\in\mathcal{Z}} \log P^{\zeta}\Big(\frac{1}{T}\{F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\} \le \kappa\Big) = -I(\kappa),$$
$$I(k) := \sup_{\theta<0}\{\theta k - \chi(\theta)\}.$$

Moreover, for $\theta(\kappa)$ such that $\chi'(\theta(\kappa)) = \kappa \in (\chi'(-\infty), \chi'(0-))$ take a strategy $\bar{h}^{(\theta(\kappa))}(t, x, \tilde{\zeta}) \equiv \bar{h}^{(\theta(\kappa),T)}(t, x, \tilde{\zeta})$. Then

$$\lim_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}\Big(\frac{1}{T}\Big\{F_T(X_\cdot, \bar{h}^{(\theta(\kappa))}(\cdot, X_\cdot, \tilde{\zeta}_\cdot)) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\,ds\Big\} \le \kappa\Big) = -I(\kappa)$$

where $\tilde{\zeta}_s = \tilde{\zeta}(X_s)$.

Proof. Let us first give an upper estimate. For a given constant $\chi'(-\infty) < \kappa < \chi'(0-)$ and $\theta < 0$,

$$\begin{aligned}&\log P^{\zeta}\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds \le \kappa T\Big)\\ &\quad= \log P^{\zeta}\Big(\exp\Big(\theta\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\Big)\Big) \ge e^{\theta\kappa T}\Big)\\ &\quad\le \log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\Big) - \theta\kappa T\Big)\Big]\\ &\quad= \log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\Big)\Big)\Big] - \theta\kappa T.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sup_{\zeta\in\mathcal{Z}} \log P^{\zeta}\Big(\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\Big) \le \kappa T\Big)\\ &\qquad\qquad\le \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\,ds\Big)\Big)\Big] - \theta\kappa T\end{aligned}$$

and so

$$\inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}}\log P^{\zeta}\Big(\Big(F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2\,ds\Big)\le\kappa T\Big)$$
$$\le\inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}}\log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2\,ds\Big)\Big)\Big]-\theta\kappa T=u(0,x;T)-\theta\kappa T,$$

which implies

$$\varliminf_{T\to\infty}\frac{1}{T}\inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}}\log P^{\zeta}\Big(F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2\,ds\le\kappa T\Big)\le\chi(\theta)-\theta\kappa$$

for all $\theta<0$. Hence, we obtain

$$\varliminf_{T\to\infty}\frac{1}{T}\inf_{h\in\Delta_{\mathcal{H}}}\sup_{\zeta\in\mathcal{Z}}\log P^{\zeta}\Big(F_T(X_\cdot,h_\cdot)+\frac{\mu}{2}\int_0^T|\zeta_s|^2\,ds\le\kappa T\Big)$$
$$\le\inf_{\theta<0}\{\chi(\theta)-\theta\kappa\}=-I(\kappa)=-\inf_{k\in(\chi'(-\infty),\kappa]}I(k).$$

Now, let us prove the converse inequality. Take a constant κ and $\epsilon>0$ such that $\chi(0-)-\epsilon>\kappa-\epsilon>\chi'(-\infty)$. Then there exists θ_ϵ such that

$$\inf_{\theta<0}\{\chi(\theta)-\theta(\kappa-\epsilon)\}=\chi(\theta_\epsilon)-\theta_\epsilon\chi'(\theta_\epsilon).$$

We write θ_ϵ as θ for simplicity in the following. Assuming the probability space $(\Omega,\mathcal{F},P)$ is the canonical one, introduce a probability measure $\tilde{P}$ with the density

$$\frac{d\tilde{P}}{dP^{\tilde{\zeta}}}\Big|_{\mathcal{F}_T}=\exp\Big(\int_0^T\{\tilde{\nu}(X_s)+\theta\delta(X_s)\tilde{h}(X_s)\}^*\,dW_s^{\tilde{\zeta}}-\frac{1}{2}\int_0^T\big|\tilde{\nu}(X_s)+\theta\delta(X_s)\tilde{h}(X_s)\big|^2\,ds\Big).$$

Then, as we have seen above, the solution X_t to (1.5) satisfies

$$dX_t=\{\beta(X_t)+\lambda(X_t)(\tilde{\zeta}_t+\tilde{\nu}_t+\theta\delta(X_t)\tilde{h}_t)\}\,dt+\lambda(X_t)\,d\tilde{W}_t,\quad X_0=x,$$

with the Brownian motion process $\tilde{W}_t=W_t^{\tilde{\zeta}}-\int_0^t\{\tilde{\nu}_s+\theta\delta(X_s)\tilde{h}_s\}\,ds$ under $\tilde{P}$, where $\tilde{h}_s=\tilde{h}(X_s)$ and $\tilde{\nu}_s=\tilde{\nu}(X_s)$. Let us first show the following lemma.

Lemma 6.1. *For each ϵ, by taking sufficiently large T,*

$$\tilde{P}\Big(\int_0^T f(X_s,h_s)\,ds+\int_0^T\varphi(X_s,h_s)^*\,dW_s+\frac{\mu}{2}\int_0^T|\tilde{\zeta}_s|^2\,ds>\kappa T\Big)<\epsilon,\tag{6.1}$$

holds for $h\in\Delta_{\mathcal{H}}$.

Proof. We see that

$$\begin{aligned}&\theta\Big\{\int_0^T f(X_s,h_s)\,ds+\int_0^T\varphi(X_s,h_s)^*\,dW_s+\frac{\mu}{2}\int_0^T|\tilde{\zeta}_s|^2\,ds\Big\}\\&=\int_0^T\Xi(X_s,\tilde{h}_s,\tilde{\nu}_s,\tilde{\zeta}_s;\theta)\,ds+\theta\int_0^T\tilde{h}_s\delta(X_s)^*\,d\tilde{W}_s\\&+\theta\int_0^T(h_s-\tilde{h}_s)^*\big(\delta(X_s)^*-S^{1/2}(X_s)E_{m,M}\big)\,d\tilde{W}_s+\frac{1}{2}\int_0^T\big|\tilde{\nu}_s+\theta\delta(X_s)\tilde{h}_s\big|^2\,ds\\&+\theta\int_0^T(h_s-\tilde{h}_s)^*S^{1/2}(X_s)E_{m,M}\,d\tilde{W}_s-\frac{\theta}{2}\int_0^T(h_s-\tilde{h}_s)^*S(X_s)(h_s-\tilde{h}_s)\,ds.\end{aligned}\tag{6.2}$$

Indeed,

$$\begin{aligned}
&\theta\int_0^T f(X_s,h_s)\,ds+\theta\int_0^T \varphi(X_s,h_s)^*\,dW_s+\frac{\theta\mu}{2}\int_0^T|\tilde\zeta_s|^2\,ds\\
&=\theta\int_0^T\big[f(X_s,h_s)+h_s^*\delta(X_s)^*\{\tilde\nu_s+\tilde\zeta_s+\theta\delta(X_s)\tilde h_s\}\big]\,ds\\
&\quad+\frac{\theta\mu}{2}\int_0^T|\tilde\zeta_s|^2\,ds+\theta\int_0^T h_s^*\delta(X_s)^*\,d\tilde W_s\\
&=\theta\int_0^T\big[f(X_s,\tilde h_s)+\tilde h_s^*\delta(X_s)^*\{\tilde\nu_s+\tilde\zeta_s+\theta\delta(X_s)\tilde h_s\}\big]\,ds\\
&\quad+\frac{\theta\mu}{2}\int_0^T|\tilde\zeta_s|^2\,ds+\theta\int_0^T\tilde h_s^*\delta(X_s)^*\,d\tilde W_s+\theta\int_0^T(h_s-\tilde h_s)^*\delta(X_s)^*\,d\tilde W_s\\
&\quad+\theta\int_0^T\big[f(X_s,h_s)-f(X_s,\tilde h_s)+(h_s-\tilde h_s)^*\delta(X_s)^*\big\{\tilde\nu_s+\tilde\zeta_s+\theta\delta(X_s)\tilde h_s\big\}\big]\,ds.
\end{aligned}$$

Then we can see that

$$\begin{aligned}
&\theta f(x,\tilde h)+\theta\tilde h^*\delta^*(\tilde\nu+\tilde\zeta+\theta\delta\tilde h)+\frac{\theta\mu}{2}|\tilde\zeta|^2\\
&\quad=-\frac{\theta}{2}\tilde h^*Q_\theta\tilde h+\theta\tilde h^*(\delta\tilde\zeta+g)+\theta U+\frac{\theta\mu}{2}|\tilde\zeta|^2-\frac12|\tilde\nu|^2+\frac12|\tilde\nu+\theta\delta\tilde h|^2\\
&\quad=\Xi(x,\tilde h,\tilde\nu,\tilde\zeta;\theta)+\frac12|\tilde\nu+\theta\delta\tilde h|^2
\end{aligned}$$

and that

$$\begin{aligned}
&\theta f(x,h)-\theta f(x,\tilde h)+\theta(h-\tilde h)^*\delta^*(\tilde\nu+\tilde\zeta+\theta\delta\tilde h)\\
&=-\frac{\theta}{2}h^*Q_\theta h+\theta h^*(g+\delta^*\tilde\nu+\delta^*\tilde\zeta)-\frac{\theta^2}{2}h^*\delta^*\delta h\\
&\quad+\frac{\theta}{2}h^*Q_\theta h-\theta h^*\big(g+\delta^*\tilde\nu+\delta^*\tilde\zeta\big)-\frac{\theta^2}{2}h^*\delta^*\delta h+\theta^2h^*\delta^+\delta\tilde h\\
&=-\frac{\theta}{2}\big\{h-Q_\theta^{-1}(g+\delta^*\tilde\nu+\delta^*\tilde\zeta)\big\}^*Q_\theta\big\{h-Q_\theta^{-1}(g+\delta^*\tilde\nu+\delta^*\tilde\zeta)\big\}\\
&\quad+\frac{\theta}{2}\big\{\tilde h-Q_\theta^{-1}(g+\delta^*\tilde\nu+\delta^*\tilde\zeta)\big\}^*Q_\theta\big\{\tilde h-Q_\theta^{-1}(g+\delta^*\tilde\nu+\delta^*\tilde\zeta)\big\}\\
&\quad-\frac{\theta^2}{2}(h-\tilde h)^*\delta^*\delta(h-\tilde h)\\
&=-\frac{\theta}{2}(h-\tilde h)^*Q_\theta(h-\tilde h)-\frac{\theta^2}{2}(h-\tilde h)^*\delta^*\delta(h-\tilde h)=-\frac{\theta}{2}(h-\tilde h)^*S(h-\tilde h)
\end{aligned}$$

since $\Xi(x,h,\nu,\zeta;\theta)=-\frac{\theta}{2}h^*Q_\theta h+\theta h^*(g+\delta^*\zeta)+\theta U+\frac{\theta\mu}{2}|\zeta|^2-\frac12|\nu|^2$, $\tilde h=Q_\theta^{-1}(g+\delta^*\tilde\zeta+\delta^*\tilde\nu)$, and $Q_\theta=S-\theta\delta^*\delta$. Therefore, we obtain (6.2).

Let $E_{m,M}$ be the $m\times M$ matrix such that $(E_{m,M})_{ij}=\delta_{i,j}$, for $i,j=1,\dots,m$ and $(E_{m,M})_{i,j}=0$, for $m+1\le j\le M$, $i=1,\dots,m$, and set

$$M_t^h=\int_0^t(h_s-\tilde h_s)^*S^{1/2}(X_s)E_{m,M}\,d\tilde W_s.$$

Then, since

$$w(X_T) - w(X_0) = \int_0^T L^w w(X_s)\,ds + \int_0^T (Dw)^*\lambda(X_s)\,d\tilde{W}_s$$
$$= \chi(\theta)T - \int_0^T \Xi\big(X_s, \tilde{h}_s, \tilde{\nu}_s, \tilde{\zeta}_s; \theta\big)\,ds + \int_0^T (Dw)^*\lambda(X_s)\,d\tilde{W}_s$$

we have

$$\begin{aligned} &\theta\Big\{\int_0^T f(X_s,h_s)\,ds + \int_0^T \varphi(X_s,h_s)^*\,dW_s + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\,ds\Big\} \\ &= \chi(\theta)T + w(X_0) - w(X_T) + \int_0^T \{(Dw)^*\lambda(X_s) + \theta\tilde{h}_s\delta(X_s)^*\}\,d\tilde{W}_s \\ &\quad + \theta\int_0^T (h_s - \tilde{h}_s)^*\big(\delta(X_s)^* - S^{1/2}(X_s)E_{m,M}\big)\,d\tilde{W}_s \\ &\quad + \frac{1}{2}\int_0^T \big|\tilde{\nu}_s + \theta\delta(X_s)\tilde{h}_s\big|^2\,ds + \theta\Big(M_T^h - \frac{1}{2}\langle M^h\rangle_T\Big). \end{aligned} \tag{6.3}$$

Thanks to (5.6), we have

$$\begin{aligned} &(w - \theta w')(X_T) - (w - \theta w')(X_0) \\ &= \int_0^T L^w(w - \theta w')(X_s)\,d\tilde{W}_s + \int_0^T D(w - \theta w')(X_s)^*\lambda(X_s)\,d\tilde{W}_s \\ &= \frac{1}{2}\int_0^T |\tilde{\nu} + \theta\delta^*\tilde{h}|^2(X_s)\,ds + (\chi(\theta) - \theta\chi'(\theta))T + \int_0^T D(w - \theta w')(X_s)^*\lambda(X_s)\,d\tilde{W}_s. \end{aligned} \tag{6.4}$$

Therefore,

$$\begin{aligned} &\theta\Big\{\int_0^T f(X_s,h_s)\,ds + \int_0^T \varphi(X_s,h_s)^*\,dW_s + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\,ds\Big\} \\ &= \theta\chi'(\theta)T + \theta w'(X_0) - \theta w'(X_T) + \theta\int_0^T \big\{(Dw')^*\lambda(X_s) + \tilde{h}_s^*\delta(X_s)^*\big\}\,d\tilde{W}_s \\ &\quad + \theta\int_0^T (h_s - \tilde{h}_s)^*(\delta(X_s)^* - S^{1/2}(X_s)E_{m,M})\,d\tilde{W}_s + \theta\Big(M_T^h - \frac{1}{2}\langle M^h\rangle_T\Big). \end{aligned} \tag{6.5}$$

Now we can prove (6.1). Since $\kappa = \chi'(\theta) + \epsilon$,

$$\begin{aligned} &\tilde{P}\Big(\int_0^T f(X_s,h_s)\,ds + \int_0^T \varphi(X_s,h_s)^*\,dW_s + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\,ds > \kappa T\Big) \\ &\le \tilde{P}\Big(w'(X_0) - w'(X_T) > \frac{\epsilon T}{5}\Big) + \tilde{P}\Big(\int_0^T (Dw')^*\lambda(X_s)\,d\tilde{W}_s > \frac{\epsilon T}{5}\Big) \\ &\qquad + \tilde{P}\Big(\int_0^T \tilde{h}_s^*\delta(X_s)^*\,d\tilde{W}_s > \frac{\epsilon T}{5}\Big) + \tilde{P}\Big(M_T^h - \frac{1}{2}\langle M^h\rangle_T > \frac{\epsilon T}{5}\Big) \\ &\qquad + \tilde{P}\Big(\int_0^T (h_s - \tilde{h}_s)^*(\delta(X_s)^* - S^{1/2}(X_s)E_{m,M})\,d\tilde{W}_s > \frac{\epsilon T}{5}\Big) \\ &\equiv Q_1 + Q_2 + Q_3 + Q_4 + Q_5. \end{aligned}$$

Then
$$Q_1 \le \frac{5^2}{\epsilon^2 T^2}\, \tilde{E}[|w'(X_T) - w'(X_0)|^2].$$
Noting that the law $P \circ \bar{X}_\cdot^{-1}$ defined in Section 3 is nothing but $\tilde{P}$ and that w' is of at most polynomial growth, we see that $\tilde{E}[|w'(X_T) - w'(X_0)|^2]$ is bounded by a positive constant C independent of T thanks to Lemma 3.1. Thus, Q_1 is seen to be small enough when T is sufficiently large. Next, since
$$Q_2 \le \frac{5^2}{\epsilon^2 T^2}\, \tilde{E}\Big[\Big|\int_0^T (Dw')^*\lambda(X_s)\, d\tilde{W}_s\Big|^2\Big] = \frac{5^2}{\epsilon^2 T^2}\Big[\int_0^T \tilde{E}(Dw')^*\lambda\lambda^* Dw'(X_s)\Big]\, ds$$
and $(Dw')^*\lambda\lambda^* Dw'$ is of at most polynomial growth again, $\tilde{E}[(Dw')^*\lambda\lambda^* Dw'(X_s)]$ is bounded by a positive constant. Therefore, Q_2 is small enough when T is sufficiently large. As for Q_3 and Q_5, the same arguments apply since $\tilde{h}_s = \tilde{h}(X_s)$ and $h_s = h(s, X_s)$ with functions h, $\tilde{h}$ which are of at most linear growth. Concerning Q_4, we have
$$Q_4 \le e^{-\epsilon T/5}\tilde{E}[e^{M_T^h - \langle M^h\rangle_T/2}] \le e^{-\epsilon T/5}.$$
Hence we obtain (6.1). ∎

Then let us complete the proof of Theorem 6.1. It can proceed in a similar way to the proof of Theorem 2.4 in [20]. Indeed, set
$$\tilde{M}_t^\theta := \int_0^t \{\tilde{\nu}_s + \theta\delta(X_s)\tilde{h}_s\}\, dW_s^{\tilde{\zeta}}$$
and
$$\begin{aligned} A_1 &= \{-\tilde{M}_T \ge -\epsilon T\} \\ A_2 &= \{-\tfrac{1}{2}\,\langle \tilde{M}^\theta\rangle_T \ge (\chi(\theta) - \theta\chi'(\theta) - \epsilon)T\} \\ A_3 &= \{F_T(X_\cdot, h_\cdot) \le \kappa T\}. \end{aligned}$$
Thanks to (6.4) we have
$$\begin{aligned} &\tilde{P}\Big(\frac{1}{2}\,\langle \tilde{M}^\theta\rangle_T + (\chi(\theta) - \theta\chi'(\theta))T > \epsilon T\Big) \\ &\qquad = \tilde{P}\Big((w - \theta w')(X_T) - (w - \theta w')(X_0) - \int_0^T D(w - \theta w')^*\lambda(X_s)\, d\tilde{W}_s > \epsilon T\Big), \end{aligned}$$
and similarly to the above we see that
$$\tilde{P}\Big(\frac{1}{2}\,\langle \tilde{M}^\theta\rangle_T + (\chi(\theta) - \theta\chi'(\theta))T > \epsilon T\Big) < \epsilon$$
holds for sufficiently large T. Thus we see that $\tilde{P}(A_2^c) < \epsilon$. We have already seen that $\tilde{P}(A_3^c) < \epsilon$. It is similar to the above showing that $\tilde{P}(A_1^c) < \epsilon$. Thus,
$$\begin{aligned} &P^{\tilde{\zeta}}\Big(F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big) \\ &\quad = \tilde{E}\Big[e^{-\tilde{M}_T^\theta - 1/2\langle \tilde{M}^\theta\rangle_T}; F_T(X_\cdot, h_\cdot) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big] \\ &\quad \ge \tilde{E}[e^{-\tilde{M}_T^\theta - 1/2\langle \tilde{M}^\theta\rangle_T}; A_1 \cap A_2 \cap A_3] \ge e^{(\chi(\theta) - \theta\chi'(\theta) - 2\epsilon)T}\tilde{P}(A_1 \cap A_2 \cap A_3) \\ &\quad \ge e^{(\chi(\theta) - \theta\chi'(\theta) - 2\epsilon)T}\{1 - \tilde{P}(A_1^c) - \tilde{P}(A_2^c) - \tilde{P}(A_3^c)\} \ge e^{(\chi(\theta) - \theta\chi'(\theta) - 2\epsilon)T}(1 - 3\epsilon). \end{aligned}$$

Therefore,

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \log P^{\tilde{\zeta}}\Big(F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big)$$
$$\ge \chi(\theta) - \theta\chi'(\theta) - 2\epsilon = \chi(\theta) - \theta(\kappa - \epsilon) - 2\epsilon \ge \inf_{\theta<0}\{\chi(\theta) - \theta(\kappa-\epsilon)\} - 2\epsilon$$

for each ϵ. Since $\chi(\theta)$ is smooth and convex, $\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}$ is convex, and thus continuous. Hence

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \sup_{\zeta.} \log P^{\zeta}\Big(F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds \le \kappa T\Big) \ge -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}$$

because

$$\sup_{\zeta.} \log P^{\zeta}\Big(F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds \le \kappa T\Big)$$
$$\ge \log P^{\tilde{\zeta}}\Big(F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big).$$

Now, for $\chi'(-\infty) < \kappa < \chi'(0-)$, take a strategy $\bar{h}^{(\theta(\kappa),T)}(\zeta)$ defined in Proposition 2.2. Then

$$\begin{aligned}
&\sup_{\zeta\in\mathcal{Z}} \log P^{\zeta}\Big(F_T(X_., \bar{h}^{(\theta(\kappa),T)}(\zeta)) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds \le \kappa T\Big)\\
&\quad\le \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_., \bar{h}^{(\theta(\kappa),T)}(\zeta)) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds\Big) - \theta\kappa T\Big)\Big]\\
&\quad= \sup_{\zeta\in\mathcal{Z}} \log E^{\zeta}\Big[\exp\Big(\theta\Big(F_T(X_., \bar{h}^{(\theta(\kappa),T)}(\zeta)) + \frac{\mu}{2}\int_0^T |\zeta_s|^2\, ds\Big)\Big)\Big] - \theta\kappa T\\
&\quad= u(0, x; T) - \theta\kappa T.
\end{aligned}$$

Therefore,

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}\Big(F_T(X_., \bar{h}^{(\theta(\kappa),T)}(\tilde{\zeta})) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big)$$
$$\le \chi(\theta(\kappa)) - \theta(\kappa) = -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}.$$

On the other hand,

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \log P^{\tilde{\zeta}}\Big(F_T(X_., \bar{h}^{(\theta(\kappa),T)}(\tilde{\zeta})) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big)$$
$$\ge \underline{\lim}_{T\to\infty} \frac{1}{T} \inf_{h.} \log P^{\tilde{\zeta}}\Big(F_T(X_., h_.) + \frac{\mu}{2}\int_0^T |\tilde{\zeta}_s|^2\, ds \le \kappa T\Big) = -\sup_{\theta<0}\{\theta\kappa - \chi(\theta)\}$$

is obvious and hence we obtain our theorem. ∎

Remark 6.1. The worst case uncertainty process is $\tilde{\zeta}(X_t)$ with

$$\tilde{\zeta}(x) = -\frac{1}{\theta\mu} R_{\theta,\mu}^{-1}(N_\theta \lambda^* Dw + \theta\delta Q_\theta^{-1} g)$$

and in the gradient estimate (3.4) for w we can see that there exists $\mu_0 > 0$ such that C_w and C'_w do not depend on μ for $\mu \geq \mu_0$ but μ_0. Thus, we see that $\|\tilde{\zeta}\|_{L^\infty_{\mathrm{loc}}} = O(\frac{1}{\mu})$ since $\|\frac{1}{\theta\mu} R^{-1}_{\theta,\mu} N_\theta\|_{L^\infty_{\mathrm{loc}}} = \frac{1}{|\theta\mu|}\|N_{\theta-1/\mu}\|_{L^\infty_{\mathrm{loc}}} = O(\frac{1}{\mu})$ and $\|\frac{1}{\mu} R^{-1}_{\theta,\mu}\delta Q_\theta^{-1}\|_{L^\infty_{\mathrm{loc}}} = \frac{1}{\mu}\|\delta Q^{-1}_{\theta-1/\mu}\|_{L^\infty_{\mathrm{loc}}} = O(\frac{1}{\mu})$ (cf. the formulae appearing in the proof of Lemma 2.1). Therefore μ is to be taken as the certainty level of the drift coefficient β and the worst case uncertainty $\tilde{\zeta}(x)$ would make the strategy $\overline{h}^{(\theta(\kappa))}(t, X_t, \tilde{\zeta}(X_t))$ in Theorem 6.1 robust to have the large deviation estimate at the certainty level. It is because, owing to the above order estimates, $\mu = \infty$ should correspond to the worst case uncertainty $\tilde{\zeta}(x) = 0$ which means β is the true drift coefficient.

7. Linear Gaussian case. We consider a linear Gaussian case where the following conditions are assumed:

$$\begin{aligned}
&\beta(x) = Bx + b, \quad g(x) = Ax + a, \quad U(x) = \tfrac{1}{2}\, x^* V x + m,\\
&a, b, m \text{ are constant vectors,}\\
&\lambda, \delta, S, A, B, V \text{ are all constant matrices}\\
&\qquad\qquad \text{such that } S \text{ is positive definite and } V \text{ is nonnegative definite.}
\end{aligned}$$

In this case the lower Isaacs equation (2.13) has the solution $u(t,x)$ of the following explicit form:

$$u(t,x) = \frac{1}{2}\, x^* P(t) x + q(t)^* x + l(t),$$

where $P(t)$ is the solution to the Riccati ordinary differential equation

$$\begin{cases}
\dot{P}(t) - \dfrac{1-\theta\mu}{\theta\mu}\, P(t)\lambda N_{\theta-1/\mu}\lambda^* P(t) + K_1^* P(t) + P(t) K_1 - C^*C + \theta V = 0\\
P(T) = 0
\end{cases}$$

with

$$K_1 = B + \Big(\theta - \frac{1}{\mu}\Big)\lambda\delta Q^{-1}_{\theta-1/\mu} A$$

$$C^*C = -\theta A^* Q^{-1}_{\theta-1/\mu} A, \quad C := \sqrt{-\theta}\sqrt{\delta Q^{-1}_{\theta-1/\mu}\delta^*}\,\delta(\delta^*\delta)^{-1} A$$

and $q(t)$ and $l(t)$ are solutions to the ordinary equations, respectively,

$$\begin{cases}
\dot{q}(t) + \Big(K_1^* - \dfrac{1-\theta\mu}{\theta\mu}\, P(t)\lambda N_{\theta-1/\mu}\lambda^*\Big) q(t) + P(t) b\\
\qquad\qquad - \theta A^*\delta Q^{-1}_{\theta-1/\mu} a - \dfrac{1-\theta\mu}{\mu}\, P(t)\lambda\delta Q^{-1}_{\theta-1/\mu} a = 0\\
q(T) = 0
\end{cases}$$

and

$$\begin{cases}
\dot{l}(t) + \dfrac{1}{2}\,\mathrm{tr}[\lambda\lambda^* P(t)] + \dfrac{1}{2}\Big(1 - \dfrac{1}{\theta\mu}\Big) q(t)\lambda N_{\theta-1/\mu}\lambda^* q(t)\\
\qquad\qquad + \Big(b + \Big(\theta - \dfrac{1}{\mu}\Big)\lambda\delta Q^{-1}_{\theta-1/\mu} a\Big)^* q(t) + \dfrac{\theta}{2}\, a^* Q^{-1}_{\theta-1/\mu} a + \theta m = 0\\
l(T) = 0.
\end{cases}$$

In this case, if we moreover assume that
(i) B is stable,
or
(ii) $\lambda\lambda^*$ and A^*A are positive definite,
then

$$P(t;T) \to \overline{P}, \quad q(t;T) \to \bar{q}, \qquad T \to \infty,$$

where $\overline{P}$ is the unique, negative semi-definite solution to the stationary equation

$$-\frac{1-\theta\mu}{\theta\mu}\,\overline{P}\lambda N_{\theta-1/\mu}\lambda^*\overline{P} + K_1^*\overline{P} + \overline{P}K_1 - C^*C + \theta V = 0 \tag{7.1}$$

such that

$$K_1 - \frac{1-\theta\mu}{\theta\mu}\,\lambda N_{\theta-1/\mu}\lambda^*\overline{P} \ \text{ is stable} \tag{7.2}$$

and $\bar{q}$ is the solution to

$$\Big(K_1^* - \frac{1-\theta\mu}{\theta\mu}\,\overline{P}\lambda N_{\theta-1/\mu}\lambda^*\Big)\bar{q} + \overline{P}b - \theta A^*\delta Q_{\theta-1/\mu}^{-1}a - \frac{1-\theta\mu}{\mu}\,\overline{P}\lambda\delta Q_{\theta-1/\mu}^{-1}a = 0. \tag{7.3}$$

Further,

$$\frac{l(0;T)}{T} \to \chi(\theta), \quad T \to \infty$$

$$\begin{aligned}\chi(\theta) = \frac{1}{2}\,\mathrm{tr}[\lambda\lambda^*\overline{P}] &+ \frac{1}{2}\Big(1 - \frac{1}{\theta\mu}\Big)\bar{q}\lambda N_{\theta-1/\mu}\lambda^*\bar{q} \\ &+ \Big(b + \Big(\theta - \frac{1}{\mu}\Big)\lambda\delta Q_{\theta-1/\mu}^{-1}a\Big)^*\bar{q} + \frac{\theta}{2}\,a^*Q_\theta^{-1}a + \theta m.\end{aligned} \tag{7.4}$$

Indeed, if condition i) is satisfied, then by setting $K = \frac{1-\theta\mu}{\mu}\,\delta Q_{\theta-1/\mu}^{-1}A$ we see that $K_1 + \lambda K = B$ is stable and therefore (K_1, λ) is stabilizable. While, by setting $K' = (\sqrt{-\theta} + \frac{1}{\sqrt{-\theta\mu}})\sqrt{\delta Q_{\theta-1/\mu}^{-1}\delta^*}\lambda^*$ we have $K_1^* + C^*K' = B^*$, which is stable. Thus, we see that (C, K_1) is detectable and it implies $(\sqrt{C^*C - \theta V}, K_1)$ is detectable. Hence, there exists a negative semi-definite solution $\overline{P}$ to (7.1) satisfying (7.2) (cf. [29]).

On the other hand, we assume ii). Then we set

$$\Gamma = K - \lambda^*(\lambda\lambda^*)^{-1}(B + I).$$

Since $K_1 + \lambda\Gamma = -I$, (K_1, λ) is seen to be stabilizable. Moreover, setting

$$\Gamma' = K' + \frac{1}{\sqrt{-\theta}}\,\sqrt{\delta Q_{\theta-1/\mu}^{-1}\delta^*}\,\delta(\delta^*\delta)^{-1}Q_{\theta-1/\mu}A(A^*A)^{-1}(-B^* - I),$$

we see that $K_1^* + C^*\Gamma' = -I$. Therefore, (C, K_1) is detectable and hence $(\sqrt{C^*C - \theta V}, K_1)$ is detectable, which ensures the existence of the solution to (7.1) satisfying (7.2) (cf. [29]).

Thus, in the both cases, we have a solution $(\chi(\theta), w(x))$ to H–J–B equation of ergodic type (3.16) such that $w(x) = \frac{1}{2}\,x^*\overline{P}x + \bar{q}^*x$ with the solution $\overline{P}$ of (7.1) and $\bar{q}$ of (7.3), and $\chi(\theta)$ is given by (7.4).

Further, differentiability of $\overline{P}$, $\bar{q}$, and $\chi(\theta)$ with respect to θ is seen in a similar way to Hata–Nagai–Sheu [10] and thus we can obtain the duality theorem.

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