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METRICS FOR MULTIVARIATE STABLE DISTRIBUTIONS

JOHN P. NOLAN

Department of Mathematics and Statistics, American University
Washington, DC 20912-8050, U.S.A.
E-mail: jpnolan@american.edu

Abstract. Metrics are proposed for the distance between two multivariate stable distributions. The first set of metrics are defined in terms of the closeness of the parameter functions of one dimensional projections of the laws. Convergence in these metrics is equivalent to convergence in distribution and an explicit bound on the uniform closeness of two stable densities is given. Another metric based on the Prokhorov metric between the spectral measures is related to the first metric. Consequences for approximation, simulation and estimation are discussed.

1. Introduction. A random variable X is stable if for all $n = 2, 3, 4, \ldots$, there are constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $X_1 + X_2 + \cdots + X_n \stackrel{d}{=} a_n X + b_n$, where $X_1, X_2, X_3, \ldots$ are i.i.d. copies of X. Univariate stable distributions are characterized by 4 parameters: two shape parameters $\alpha \in (0, 2]$ and $\beta \in [-1, 1]$ and a scale $\gamma > 0$ and location $\delta \in \mathbb{R}$. Because there are no closed formulas for general stable densities or distribution functions, they are usually specified by their characteristic function $\phi(u) = E \exp(iuX)$. There are multiple parameterizations used for stable distributions; we will discuss two. They are based on Zolotarev's (M) and (A) parameterizations, see [Z]. We will focus on a scale and location family that is continuous in all four parameters: $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 0)$ if

$$\phi(u) = \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 + i\beta \tan\frac{\pi\alpha}{2}(\operatorname{sign} u)(|\gamma u|^{1-\alpha} - 1)\right] + i\delta u\right), & \alpha \neq 1, \\ \exp\left(-\gamma |u| \left[1 + i\beta \frac{2}{\pi}(\operatorname{sign} u) \ln(\gamma |u|)\right] + i\delta u\right), & \alpha = 1. \end{cases}$$

The second parameterization is $X \sim \mathbf{S}(\alpha, \beta, \gamma, \delta; 1)$ if

$$\phi(u) = \begin{cases} \exp\left(-\gamma^\alpha |u|^\alpha \left[1 - i\beta \tan\frac{\pi\alpha}{2}(\operatorname{sign} u)\right] + i\delta u\right), & \alpha \neq 1, \\ \exp\left(-\gamma |u| \left[1 + i\beta \frac{2}{\pi}(\operatorname{sign} u) \ln |u|\right] + i\delta u\right), & \alpha = 1. \end{cases}$$

This second parameterization is more frequently used in the literature; it is more compact and has simple algebraic properties. However, it is poorly suited for our purposes here

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because there is a discontinuity at $\alpha = 1$. Also, it is not a scale and location distribution when $\alpha = 1$, which results in technical complications in the multivariate case. These ideas are discussed further in Section 3. For brevity, these different parameterizations will be called the 0-parameterization and the 1-parameterization.

A d-dimensional random vector $\mathbf{X} = (X^1, \ldots, X^d)$ is said to be stable if for all $n = 2, 3, 4, \ldots$, there is a constant $a_n > 0$ and a vector $\mathbf{b}_n \in \mathbb{R}^d$ such that $\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n \stackrel{d}{=} a_n\mathbf{X} + \mathbf{b}_n$, where $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \ldots$ are i.i.d. copies of $\mathbf{X}$. If $\mathbf{X}$ is a stable vector, then every one dimensional projection $\langle \mathbf{u}, \mathbf{X}\rangle = u_1X_1 + u_2X_2 + \cdots + u_dX_d$ has a univariate stable distribution, with a constant index of stability α and skewness $\beta(\mathbf{u})$, scale $\gamma(\mathbf{u})$ and shift $\delta(\mathbf{u})$ that depend on the direction $\mathbf{u}$, see [ST], Section 2.1. (The converse is true if $\alpha \geq 1$; when $\alpha < 1$ an extra condition is needed for the converse, see the discussion after Lemma 4.1 below.) We will call the functions $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot)$ the projection parameter functions. Since they uniquely determine all one dimensional projections, they determine the joint distribution via the Cramér-Wold device. As in the univariate case, there are multiple parameterizations possible for multivariate stable laws. We will say $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); k)$ for $k = 0$ or $k = 1$ if for every $\mathbf{u} \in \mathbb{R}^d$, $\langle \mathbf{u}, \mathbf{X}\rangle \sim \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}); k)$. The 0-parameterization is jointly continuous in the parameter functions, but the discontinuity of the 1-parameterization carries over to d-dimensions, see Section 3. Scaling properties (see below) of these parameter functions make it sufficient to know them on the unit sphere $\mathbb{S} = \{\mathbf{s} \in \mathbb{R}^d : |\mathbf{s}| = 1\}$.

Let $\mathbf{X}_1 \sim \mathbf{S}(\alpha_1, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0)$, $\mathbf{X}_2 \sim \mathbf{S}(\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_2(\cdot); 0)$, and for $1 \leq p \leq \infty$ define

$$\Delta_p(\mathbf{X}_1, \mathbf{X}_2) = |\alpha_1 - \alpha_2| + \|\beta_1(\cdot) - \beta_2(\cdot)\|_p + \|\gamma_1(\cdot) - \gamma_2(\cdot)\|_p + \|\delta_1(\cdot) - \delta_2(\cdot)\|_p,$$

where $\|\cdot\|_p$ is the $L^p(\mathbb{S}, d\mathbf{s})$ norm and $d\mathbf{s}$ is (unnormalized) surface area measure on $\mathbb{S}$ (not on $\mathbb{R}^d$). Each Δ_p is a metric since every term on the right hand side is a metric. The first result is that Δ_∞ meterizes convergence in distribution.

THEOREM 1.1. *Let* $\mathbf{X}_j \sim \mathbf{S}(\alpha_j, \beta_j(\cdot), \gamma_j(\cdot), \delta_j(\cdot); 0)$, $j = 1, 2, \ldots, \infty$. *Then* $\mathbf{X}_j \xrightarrow{d} \mathbf{X}_\infty$ *if and only if* $\Delta_\infty(\mathbf{X}_j, \mathbf{X}_\infty) \to 0$.

Proof. Lemma 4.1 below shows that all the projection parameter functions are uniformly continuous on compact $\mathbb{S}$ and have simple scaling properties. These scaling results show that the ch. f. $\phi_{\mathbf{X}_j}$ converges to $\phi_{\mathbf{X}_\infty}$ on $\mathbb{S}$ if and only if the convergence is uniform on any compact set. Hence $\Delta_\infty(\mathbf{X}_j, \mathbf{X}_\infty) \to 0$ if and only if $\phi_{\mathbf{X}_j} \to \phi_{\mathbf{X}_\infty}$ uniformly on any compact set, which is equivalent to $\mathbf{X}_j \xrightarrow{d} \mathbf{X}_\infty$. ∎

If two stable random vectors have densities, then the metric Δ_1 gives an explicit measure of closeness between the densities.

THEOREM 1.2. *Let* $\mathbf{X}_j \sim \mathbf{S}(\alpha_j, \beta_j(\cdot), \gamma_j(\cdot), \delta_j(\cdot); 0)$, $j = 1, 2$ *with*

$$\underline{\gamma} := \min(\inf_{\mathbf{s}\in\mathbb{S}} \gamma_1(\mathbf{s}), \inf_{\mathbf{s}\in\mathbb{S}} \gamma_2(\mathbf{s})) > 0. \tag{1}$$

Then the respective densities $f_1(\mathbf{x})$ *and* $f_2(\mathbf{x})$ *exist and*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \leq c_1\, \Delta_1(\mathbf{X}_1, \mathbf{X}_2),$$

where $c_1 = c_1(\alpha_1, \alpha_2, \underline{\gamma}, d)$ *is a positive constant.*

This implies uniform bounds on the difference of the cumulative distribution functions: the proof of Theorem 1(b) in [BNR] shows that if (1) holds, then for all Borel sets $A \subset \mathbb{R}^d$

$$|P(\mathbf{X}_1 \in A) - P(\mathbf{X}_2 \in A)| \le c(\alpha_1, \alpha_2, \underline{\gamma}, d)\Delta_1(\mathbf{X}_1, \mathbf{X}_2).$$

Since $(\mathbb{S}, d\mathbf{s})$ is a finite measure space, with total mass $\text{Area}(\mathbb{S}) = 2\pi^{d/2}/\Gamma(d/2)$, $\|\cdot\|_1 \le (2\pi^{d/2}/\Gamma(d/2))^{1-1/p}\|\cdot\|_p$ for any $p \in (1,\infty]$. So when (1) holds, we automatically get

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \le c_1'\, \Delta_p(\mathbf{X}_1, \mathbf{X}_2), \tag{2}$$

where $c_1' = \max(1, (2\pi^{d/2}/\Gamma(d/2))^{1-1/p})c_1$. Likewise, all the results below hold when Δ_1 is replaced by Δ_p, $p \in (1,\infty]$, with modified constants.

A multivariate stable distribution can also be described by a spectral measure Λ, a finite Borel measure on the unit sphere $\mathbb{S}$, and a shift vector $\boldsymbol{\delta} \in \mathbb{R}^d$. We will use the analog of the 0-parameterization defined above and write $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 0)$ to specify the distribution. (The precise meaning of this and the 1-parameterization in terms of spectral measures are defined in Section 4.) To compare two measures on $\mathbb{S}$ that may have different masses, define the extended Prokhorov distance between them by

$$\pi^*(\Lambda_1, \Lambda_2) = |\lambda_1 - \lambda_2| + \min(\lambda_1, \lambda_2)\pi\left(\lambda_1^{-1}\Lambda_1(\cdot), \lambda_2^{-1}\Lambda_2(\cdot)\right), \tag{3}$$

where $\lambda_i := \Lambda_i(\mathbb{S})$ and $\pi(\cdot,\cdot)$ is the Prokhorov metric, see [DP] and [DN]. The next result shows that if two spectral measures Λ_1 and Λ_2 are close in this metric and α_1 and α_2 are close, then the densities of the corresponding stable distributions are uniformly close.

THEOREM 1.3. *Let $\mathbf{X}_j \sim \mathbf{S}(\alpha_j, \Lambda_j, \boldsymbol{\delta}_j; 0)$, $j = 1, 2$. If (1) holds, then the respective densities exist and satisfy*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \le c_2 \left[|\alpha_1 - \alpha_2| + \pi^*(\Lambda_1, \Lambda_2)^{\max(\alpha_1,\alpha_2)/2} + |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2|\right],$$

where $c_2 = c_2(\alpha_1, \alpha_2, d, \underline{\gamma}, \lambda_1, \lambda_2)$ is a positive constant.

The next section examines the symmetric stable case of Theorem 1.2, where sharper results are possible. The third section proves Theorem 1.2 for the general, non-symmetric case. The fourth section relates $\Delta_\infty(\mathbf{X}_1, \mathbf{X}_2)$ to the Prokhorov metric between the respective spectral measures to prove Theorem 1.3 and some related results. The proofs of technical lemmas are collected in the last section.

We end this section with a brief discussion of previous work in this area and some motivation for our approach. In [BNR], it was shown that any multivariate stable distribution can be approximated by one with a discrete spectral measure. The original motivation for this was to give a numerically simpler case to work with when calculating multivariate stable densities. An algorithm to simulate exactly from a multivariate stable law with a discrete spectral measure is given in [MN], so showing that this class is dense is of practical interest. [DP] and [DN] showed that if two symmetric spectral measures were close in the extended Prokhorov metric, then their respective stable densities were uniformly close. They also compared symmetric stable densities with different α's. To handle the non-symmetric case, allowing different α's and non-symmetry, some parameterization like the 0-parameterization is needed. Because the 0-parameterization is used, the constants c_1 in Theorem 1.2 and c_2 in Theorem 1.3 are bounded in a neighborhood of $\alpha = 1$, unlike the results in [BNR]. It is impossible to do this in the 1-parameterization.

We emphasize the role of the parameter functions rather than the spectral measure for several reasons. First, the basic argument used in [BNR], [DP], and [DN] and here is to show that the characteristic functions of the two distributions are close in $L^1(\mathbb{R}^d, d\mathbf{x})$, which implies that the respective densities are close. This argument is conceptually simpler if phrased in terms of the parameter functions, and then those results are used to derive the ones for the spectral measure. Second, some estimation methods for multivariate stable distributions, e.g. [NPM], work by estimating the projection functions in multiple directions using univariate estimation on the projected data. If those are estimated accurately, then the approximation of the distribution is close. Third, if the distributions are specified in terms of the a stochastic integral representation, then the spectral measure is not known directly, but the parameter functions are. Finally, the only way in which the spectral measure enters the characteristic function is through the parameter functions. In particular, its nature as a measure, e. g. discrete or continuous, isn't crucial for closeness of the respective densities. (The tail behavior of stable laws is dependent on the nature of the spectral measure, but since the tails of the densities are uniformly small, that does not enter here.)

There are numerous positive constants used in the proofs below, which we denote by c_1, c_2, etc. While the bounds given are generally not optimal, the constants given are bounded in the neighborhood of $\alpha = 1$ and work for all dimensions $d \geq 1$. Finally, because the 0-parameterization used here is not standard, some detail is given about the relationship between this continuous parameterization and the standard one in Section 4.

2. Symmetric case. The symmetric α-stable case is a common special case, where the index α and the scale function $\gamma(\cdot)$ completely determine the distribution: the joint characteristic function is $\phi(\mathbf{u}) = \exp(-\gamma^\alpha(\mathbf{u}))$.

A necessary and sufficient condition for a multivariate stable random vector (symmetric or non-symmetric) $\mathbf{X}$ to have a density is that

$$\gamma_{\min} := \min_{\mathbf{u} \in \mathbb{S}} \gamma(\mathbf{u}) > 0. \tag{4}$$

If $\gamma_{\min} > 0$, then $\int_{\mathbb{R}^d} |\phi(\mathbf{u})| d\mathbf{u} \leq \int_{\mathbb{R}^d} \exp(-\gamma_{\min}^\alpha |\mathbf{u}|^\alpha) d\mathbf{u} < \infty$, so a density exists. Conversely, if $\gamma_{\min} = 0$, then by continuity of $\gamma(\cdot)$, there is a $\mathbf{u}_0 \in \mathbb{S}$ where $\gamma(\mathbf{u}_0) = 0$. By scaling (Lemma 4.1 below), $\gamma(r\mathbf{u}_0) = r\gamma(\mathbf{u}_0) = 0$ for all r, and thus the ch. f. of the univariate r.v. $\langle \mathbf{u}_0, \mathbf{X} \rangle$ satisfies $|E \exp(ir\langle \mathbf{u}_0, \mathbf{X} \rangle)| = 1$. Hence $\langle \mathbf{u}_0, \mathbf{X} \rangle$ is degenerate, i.e. the components of $\mathbf{X}$ are linearly dependent so $\mathbf{X}$ is supported on a proper subset of $\mathbb{R}^d$.

THEOREM 2.1. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be α-symmetric stable d-dimensional random vectors with respective scale functions $\gamma_1(\cdot)$ and $\gamma_2(\cdot)$ and respective densities $f_1(\cdot)$ and $f_2(\cdot)$. Then*

$$\sup_{\mathbf{x} \in \mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \leq \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \int_{\mathbb{S}} \left| \gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s}) \right| d\mathbf{s}.$$

Proof. The assumption that both densities exist is equivalent to the statements $\gamma_{\min}(\mathbf{X}_1) > 0$ and $\gamma_{\min}(\mathbf{X}_2) > 0$, so all the terms below are well defined. By the inversion formula, the density is given by

$$f(\mathbf{x}) = f(\mathbf{x}|\alpha, \gamma(\cdot)) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i\langle \mathbf{x}, \mathbf{u}\rangle} e^{-\gamma^\alpha(\mathbf{u})} d\mathbf{u}. \tag{5}$$

The proof uses the algebraic fact $|a-b| = [\text{sign}\,(a-b)](a-b)$. For any $\mathbf{x}$, a change to polar coordinates $\mathbf{u} = r\mathbf{s}$ in the inversion formula and the scaling property $\gamma(r\mathbf{s}) = r\gamma(\mathbf{s})$ show

$$\begin{aligned}
&|f_1(\mathbf{x}) - f_2(\mathbf{x})| \\
&\quad \le (2\pi)^{-d} \int_{\mathbb{S}} \int_0^\infty |e^{-\gamma_1^\alpha(\mathbf{s}) r^\alpha} - e^{-\gamma_2^\alpha(\mathbf{s}) r^\alpha}| r^{d-1} dr d\mathbf{s} \\
&\quad = (2\pi)^{-d} \int_{\mathbb{S}} \text{sign}\,[\gamma_2^\alpha(\mathbf{s}) - \gamma_1^\alpha(\mathbf{s})] \int_0^\infty \left(e^{-\gamma_1^\alpha(\mathbf{s}) r^\alpha} - e^{-\gamma_2^\alpha(\mathbf{s}) r^\alpha}\right) r^{d-1} dr d\mathbf{s} \\
&\quad = (2\pi)^{-d} \int_{\mathbb{S}} \text{sign}\,[\gamma_2^\alpha(\mathbf{s}) - \gamma_1^\alpha(\mathbf{s})] \left(\frac{\Gamma(d/\alpha)}{\alpha(\gamma_1^\alpha(\mathbf{s}))^{d/\alpha}} - \frac{\Gamma(d/\alpha)}{\alpha(\gamma_2^\alpha(\mathbf{s}))^{d/\alpha}}\right) d\mathbf{s} \\
&\quad = \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \int_{\mathbb{S}} \text{sign}\,\left(\gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s})\right)\left(\gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s})\right) d\mathbf{s} \\
&\quad = \frac{\Gamma(d/\alpha)}{\alpha(2\pi)^d} \int_{\mathbb{S}} \left|\gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s})\right| d\mathbf{s}. \ \blacksquare
\end{aligned}$$

We note that this bound is sharp: Corollary 4 in [AN] shows $f_j(0) = \Gamma(d/\alpha)/(\alpha(2\pi)^d) \cdot \int_{\mathbb{S}} \gamma_j^{-d}(\mathbf{s}) d\mathbf{s}$, so for any two scale functions with $\gamma_1(\mathbf{s}) \le \gamma_2(\mathbf{s})$, $f_1(0) - f_2(0)$ equals the upper bound in Theorem 2.1.

For a Gaussian r. vectors we have the following sharp bound.

Corollary 2.2. *Let $\mathbf{X}_i \sim \mathrm{N}(\mathbf{0}, \Sigma_i)$, $i = 1, 2$ be nondegenerate Gaussian, then*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \le \frac{\Gamma(d/2)}{2^{1+d/2}\pi^d} \int_{\mathbb{S}} |\langle \mathbf{s}, \Sigma_1 \mathbf{s}\rangle^{-d/2} - \langle \mathbf{s}, \Sigma_2 \mathbf{s}\rangle^{-d/2}| d\mathbf{s}.$$

Proof. For Gaussian laws, $\gamma_i(\mathbf{s}) = (\langle \mathbf{s}, \Sigma_i \mathbf{s}\rangle/2)^{1/2}$. $\blacksquare$

Corollary 2.3. *Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be symmetric α-stable with* (1) *holding. Then*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \le c_3(\alpha, d)\underline{\gamma}^{-d-1} \Delta_1(\mathbf{X}_1, \mathbf{X}_2).$$

Proof. In the symmetric case with $\alpha_1 = \alpha_2$, $\Delta_1(\mathbf{X}_1, \mathbf{X}_2) = \|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1$. The function $f(\gamma) = \gamma^{-d}$ is decreasing with $|f'(\gamma)| \le d\underline{\gamma}^{-d-1}$ on the interval $[\underline{\gamma}, \infty)$. Hence $|\gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s})| \le d\underline{\gamma}^{-d-1}|\gamma_1(\mathbf{s}) - \gamma_2(\mathbf{s})|$, and $\int_{\mathbb{S}} |\gamma_1^{-d}(\mathbf{s}) - \gamma_2^{-d}(\mathbf{s})| d\mathbf{s} \le d\underline{\gamma}^{-d-1} \int_{\mathbb{S}} |\gamma_1(\mathbf{s}) - \gamma_2(\mathbf{s})| d\mathbf{s}$. Using the previous theorem, we may take $c_3(\alpha, d) = d\Gamma(d/\alpha)/(\alpha(2\pi)^d)$. $\blacksquare$

Next we compare two symmetric stable distributions with different α's. The proof uses the following lemma.

Lemma 2.4. $\int_0^\infty |e^{-u^{\alpha_1}} - e^{-u^{\alpha_2}}| u^{d-1} du \le c_4(\min(\alpha_1, \alpha_2), d)|\alpha_1 - \alpha_2|$.

Theorem 2.5. *Let $\mathbf{X}_j$, $j = 1, 2$, be a symmetric α_i-stable random vectors satisfying* (1). *Then the respective densities $f_1(\cdot)$ and $f_2(\cdot)$ satisfy*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f_1(\mathbf{x}) - f_2(\mathbf{x})| \le c_5(\alpha_1, \alpha_2, d, \underline{\gamma}) \Delta_1(\mathbf{X}_1, \mathbf{X}_2).$$

Proof. For notational simplicity, assume $\alpha_1 < \alpha_2$. By symmetry, $\Delta_1(\mathbf{X}_1, \mathbf{X}_2) = |\alpha_1 - \alpha_2| + \|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1$. Then $|f_1(\mathbf{x}) - f_2(\mathbf{x})| \le |f(\mathbf{x}|\alpha_1, \gamma_1(\cdot)) - f(\mathbf{x}|\alpha_2, \gamma_1(\cdot))| + |f(\mathbf{x}|\alpha_2, \gamma_1(\cdot)) - f(\mathbf{x}|\alpha_2, \gamma_2(\cdot))|$. We note that $f(\mathbf{x}|\alpha_2, \gamma_1(\cdot))$ may not be a density, because while $\gamma_1(\cdot)$ is a valid scale function for an α_1-stable random vector, it may not be a valid scale function for an α_2-stable random vector. However, as functions these terms are defined by (5). For the first term above, use the preceding Lemma to show

$$
\begin{aligned}
&|f(\mathbf{x}|\alpha_1, \gamma_1(\cdot)) - f(\mathbf{x}|\alpha_2, \gamma_1(\cdot))| \\
&\qquad \le (2\pi)^{-d} \int_{\mathbb{R}^d} |e^{-\gamma_1^{\alpha_1}(\mathbf{u})} - e^{-\gamma_1^{\alpha_2}(\mathbf{u})}| d\mathbf{u} \\
&\qquad = (2\pi)^{-d} \int_{\mathbb{S}} \int_0^\infty |e^{-(\gamma_1(\mathbf{s})r)^{\alpha_1}} - e^{-(\gamma_1(\mathbf{s})r)^{\alpha_2}}| r^{d-1} dr d\mathbf{s} \\
&\qquad = (2\pi)^{-d} \int_{\mathbb{S}} \gamma_1(\mathbf{s})^{-d} \int_0^\infty |e^{-t^{\alpha_1}} - e^{-t^{\alpha_2}}| t^{d-1} dt d\mathbf{s} \\
&\qquad \le (2\pi)^{-d} \left(\int_{\mathbb{S}} \gamma_1(\mathbf{s})^{-d} d\mathbf{s} \right) c_4(\alpha_1, d)|\alpha_1 - \alpha_2| \le k(\alpha_1, d, \underline{\gamma})|\alpha_1 - \alpha_2|
\end{aligned}
$$

where $k = (2\pi)^{-d}\underline{\gamma}^{-d}\text{Area}(\mathbb{S})c_4(\alpha_1, d))$. Combining this with Corollary 2.3 yields the result:

$$
\begin{aligned}
|f_1(\mathbf{x}) - f_2(\mathbf{x})| &\le k(\alpha_1, d, \underline{\gamma})|\alpha_1 - \alpha_2| + c_3(\alpha_2, d)\underline{\gamma}^{-d-1}\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1 \\
&\le c_5 \left(|\alpha_1 - \alpha_2| + \|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1 \right),
\end{aligned}
$$

where $c_5 = \max(k, c_3\underline{\gamma}^{-d-1})$. ∎

COROLLARY 2.6. (a) *If $X_j \sim \mathbf{S}(\alpha, 0, \gamma_j, 0; 0)$, $j = 1, 2$ are non-degenerate univariate symmetric stable distributions, then their densities satisfy*

$$
\sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)| \le \frac{\Gamma(1/\alpha)}{\alpha\pi} |\gamma_1^{-1} - \gamma_2^{-1}| \le \frac{\Gamma(1/\alpha)}{\alpha\pi \min(\gamma_1, \gamma_2)^2} |\gamma_1 - \gamma_2|.
$$

(b) *If $X_j \sim \mathbf{S}(\alpha_j, 0, \gamma_j, 0; 0)$, $j = 1, 2$ are non-degenerate univariate symmetric stable distributions. Then their densities satisfy*

$$
\sup_{x \in \mathbb{R}} |f_1(x) - f_2(x)| \le c_5(\alpha_1, \alpha_2, 1, \min(\gamma_1, \gamma_2)) \left[|\alpha_1 - \alpha_2| + 2|\gamma_1 - \gamma_2| \right].
$$

As above, the first bound (a) is achieved at the origin, where $f_j(0) = \Gamma(1/\alpha)/(\alpha\pi\gamma_j)$.

3. Non-symmetric case. To compare non-symmetric univariate stable distributions with different α's in a uniform way requires a continuous parameterization of stable laws. (In the symmetric case, the 0- and 1-parameterizations coincide.) Using the standard parameterization makes it impossible to compare non-symmetric distributions with $\alpha_1 = 1$ and any nearby $\alpha_2 \ne 1$. While one can restrict the α's to be in either $(0, 1-\epsilon]$ or $[1+\epsilon, 2]$, this limits the generality of results and introduces constants that involve $\tan(\pi\alpha/2)$, which tend to ∞ as $\alpha \to 1$.

This problem carries over to non-symmetric multivariate stable distributions: roughly speaking when $\alpha \uparrow 1$, the center of the distribution shifts toward infinity in the direction where the spectral measure has the most mass; when $\alpha \downarrow 1$, the center shifts away from the direction where the spectral measure has the most mass. (See Lemma 4.1 below.)

To prove Theorem 1.2 with constants that are bounded in a neighborhood of $\alpha = 1$, the function

$$\eta(r,\alpha) = \begin{cases} \tan(\pi\alpha/2)(r - r^{<\alpha>}), & \alpha \neq 1, \\ \frac{2}{\pi} r \ln |r|, & \alpha = 1, \end{cases}$$

is used, where $r^{<\alpha>} := (\operatorname{sign} r)|r|^\alpha$ is the signed power function. The function η has the following properties.

Lemma 3.1. *The function $\eta(r,\alpha)$ is jointly continuous on $(-\infty,\infty) \times (0,2)$. Furthermore for all $0 < \alpha < 2$,*

(a) $|\eta(r,\alpha)| \leq \frac{2}{\pi}(1+r^2)$, $-\infty < r < \infty$
(b) $\left| r\frac{\partial \eta}{\partial r} \right| \leq \frac{4}{\pi}(1+r^2)$, $-\infty < r < \infty$
(c) $\left| \frac{\partial \eta}{\partial \alpha} \right| \leq \frac{\pi}{2}(1+r^2)$, $-\infty < r < \infty$
(d) $|\eta(r_1,\alpha) - \eta(r_2,\alpha)| \leq |r_1 - r_2|^{\alpha/2}$, $r_1, r_2 \in [-1,1]$

The proof of Theorem 1.2 is based on the following formula for multivariate stable densities from Theorem 1(b) (when $\alpha = 1$) and Theorem 2 (when $\alpha \neq 1$) of [AN]. Let $\mathbf{X} \sim \mathbf{S}(\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 0)$ be d dimensional with $\gamma_{\min}(\mathbf{X}) > 0$. Then the density of $\mathbf{X}$ is given by

$$f(\mathbf{x}) = f(\mathbf{x}|\alpha, \beta(\cdot), \gamma(\cdot), \delta(\cdot); 0) = \int_{\mathbb{S}} g\left(\frac{\langle \mathbf{x}, \mathbf{s} \rangle - \delta(\mathbf{s})}{\gamma(\mathbf{s})}, \alpha, \beta(\mathbf{s}) \right) \gamma^{-d}(\mathbf{s}) d\mathbf{s} \tag{6}$$

where

$$g(v,\alpha,\beta) = (2\pi)^{-d} \int_0^\infty \cos\left(vr + \beta\eta(r,\alpha)\right) r^{d-1} e^{-r^\alpha} dr.$$

Note that $g(v,\alpha,\beta) = g_d(v,\alpha,\beta;0)$, i.e. the formulas above depend on the dimension d and are phrased in the 0-parameterization, we suppress the d and the 0 to simplify notation. The following technical lemma gives bounds on the behavior of g.

Lemma 3.2. *The function $g(v,\alpha,\beta)$ has the following properties: for all $v \in \mathbb{R}$, $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $d \geq 1$,*

(a) $|g(v,\alpha,\beta)| \leq c_6(\alpha,d) := \Gamma(d/\alpha)/(\alpha(2\pi)^d)$.
(b) $|(\partial g/\partial v)(v,\alpha,\beta)| \leq c_7(\alpha,d) := \Gamma((d+1)/\alpha)/(\alpha(2\pi)^d)$.
(c) $|(\partial g/\partial \alpha)(v,\alpha,\beta)| \leq c_8(\alpha,d) := [(\pi/2)(\Gamma(d/\alpha) + \Gamma((d+2)/\alpha)) + \Gamma(1/\alpha)/(e(d+\alpha-1)) + \Gamma((d+3)/\alpha)]/(\alpha(2\pi)^d)$, *and* $c_8(\cdot,d)$ *is decreasing in* α.
(d) $|(\partial g/\partial \beta)(v,\alpha,\beta)| \leq c_9(\alpha,d) := (\Gamma(d/\alpha) + \Gamma((d+2)/\alpha))/(\alpha(2\pi)^d)$.
(e) $|v(\partial g/\partial v)(v,\alpha,\beta)| \leq c_{10}(\alpha,d) := [(d + (4/\pi))\Gamma(d/\alpha) + \Gamma((d+\alpha)/\alpha) + (4/\pi)\Gamma((d+2)/\alpha)]/\alpha$.

Theorem 3.3. *Let $\mathbf{X}_j \sim \mathbf{S}(\alpha_j, \beta_j(\cdot), \gamma_j(\cdot), \delta_j(\cdot); 0)$, $j = 1,2$ with $\alpha_1 \leq \alpha_2$ and (1) holds. Then*

$$\begin{aligned} \sup_{\mathbf{x} \in \mathbb{R}^d} & |f(\mathbf{x}|\alpha_1, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_2(\cdot); 0)| \\ & \leq c_{11}(\alpha_1, \underline{\gamma}, d)|\alpha_1 - \alpha_2| + c_{12}(\alpha_2, \underline{\gamma}, d)\|\beta_1(\cdot) - \beta_2(\cdot)\|_1 \\ & \quad + c_{13}(\alpha_2, \underline{\gamma}, d)\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1 + c_{14}(\alpha_2, \underline{\gamma}, d)\|\delta_1(\cdot) - \delta_2(\cdot)\|_1. \end{aligned}$$

Proof. For any $\mathbf{x} \in \mathbb{R}^d$,

$$\begin{aligned}
&|f(\mathbf{x}|\alpha_1, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_2(\cdot); 0)| \\
&\le |f(\mathbf{x}|\alpha_1, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0)| \\
&\quad + |f(\mathbf{x}|\alpha_2, \beta_1(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0)| \\
&\quad + |f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_1(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_1(\cdot); 0)| \\
&\quad + |f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_1(\cdot); 0) - f(\mathbf{x}|\alpha_2, \beta_2(\cdot), \gamma_2(\cdot), \delta_2(\cdot); 0)| \\
&\le \int_{\mathbb{S}} \left| g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_1(\mathbf{s})}, \alpha_1, \beta_1(\mathbf{s})\right) - g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_1(\mathbf{s})}, \alpha_2, \beta_1(\mathbf{s})\right)\right| \gamma_1^{-d}(\mathbf{s}) d\mathbf{s} \\
&\quad + \int_{\mathbb{S}} \left| g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_1(\mathbf{s})}, \alpha_2, \beta_1(\mathbf{s})\right) - g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_1(\mathbf{s})}, \alpha_2, \beta_2(\mathbf{s})\right)\right| \gamma_1^{-d}(\mathbf{s}) d\mathbf{s} \\
&\quad + \int_{\mathbb{S}} \left| g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_1(\mathbf{s})}, \alpha_2, \beta_2(\mathbf{s})\right) \gamma_1^{-d}(\mathbf{s}) - g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_2(\mathbf{s})}, \alpha_2, \beta_2(\mathbf{s})\right) \gamma_2^{-d}(\mathbf{s})\right| d\mathbf{s} \\
&\quad + \int_{\mathbb{S}} \left| g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_2(\mathbf{s})}, \alpha_2, \beta_2(\mathbf{s})\right) - g\left(\frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_2(\mathbf{s})}{\gamma_2(\mathbf{s})}, \alpha_2, \beta_2(\mathbf{s})\right)\right| \gamma_2^{-d}(\mathbf{s}) d\mathbf{s} \\
&:= A + B + C + D.
\end{aligned}$$

As noted in the proof of Theorem 2.5, some of the intermediate terms may not be densities, but are defined as functions by (6). Using Lemma 3.2(c),

$$A \le \int_{\mathbb{S}} \sup_{\alpha_1 \le \alpha \le \alpha_2} c_8(\alpha, d)|\alpha_1 - \alpha_2| \underline{\gamma}^{-d} d\mathbf{s} = c_{11}(\alpha_1, \underline{\gamma}, d)|\alpha_1 - \alpha_2|,$$

where $c_{11}(\alpha_1, \underline{\gamma}, d) = c_8(\alpha_1, d)\mathrm{Area}(\mathbb{S})\underline{\gamma}^{-d}$. Using Lemma 3.2(d),

$$B \le \int_{\mathbb{S}} c_9(\alpha_2, d)|\beta_1(\mathbf{s}) - \beta_2(\mathbf{s})| \underline{\gamma}^{-d} d\mathbf{s} = c_9(\alpha_2, d)\underline{\gamma}^{-d} \|\beta_1(\cdot) - \beta_2(\cdot)\|_1,$$

so $c_{12}(\alpha_2, \underline{\gamma}, d) = c_9(\alpha_2, d)\underline{\gamma}^{-d}$. For the next term, fix v,α, and β and consider the function defined by $h(\gamma) := g(v/\gamma, \alpha, \beta)\gamma^{-d}$. Then

$$\begin{aligned}
|h'(\gamma)| &= |-(\partial g/\partial v)(v/\gamma, \alpha, \beta) v \gamma^{-d-2} - d g(v/\gamma, \alpha, \beta)\gamma^{-d-1}| \\
&\le c_{10}(\alpha_2, d)\underline{\gamma}^{-d-2} + d c_6(\alpha_2, d)\underline{\gamma}^{-d-1} := c_{13}(\alpha_2, \underline{\gamma}, d)
\end{aligned}$$

where the last step uses Lemma 3.2. Hence

$$C \le \int_{\mathbb{S}} c_{13}(\alpha_2, \underline{\gamma}, d)|\gamma_1(\mathbf{s}) - \gamma_2(\mathbf{s})| d\mathbf{s} = c_{13}(\alpha_2, \underline{\gamma}, d)\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1.$$

For the last term, using Lemma 3.2(b),

$$\begin{aligned}
D &\le \int_{\mathbb{S}} c_7(\alpha_2, d) \left| \frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_1(\mathbf{s})}{\gamma_2(\mathbf{s})} - \frac{\langle \mathbf{x}, \mathbf{s}\rangle - \delta_2(\mathbf{s})}{\gamma_2(\mathbf{s})} \right| \gamma_2(\mathbf{s})^{-d} d\mathbf{s} \\
&\le c_7(\alpha_2, d)\underline{\gamma}^{-d-1} \int_{\mathbb{S}} |\delta_1(\mathbf{s}) - \delta_2(\mathbf{s})| d\mathbf{s} = c_{14}(\alpha_2, \underline{\gamma}, d)\|\delta_1(\mathbf{s}) - \delta_2(\mathbf{s})\|_1.
\end{aligned}$$

Note that for fixed d, the constants $c_6(\alpha, d)$, $c_7(\alpha, d)$, $c_8(\alpha, d)$, and $c_9(\alpha, d)$ are decreasing in α, so that $\alpha_1 \le \alpha_2$ will yield smaller constants. ∎

Theorem 1.2 follows from Theorem 3.3 with $c_1 = \max(c_{11}, c_{12}, c_{13}, c_{14})$. Also, the univariate case follows as in the symmetric case.

COROLLARY 3.4. *Let $X_j \sim \mathbf{S}(\alpha_j, \beta_j, \gamma_j, \delta_j; 0)$, $j = 1, 2$ be non-degenerate univariate stable distributions with $\alpha_1 \leq \alpha_2$ and $\underline{\gamma} = \min(\gamma_1, \gamma_2)$. Then their respective densities satisfy*

$$\begin{aligned}\sup_{x\in\mathbb{R}} &|f(x|\alpha_1, \beta_1, \gamma_1, \delta_1; 0) - f(x|\alpha_2, \beta_2, \gamma_2, \delta_2; 0)| \\ &\leq c_{11}(\alpha_1, \underline{\gamma}, 1)|\alpha_1 - \alpha_2| + c_{12}(\alpha_2, \underline{\gamma}, 1)|\beta_1 - \beta_2| \\ &\quad + c_{13}(\alpha_2, \underline{\gamma}, 1)|\gamma_1 - \gamma_2| + c_{14}(\alpha_2, \underline{\gamma}, 1)|\delta_1 - \delta_2|.\end{aligned}$$

We end this section by using the ideas above to define a distance from independence $\tau_{\perp,p}$ and a distance from sub-Gaussianity $\tau_{SG(R),p}$. For simplicity we will consider a symmetric stable r. vector $\mathbf{X} = (X_1, \ldots, X_d) \sim \mathbf{S}(\alpha, 0, \gamma(\cdot), 0; k)$. For the distance from independence, let $\mathbf{e}_j$ = the j^{th} standard unit basis vector for $\mathbb{R}^d$ and set $\gamma_j = \gamma(\mathbf{e}_j)$ = scale of X_j. For $0 < p \leq \infty$, define

$$\tau_{\perp,p} = \left\| \gamma^\alpha(\mathbf{u}) - \sum_{j=1}^{d} \gamma_j^\alpha |u_j|^\alpha \right\|_{L^p(\mathbb{S},d\mathbf{u})}.$$

It is clear from the proofs above that $\tau_{\perp,p} = 0$ for some (all) p if and only if the components of $\mathbf{X}$ are independent. And $\tau_{\perp,p}$ increases as $\mathbf{X}$ gets "further away" from independence.

For the second quantity, let R be a nonnegative definite $d \times d$ matrix and define

$$\tau_{SG(R),p} = \|\gamma^\alpha(\mathbf{u}) - \langle \mathbf{u}, R\mathbf{u}\rangle^{\alpha/2}\|_{L^p(\mathbb{S},d\mathbf{u})}.$$

Then $\tau_{SG(R),p} = 0$ for some (all) p if and only if $\mathbf{X}$ is sub-Gaussian (SG) α-stable, with shape matrix R. In particular, $\tau_{SG(\gamma_0 I),p} = 0$ if and only if $\mathbf{X}$ is isotropic with scale parameter γ_0.

Note that unlike other measures of dependence (covariation, codifference, James orthogonality, etc.), $\tau_{\perp,p}$ characterizes independence for all α. Likewise, $\tau_{SG(R),p}$ characterizes sub-Gaussianity. In the Gaussian case with standardized components, independence is equivalent to isotropic; in the non-Gaussian stable case the two concepts are distinct. There are many types of dependence possible in the stable case.

Sample analogs of $\tau_{\perp,p}$ and $\tau_{SG(R),p}$ are defined by taking sample estimates of $\widehat{\alpha}$, $\widehat{\gamma}_j$, $j = 1, \ldots, d$ for the first case and $\widehat{\alpha}$ and $\widehat{R}$ for the second, and then approximating the integrals in the L^p norm by Riemann sums (or max if $p = \infty$). It is also possible to use this approach for other specified form of dependence, including skewness in the distribution. These ideas are explored in a related paper.

4. Closeness in terms of spectral measures. The standard way to characterize a stable vector is in terms of an index of stability α, a spectral measure Λ (a finite Borel measure on the unit sphere $\mathbb{S} = \{\mathbf{s} \in \mathbb{R}^d : |\mathbf{s}| = 1\}$), and a shift vector $\boldsymbol{\delta} \in \mathbb{R}^d$. There are multiple parameterizations possible; we describe the two that correspond to the one dimensional parameterizations described above. We will say $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; k)$, $k = 0, 1$ if its joint characteristic function is

$$\phi(\mathbf{u}) = E\exp(i\langle \mathbf{u}, \mathbf{X}\rangle) = \exp\left(-\int_{\mathbb{S}} \omega(\langle \mathbf{u}, \mathbf{s}\rangle|\alpha; k)\Lambda(d\mathbf{s}) + i\langle \mathbf{u}, \boldsymbol{\delta}\rangle\right),$$

where

$$\omega(u|\alpha;k) = \begin{cases} |u|^\alpha[1 + i(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}(|u|^{1-\alpha} - 1)] & k = 0,\ \alpha \neq 1 \\ |u|^\alpha[1 - i(\operatorname{sign} u)\tan\frac{\pi\alpha}{2}] & k = 1,\ \alpha \neq 1 \\ |u| + i\frac{2}{\pi}u\ln|u| & \alpha = 1, \end{cases}$$

see Section 2.3 of [ST] and [N]. These parameterizations are identical when $\alpha = 1$, and shifts of each other when $\alpha \neq 1$. The next result makes this precise and describes the projections of each parameterization in terms of the respective univariate parameterization.

LEMMA 4.1. *Let* $\mathbf{X}_0 \sim \mathbf{S}(\alpha, \Lambda_0, \boldsymbol{\delta}_0; 0)$ *and* $\mathbf{X}_1 \sim \mathbf{S}(\alpha, \Lambda_1, \boldsymbol{\delta}_1; 1)$.

(a) $\mathbf{X}_0 \stackrel{d}{=} \mathbf{X}_1$ *if and only if* $\Lambda_0 = \Lambda_1$ *and*

$$\boldsymbol{\delta}_1 = \begin{cases} \boldsymbol{\delta}_0 - \tan(\pi\alpha/2)\ \boldsymbol{\mu}, & \alpha \neq 1, \\ \boldsymbol{\delta}_0, & \alpha = 1, \end{cases}$$

where $\boldsymbol{\mu} = \boldsymbol{\mu}(\Lambda) = \int_{\mathbb{S}} \mathbf{s}\,\Lambda(d\mathbf{s}) = \left(\int_{\mathbb{S}} s_1\Lambda(d\mathbf{s}), \ldots, \int_{\mathbb{S}} s_d\Lambda(d\mathbf{s})\right)$.

(b) *Let* $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_0; 0) = \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}_1; 1)$. *For* $\mathbf{u} \in \mathbb{R}^d$, $\langle \mathbf{u}, \mathbf{X}\rangle \sim \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}; 0); 0) = \mathbf{S}(\alpha, \beta(\mathbf{u}), \gamma(\mathbf{u}), \delta(\mathbf{u}; 1); 1)$, *where*

$$\beta(\mathbf{u}) = \gamma^{-\alpha}(\mathbf{u}) \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} \Lambda(d\mathbf{s}),$$

$$\gamma^\alpha(\mathbf{u}) = \int_{\mathbb{S}} |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha \Lambda(d\mathbf{s}),$$

$$\delta(\mathbf{u}; 0) = \begin{cases} \langle \boldsymbol{\delta}_0, \mathbf{u}\rangle + \tan\frac{\pi\alpha}{2}\,(\beta(\mathbf{u})\gamma(\mathbf{u}) - \langle\ \boldsymbol{\mu}, \mathbf{u}\rangle), & \alpha \neq 1, \\ \langle \boldsymbol{\delta}_0, \mathbf{u}\rangle + \frac{2}{\pi}\beta(\mathbf{u})\gamma(\mathbf{u})\ln\gamma(\mathbf{u}) - \frac{2}{\pi}\int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \ln|\langle \mathbf{u}, \mathbf{s}\rangle| \Lambda(d\mathbf{s}), & \alpha = 1, \end{cases}$$

$$\delta(\mathbf{u}; 1) = \begin{cases} \langle \boldsymbol{\delta}_1, \mathbf{u}\rangle, & \alpha \neq 1, \\ \langle \boldsymbol{\delta}_1, \mathbf{u}\rangle - \frac{2}{\pi}\int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \ln|\langle \mathbf{u}, \mathbf{s}\rangle| \Lambda(d\mathbf{s}), & \alpha = 1. \end{cases}$$

(c) *The parameter functions have the following scaling properties: for* $r \in \mathbb{R}$, $\mathbf{u} \in \mathbb{R}^d$

$$\begin{aligned} \beta(r\mathbf{u}) &= (\operatorname{sign} r)\beta(\mathbf{u}), \\ \gamma(r\mathbf{u}) &= |r|\gamma(\mathbf{u}), \\ \delta(r\mathbf{u}; 0) &= r\delta(\mathbf{u}; 0), \\ \delta(r\mathbf{u}; 1) &= \begin{cases} r\delta(\mathbf{u}; 1), & \alpha \neq 1, \\ r\delta(\mathbf{u}; 1) - \frac{2}{\pi}(r\ln|r|)\langle \mathbf{u},\ \boldsymbol{\mu}\rangle, & \alpha = 1. \end{cases} \end{aligned}$$

Let $\mathbf{X}$ be a random vector with all univariate projections α-stable with $\alpha < 1$. Section 2.2 of [ST] shows that there are vectors $\mathbf{X}$ with this property that are not jointly stable. However, an extra simple condition does guarantee joint stability: let $\delta(\mathbf{u}; 1)$ be the location parameter in the 1-parameterization of the projection $\langle \mathbf{u}, \mathbf{X}\rangle$. If

$$\delta(\mathbf{u}; 1) = \langle \mathbf{u}, \boldsymbol{\delta}_1\rangle \tag{7}$$

for some $\boldsymbol{\delta}_1 \in \mathbb{R}^d$, then $\mathbf{X} - \boldsymbol{\delta}_1$ is strictly stable (recall $\alpha \neq 1$). Hence by [ST], Section 2.1, (7) and all one dimensional projections univariate stable implies $\mathbf{X}$ is jointly stable.

Fixing a spectral measure Λ and varying α illustrates the advantages of the 0-parameterization and disadvantages of the 1-parameterization for our purposes. As in the univariate case, the distribution is centered near $\boldsymbol{\delta}_0$ in the 0-parameterization for all α, but in the 1-parameterization it is shifted by $\tan(\pi\alpha/2)\, \boldsymbol{\mu}$ as $\alpha \to 1$. So, as $\alpha \to 1$, the location parameter functions $\delta(\cdot;1)$ do not converge to the $\alpha = 1$ values when the spectral measure is not symmetric. In contrast, the location functions do converge in the 0-parameterization: for a fixed spectral measure Λ, $\delta(\cdot;0)$ converges to the $\alpha = 1$ case as $\alpha \to 1$:

$$\begin{aligned}
&\tan \tfrac{\pi\alpha}{2} \left(\beta(\mathbf{u})\gamma(\mathbf{u}) - \langle\, \boldsymbol{\mu}, \mathbf{u}\rangle\right) \\
&= \tan \tfrac{\pi\alpha}{2} \left(\int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} \Lambda(d\mathbf{s}) \gamma(\mathbf{u})^{1-\alpha} - \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \Lambda(d\mathbf{s}) \right) \\
&= \int_{\mathbb{S}} \tan \tfrac{\pi\alpha}{2} \left[\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} - \langle \mathbf{u}, \mathbf{s}\rangle \right] \Lambda(d\mathbf{s}) \\
&\qquad\qquad\qquad - \tan \tfrac{\pi\alpha}{2} (1 - \gamma(\mathbf{u})^{1-\alpha}) \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} \Lambda(d\mathbf{s}) \\
&\to -\frac{2}{\pi} \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \ln |\langle \mathbf{u}, \mathbf{s}\rangle| \Lambda(d\mathbf{s}) + \frac{2}{\pi} \ln \gamma(\mathbf{u}) \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle \Lambda(d\mathbf{s}) \qquad \text{as } \alpha \to 1. \qquad (8)
\end{aligned}$$

As in the proof of Lemma 4.1, the integral in the last term is $\beta(\mathbf{u})\gamma(\mathbf{u})$ when $\alpha = 1$.

Lemma 4.1 tells how $\beta(\cdot)$, $\gamma(\cdot)$, $\delta(\cdot;0)$ and $\delta(\cdot;1)$ are defined in terms of Λ and $\boldsymbol{\delta}_0$ or $\boldsymbol{\delta}_1$. Conversely, given the parameter functions, a spectral measure Λ and shift are uniquely defined. Unfortunately, there is no explicit formula for Λ. In [NPM], a numerical method is described to get an approximate inverse, i.e. a discrete measure Λ^* which is close, in the metric π^*, to Λ. If Λ and the parameter functions are known, then the shift vectors can be recovered explicitly. Let $\boldsymbol{\mu} = (\mu^1, \ldots, \mu^d)$ be as in Lemma 4.1 and define $\boldsymbol{\nu} = (\nu^1, \ldots, \nu^d)$ by $\nu^j = \int_{\mathbb{S}} s_j \ln |s_j| \Lambda(d\mathbf{s})$. Substituting $\mathbf{u} = \mathbf{e}_j$ into Lemma 4.1 (b) shows

$$\delta(\mathbf{e}_j;0) = \begin{cases} \delta_0^j + \tan \frac{\pi\alpha}{2} \left[\beta(\mathbf{e}_j)\gamma(\mathbf{e}_j) - \mu^j\right], & \alpha \neq 1, \\ \delta_0^j + \frac{2}{\pi} \left[\nu^j - \beta(\mathbf{e}_j)\gamma(\mathbf{e}_j) \ln \gamma(\mathbf{e}_j)\right], & \alpha = 1, \end{cases}$$

$$\delta(\mathbf{e}_j;1) = \begin{cases} \delta_1^j, & \alpha \neq 1, \\ \delta_0^j + \frac{2}{\pi}\nu^j, & \alpha = 1, \end{cases}$$

which can be solved for $\boldsymbol{\delta}_0$ or $\boldsymbol{\delta}_1$.

The extended Prokhorov metric π^* defined by (3) is defined for all finite Borel measures on a Polish space (S, ρ) and has similar properties as the regular Prokhorov metric has on the space of probability measures. Propositions 1 and 2 of [DP] establish the following lemma. The statement below uses operator notation: for a Borel measure Λ and integrable function f, $\Lambda f := \int_S f(s)\Lambda(ds)$.

Lemma 4.2. *Let (S, ρ) be a Polish space.*

(a) *π^* metrizes vague convergence in the space of finite Borel measures on S.*

(b) *Let Λ_1 and Λ_2 be finite Borel measures on S with $\lambda_j := \Lambda_j(S)$. Assume for notational convenience that $\lambda_1 \leq \lambda_2$. If f is a complex valued function on S that is bounded, $|f(x)| \leq$*

$M < \infty$, and if $|f(s) - f(t)| \le c\rho(s,t)^p$ for some $0 < p \le 1$, then

$$|\Lambda_1 f - \Lambda_2 f| \le M(\lambda_2^{1-p} + c\lambda_1^{1-p})\pi^*(\Lambda_1, \Lambda_2)^p.$$

In particular, if f satisfies a Lipschitz condition $|f(s) - f(t)| \le c\rho(s,t)$, then

$$|\Lambda_1 f - \Lambda_2 f| \le M(1+c)\pi^*(\Lambda_1, \Lambda_2).$$

The above result can be applied to yield pointwise bounds on the closeness of the projection parameter functions.

LEMMA 4.3. *Let $\mathbf{X}_j \sim \mathbf{S}(\alpha, \Lambda_j, \boldsymbol{\delta}_j; 0)$, $j = 1,2$ with respective parameter functions $\beta_j(\cdot)$, $\gamma_j(\cdot)$, and $\delta_j(\cdot; 0)$. Let $\underline{\gamma} > 0$ be defined by* (1), *then*

$$\begin{aligned}
\|\beta_1(\cdot) - \beta_2(\cdot)\|_\infty &\le c_{15}(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\min(1,\alpha)},\\
\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_\infty &\le c_{16}(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\min(1,\alpha)},\\
\|\delta_1(\cdot;0) - \delta_2(\cdot;0)\|_\infty &\le |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2| + c_{17}(\alpha, \underline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\alpha/2}.
\end{aligned}$$

Proof of Theorem 1.3. Assume for notational convenience that $\alpha_1 \le \alpha_2$. On the finite measures space $\mathbb{S}$: $\|f\|_1 \le k_3\|f\|_\infty$, where $k_3 =$Area($\mathbb{S}$). Thus the L^∞ bounds on the parameter functions in Lemma 4.3 together with Theorem 3.3 show that for all $\mathbf{x}$,

$$\begin{aligned}
|f_1(\mathbf{x}) - f_2(\mathbf{x})| &\le c_{11}|\alpha_1 - \alpha_2| + c_{12}\|\beta_1(\cdot) - \beta_2(\cdot)\|_1 + c_{13}\|\gamma_1(\cdot) - \gamma_2(\cdot)\|_1\\
&\qquad + c_{14}\|\delta_1(\cdot;0) - \delta_2(\cdot;0)\|_1\\
&\le c_{11}|\alpha_1 - \alpha_2| + (c_{12}c_{15} + c_{13}c_{16})k_3\pi^*(\Lambda_1, \Lambda_2)^{\min(1,\alpha_2)}\\
&\qquad + c_{14}[c_{17}k_3\pi^*(\Lambda_1, \Lambda_2)^{\alpha_2/2} + k_3|\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2|]\\
&\le c_2[|\alpha_1 - \alpha_2| + \pi^*(\Lambda_1, \Lambda_2)^{\alpha_2/2} + |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2|],
\end{aligned}$$

where $c_2 = \max(c_{11}, (c_{12}c_{15} + c_{13}c_{16})k_3(|\lambda_1 - \lambda_2| + \min(\lambda_1, \lambda_2)), c_{14}c_{17}k_3, k_3c_{14})$ and we have used the fact that $\pi(\cdot,\cdot) \le 1$ for all probability measures, so $\pi^*(\Lambda_1, \Lambda_2) \le |\lambda_1 - \lambda_2| + \min(\lambda_1, \lambda_2)$. ∎

A case of particular interest is when the spectral measure is "discretized". While this is a special case of the above, it is possible to give a sharper result by treating it separately. Again let (S, ρ) be a Polish space. Let $\{A_j\}_{j\in J}$ be a partition of S and for each $j \in J$, let s_j be a point in A_j. The pair $(\{A_j\}, \{s_j\})$ is called a tagged partition of S. For any Borel measure Λ on S, define the discretization of Λ with respect to $(\{A_j\}, \{s_j\})$ as

$$\Lambda_{\text{disc}}(B) = \Lambda_{\text{disc}}(B; \{A_j\}, \{s_j\}) := \sum_{j\in J} 1_B(s_j)\Lambda(A_j).$$

Λ_{disc} is a discrete measure with mass $\Lambda(A_j)$ concentrated at the point s_j. Related to the tagged partition, define the radii $\rho_j = \sup_{s\in A_j} \rho(s, s_j)$. The following is a special case of Lemma 4.2, it is similar to Lemma 1 in [BNR].

LEMMA 4.4. *Using the notations above, if f satisfies $|f(s) - f(t)| \le c\rho(s,t)^p$ for some $0 < p \le 1$, then*

$$|\Lambda f - \Lambda_{\text{disc}} f| \le c\sum_{j\in J} \rho_j^p \Lambda(A_j).$$

If Λ is a finite measure with $\overline{\rho} = \sup_{j\in J} \rho_j < \infty$, then

$$|\Lambda f - \Lambda_{\mathrm{disc}} f| \le c\overline{\rho}^p \Lambda(S).$$

We now apply this to prove a sharpened version of Theorem 1.3 for the discretization of a spectral measure.

THEOREM 4.5. *Let Λ be any finite Borel measure on $\mathbb{S}$ and let Λ_{disc} be its discretization w.r.t. the tagged partition $(\{A_j\}, \{\mathbf{s}_j\})_{j=1}^m$. Let $\overline{\rho}$ be as in the previous lemma. Then the densities $f(\mathbf{x})$ of $\mathbf{X} \sim \mathbf{S}(\alpha, \Lambda, \boldsymbol{\delta}; 0)$ and $f_{\mathrm{disc}}(\mathbf{x})$ of $\mathbf{X}_{\mathrm{disc}} \sim \mathbf{S}(\alpha, \Lambda_{\mathrm{disc}}, \boldsymbol{\delta}; 0)$ satisfy*

$$\sup_{\mathbf{x}\in\mathbb{R}^d} |f(\mathbf{x}) - f_{\mathrm{disc}}(\mathbf{x})| \le c_{18}(\alpha, d, \underline{\gamma}, \Lambda(\mathbb{S}))\overline{\rho}^{\alpha/2}.$$

Proof. Let $\lambda = \Lambda(\mathbb{S}) = \Lambda_{\mathrm{disc}}(\mathbb{S})$, $\alpha_0 = \min(\alpha, 1)$, $\beta(\cdot), \gamma(\cdot), \delta(\cdot; 0)$ be the parameter functions for $\mathbf{X}$ and $\beta_{\mathrm{disc}}(\cdot), \gamma_{\mathrm{disc}}(\cdot), \delta_{\mathrm{disc}}(\cdot; 0)$ be the parameter functions for $\mathbf{X}_{\mathrm{disc}}$. Here $\underline{\gamma} = \min(\inf_{\mathbf{s}\in\mathbb{S}} \gamma(\mathbf{s}), \inf_{\mathbf{s}\in\mathbb{S}} \gamma_{\mathrm{disc}}(\mathbf{s}))$ and $\overline{\gamma} := \max(\sup_{\mathbf{s}\in\mathbb{S}} \gamma(\mathbf{s}), \sup_{\mathbf{s}\in\mathbb{S}} \gamma_{\mathrm{disc}}(\mathbf{s})) \le \lambda^{1/\alpha}$. The same approach and notation as in Lemma 4.3 yields the following bounds for the distances between the parameter functions.

For the scale functions,

$$\begin{aligned}
|\gamma(\mathbf{u}) - \gamma_{\mathrm{disc}}(\mathbf{u})| &\le \begin{cases} \alpha^{-1}\overline{\gamma}^{1-\alpha}|\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_{\mathrm{disc}}(\mathbf{u})|, & \alpha < 1 \\ \alpha^{-1}\underline{\gamma}^{1-\alpha}|\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_{\mathrm{disc}}(\mathbf{u})|, & \alpha \ge 1 \end{cases} \\
&\le \begin{cases} \alpha^{-1}\overline{\gamma}^{1-\alpha}\overline{\rho}^\alpha\lambda, & \alpha < 1 \\ \alpha^{-1}\underline{\gamma}^{1-\alpha}\overline{\rho}\lambda, & \alpha \ge 1 \end{cases} \le \begin{cases} \alpha^{-1}\lambda^{1/\alpha}\overline{\rho}^\alpha, & \alpha < 1 \\ \alpha^{-1}\underline{\gamma}^{1-\alpha}\lambda\overline{\rho}, & \alpha \ge 1 \end{cases} \\
&:= c_{19}(\alpha, \underline{\gamma}, \lambda)\overline{\rho}^{\alpha_0}
\end{aligned}$$

For the skewness functions,

$$\begin{aligned}
|\beta(\mathbf{u}) - \beta_{\mathrm{disc}}(\mathbf{u})| &= |h(\psi^\alpha(\mathbf{u}, \gamma^\alpha(\mathbf{u})) - h(\psi^\alpha_{\mathrm{disc}}(\mathbf{u}, \gamma^\alpha_{\mathrm{disc}}(\mathbf{u}))| \\
&\le \underline{\gamma}^{-\alpha}|\psi^\alpha(\mathbf{u}) - \psi^\alpha_{\mathrm{disc}}(\mathbf{u})| + \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha|\gamma^\alpha(\mathbf{u}) - \gamma^\alpha_{\mathrm{disc}}(\mathbf{u})| \\
&\le \begin{cases} \underline{\gamma}^{-\alpha}2\overline{\rho}^\alpha\lambda + \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha c_{19}\overline{\rho}, & \alpha < 1 \\ \underline{\gamma}^{-\alpha}\alpha\overline{\rho}\lambda + \underline{\gamma}^{-2\alpha}\lambda c_{19}\overline{\rho}, & \alpha \ge 1 \end{cases} \\
&:= c_{20}(\alpha, \underline{\gamma}, \lambda)\overline{\rho}^{\alpha_0}.
\end{aligned}$$

For the location functions, $|\delta(\mathbf{u}; 0) - \delta_{\mathrm{disc}}(\mathbf{u}; 0)| \le A + B$, where $A = |\int \eta(\langle\mathbf{u}, \mathbf{s}\rangle)\Lambda - \int \eta(\langle\mathbf{u}, \mathbf{s}\rangle)\Lambda_{\mathrm{disc}}| \le \lambda\overline{\rho}^{\alpha/2}$ and

$$\begin{aligned}
B &= |\eta_1(1/\gamma(\mathbf{u}))\psi^\alpha(\mathbf{u}) - \eta_1(1/\gamma_{\mathrm{disc}}(\mathbf{u}))\psi^\alpha_{\mathrm{disc}}(\mathbf{u})| \\
&\le \begin{cases} |\eta_1(1/\gamma(\mathbf{u}))|2\lambda\overline{\rho}^\alpha + \frac{2}{\pi}\underline{\gamma}^{-\alpha}|\gamma(\mathbf{u}) - \gamma_{\mathrm{disc}}(\mathbf{u})| \\ |\eta_1(1/\gamma(\mathbf{u}))|\alpha\lambda\overline{\rho} + \frac{2}{\pi}\underline{\gamma}^{-\alpha}|\gamma(\mathbf{u}) - \gamma_{\mathrm{disc}}(\mathbf{u})| \end{cases} \\
&\le \begin{cases} \left[\max(|\eta_1(1/\underline{\gamma})|, |\eta_1(\lambda^{-1/\alpha})|)2\lambda + \frac{2}{\pi}\underline{\gamma}^{-\alpha}c_{19}\right]\overline{\rho}^\alpha \\ \left[\max(|\eta_1(1/\underline{\gamma})|, |\eta_1(\lambda^{-1/\alpha})|)\alpha\lambda + \frac{2}{\pi}\underline{\gamma}^{-\alpha}c_{19}\right]\overline{\rho} \end{cases} \\
&:= c_{21}(\alpha, \underline{\gamma}, \lambda)\overline{\rho}^{\alpha_0}.
\end{aligned}$$

Thus $|\delta(\mathbf{u};0)-\delta_{\text{disc}}(\mathbf{u};0)|\le\lambda\overline{\rho}^{\alpha/2}+c_{21}\overline{\rho}^{\alpha_0}$. Hence, by Theorem 3.3, with $k=\text{Area}(\mathbb{S})$,

$$\begin{aligned}|f(\mathbf{x})-f_{\text{disc}}(\mathbf{x})| &\le c_{12}\|\beta(\cdot)-\beta_{\text{disc}}(\cdot)\|_1+c_{13}\|\gamma(\cdot)-\gamma_{\text{disc}}(\cdot)\|_1\\ &\qquad\qquad\qquad +c_{14}\|\delta(\cdot;0)-\delta_{\text{disc}}(\cdot;0)\|_1\\ &\le k(c_{12}c_{20}\overline{\rho}^{\alpha_0}+c_{13}c_{19}\overline{\rho}^{\alpha_0}+c_{14}(\lambda\overline{\rho}^{\alpha/2}+c_{21}\overline{\rho}^{\alpha_0}))\\ &\le k((c_{12}c_{20}+c_{13}c_{19}+c_{14}c_{21})\lambda^{\alpha_0-\alpha/2}+c_{14}\lambda)\overline{\rho}^{\alpha/2}\\ &:= c_{18}(\alpha,d,\underline{\gamma},\Lambda(\mathbb{S}))\overline{\rho}^{\alpha/2}.\end{aligned}$$

If Λ is symmetric, choosing a symmetrical partition makes both $\mathbf{X}$ and $\mathbf{X}_{\text{disc}}$ symmetric around $\boldsymbol{\delta}$, in which case $\beta(\mathbf{u})=\beta_{\text{disc}}(\mathbf{u})=0$ and $\delta(\cdot;0)=\delta_{\text{disc}}(\cdot;0)$, so Corollary 2.3 shows

$$\sup_{\mathbf{x}}|f(\mathbf{x})-f_{\text{disc}}(\mathbf{x})|\le c_3\underline{\gamma}^{-d-1}\|\gamma(\cdot)-\gamma_{\text{disc}}(\cdot)\|_1\le c_3\underline{\gamma}^{-d-1}c_{19}\overline{\rho}^{\min(\alpha,1)}\ \blacksquare \tag{9}$$

To get an idea of how many terms are needed in a discrete approximating measure, consider the two dimensional case where Λ is uniform measure with total mass λ. Take the symmetric uniform partition $A_j=\{(\cos\theta,\sin\theta):(2j-3)\pi/n\le\theta<(2j-1)\pi/n\}$, $j=1,\dots,2n$, with tags $\mathbf{s}_j$ the midpoint of the arc A_j. Then some approximations and numerical bounds show $\overline{\rho}\le\pi/n$, $\underline{\gamma}\approx0.85(.75)^{\alpha}\lambda\ge\lambda/2$ and by (9)

$$\begin{aligned}\sup_{\mathbf{x}\in\mathbb{R}^d}|f(\mathbf{x})-f_{\text{disc}}(\mathbf{x})| &\le c_3(\alpha,2)\underline{\gamma}^{-d-1}c_{19}(\alpha,\underline{\gamma},\lambda)\overline{\rho}^{\min(\alpha,1)}\\ &\le\begin{cases}\left(\frac{\Gamma(2/\alpha)}{\alpha^2}\right)\lambda^{1/\alpha}\left(\frac{\pi}{n}\right)^{\alpha}, & \alpha<1,\\ \frac{\lambda\pi}{n}, & \alpha\ge1.\end{cases}\end{aligned}$$

This bound is not very sharp for two reasons: non-optimal constants in the proofs, and the fact that bounds on the maximum difference between parameter functions were used, whereas the exact difference between two stable densities at any point is the difference of an average of expressions involving the parameter functions (see Theorem 2.1 in the symmetric case and (6) in the non-symmetric case).

We end with a few general comments. If it is known that $|\alpha-1|\ge c>0$, then it is possible to replace the $\alpha/2$ power of π^* with the $\min(\alpha,1)$ power in Lemma 3.1. However, the constants involved in such an expression seem to be unbounded as $\alpha\to1$. Also, if $\alpha\to0$ or $\underline{\gamma}\to0$, many of the constants tend to ∞. The reason for this is that the densities of α-stable laws have very sharp peaks when either quantity tends to 0. Finally, these arguments are based on the form of the stable characteristic function and it is not clear how to measure closeness to a multivariate stable law for a random vector in its domain of attraction.

5. Proofs of lemmas. This section contains the proofs of the technical lemmas used above. To save space, some calculus details are left out of these arguments.

Proof of Lemma 2.4. For notational simplicity, assume $\alpha_1<\alpha_2$. For fixed $u>0$, set $h(\alpha)=\exp(-u^{\alpha})$. Then $h'(\alpha)=-(\ln u)u^{\alpha}\exp(-u^{\alpha})$. For $u>0$ and $0<\alpha_1\le\alpha\le\alpha_2\le2$, $|u\ln u|=|\eta(u,1)|\le(2/\pi)(1+u^2)$ (Lemma 3.1(a)) and $u^{\alpha-1}\exp(-u^{\alpha})\le u^{\alpha_1-1}\exp(-u^{\alpha_1})$, and thus $|h'(\alpha)|\le(2/\pi)(1+u^2)u^{\alpha_1-1}\exp(-u^{\alpha_1})$. Thus $|h(\alpha_1)-$

$h(\alpha_2)| \le (2/\pi)(1+u^2)u^{\alpha_1-1}\exp(-u^{\alpha_1})|\alpha_1-\alpha_2|$, and therefore

$$\begin{aligned} A &= \int_0^\infty |e^{-u^{\alpha_1}} - e^{-u^{\alpha_2}}|u^{d-1}du \\ &\le (2/\pi)|\alpha_1-\alpha_2| \int_0^\infty (1+u^2)u^{d+\alpha_1-2}e^{-u^{\alpha_1}}du = c_4(\alpha_1,d)|\alpha_1-\alpha_2|, \end{aligned}$$

where $c_4(\alpha_1,d) = 2/(\alpha_1\pi)\,(\Gamma((d+\alpha_1-1)/\alpha_1) + \Gamma((d+\alpha_1+1)/\alpha_1))$. ■

Proof of Lemma 3.1. Set $k(\alpha) = (1-\alpha)\tan(\pi\alpha/2)$ if $\alpha \ne 1$; $k(1) = 2/\pi$, and note $k(\alpha) \le k(1)$ and $k(\alpha) \to k(1)$ as $\alpha \to 1$. For $r > 0$, define

$$\eta_1(r,\alpha) = \frac{\eta(r,\alpha)}{r} = \begin{cases} \tan\frac{\pi\alpha}{2}(1-r^{\alpha-1}), & \alpha \ne 1, \\ (2/\pi)\ln r, & \alpha = 1. \end{cases}$$

For any $r > 0$, $\alpha \ne 1$,

$$\eta_1(r,\alpha) = k(\alpha)\frac{1-r^{\alpha-1}}{1-\alpha} \to k(1)\,\frac{d}{d\alpha}r^{\alpha-1}\bigg|_{\alpha=1} = \frac{2}{\pi}\ln r$$

as $\alpha \to 1$, showing continuity of η_1 in α. Since $\eta(r,\alpha) = r\eta_1(|r|,\alpha)$ for all r and all α, $(\eta(0,\alpha) := 0)$, joint continuity of η follows.

(a) Differentiation shows that for all $0 < \alpha < 2$ and all $r > 0$,

$$\partial\eta_1/\partial r = k(\alpha)r^{\alpha-2}, \tag{10}$$

$$\partial\eta/\partial r = \partial(r\eta_1)/\partial r = \eta_1(r,\alpha) + k(\alpha)r^{\alpha-1},$$

$$r\partial\eta/\partial r = \eta(r,\alpha) + k(\alpha)r^\alpha. \tag{11}$$

For all α, $\eta(0,\alpha) = \eta(1,\alpha) = 0$, with $\partial\eta/\partial r < 0$ for $r < r^*$ and $\partial\eta/\partial r > 0$ for $r > r^*$, where $r^* \in (0,1)$ is the unique root of $\partial\eta/\partial r$. It can be found explicitly:

$$r^* = r^*(\alpha) = \begin{cases} \alpha^{1/(1-\alpha)}, & \alpha \ne 1, \\ e^{-1}, & \alpha = 1. \end{cases}$$

Some calculation shows $|\eta(r^*,\alpha)| \le 1/2$ for all $\alpha \in (0,2)$ and thus for all $0 \le r \le 1$ and all $0 < \alpha < 2$, $|\eta(r,\alpha)| \le 1/2$. For $r \ge 1$, we claim $(1-r^{\alpha-1})/(1-\alpha) \le r-1$ for all $\alpha \ne 1$. To see this, set $h(r) = (r-1) - (1-r^{\alpha-1})/(1-\alpha)$, then $h(1) = 0$ and $h'(r) > 0$ for all $r \ge 1$ and all $\alpha \ne 1$. Hence

$$\eta(r,\alpha) = k(\alpha)r(1-r^{\alpha-1})/(1-\alpha) \le k(1)r(r-1).$$

When $\alpha = 1$, $\ln r \le r-1$ for all $r > 0$, so we also have $\eta(r,1) \le k(1)r(r-1)$. Thus for all $r \ge 1$ and all α, $\eta(r,1) \le (2/\pi)r(r-1)$. Adding the $0 \le r \le 1$ bound and the $r \ge 1$ bound, we conclude part (a) of the lemma:

$$|\eta(r,\alpha)| \le 1/2 + (2/\pi)r(r-1) \le (2/\pi)(1+r^2).$$

(b) Using (11), the above bound on $|\eta(r,\alpha)|$ and a simple bound for $|r|^\alpha$: $|r\partial\eta/\partial r| \le |\partial\eta/\partial r| + k(a)|r|^\alpha \le (2/\pi)(1+r^2) + (2/\pi)(1+r^2) = (4/\pi)(1+r^2)$.

(c) When $\alpha \neq 1$,

$$\begin{aligned}\frac{\partial \eta}{\partial \alpha}(r,\alpha) &= \sec^2(\pi\alpha/2)\left[\frac{\pi}{2}(r-r^\alpha)\right] - \tan\tfrac{\pi\alpha}{2} r^\alpha \ln r \\ &= \sec^2(\pi\alpha/2)\left[\frac{\pi}{2}(r-r^\alpha) - \cos(\pi\alpha/2)\sin(\pi\alpha/2) r^\alpha \ln r\right] \\ &= \frac{1}{2}\sec^2(\pi\alpha/2)\left[\pi(r-r^\alpha) - \sin(\pi\alpha) r^\alpha \ln r\right]\end{aligned}$$

Define $(\partial\eta/\partial\alpha)(r,1) = (1/\pi) r (\ln r)^2$ by continuity: for α near 1, $\sec(\pi\alpha/2) \approx \pi(1-\alpha)$, $\sin(\pi\alpha) \approx (\pi/2)(1-\alpha)$, and $1 - r^{\alpha-1} \approx (1-\alpha)\ln r - (1/2)((1-\alpha)\ln r)^2$.

For all $0 < \alpha \leq 2$ and $0 \leq r \leq 1$, $|(\partial\eta/\partial\alpha)(r,\alpha)| \leq |(\partial\eta/\partial\alpha)(r,0)| = (\pi/2)(1-r) \leq (\pi/2)$, while for $r \geq 1$ $|(\partial\eta/\partial\alpha)(r,\alpha)| \leq |(\partial\eta/\partial\alpha)(r,2)| = (\pi/2)(r^2-r) \leq (\pi/2)r^2$. Hence for all $r \geq 0$ and all $0 < \alpha \leq 2$, $|(\partial\eta/\partial\alpha)(r,\alpha)| \leq (\pi/2)(1+r^2)$.

(d) Since $\eta(\cdot,\alpha)$ is odd, and has maximal rate of change on $[-1,1]$ at the origin, $|\eta(r_1,\alpha) - \eta(r_2,\alpha)| \leq |\eta(|r_1-r_2|/2,\alpha) - \eta(-|r_1-r_2|/2,\alpha)| = 2|\eta(|r_1-r_2|/2,\alpha)|$. On $r \in [0,1]$, $|\eta(r,\alpha)| \leq (4/(\pi e)) r^{\alpha/2}$ (equality is achieved at $\alpha = 1$), so

$$|\eta(r_1,\alpha) - \eta(r_2,\alpha)| \leq 2(4/(\pi e))(|r_1-r_2|/2)^{\alpha/2} \leq |r_1-r_2|^{\alpha/2}. \blacksquare$$

Proof of Lemma 3.2. (a) $|g(v,\alpha,\beta)| \leq (2\pi)^{-d}\int_0^\infty r^{d-1}e^{-r^\alpha}\,dr = g(0,\alpha,0) = c_6(\alpha,d)$.

(b) For any α,

$$\begin{aligned}\left|\frac{\partial g}{\partial v}\right| &= \left|(2\pi)^{-d}\int_0^\infty -\sin(vr + \beta\eta(r,\alpha)) r^d e^{-r^\alpha}\,dr\right| \\ &\leq (2\pi)^{-d}\int_0^\infty r^d e^{-r^\alpha}\,dr = c_7(\alpha,d).\end{aligned}$$

(c) For $0 \leq r \leq 1$, $r^{d+\alpha-1}\ln r \leq k := (1/(e(d+\alpha-1)))$, while for $r > 1$, $r^{d+\alpha-1}\ln r \leq r^{d+\alpha}$; hence for all $r \geq 0$, $r^{d+\alpha-1}\ln r \leq k + r^{d+\alpha}$. Using Lemma 3.1(c) to bound $|\partial\eta/\partial\alpha|$ and the above shows

$$\begin{aligned}\left|\frac{\partial g}{\partial \alpha}\right| &= (2\pi)^{-d}\left|\int_0^\infty [-\sin(vr+\beta\eta(r,\alpha))\beta(\partial\eta/\partial\alpha) + \cos(vr+\beta\eta(r,\alpha)) \right. \\ &\qquad\qquad \left. \times(-r^\alpha \ln r)] r^{d-1} e^{-r^\alpha}\,dr\right| \\ &\leq (2\pi)^{-d}\int_0^\infty \left[(\pi/2)(1+r^2)r^{d-1} + k + r^{d+2}\right] e^{-r^\alpha}\,dr \\ &= \alpha^{-1}(2\pi)^{-d}\left[(\pi/2)(\Gamma(d/\alpha) + \Gamma((d+2)/\alpha)) + k\Gamma(1/\alpha) + \Gamma((d+3)/\alpha)\right].\end{aligned}$$

(d) For any α, Lemma 3.1(a) shows

$$\begin{aligned}\left|\frac{\partial g}{\partial \beta}\right| &= \left|(2\pi)^{-d}\int_0^\infty -\sin(vr+\beta\eta(r,\alpha))\eta(r,\alpha) r^{d-1} e^{-r^\alpha}\,dr\right| \\ &\leq (2\pi)^{-d}\int_0^\infty (1+r^2) r^{d-1} e^{-r^\alpha}\,dr = c_9(\alpha,d).\end{aligned}$$

(e) To get a bound on $v(\partial g_d/\partial v)$, start with

$$\frac{d}{dr}\cos(vr+\beta\eta(r,\alpha)) = -\sin(vr+\beta\eta(r,\alpha))\left[v + \beta\partial\eta/\partial r\right].$$

Solving this for $-v\sin(vr+\beta\eta(r,\alpha))$ and substituting in the first integral below gives

$$\begin{aligned}
v\frac{\partial g_d}{\partial v}(v,\alpha,\beta) &= \int_0^\infty -v\sin(vr+\beta\eta(r,\alpha))r^d e^{-r^\alpha}\,dr \\
&= \int_0^\infty \frac{d}{dr}\cos(vr+\beta\eta(r,\alpha))r^d e^{-r^\alpha}\,dr \\
&\quad + \beta\int_0^\infty \sin(vr+\beta\eta(r,\alpha))\frac{\partial\eta}{\partial r} r^d e^{-r^\alpha}\,dr \\
&= I_1 + I_2.
\end{aligned}$$

To bound I_1, integrate by parts

$$\begin{aligned}
I_1 &= \int_0^\infty \left[\frac{d}{dr}\cos(vr+\beta\eta(r,\alpha))\right] r^d e^{-r^\alpha}\,dr \\
&= \cos(vr+\beta\eta(r,\alpha))r^d e^{-r^\alpha}\Big|_0^\infty - \int_0^\infty \cos(vr+\beta\eta(r,\alpha))\left(dr^{d-1}-\alpha r^{d+\alpha-1}\right)e^{-r^\alpha}\,dr \\
&= 0 - d\int_0^\infty \cos(vr+\beta\eta(r,\alpha))r^{d-1}e^{-r^\alpha}\,dr + \alpha\int_0^\infty \cos(vr+\beta\eta(r,\alpha))r^{d+\alpha-1}e^{-r^\alpha}\,dr,
\end{aligned}$$

$$|I_1| \le d\int_0^\infty r^{d-1}e^{-r^\alpha}\,dr + \alpha\int_0^\infty r^{d+\alpha-1}e^{-r^\alpha}\,dr = (d/\alpha)\Gamma(d/\alpha) + \Gamma((d+\alpha)/\alpha).$$

To bound I_2, use Lemma 3.1(b):

$$\begin{aligned}
|I_2| &\le 1\cdot\int_0^\infty \left|r\frac{\partial\eta}{\partial r}\right| r^{d-1}e^{-r^\alpha}\,dr \\
&\le \int_0^\infty \frac{4}{\pi}(1+r^2)r^{d-1}e^{-r^\alpha}\,dr = \frac{4}{\alpha\pi}\left(\Gamma(d/\alpha)+\Gamma((d+2)/\alpha)\right).
\end{aligned}$$

Hence $v\partial g/\partial v(v,\alpha,\beta) \le c_{10}(\alpha,d)$ where $c_{10}(\alpha,d) = (d/\alpha)\Gamma(d/\alpha)+\Gamma((d+\alpha)/\alpha)+4/(\alpha\pi)\cdot(\Gamma(d/\alpha)+\Gamma((d+2)/\alpha))$ is independent of β and v. ∎

Proof of Lemma 4.1. (a) When $\alpha = 1$, there is nothing to prove. When $\alpha \ne 1$, $\omega(u|\alpha;0) = \omega(u|\alpha;1) + i\tan(\pi\alpha/2)u$, so

$$\begin{aligned}
&-\int_{\mathbb{S}} \omega(\langle\mathbf{u},\mathbf{s}\rangle;0)\Lambda_0(d\mathbf{s}) + i\langle\mathbf{u},\boldsymbol{\delta}_0\rangle \\
&= -\int_{\mathbb{S}} \left(\omega(\langle\mathbf{u},\mathbf{s}\rangle;1) + i\tan\tfrac{\pi\alpha}{2}(\langle\mathbf{u},\mathbf{s}\rangle)\right)\Lambda_0(d\mathbf{s}) + i\langle\mathbf{u},\boldsymbol{\delta}_0\rangle \\
&= -\int_{\mathbb{S}} \omega(\langle\mathbf{u},\mathbf{s}\rangle;1)\Lambda_0(d\mathbf{s}) + i\left(-\tan\tfrac{\pi\alpha}{2}\int_{\mathbb{S}}\langle\mathbf{u},\mathbf{s}\rangle\Lambda_0(d\mathbf{s}) + \langle\mathbf{u},\boldsymbol{\delta}_0\rangle\right) \\
&= -\int_{\mathbb{S}} \omega(\langle\mathbf{u},\mathbf{s}\rangle;1)\Lambda_0(d\mathbf{s}) + i\langle\mathbf{u}, -\tan\tfrac{\pi\alpha}{2}\,\boldsymbol{\mu}+\boldsymbol{\delta}_0\rangle.
\end{aligned}$$

If $\Lambda_0 = \Lambda_1$ and $\boldsymbol{\delta}_1 = -\tan\frac{\pi\alpha}{2}\,\boldsymbol{\mu}+\boldsymbol{\delta}_0$, then $\mathbf{X}_0$ and $\mathbf{X}_1$ have the same characteristic function, so $\mathbf{X}_0 \overset{d}{=} \mathbf{X}_1$. Conversely, if $\mathbf{X}_0 \overset{d}{=} \mathbf{X}_1$, then they have the same characteristic functions, which requires that $\boldsymbol{\delta}_1$ is related to $\boldsymbol{\delta}_0$ as above and by the uniqueness of spectral measures, the family $\{\langle\mathbf{u},\cdot\rangle, \mathbf{u}\in\mathbb{R}^d\}$ is a separating family, and hence $\Lambda_0 = \Lambda_1$.
(b) The formulas for $\beta(\cdot)$, $\gamma(\cdot)$ and $\delta(\cdot;1)$ are from Example 2.3.4 of [ST]. The formulas for $\delta(\cdot;0)$ follow from (a) and the relation for univariate parameterizations (see [N]): if

univariate $Y \sim \mathbf{S}(\alpha, \beta, \gamma, \delta_0; 0) = \mathbf{S}(\alpha, \beta, \gamma, \delta_1; 1)$, then

$$\delta_1 = \begin{cases} \delta_0 - \tan(\pi\alpha/2)\beta\gamma, & \alpha \neq 1, \\ \delta_0 - \frac{2}{\pi}\beta\gamma\ln\gamma, & \alpha = 1. \end{cases}$$

(c) Substituting $r\mathbf{u}$ into the definitions and simplifying yields these formulas. When $\alpha = 1$, use the fact that $\beta(\mathbf{u})\gamma(\mathbf{u}) = \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<1>} \Lambda(d\mathbf{s}) = \langle \mathbf{u}, \boldsymbol{\mu}\rangle$. ■

Proof of Lemma 4.3. For notational convenience, assume $\lambda_1 \leq \lambda_2$ and set $\alpha_0 = \min(1, \alpha)$. First consider the term involving the scale functions. For any $\mathbf{u} \in \mathbb{S}$, the function $f(\mathbf{s}) := |\langle \mathbf{u}, \mathbf{s}\rangle|^\alpha$ is bounded by 1 and for $\mathbf{s}, \mathbf{t} \in \mathbb{S}$ satisfies

$$|f(\mathbf{s}) - f(\mathbf{t})| \leq \begin{cases} |\mathbf{s} - \mathbf{t}|^\alpha, & \alpha < 1, \\ \alpha|\mathbf{s} - \mathbf{t}|, & \alpha \geq 1. \end{cases}$$

Lemma 4.2(b) applied to this f shows that uniformly in $\mathbf{u}$

$$|\gamma_1^\alpha(\mathbf{u}) - \gamma_2^\alpha(\mathbf{u})| \leq \begin{cases} (\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})\pi^*(\Lambda_1, \Lambda_2)^\alpha, & \alpha < 1, \\ (1+\alpha)\pi^*(\Lambda_1, \Lambda_2), & \alpha \geq 1. \end{cases} \tag{12}$$

Define $\overline{\gamma} = \max(\sup_{\mathbf{s}\in\mathbb{S}} \gamma_1(\mathbf{s}), \sup_{\mathbf{s}\in\mathbb{S}} \gamma_2(\mathbf{s}))$, so $\gamma_j(\mathbf{u}) \in [\underline{\gamma}, \overline{\gamma}]$, $j = 1, 2$. On the interval $[\underline{\gamma}^\alpha, \overline{\gamma}^\alpha]$, the derivative of the function $x \mapsto x^{1/\alpha}$ has derivative bounded by $\alpha^{-1}\overline{\gamma}^{1-\alpha}$ if $\alpha < 1$ and by $\alpha^{-1}\underline{\gamma}^{1-\alpha}$ if $\alpha \geq 1$, so we conclude that uniformly in $\mathbf{u}$,

$$|\gamma_1(\mathbf{u}) - \gamma_2(\mathbf{u})| \leq c_{16}^*(\alpha, \underline{\gamma}, \overline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0},$$

with

$$c_{16}^* = \begin{cases} \alpha^{-1}\overline{\gamma}^{1-\alpha}(\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha}), & \alpha < 1, \\ \alpha^{-1}\underline{\gamma}^{1-\alpha}(1+\alpha), & \alpha \geq 1. \end{cases}$$

To minimize the number of parameters, eliminate the dependence on $\overline{\gamma}$: $\gamma_j^\alpha(\mathbf{u}) \leq \int 1\Lambda_j(d\mathbf{s}) \leq \lambda_j$, so $\overline{\gamma} \leq \max(\lambda_1, \lambda_2)^{1/\alpha}$. Thus $c_{16}(\alpha, \underline{\gamma}, \lambda_1, \lambda_2) = c_{16}^*(\alpha, \underline{\gamma}, \max(\lambda_1, \lambda_2)^{1/\alpha})$ proves the result for the scale functions.

For the skewness term, adapt the previous argument to the signed power. For $x, y \in [-R, R]$,

$$|x^{<\alpha>} - y^{<\alpha>}| \leq \begin{cases} 2|x - y|^\alpha, & \alpha < 1, \\ \alpha R^{\alpha-1}|x - y|, & \alpha \geq 1. \end{cases}$$

Then for any $\mathbf{s}, \mathbf{t}, \mathbf{u} \in \mathbb{S}$,

$$|\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} - \langle \mathbf{u}, \mathbf{t}\rangle^{<\alpha>}| \leq \begin{cases} 2|\mathbf{s} - \mathbf{t}|^\alpha, & \alpha < 1, \\ \alpha|\mathbf{s} - \mathbf{t}|, & \alpha \geq 1. \end{cases}$$

Mimic the argument above for $\psi_j^\alpha(\mathbf{u}) := \int_{\mathbb{S}} \langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} \Lambda_j(d\mathbf{s})$ to show

$$|\psi_1^\alpha(\mathbf{u}) - \psi_2^\alpha(\mathbf{u})| \leq k_1(\alpha, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0}, \tag{13}$$

where

$$k_1(\alpha, \lambda_1, \lambda_2) = \begin{cases} 2(\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha}), & \alpha < 1, \\ (1+\alpha), & \alpha \geq 1. \end{cases}$$

Note that $\psi_j(\mathbf{u}) \in [-\overline{\gamma}, \overline{\gamma}]$. On the rectangle $\{(v,w) : -\overline{\gamma}^\alpha \le v \le \overline{\gamma}^\alpha, \underline{\gamma}^\alpha \le w \le \overline{\gamma}^\alpha\}$, the function $h(v,w) = v/w$ satisfies $|\partial h/\partial v| \le \underline{\gamma}^{-\alpha}$ and $|\partial h/\partial w| \le \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha$. Thus

$$
\begin{aligned}
&|\beta_1(\mathbf{u}) - \beta_2(\mathbf{u})| \\
&= |h(\psi_1^\alpha(\mathbf{u}), \gamma_1^\alpha(\mathbf{u})) - h(\psi_2^\alpha(\mathbf{u}), \gamma_2^\alpha(\mathbf{u}))| \\
&\le \underline{\gamma}^{-\alpha}|\psi_1^\alpha(\mathbf{u}) - \psi_2^\alpha(\mathbf{u})| + \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha|\gamma_1^\alpha(\mathbf{u}) - \gamma_2^\alpha(\mathbf{u})| \\
&\le \begin{cases} \left(\underline{\gamma}^{-\alpha}2(\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha}) + \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha(\lambda_1^{1-\alpha} + \lambda_2^{1-\alpha})\right)\pi^*(\Lambda_1, \Lambda_2)^\alpha, & \alpha < 1 \\ (\underline{\gamma}^{-\alpha} + \underline{\gamma}^{-2\alpha}\overline{\gamma}^\alpha)(1+\alpha)\pi^*(\Lambda_1, \Lambda_2), & \alpha \ge 1 \end{cases} \\
&=: c_{15}^*(\alpha, \underline{\gamma}, \overline{\gamma}, \lambda_1, \lambda_2)\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0}.
\end{aligned}
$$

To eliminate the value of $\overline{\gamma}$ from this constant, and the assumption that $\lambda_1 \le \lambda_2$, and we can set $c_{15}(\alpha, \underline{\gamma}, \lambda_1, \lambda_2) = c_{15}^*(\alpha, \underline{\gamma}, \max(\lambda_1, \lambda_2)^{1/\alpha}, \min(\lambda_1, \lambda_2), \max(\lambda_1, \lambda_2))$.

For the shift term, we have to consider the $\alpha \ne 1$ and the $\alpha = 1$ case separately. When $\alpha \ne 1$, using (8)

$$
\begin{aligned}
&|\delta_1(\mathbf{u}; 0) - \delta_2(\mathbf{u}; 0)| \\
&= \big|\langle \boldsymbol{\delta}_1 - \boldsymbol{\delta}_2, \mathbf{u}\rangle - \tan\tfrac{\pi\alpha}{2}\left(\beta_1(\mathbf{u})\gamma_1(\mathbf{u}) - \langle\ \boldsymbol{\mu}_1, \mathbf{u}\rangle\right) \\
&\qquad\qquad + \tan\tfrac{\pi\alpha}{2}\left(\beta_2(\mathbf{u})\gamma_2(\mathbf{u}) - \langle\ \boldsymbol{\mu}_2, \mathbf{u}\rangle\right)\big| \\
&\le |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2| \\
&\quad + \bigg|\int_{\mathbb{S}} \tan\tfrac{\pi\alpha}{2}\left[\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} - \langle \mathbf{u}, \mathbf{s}\rangle\right]\Lambda_1(d\mathbf{s}) \\
&\qquad\qquad - \int_{\mathbb{S}} \tan\tfrac{\pi\alpha}{2}\left[\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>} - \langle \mathbf{u}, \mathbf{s}\rangle\right]\Lambda_2(d\mathbf{s})\bigg| \\
&\quad + \bigg|\tan\tfrac{\pi\alpha}{2}(1 - \gamma_1(\mathbf{u})^{1-\alpha})\int_{\mathbb{S}}\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_1(d\mathbf{s}) \\
&\qquad\qquad - \tan\tfrac{\pi\alpha}{2}(1 - \gamma_2(\mathbf{u})^{1-\alpha})\int_{\mathbb{S}}\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_2(d\mathbf{s})\bigg| \\
&:= |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2| + A + B.
\end{aligned}
$$

Using Lemma 4.2 with the bounds on $\eta(\cdot, \alpha)$ in Lemma 3.1 shows

$$
A = \left|\int \eta(\langle \mathbf{u}, \mathbf{s}\rangle)\Lambda_1(d\mathbf{s}) - \int \eta(\langle \mathbf{u}, \mathbf{s}\rangle)\Lambda_2(d\mathbf{s})\right| \le 2(\lambda_2^{1-\alpha/2} + 1 \cdot \lambda_1^{1-\alpha/2})\pi^*(\Lambda_1, \Lambda_2)^{\alpha/2}.
$$

To bound B, use $\eta_1(\cdot) = \eta_1(\cdot, \alpha)$ from Lemma 3.1 with Lemma 4.2 and (13) to show

$$
\begin{aligned}
B &= \left|\eta_1(1/\gamma_1(\mathbf{u}))\int\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_1(d\mathbf{s}) - \eta_1(1/\gamma_2(\mathbf{u}))\int\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_2(d\mathbf{s})\right| \\
&= \bigg|\eta_1(1/\gamma_1(\mathbf{u}))\left(\int\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_1(d\mathbf{s}) - \int\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_2(d\mathbf{s})\right) \\
&\qquad + (\eta_1(1/\gamma_1(\mathbf{u})) - \eta_1(1/\gamma_2(\mathbf{u})))\int\langle \mathbf{u}, \mathbf{s}\rangle^{<\alpha>}\Lambda_2(d\mathbf{s})\bigg| \\
&\le |\eta_1(1/\gamma_1(\mathbf{u}))|k_1\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0} + \frac{2}{\pi}\underline{\gamma}^{2-\alpha}\left|\frac{1}{\gamma_1(\mathbf{u})} - \frac{1}{\gamma_2(\mathbf{u})}\right|\lambda_2
\end{aligned}
$$

$$\leq k_1 \max(|\eta_1(1/\underline{\gamma})|, |\eta(1/\overline{\gamma})|)\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0} + \frac{2}{\pi}\underline{\gamma}^{-\alpha}\lambda_2|\gamma_1(\mathbf{u}) - \gamma_2(\mathbf{u})|$$
$$\leq \left(k_1 \max(|\eta_1(1/\underline{\gamma})|, |\eta(\max(\lambda_1, \lambda_2)^{-1/\alpha})|) + \frac{2\lambda_2 c_{16}}{\pi\underline{\gamma}^{\alpha}}\right)\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0}.$$

Since $\pi^*(\Lambda_1, \Lambda_2) \leq k_2 := |\lambda_1 - \lambda_2| + \min(\lambda_1, \lambda_2)$, $\pi^*(\Lambda_1, \Lambda_2)^{\alpha_0} \leq k_2^{\alpha_0 - \alpha/2}\pi^*(\Lambda_1, \Lambda_2)^{\alpha/2}$. Hence $|\delta_1(\mathbf{u}; 0) - \delta_2(\mathbf{u}; 0)| \leq |\boldsymbol{\delta}_1 - \boldsymbol{\delta}_2| + c_{17}\pi^*(\Lambda_1, \Lambda_2)^{\alpha/2}$. ■

Proof of Lemma 4.4. Using the fact that $\{A_j\}$ partitions S,

$$\begin{aligned}|\Lambda f - \Lambda_{\mathrm{disc}} f| &= \left|\int_S f(t)\Lambda(dt) - \int_S f(t)\Lambda_{\mathrm{disc}}(dt)\right| \\ &= \left|\sum_{j\in J}\int_{A_j} f(t)\Lambda(dt) - \sum_{j\in J} f(s_j)\Lambda(A_j)\right| \\ &\leq \sum_{j\in J}\left|\int_{A_j} f(t)\Lambda(dt) - \int_{A_j} f(s_j)\Lambda(dt)\right| \\ &\leq \sum_{j\in J}\int_{A_j} |f(t) - f(s_j)|\Lambda(dt) \leq c\sum_{j\in J}\rho_j^p\Lambda(A_j).\end{aligned}$$

The case where Λ is finite follows directly. ■

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