ON TWO QUANTUM VERSIONS
OF THE DETAILED BALANCE CONDITION

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Abstract. Quantum detailed balance conditions are often formulated as relationships between the generator of a quantum Markov semigroup and the generator of a dual semigroup with respect to a certain scalar product defined by an invariant state. In this paper we survey some results describing the structure of norm continuous quantum Markov semigroups on $\mathcal{B}(\mathsf{h})$ satisfying a quantum detailed balance condition when the duality is defined by means of pre-scalar products on $\mathcal{B}(\mathsf{h})$ of the form $\langle x, y \rangle_s := \operatorname{tr}(\rho^{1-s} x^* \rho^s y)$ $(s \in [0,1])$ in order to compare the resulting quantum versions of the classical detailed balance condition. Moreover, we discuss the structure of generators of a quantum Markov semigroup which commute with the modular automorphism because this condition appears when we consider pre-scalar products with $s \neq 1/2$.

1. Introduction. Quantum detailed balance conditions generalise the well known notion of reversibility for classical Markov semigroups which is expressed by the symmetry in the L^2 space of an invariant measure.

The formulation for quantum open systems is not yet established and there are several versions involving different types of non-commutative symmetry and possibly also time reversal operations. The best known one for a norm continuous quantum Markov semigroup (QMS) $\mathcal{T}$ on the algebra $\mathcal{B}(\mathsf{h})$ of linear bounded operators on a complex separable Hilbert space h, with a faithful invariant normal state ρ was introduced by Alicki [4], [5]

(see also Frigerio, Gorini, Kossakowski, and Verri [16], [19]): a QMS $\mathcal{T}$ generated by $\mathcal{L}$ satisfies the quantum detailed balance condition if there exists a linear bounded operator $\widetilde{\mathcal{L}}$ on $\mathcal{B}(\mathsf{h})$ such that

$$\mathrm{tr}(\rho x \mathcal{L}(y)) = \mathrm{tr}(\rho \widetilde{\mathcal{L}}(x) y) \qquad \text{for all } x, y \in \mathcal{B}(\mathsf{h}) \tag{1}$$

$$\text{and} \qquad \mathcal{L} - \widetilde{\mathcal{L}} = 2i[K, \cdot] \tag{2}$$

for some self-adjoint operator $K \in \mathcal{B}(\mathsf{h})$.

Condition (1) means that $\widetilde{\mathcal{L}}$ is the dual operator of $\mathcal{L}$ with respect to the pre-scalar product $\langle x, y \rangle_0 := \mathrm{tr}(\rho x^* y)$ on $\mathcal{B}(\mathsf{h})$. The norm continuous semigroup $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{T}}_t)_{t \geq 0}$ generated by $\widetilde{\mathcal{L}}$ is called the *dual semigroup* of $\mathcal{T}$ and satisfies the corresponding equation $\mathrm{tr}(\rho x \mathcal{T}_t(y)) = \mathrm{tr}(\rho \widetilde{\mathcal{T}}_t(x) y)$ for all $t \geq 0$. Since $\mathcal{L}$ is conditionally completely positive by condition (2), the quantum detailed balance condition implies that the dual semigroup $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ is still a QMS. As a consequence, all the maps $\widetilde{\mathcal{T}}_t$ commute with the modular group $(\sigma_t)_{t \in \mathbb{R}}$ associated with ρ (see Prop. 2.1 of [19]), $\sigma_t(a) = \rho^{it} a \rho^{-it}$; this is a quite restrictive condition. Indeed, Theorem 8 and the following remark show that it determines the structure of the generator of the QMS and quantum detailed balance follows under a simple additional condition.

Other pre-scalar products on the algebra, that are equivalent to the previous one in the commutative case, can be defined through the state ρ. The most natural are $\langle x, y \rangle_s := \mathrm{tr}(\rho^{1-s} x^* \rho^s y)$ with $s \in [0, 1]$, and we call $s-$dual semigroup of $\mathcal{T}$ a norm continuous semigroup $\widetilde{\mathcal{T}}^{(s)}$ satisfying $\mathrm{tr}(\rho^{1-s} \widetilde{\mathcal{T}}_t^{(s)}(x) \rho^s y) = \mathrm{tr}(\rho^{1-s} x \rho^s \mathcal{T}_t(y))$ for all $t \geq 0$ and $x, y \in \mathcal{B}(\mathsf{h})$.

Clearly $\widetilde{\mathcal{T}}^{(0)} = \widetilde{\mathcal{T}}$. The other natural and popular choice is $s = 1/2$ (see for example [3], [17] and [18]); the $1/2$-dual semigroup $\mathcal{T}'$ is called *symmetric dual* (or *standard dual* or *KMS dual*) and is always a QMS i.e. the maps $\mathcal{T}'_t$ are completely positive irrespectively of the commutation with $(\sigma_t)_{t \geq 0}$. Another quantum detailed balance condition can be defined immediately by (1) and (2) replacing $\widetilde{\mathcal{L}}$ by the generator $\mathcal{L}'$ of the dual semigroup.

In this paper we compare the structure of generators of norm-continuous QMS on $\mathcal{B}(\mathsf{h})$ satisfying one of the above quantum versions of the classical detailed balance condition in order to understand which is the most natural and flexible. Moreover, we discuss the structure of generators of QMSs which commute with the modular group or, in an equivalent way, the modular automorphism σ_{-i}, because this condition appears as a consequence of the Alicki et al. quantum detailed balance condition.

Given a special GKSL representation of $\mathcal{L}$ of the form

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2} \sum_k \left(L_k^* L_k x - 2 L_k^* x L_k + x L_k^* L_k \right) \tag{3}$$

with $H = H^*, L_k \in \mathcal{B}(\mathsf{h})$ (see Section 2 for the precise definition), we first describe the structure of generators of QMSs whose dual semigroup is still a QMS: we find that the Hamiltonian H is forced to commute with ρ, and the operators L_k can be chosen as eigenvectors of the modular automorphism, i.e. $\rho L_k \rho^{-1} = \lambda_k L_k$ for some $\lambda_k > 0$ (see Theorem 4). A special GKSL representation of $\mathcal{L}$ with these properties is called *privileged*.

We stress the close relationship between the form of the L_k's and the structure of the spectrum of the modular automorphism (see Proposition 7). We also show the expression of the generator $\mathcal{L}$ in terms of the so-called *ladder operators* $(|e_j\rangle\langle e_k|)_{j,k}$, where $(e_i)_i$ is an orthonormal basis of eigenvectors of ρ (see Theorem 8). This result was obtained before by Frigerio, Kossakowski, Gorini and Verri in [19] under the stronger assumption of quantum detailed balance.

A QMS commuting with $(\sigma_t)_{t\geq 0}$ satisfies the quantum detailed balance condition $\mathcal{L} - \widetilde{\mathcal{L}} = 2i[K,\cdot\,]$ if and only if $H - K = c$ for some real constant c and there exists a unitary matrix $(u_{kl})_{kl}$ such that $\lambda_k^{-1/2} L_k^* = \sum_l u_{kl} L_l$ (Theorem 14). The structure of the generator of a detailed balance QMS was already analyzed in [4] under the additional hypothesis that $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ commute; here, we prove that this assumption is not necessary (see example 18). Moreover, we further characterize the generators of quantum detailed balance QMSs which commute with their dual map $\widetilde{\mathcal{L}}$ (Proposition 16), since in this case the generator $\mathcal{L}$ commutes with its derivation part $\delta_H(x) = [H,x]$ and it is possible to express the dual semigroup of $\mathcal{T}$ as $\widetilde{\mathcal{T}}_t(x) = \mathrm{e}^{-2it\delta_H}(\mathcal{T}_t(x)) = \mathcal{T}_t(\mathrm{e}^{-2it\delta_H}(x))$.

We also characterize QMSs satisfying the "symmetric" (also called standard [10] or KMS [8], [18]) quantum detailed balance condition $\mathcal{L} - \mathcal{L}' = 2i[K,\cdot\,]$, where $\mathcal{L}'$ is the generator of $\mathcal{T}'$ (Theorem 19): in particular, we show that this is equivalent to the symmetry of the completely positive part of $\mathcal{L}$, i.e. to the property

$$\mathrm{tr}(\rho^{1/2}\Phi(x)\rho^{1/2}y) = \mathrm{tr}(\rho^{1/2}x\rho^{1/2}\Phi(y)), \tag{4}$$

where $\Phi(x) := \sum_k L_k^* x L_k$ (see also [13] and [15]).

Although we do not find a definite answer the above results seem to indicate that the best known Alicki et al. quantum detailed balance condition is too restrictive and the "standard" or "symmetric" detailed balance condition is too weak because it involves only the completely positive part of the generator.

As a corollary of our results, we can also characterise (Theorem 21) generators of norm-continuous KMS-symmetric QMS, i.e. satisfying $\mathcal{L} = \mathcal{L}'$, investigated by Accardi and Mohari [2], Goldstein and Lindsay [18], Cipriani [8], Park [23] and the references therein.

2. General setting. Let h be a complex separable Hilbert space and consider a norm continuous quantum Markov semigroup (QMS) $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ on $\mathcal{B}(\mathsf{h})$ with a faithful normal invariant state $\rho = \sum_{j\geq 1} \rho_j |e_j\rangle\langle e_j|$ with $(e_j)_{j\geq 1}$ an orthonormal basis of h. We recall (Theorem 30.10 [24]) that the generator $\mathcal{L}$ of $\mathcal{T}$ admits a *special* GKSL representation with respect to ρ of the form (3) by means of operators $H = H^*$ and $(L_k)_{k\geq 1}$ in $\mathcal{B}(\mathsf{h})$ such that

  (i) $\mathrm{tr}(\rho L_k) = 0$ for each $k \geq 1$,
  (ii) $\sum_{k\geq 1} L_k^* L_k$ is strongly convergent,
  (iii) if $\sum_{k\geq 0} |c_k|^2 < \infty$ and $c_0 + \sum_{k\geq 1} c_k L_k = 0$ for scalars $(c_k)_{k\geq 0}$ then $c_k = 0$ for every $k \geq 0$.

Moreover, another family $\{H', L_k' : k \geq 1\}$ of bounded operators in $\mathcal{B}(\mathsf{h})$ with H' self-adjoint, satisfies equation (3) and conditions (i) $-$ (iii) if and only if the lengths of the

sequences $(L_k)_{k \geq 1}, (L'_k)_{k \geq 1}$ are equal and

$$H' = H + \alpha, \qquad L'_k = \sum_j u_{kj} L_j$$

for some scalar α and a unitary matrix $(u_{kj})_{kj}$.

Introducing the bounded operator

$$G = -iH - \frac{1}{2} \sum_k L_k^* L_k$$

on h, we can also write $\mathcal{L}$ as

$$\mathcal{L}(a) = G^* a + \sum_k L_k^* a L_k + aG; \tag{5}$$

we say that such a representation is *special* if the operators $H = (G^* - G)/2i$ and L_k give a special GKSL representation of $\mathcal{L}$. In this case, G is unique up to a purely imaginary multiple of the identity operator. Indeed, if $G', \{L'_k\}$ give another special representation of $\mathcal{L}$, then $G' = -iH' - \frac{1}{2} \sum_k L_k'^* L'_k$ fulfills

$$G' = -iH - i\alpha - \frac{1}{2} \sum_{k,j,m} \bar{u}_{kj} u_{km} L_j^* L_m = -iH - i\alpha - \frac{1}{2} \sum_j L_j^* L_j = G - i\alpha$$

for some $\alpha \in \mathbb{R}$, since the matrix $(u_{kj})_{kj}$ is unitary.

Finally, let k be a Hilbert space with Hilbertian dimension equal to the length of the sequence $(L_k)_k$ (the multiplicity space of the completely positive part of the generator) and let (f_k) be an orthonormal basis of k. Defining a linear bounded operator $L : \mathsf{h} \to \mathsf{h} \otimes \mathsf{k}$ by $Lu = \sum_k L_k u \otimes f_k$, any special GKSL representation of $\mathcal{L}$ can be expressed as (see Theorem 30.12 [24])

$$\mathcal{L}(a) = i[H, a] - \frac{1}{2} \left(L^* L a - 2L^* (a \otimes \mathbb{1}_\mathsf{k}) L + a L^* L \right) \tag{6}$$

for all $a \in \mathcal{B}(\mathsf{h})$, with $\mathrm{tr}(\rho L_k) = 0$ for all k and the set

(ii′) $\{(a \otimes \mathbb{1}_\mathsf{k}) L u : a \in \mathcal{B}(\mathsf{h}), \ u \in \mathsf{h}\}$ total in $\mathsf{h} \otimes \mathsf{k}$.

If H' and $L' : \mathsf{h} \to \mathsf{h} \otimes \mathsf{k}'$ give another special representation of $\mathcal{L}$ with the above properties, then

$$H' = H + \alpha, \qquad L' = (\mathbb{1}_\mathsf{h} \otimes V) L$$

for some $\alpha \in \mathbb{R}$ and $V : \mathsf{k} \to \mathsf{k}'$ a unitary isomorphism.

DEFINITION 1. The semigroup $\mathcal{T}$ is *s-symmetric* (with respect to ρ) if

$$\mathrm{tr}(\rho^{1-s} \mathcal{T}_t(a) \rho^s b) = \mathrm{tr}(\rho^{1-s} a \rho^s \mathcal{T}_t(b)) \tag{7}$$

for all $a, b \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$.

We stress that for $s = 1/2$ this is exactly the definition of KMS-symmetry with respect to the faithful state $\omega_\rho = \mathrm{tr}(\rho \cdot)$ (see for example [8], [18] and [23]). We refer to the lecture notes [9] for a discussion on this point.

DEFINITION 2. We say that $\mathcal{T}$ admits the *s-dual semigroup* with respect to ρ if there exists a norm continuous semigroup $\widetilde{\mathcal{T}}^{(s)} = (\widetilde{\mathcal{T}}_t^{(s)})_{t \geq 0}$ on $\mathcal{B}(\mathsf{h})$ such that

$$\mathrm{tr}(\rho^{1-s}\widetilde{\mathcal{T}}_t^{(s)}(a)\rho^s b) = \mathrm{tr}(\rho^{1-s}a\rho^s \mathcal{T}_t(b)) \tag{8}$$

for all $a, b \in \mathcal{B}(\mathsf{h})$ and $t \geq 0$.

When $s = 0$ (resp. $s = 1/2$) we denote $\widetilde{\mathcal{T}}^{(0)}$ (resp. $\widetilde{\mathcal{T}}^{(1/2)}$) by $\widetilde{\mathcal{T}}$ (resp. $\mathcal{T}'$) and we call it the *dual semigroup* (resp. *symmetric dual semigroup*).

The s-dual semigroup $\widetilde{\mathcal{T}}^{(s)}$ is uniquely determined by condition (8); moreover, $\widetilde{\mathcal{T}}^{(s)}$ and its generator $\widetilde{\mathcal{L}}^{(s)}$ satisfy the following properties (see [13], Proposition 3.1 and 3.2):

1. $\rho^{1-s}\widetilde{\mathcal{T}}_t^{(s)}(a)\rho^s = \mathcal{T}_{*t}(\rho^{1-s}a\rho^s)$ for all $t \geq 0$, and $a, b \in \mathcal{B}(\mathsf{h})$;
2. $\widetilde{\mathcal{T}}_t^{(s)}(\mathbf{1}) = \mathbf{1}$ and $\widetilde{\mathcal{T}}_{*t}^{(s)}(\rho) = \rho$ for all $t \geq 0$;
3. if $\widetilde{\mathcal{T}}_t^{(s)}$ is positive, then it is also normal;
4. $\rho^{1-s}\widetilde{\mathcal{L}}^{(s)}(a)\rho^s = \mathcal{L}_*(\rho^{1-s}a\rho^s)$ for all $a, b \in \mathcal{B}(\mathsf{h})$.

It is clear by 1. that

$$\widetilde{\mathcal{T}}_t^{(s)}(a) = \rho^{-(1-s)}\mathcal{T}_{*t}(\rho^{1-s}a\rho^s)\rho^{-s} \tag{9}$$

on the dense subset $\rho^s(\mathsf{h}) = \rho(\mathsf{h})$ of h; therefore, the symmetric dual semigroup (case $s = 1/2$) is completely positive and then it is a QMS thanks to property 3. However, if $s \neq 1/2$ the maps $\widetilde{\mathcal{T}}_t^{(s)}$ might not be positive and a fortiori not a QMS (see example 4.1 in [13]).

Moreover, if h is finite dimensional, equation (9) defines a norm continuous semigroup of bounded operators on $\mathcal{B}(\mathsf{h})$ fulfilling relation (8), and so the s-dual semigroup exists. In the general case the situation is more difficult and it is not yet clear when $\mathcal{T}$ admits its dual semigroup: namely, we showed in [15] that there exists a QMS which satisfies equation (8) for $s = 1/2$, but such a semigroup might not be norm continuous.

In the rest of paper, whenever we fix a value of s, we always assume that $\mathcal{T}$ admits its s-dual semigroup $\widetilde{\mathcal{T}}^{(s)}$.

We now focus our attention on the cases $s = 0$ and $s = 1/2$. Indeed all the cases $s \neq 1/2$ have the same features, as shown in [13], Section 8.

2.1. The quantum dual semigroup. In this section we analyze the duality for $s = 0$: by the general properties of $\widetilde{\mathcal{T}}^{(s)}$ we have

$$\rho\widetilde{\mathcal{T}}_t(a) = \mathcal{T}_{*t}(\rho a) \quad \text{and} \quad \rho\widetilde{\mathcal{L}}(a) = \mathcal{L}_*(\rho a). \tag{10}$$

Since $\widetilde{\mathcal{T}}_t$ is not in general a positive map, we start finding those QMSs with dual which is still a QMS and describing the relationship between special GKSL representations of $\mathcal{L}$ and $\widetilde{\mathcal{L}}$.

Adapting an argument from Majewski and Streater (see [20], proof of Theorem 6), we can show (Theorem 3.1 of [13]) the following:

THEOREM 3. *Let $\mathcal{A}$ be the set of analytic elements for $(\sigma_t)_{t \in \mathbb{R}}$. The dual semigroup $\widetilde{\mathcal{T}}$ is a QMS if and only if one of the equivalent conditions holds:*

(1) *each $\mathcal{T}_t$ commutes with the modular automorphism σ_{-i} on $\mathcal{A}$;*
(2) *$\mathcal{L}$ commutes with σ_{-i} on $\mathcal{A}$.*

In terms of operators H and L_k which give a special representation of $\mathcal{L}$, we have the following result (see Theorem 4.3 of [13]).

THEOREM 4. *The quantum dual semigroup* $\widetilde{\mathcal{T}}$ *is a QMS if and only if there exists a special GKSL representation of* $\mathcal{L}$ *by means of operators* $H, (L_k)_k$ *such that*

(1) $\rho L_k = \lambda_k L_k \rho$ *for some* $\lambda_k > 0$,
(2) H *commutes with* ρ.

In particular, each of the operators $L_k^* L_k$, G *and* H *commutes with* ρ.

DEFINITION 5. A special GKSL representation of $\mathcal{L}$ by means of operators $H, (L_k)_k$ is *privileged* if it satisfies conditions $(1), (2)$ of Theorem 4.

REMARK 6. Introducing the multiplicity space k of the completely positive part of $\mathcal{L}$, condition (1) of Theorem 4 is equivalent to

$$(\rho \otimes \mathbb{1}_{\mathsf{k}})L = (\mathbb{1}_{\mathsf{h}} \otimes D)L\rho,$$

with $D = diag(\lambda_1, \lambda_2, \dots) = \sum_k \lambda_k |f_k\rangle\langle f_k|$, where $(f_k)_k$ is an orthonormal basis of k.

Therefore, if H', L'_k give another privileged representation of $\mathcal{L}$, putting $L' : \mathsf{h} \to \mathsf{h} \otimes \mathsf{k}$, $L'u = \sum_k L'_k u \otimes f_k$, and $D' = diag(\lambda'_1, \lambda'_2, \dots) = \sum_k \lambda'_k |f_k\rangle\langle f_k|$, we get

$$H' = H + \alpha, \qquad L' = (\mathbb{1}_{\mathsf{h}} \otimes V)L, \quad D' = VDV^*.$$

for some unitary operator V on k and $\alpha \in \mathbb{R}$.

The identity $D' = VDV^*$ means, roughly speaking, that V is a change of coordinates that transforms D into another diagonal matrix. Hence, when the λ_k are all different, privileged GKSL representations of $\mathcal{L}$, if any, are unique up to a permutation of the operators L_k, a multiplication of each L_k by a phase $e^{i\theta_k}$ and a constant α in the Hamiltonian H. On the other hand, if some λ_k's are equal, then also unitary transformations of subspaces of k associated with the same λ_k's are allowed.

Condition (1) in Theorem 4 means that the operators L_k are eigenvectors of the modular automorphism σ_{-i}; since $L_k = \sum_{i,j} \langle e_i, L_k e_j \rangle |e_i\rangle\langle e_j|$ and the so-called *ladder operators* $|e_i\rangle\langle e_j|$ are always eigenvectors of σ_{-i} (with eigenvalues $\rho_i \rho_j^{-1}$), the form of the L_k's is then strictly connected with the structure of the spectrum of the modular automorphism.

PROPOSITION 7. *Suppose that* $\mathcal{T}$ *commutes with the modular automorphism and* $\rho L_k = \lambda L_k \rho$ *for a fixed* $\lambda > 0$.

1. *If there exists a unique pair* (i, j) *such that* $\rho_i \rho_j^{-1} = \lambda$, *i.e.* $\sigma_{-i}(|e_i\rangle\langle e_j|) = \lambda|e_i\rangle\langle e_j|$ *we have* $L_k = \alpha |e_i\rangle\langle e_j|$ *for some* $\alpha \in \mathbb{C}$.
2. *If there exists* $(i_1, j_1), \dots, (i_d, j_d)$ *such that* $\sigma_{-i}(|e_{i_l}\rangle\langle e_{j_l}|) = \lambda|e_{i_l}\rangle\langle e_{j_l}|$ *for* $l = 1, \dots, d$, *then* $L_k = \alpha_1 |e_{i_1}\rangle\langle e_{j_1}| + \cdots + \alpha_d |e_{i_d}\rangle\langle e_{j_d}|$ *for some* $\alpha_1, \dots, \alpha_d \in \mathbb{C}$.

Proof. Since $\rho = \sum_l \rho_l |e_l\rangle\langle e_l|$, the condition $\sigma_{-i}(|e_i\rangle\langle e_j|) = \lambda|e_i\rangle\langle e_j|$ is equivalent to $\rho_i \rho_j^{-1} = \lambda$. Moreover, by $\rho L_k = \lambda L_k \rho$ we have

$$\rho_l \langle e_l, L_k e_m \rangle = \lambda \rho_m \langle e_l, L_k e_m \rangle \tag{11}$$

for all l, m, so that $\langle e_l, L_k e_m \rangle \neq 0$ if and only if $\rho_l \rho_m^{-1} = \lambda$. ∎

It is worth noticing that, in case 1, the self-adjoint operator $-\log(\rho)$ is *generic* in the sense of Accardi et al. (see e.g. [1]).

The generator of a QMS $\mathcal{T}$ commuting with σ_{-i} can be expressed in terms of the ladder operators in the following way:

THEOREM 8. *The QMS $\mathcal{T}$ commutes with the modular automorphism if and only if its generator assumes the form*

$$\mathcal{L}(x) = i[H, x] + \text{uw-}\lim_{N} \sum_{rr'\,ss'=1}^{N} C_{rr'\,ss'}\left(P_{rr'}xP_{s's} - \frac{1}{2}\{P_{rr'}P_{s's}, x\}\right), \qquad (12)$$

where

(a) $P_{rr'} = |e_r\rangle\langle e_{r'}|$;
(b) $C_{rr'\,ss'}$ *satisfies* $\sum_{rr'\,ss'} \bar{\alpha}_{rr'} C_{rr'\,ss'} \alpha_{ss'} \geq 0$ *for all sequences of scalars* $(\alpha_{rr'})_{rr'}$ *for which the series converges;*
(c) $\sum_{rr'\,ss'=1}^{N} C_{rr'\,ss'} P_{rr'} P_{ss'}$ *is ultraweakly convergent as $N \to \infty$;*
(d) $C_{rr'\,ss'}\rho_{r'}\rho_s = C_{rr'\,ss'}\rho_r\rho_{s'}$;
(e) $H = H^* \in \mathcal{B}(\mathsf{h})$ *commutes with ρ.*

Proof. Suppose that $\mathcal{L}$ commutes with the modular automorphism and let

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\left(\{\Phi(\mathbb{1}), x\} - 2\Phi(x)\right)$$

be a privileged representation of $\mathcal{L}$, with $\Phi(x) = \sum_k L_k^* x L_k$. Defining

$$C_{rr'\,ss'} := \text{tr}(P_{sr}\Phi(P_{r's'})) = \sum_k \langle e_r, L_k^* e_{r'}\rangle\langle e_{s'}, L_k e_s\rangle < \infty,$$

property (b) follows immediately for finite sequences. Then it can be extended to sequences of scalars $(\alpha_{rr'})_{rr'}$ for which the series converges exploiting positivity of finite sums.

Since Φ commutes with the modular automorphism σ_{-i} and ladder operator P_{ij} are analytic, property (d) follows from

$$C_{rr'\,ss'} = \text{tr}(P_{sr}\rho\Phi(\rho^{-1}P_{r's'}\rho)\rho^{-1}) = \text{tr}(\rho_s^{-1}P_{sr}\rho_r\Phi(\rho_{r'}^{-1}P_{r's'}\rho_{s'}))$$
$$= C_{rr'\,ss'}\frac{\rho_r}{\rho_{r'}}\frac{\rho_{s'}}{\rho_s}.$$

Set $Q_N = \sum_{r=1}^{N} P_{rr}$, $E_N(x) = Q_N x Q_N$ and

$$\Phi_N(x) := E_N\Phi E_N(x) = \sum_k Q_N L_k^* Q_N x Q_N L_k Q_N = \sum_{rr'\,ss'=1}^{N} C_{rr'\,ss'} P_{rr'} x P_{s's}$$

for all $x \in \mathcal{B}(\mathsf{h})$, equation (12) is equivalent to

$$\text{uw-}\lim_{N}\left(\Phi_N(x) - \frac{1}{2}\{\Phi_N(\mathbb{1}), x\}\right) = \Phi(x) - \frac{1}{2}\{\Phi(\mathbb{1}), x\} \qquad (13)$$

for all $x \in \mathcal{B}(\mathsf{h})$.

Since $\text{uw-}\lim_N E_N(x) = x$ and Φ is normal, we get

$$\text{tr}(P_{ij}(\Phi_N(x) - \Phi(x))) = \text{tr}(P_{ij}(\Phi(E_N(x)) - \Phi(x))) \to_N 0$$

for all $i, j \in \{1, \ldots, N\}$, so that $\lim_N \operatorname{tr}(A\, (\Phi_N(x) - \Phi(x))) = 0$ for all A in the linear span of the P_{rs}'s; by the density of this set in the space of trace class operators on h, we can conclude that uw-$\lim_N \Phi_N(x) = \Phi(x)$, due to the boundedness of the sequence $(|\operatorname{tr}(A\, (\Phi_N(x) - \Phi(x))|)|)_N$.

In particular, taking $x = \mathbb{1}$ we obtain property (c), and equation (13) follows then immediately.

Conversely, assume that $\mathcal{L}$ is written in the form (12) and satisfies conditions (a)–(e). Thus we can write it as

$$\mathcal{L}(x) = i[H, x] - \frac{1}{2}\left(\{\Phi(\mathbb{1}), x\} - 2\Phi(x)\right)$$

with $\Phi(x) := \text{uw-}\lim_N \sum_{rr'\,ss'=1}^{N} C_{rr'\,ss'}\,(P_{rr'} x P_{s's}) = \text{uw-}\lim_N \Phi_N(x)$.

Φ is well defined and normal by (c); it is completely positive since any Φ_N is a completely positive map by (b). Therefore, $\Phi(x) = \sum_k L_k^* x L_k$ for some $L_k \in \mathcal{B}(\mathsf{h})$ such that $\sum_k L_k^* L_k$ is strongly convergent and $\mathcal{L}$ is represented in a GKSL form. To conclude we have to show that it commutes with the modular automorphism.

First of all, we note that

$$\sum_k \langle e_r, L_k^* e_{r'}\rangle\langle e_{s'}, L_k e_s\rangle = \operatorname{tr}(P_{sr}\Phi(P_{r's'})) = \lim_N \operatorname{tr}(P_{sr}\Phi_N(P_{r's'})) = C_{rr'\,ss'};$$

hence, by (d) follows $\operatorname{tr}(P_{sr}\Phi(P_{r's'})) = \operatorname{tr}(P_{sr}\rho\Phi(\rho^{-1}P_{r's'}\rho)\rho^{-1})$ for every r, r', s, s', which implies

$$\operatorname{tr}(P_{sr}\Phi(x)) = \operatorname{tr}(P_{sr}\rho\Phi(\rho^{-1}x\rho)\rho^{-1})$$

for all $x \in \mathcal{a}$ and r, s. This clearly means that Φ commutes with the modular automorphism, and so we can conclude by condition (e). ∎

REMARK 9. Notice that, according to [19] Theorem 2.2 p.99, the map $\mathcal{L}$ defined by (12) satisfies the quantum detailed balance condition in the sense of Alicki et al. (see Definition 12 with $s = 0$) $C_{rr'\,ss'}\rho_r = C_{s'sr'r}\rho_{r'}$ or, equivalently, $C_{rr'\,ss'}\rho_s = C_{s'sr'r}\rho_{s'}$. These identities yield immediately $C_{rr'\,ss'}\rho_{r'}\rho_s = C_{s'sr'r}\rho_{r'}\rho_{s'} = C_{rr'\,ss'}\rho_r\rho_{s'}$.

We end this section establishing the relationship between the privileged GKSL representations of a generator $\mathcal{L}$ and a privileged representation of its 0-dual $\widetilde{\mathcal{L}}$ (see Proposition 4.3 and Theorem 4.4 in [13]).

THEOREM 10. If $\widetilde{\mathcal{T}}$ is a QMS, then, for every privileged GKSL representation of $\mathcal{L}$, by means of operators H, L_k, there exists a privileged GKSL representation of $\widetilde{\mathcal{L}}$, by means of operators $\widetilde{H}, \widetilde{L}_k$ such that:

(1) $\widetilde{H} = -H - \alpha$ for some $\alpha \in \mathbb{R}$;
(2) $\widetilde{L}_k = \lambda_k^{-1/2} L_k^*$ for some $\lambda_k > 0$.

As a consequence, $\widetilde{G} = G^* + ic$ for some $c \in \mathbb{R}$.

2.2. The symmetric dual semigroup. We now study the case $s = 1/2$. We recall that the symmetric dual semigroup $\mathcal{T}'$ of $\mathcal{T}$ is defined by

$$\mathcal{T}'_t(a) = \rho^{-1/2}\mathcal{T}_{*t}(\rho^{1/2}a\rho^{1/2})\rho^{-1/2} \tag{14}$$

for all $a \in \mathcal{B}(\mathsf{h})$. The name symmetric is justified by the left-right symmetry of multiplication by $\rho^{1/2}$ and $\rho^{-1/2}$. Equation (14) ensures that any map $\mathcal{T}'_t$ is completely positive, contrary to the case $s = 0$. Therefore the symmetric dual semigroup $\mathcal{T}'$ is always a QMS with generator determined by

$$\mathcal{L}'(a) = \rho^{-1/2}\mathcal{L}_*(\rho^{1/2}a\rho^{1/2})\rho^{-1/2}.$$

It is well-known (see e.g. [8] or [13] Theorem 7.1 p. 358) that the symmetric dual $\mathcal{T}'$ coincides with the 0-dual $\widetilde{\mathcal{T}}$ if and only if each map $\mathcal{T}_t$ commutes with the modular automorphism σ_{-i}.

The relationship between the GKSL representation of generators $\mathcal{L}$ and $\mathcal{L}'$ is the following.

THEOREM 11. *For every special GKSL representation (5) of $\mathcal{L}$ there exists a special GKSL representation of $\mathcal{L}'$ by means of operators G', L'_k such that*

(1) $G'\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ *for some $c \in \mathbb{R}$,*
(2) $L'_k\rho^{1/2} = \rho^{1/2}L_k^*$.

Condition (2) ensures that the completely positive parts Φ and Φ' of generators $\mathcal{L}$ and $\mathcal{L}'$, respectively are in symmetric duality wrt ρ, i.e.

$$\mathrm{tr}(\rho^{1/2}\Phi'(a)\rho^{1/2}b) = \mathrm{tr}(\rho^{1/2}a\rho^{1/2}\Phi(b)) \tag{15}$$

for all $a, b \in \mathcal{B}(\mathsf{h})$. In particular, this is equivalent to $\rho^{1/2}\Phi'(a)\rho^{1/2} = \Phi_*(\rho^{1/2}a\rho^{1/2})$, Φ_* being the predual map of Φ.

3. Quantum s-detailed balance condition. Considering the pre-scalar products $\langle \cdot, \cdot \rangle_s$ on $\mathcal{B}(\mathsf{h})$ the usual definition of quantum detailed balance introduced in [4], [5], [16] and [19], can be generalised as follows:

DEFINITION 12. A QMS $\mathcal{T}$ on $\mathcal{B}(\mathsf{h})$ satisfies the *s-quantum detailed balance* condition with respect to a normal faithful invariant state ρ, if its generator $\mathcal{L}$ and the generator $\widetilde{\mathcal{L}}^{(s)}$ of the s-dual semigroup $\widetilde{\mathcal{T}}^{(s)}$ fulfill

$$\mathcal{L}(a) - \widetilde{\mathcal{L}}^{(s)}(a) = 2i\,[K, a] \tag{16}$$

for some bounded selfadjoint operator K on h and for all $a \in \mathcal{B}(\mathsf{h})$.

Clearly the identity (16) implies that $\widetilde{\mathcal{L}}^{(s)}$ is conditionally completely positive, so that the s-dual semigroup of a QMS satisfying the s-detailed balance condition is always a QMS. Moreover the selfadjoint operator K commutes with ρ, since ρ is an invariant state for both $\mathcal{T}$ and $\widetilde{\mathcal{T}}^{(s)}$.

The s-quantum detailed balance condition is equivalent to $\mathcal{L} - i[K, \cdot] = \widetilde{\mathcal{L}}^{(s)} + i[K, \cdot]$ for some selfadjoint operator K; therefore, by comparing the special GKSL representations of $\mathcal{L} - i[K, \cdot]$ and $\widetilde{\mathcal{L}}^{(s)} + i[K, \cdot]$ we find

$$G - iK = \widetilde{G}^{(s)} + iK + ic, \qquad \widetilde{L}_k^{(s)} = \sum_j u_{kj}L_j$$

for some unitary matrix $(u_{kj})_{kj}$ and some $c \in \mathbb{R}$.

Hence, a first characterization of the *s*-quantum detailed balance condition is the following:

THEOREM 13. *$\mathcal{T}$ satisfies the quantum s-detailed balance condition $\mathcal{L} - \widetilde{\mathcal{L}}^{(s)} = 2i[K, \cdot]$ if and only if the equivalent facts below hold:*

(i) *for all special GKSL representations of generators $\mathcal{L}$ and $\widetilde{\mathcal{L}}^{(s)}$ by means of operators G, L_k and $\widetilde{G}^{(s)}, \widetilde{L}_k^{(s)}$ respectively, we have*

   (1) *$\widetilde{G}^{(s)} = G - i(2K + c)$ for some $c \in \mathbb{R}$,*

   (2) *$\widetilde{L}_k^{(s)} = \sum_j u_{kj} L_j$ for some unitary matrix $(u_{kj})_{kj}$;*

(ii) *for all special GKSL representations of generators $\mathcal{L}$ and $\widetilde{\mathcal{L}}^{(s)}$ by means of operators H, L_k and $\widetilde{H}^{(s)}, \widetilde{L}_k^{(s)}$ respectively, we have*

   (1') *$\widetilde{H}^{(s)} = H - (2K + c)$ for some $c \in \mathbb{R}$,*

   (2') *$\widetilde{L}_k^{(s)} = \sum_j u_{kj} L_j$ for some unitary matrix $(u_{kj})_{kj}$.*

In the following we will specialize this result to the cases $s = 0$ and $s = 1/2$.

3.1. Quantum detailed balance condition. Assume now $s = 0$; in this case, we speak of *quantum detailed balance condition*.

Comparing Theorems 13 and 10, we can characterize the generator of a QMS satisfying the quantum detailed balance condition as follows:

THEOREM 14. *The following conditions are equivalent:*

(i) *$\mathcal{T}$ satisfies the detailed balance condition $\mathcal{L} - \widetilde{\mathcal{L}} = 2i[K, \cdot]$;*

(ii) *there exists a privileged GKSL representation of $\mathcal{L}$, by means of operators H, L_k, such that*

   (1) *$H = K + c$ for some $c \in \mathbb{R}$,*

   (2) *$\lambda_k^{-1/2} L_k^* = \sum_j u_{kj} L_j$ for some $\lambda_k > 0$ and some unitary operator $U = (u_{kj})_{kj}$ on k;*

(iii) *every privileged GKSL representation of $\mathcal{L}$ is given by operators H, L_k satisfying conditions (1), (2).*

In particular, the unitary matrix U satisfies $\bar{U} = U^*$.

Proof. For the equivalence of items (i)–(iii) see the proof of Theorem 5.1 in [13]. We show now only the last statement.

Since $\lambda_j^{-1/2} L_j = \sum_i \bar{u}_{ji} L_i^*$ by condition (2) in (ii), substituting in $\lambda_k^{-1/2} L_k^* = \sum_j u_{kj} L_j$ we have

$$\lambda_k^{-1/2} L_k^* = \sum_{ij} u_{kj} \bar{u}_{ji} \lambda_j^{1/2} L_i^*,$$

so that $\lambda_k^{-1/2} \delta_{ki} = \sum_j \lambda_j^{1/2} u_{kj} \bar{u}_{ji}$ from the linear independence of the L_k. This clearly means

$$D^{-1/2} = U D^{1/2} \bar{U}, \tag{17}$$

where $D = diag(\lambda_1, \lambda_2, \ldots)$. Moreover, set $L'_k := \lambda_k^{-1/2} L_k^*$, equation (2) in (ii) implies that the GKSL representation given by H, L'_k is also privileged and $\rho L'_k = \lambda_k^{-1} L'_k \rho$, that is $\lambda'_k = \lambda_k^{-1}$. Therefore, by Remark 6 we get

$$D^{-1} = U D U^*, \tag{18}$$

so that $D^{-1/2} = U D^{1/2} U^*$ by spectral calculus, since U is unitary.

The conclusion follows by substituting in equation (17). ∎

REMARK 15. As a consequence of the previous Theorem, if $\mathcal{T}$ satisfies the quantum detailed balance condition, the completely positive part $\Phi(x) = \sum_k L_k^* x L_k$ in a privileged representation of $\mathcal{L}$ is symmetric wrt ρ. Indeed,

$$\operatorname{tr}(\rho \Phi(x) y) = \sum_k \operatorname{tr}(\lambda_k^{-1} L_k^* \rho x L_k y)$$

$$= \sum_k \operatorname{tr}(\rho x (\lambda_k^{-1/2} L_k) y (\lambda_k^{-1/2} L_k^*)) = \operatorname{tr}(\rho x \Phi(y)),$$

since $\lambda_k^{-1/2} L_k^* = \sum_j u_{kj} L_j$ (and so $\lambda_k^{-1/2} L_k = \sum_j \bar{u}_{kj} L_j^*$) with $(u_{kj})_{kj}$ a unitary matrix.

Theorem 14 generalizes a result of Alicki ([4], Theorem 3), in which he characterized (in the finite-dimensional case) generators $\mathcal{L}$ satisfying the quantum detailed balance condition under the additional hypothesis that the operator $\mathcal{L}$ is normal, i.e. it commutes with $\widetilde{\mathcal{L}}$. This assumption implies that $\mathcal{T}$ commutes with the modular automorphism (see Lemma 3 of [4]), but it is not necessary, as shown by Example 18.

Indeed, note first that

PROPOSITION 16. *Assume $\mathcal{T}$ satisfies the quantum detailed balance condition and consider a privileged GKSL representation of $\mathcal{L}$ by means of operators $H, (L_k)_k$. Set $\delta_H = [H, \cdot]$, $\mathcal{L}_0 = \mathcal{L} - i\delta_H$ and $\Phi(x) = \sum_k L_k^* x L_k$, the statements below are equivalent:*

(i) *$\mathcal{L}\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}\mathcal{L}$;*
(ii) *$\mathcal{L}_0 \delta_H = \delta_H \mathcal{L}_0$;*
(iii) *$\Phi\delta_H = \delta_H \Phi$.*

Proof. By Theorem 14 we have $\widetilde{\mathcal{L}} = \mathcal{L} - 2i\delta_H$, so that

$$\mathcal{L}\widetilde{\mathcal{L}}(x) = \mathcal{L}^2(x) - 2i\mathcal{L}\delta_H(x) = \mathcal{L}^2(x) - 2i\mathcal{L}_0\delta_H(x) + 2\delta_H^2(x)$$
$$\widetilde{\mathcal{L}}\mathcal{L}(x) = \mathcal{L}^2(x) - 2i\delta_H\mathcal{L}(x) = \mathcal{L}^2(x) - 2i\delta_H\mathcal{L}_0(x) + 2\delta_H^2(x).$$

The equivalence (i) $\Leftrightarrow$ (ii) is now self-evident.

To conclude it is enough to note that $\mathcal{L}_0(x) = \Phi(x) - \frac{1}{2}\{\Phi(\mathbb{1}), x\}$ and $\Phi(\mathbb{1}) = \sum_k L_k^* L_k$ commutes with H by Theorem 4. ∎

REMARK 17. The previous proposition allows us to know when the quantum detailed balance condition can be expressed in a simple way in terms of $\mathcal{T}$ and its dual $\widetilde{\mathcal{T}}$; indeed, when $\mathcal{L}_0$ and the derivation δ_H commute, the property $\mathcal{L} - \widetilde{\mathcal{L}} = 2i[H, \cdot]$ is equivalent to

$$\widetilde{\mathcal{T}}_t = e^{t\widetilde{\mathcal{L}}} = e^{t\mathcal{L}} e^{-2i\delta_H} = \mathcal{T}_t e^{-2i\delta_H}.$$

Unfortunately, the following example shows that such a commutation is not, in general, a consequence of the quantum detailed balance condition.

EXAMPLE 18. Let $\mathsf{h} = \mathbb{C}^3$ with canonical basis $(|e_j\rangle)_{j=1}^3$ and let D be the operator $z|e_1\rangle\langle e_2| + w|e_2\rangle\langle e_3|$ on h with $z, w \in \mathbb{C} - \{0\}$. Put

$$L_1 = D, \qquad \text{and} \qquad L_2 = \lambda D^*$$

for some $\lambda > 0$, $\lambda \neq 1$, and consider the conditionally completely positive map $\mathcal{L}_0$ on $\mathcal{B}(\mathbb{C}^3)$ defined as in (3) with $H = 0$. A straightforward computation shows that a faithful normal state $\rho = \sum_{j=1}^3 \rho_j |j\rangle\langle j|$ satisfies $\mathcal{L}_{0*}(\rho) = 0$ if and only if

$$\lambda^2 = \rho_2/\rho_1 = \rho_3/\rho_2, \tag{19}$$

i.e., by the normalisation $\rho_1 + \rho_2 + \rho_3 = 1$,

$$\rho_j = \lambda^{2(j-1)}/(1 + \lambda^2 + \lambda^4), \qquad \text{for} \quad j = 1, 2, 3.$$

The operator D satisfies $\rho D = \lambda^{-2} D \rho$.

Consider the generator $\mathcal{L}$ of a QMS defined by

$$\mathcal{L}(x) = i[\rho, x] - \frac{1}{2} \sum_{k=1}^2 (L_k^* L_k x - 2 L_k^* x L_k + x L_k^* L_k) \tag{20}$$

that clearly admits ρ as a faithful invariant state. Moreover, since $\rho L_1 = \lambda^{-2} L_1 \rho$ and $\rho L_2 = \lambda^2 L_2 \rho$, with our notation we have $\lambda_1 = \lambda^{-2}$ and $\lambda_2 = \lambda^2$. Therefore equation (20) defines a privileged GKSL representation of $\mathcal{L}$ with $H = \rho$, and the quantum detailed balance condition is fulfilled because

$$\begin{pmatrix} \lambda_1^{-1/2} L_1^* \\ \lambda_2^{-1/2} L_2^* \end{pmatrix} = \begin{pmatrix} \lambda D^* \\ D \end{pmatrix} = \begin{pmatrix} L_2 \\ L_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix}$$

We finally show that $\mathcal{L}$ is not normal, i.e. by Proposition 16, that the completely positive part $\Phi(x) = \sum_{k=1}^2 L_k^* x L_k$ does not commute with δ_H. We start computing

$$\Phi \delta_H(x) = D^* \rho x D - D^* x \rho D + \lambda^2 D \rho x D^* - \lambda^2 D x \rho D^*,$$
$$\delta_H \Phi(x) = \rho D^* x D - D^* x D \rho + \lambda^2 \rho D x D^* - \lambda^2 D x D^* \rho$$
$$= \lambda^2 D^* \rho x D - \lambda^2 D^* x \rho D + D \rho x D^* - D x \rho D^*$$

and find that

$$\Phi(\delta_H(x)) - \delta_H(\Phi(x)) = (\lambda^2 - 1)(D[\rho, x]D^* - D^*[\rho, x]D)$$

which is non-zero, e.g. for $x = |e_1\rangle\langle e_2|$.

Note that ρ is the *unique* invariant state because the QMS is irreducible because the commutant of $\{H, D, D^*\}$, i.e. the fixed point algebra of the QMS generated by $\mathcal{T}$, is trivial (see [12]).

3.2. Quantum symmetric detailed balance conditions.

We conclude this discussion dealing with the 1/2-quantum detailed balance condition, or *symmetric quantum detailed balance condition*

$$\mathcal{L} - \mathcal{L}' = 2i[K, \cdot].$$

If $\mathcal{T}$ satisfies this condition, by Theorem 10, there exists a special GKSL representation of $\mathcal{L}'$ with $G' = G - i(2K + c)$ and $L_k' = \sum_j u_{kj} L_j$ for some unitary matrix $(u_{kj})_{kj}$; therefore, by Theorem 11 we get

1. $G\rho^{1/2} = \rho^{1/2}G^* - (2iK + ic)\,\rho^{1/2}$ for some $c \in \mathbb{R}$,
2. $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$, for all k, for some unitary matrix $(u_{k\ell})_{k\ell}$.

Conversely, these conditions ensure the symmetric quantum detailed balance condition; in fact, we proved in [15] that condition 1 easily follows by 2 and by the invariance of ρ. Hence, we obtain:

THEOREM 19. $\mathcal{T}$ *satisfies the symmetric quantum detailed balance condition if and only if there exists a special GKSL representation of the generator* $\mathcal{L}$ *by means of operators* H, L_k *such that* $\{L_k\}$ *satisfies*

$$\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}, \tag{21}$$

for all k, *for some unitary matrix* $(u_{m\ell})_{m\ell}$.

This means that the symmetric quantum detailed balance condition depends only on the completely positive part of the generator $\mathcal{L}$. More precisely:

THEOREM 20. *Let* $\mathcal{L}(a) = Ga + \Phi(a) + aG^*$ *be a special representation of* $\mathcal{L}$. *The following conditions are equivalent:*

1. $\mathcal{T}$ *satisfies the quantum symmetric detailed balance condition;*
2. Φ *is* $1/2$-*symmetric wrt* ρ.

A condition on G (with $K = 0$) is necessary to find $\mathcal{T}$ $1/2$-symmetric.

THEOREM 21. $\mathcal{T}$ *is KMS-symmetric if and only if one of the equivalent conditions below holds:*

(i) *there exists a special GKSL representation of* $\mathcal{L}$ *by means of operators* G, L_k *with*

    (1) $G\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ *for some* $c \in \mathbb{R}$,

    (2) $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$, *for all* k, *for some unitary matrix* $(u_{k\ell})_{k\ell}$;

(ii) *there exists a special GKSL representation of* $\mathcal{L}$, $\mathcal{L}(a) = Ga + \Phi(a) + aG^*$ *such that*

    (1') $G\rho^{1/2} = \rho^{1/2}G^* + ic\rho^{1/2}$ *for some* $c \in \mathbb{R}$,

    (2') Φ *is* $1/2$-*symmetric wrt* ρ.

REMARK 22. Using the modular theory, we showed in [15] that (1), (2) in item (i) of Theorem 21 can be replaced by the following conditions:

(1'') $H = \sum_k H_k + c$ for some $c \in \mathbb{R}$, where

$$H_k := i \int \sigma_t \left(\rho^{-1/4}L_k^*L_k\rho^{1/4} - \rho^{1/4}L_k^*L_k\rho^{-1/4} \right) (e^{2\pi t} + e^{-2\pi t})^{-1}\, dt;$$

(2'') $\rho^{1/2}L_k^* = \sum_\ell u_{k\ell}L_\ell\rho^{1/2}$, for all k, for some unitary matrix $(u_{k\ell})_{k\ell}$.

This is exactly the result proved by Park in Theorem 2.2 of [23].

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