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NONNEGATIVE LINEARIZATION FOR ORTHOGONAL POLYNOMIALS WITH EVENTUALLY CONSTANT JACOBI PARAMETERS

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Abstract. We study the nonnegative product linearization property for polynomials with eventually constant Jacobi parameters. For some special cases a necessary and sufficient condition for this property is provided.

1. Introduction. Let $\{P_n\}_{n=0}^{\infty}$ be a sequence of polynomials defined by the following recurrence relation: $P_0(x) = 1$ and for $n \geq 0$

$$xP_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x) \tag{1}$$

(under the convention that $P_{-1}(x) = 0$), where $\{\gamma_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers, called *Jacobi parameters*, and $\gamma_n > 0$. Then all P_n are monic and $\deg P_n = n$. We denote by $\mathcal{L}$ the linear functional on $\mathbb{R}[x]$ defined by: $\mathcal{L}(P_0) = 1$ and $\mathcal{L}(P_n) = 0$ for $n \geq 1$. Then we have

$$\mathcal{L}(P_m P_n) = \delta_{m,n} \cdot \gamma_0 \gamma_1 \ldots \gamma_{m-1}. \tag{2}$$

In view of Favard's theorem [5] $\mathcal{L}$ can be expressed as an integral with respect to a probability measure on the real line.

Our aim is to study the *linearization coefficients* which are uniquely defined by

$$P_m(x) P_n(x) = \sum_j c(j,m,n) P_j(x). \tag{3}$$

It is known [2] that many classical orthogonal polynomials admit *nonnegative product linearization*, i.e. all $c(j,m,n)$ are nonnegative even though their exact values can be unknown. In this case one can define a *hypergroup structure* [2] on the set of nonnegative

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integers by putting

$$\delta_m * \delta_n := \sum_k \frac{c(k,m,n)P_k(x_0)}{P_m(x_0)P_n(x_0)}\,\delta_k\,, \tag{4}$$

where the *normalizing point* x_0 is choosen such that $x_0 \geq \sup(\mathrm{supp}\mu)$, so that $P_m(x_0) > 0$ for all m. Extending this to convex combinations one obtains an associative and commutative convolution on the class of probability measures on the set $\{0,1,2,\dots\}$. There are also some general criteria for nonnegative product linearization, stated in terms of the parameters γ_n, β_n [1, 9, 11].

Multiplying both sides of (3) by P_k and applying $\mathcal{L}$ we get

$$\mathcal{L}(P_k P_m P_n) = c(k,m,n)\gamma_0\gamma_1\dots\gamma_{k-1}. \tag{5}$$

We denote $L(k,m,n) := \mathcal{L}(P_k P_m P_n)$ and $\Gamma(k) := \gamma_0\gamma_1\dots\gamma_{k-1}$.

The following properties of the coefficients $L(k,m,n)$ are easy to verify [9]:

$$L(k,m,k+m) = \Gamma(k+m), \tag{6}$$

$$L(k_1,k_2,k_3) = L(k_{\sigma_1},k_{\sigma_2},k_{\sigma_3}) \tag{7}$$

$$L(k,m,n) = 0 \qquad \text{whenever } n > k+m, \tag{8}$$

$$\begin{aligned} L(k,m,n) &= L(k-1,m,n+1) + (\beta_n - \beta_{k-1})L(k-1,m,n) \\ &\quad + \gamma_{n-1}L(k-1,m,n-1) - \gamma_{k-2}L(k-2,m,n) \end{aligned} \tag{9}$$

for any k,m,n and any permutation σ of the set $\{1,2,3\}$.

In particular, one can check that for $1 \leq k \leq m$ we have:

$$L(k,m,k+m-1) = \Gamma(k+m-1)\Big[\sum_{i=0}^{k-1}(\beta_{m+i} - \beta_i)\Big] \tag{10}$$

and

$$\begin{aligned} &L(k,m,k+m-2) = \Gamma(k+m-2) \\ &\times\Big[\gamma_{m-1} + \sum_{i=0}^{k-2}(\gamma_{m+i} - \gamma_i) + \sum_{0\leq i<j\leq k-1}(\beta_{m+i} - \beta_i)(\beta_{m+j-1} - \beta_j)\Big]. \end{aligned} \tag{11}$$

2. Orthogonal polynomials with eventually constant Jacobi parameters. From now on we will assume that the coefficients γ_n, β_n are constant from some place. Orthogonal polynomials of this kind and the corresponding probability measures are sometimes encountered in various limit theorems in noncommutative probability [3, 7]. The basic example is the Wigner measure

$$\mu = \frac{1}{2\pi\gamma}\sqrt{4\gamma - (t-\beta)^2}dt \tag{12}$$

on $[\beta - 2\sqrt{\gamma}, \beta + 2\sqrt{\gamma}]$, having both sequences $\gamma_n \equiv \gamma$, $\beta_n \equiv \beta$ constant, which plays the role of the Gaussian measure in free probability.

ASSUMPTION. *For $k \geq M$ we have $\beta_k = \beta$, $\gamma_k = \gamma$.*

The case $M = 1$ is quite interesting. For example, the related four parameter family of measures is closed under the free and boolean powers [6, 10] and $\{P_n\}$ admits nonnegative product linearization if and only if $\gamma_0 \leq 2\gamma$ and $\beta_0 \leq \beta$ [8]. This class of probability measures contains the Marchenko-Pastur measures (where $\gamma_0 = \gamma$, $\beta_0 = \gamma$, $\beta = \gamma + 1$), which in free probability play the role of Poisson measures, as well as the limit measures related to conditional freeness (with $\gamma_0 = a^2$, $\gamma = b^2$, $\beta_0 = \beta = 0$ in the central limit theorem, and with $\gamma_0 = a$, $\gamma = b$, $\beta_0 = a$, $\beta = b + 1$ in the Poisson limit theorem [3]).

We will base on the following lemma, which will be used together with relation (7).

LEMMA 2.1. *If* $n + k - m > 2M$ *then* $L(k, m, n) = \gamma L(k-1, m, n-1)$.

Proof. Without loss of generality we can assume that $k \leq n$. We proceed by induction on $j := k + m - n$. Note that then

$$2k > 2M + j \tag{13}$$

because

$$n > 2M + m - k = 2M + (n + j - k) - k = 2M - 2k + j + n.$$

In particular, $k > M$ and hence $\beta_n - \beta_{k-1} = \beta - \beta = 0$, so we can ignore the second summand in (9).

If $j = 0$ then $n = k + m$ and

$$L(k, m, k+m) = \Gamma(k+m) = \gamma\Gamma(k+m-1) = \gamma L(k-1, m, k+m-1).$$

If $j = 1$ then, by (13), $M < k \leq n$. Since $n = k + m - 1$ we have $L(k-1, m, n+1) = L(k-2, m, n) = 0$. Then, by (9), we get $L(k, m, n) = \gamma L(k-1, m, n-1)$.

Finally, assume that $j \geq 2$. Then $k > M + 1$ and hence $\gamma_{k-2} = \gamma$. Therefore, by induction, we have $L(k-1, m, n+1) = \gamma L(k-2, m, n)$ so in (9) the first summand cancels with the last one, so we get $L(k, m, n) = \gamma L(k-1, m, n-1)$, which concludes the proof. ∎

DEFINITION 2.2. We write $(k', m', n')\overrightarrow{\mathcal{R}}_0(k, m, n)$ if either
$k' = k$, $m' = m - 1$, $n' = n - 1$ and $m + n - k > 2M$ or
$k' = k - 1$, $m' = m$, $n' = n - 1$ and $k + n - m > 2M$ or
$k' = k - 1$, $m' = m - 1$, $n' = n$ and $k + m - n > 2M$.

Denote by $\overrightarrow{\mathcal{R}}$ be the smallest reflexive and transitive relation containing $\overrightarrow{\mathcal{R}}_0$. Since $(k', m', n')\overrightarrow{\mathcal{R}}_0(k, m, n)$ implies $k' \leq k$, $m' \leq m$, $n' \leq n$ we see that $\overrightarrow{\mathcal{R}}$ is weakly antisymmetric and hence is a partial order. Note also that if $(k', m', n')\overrightarrow{\mathcal{R}}(k, m, n)$ then $k + m + n - k' - m' - n' := 2r$ is even and, in view of the Lemma 2.1,

$$L(k, m, n) = \gamma^r L(k', m', n').$$

Therefore it is sufficient to examine $L(k, m, n)$ for those triples (k, m, n) which satisfy $0 < k \leq m \leq n < k + m$ and $m + n - k \leq 2M$. The set of such triples will be denoted by Σ.

PROPOSITION 2.3. *The number of elements in* Σ *is equal to*

$$\frac{M(M+1)(M+2)}{3}. \tag{14}$$

Proof. Set $r := k + m - n$, $1 \leq r \leq 2M$. Then the inequalities $k \leq m \leq n$ and $m + n - k \leq 2M$ are equivalent to

$$r \leq k \leq m \leq M + \frac{r}{2}\ . \tag{15}$$

Note that for fixed integers $r \le s$ there are exactly

$$\frac{(s-r+1)(s-r+2)}{2}$$

pairs (k,m) such that $r \le k \le m \le s$. Therefore for fixed r there are exactly

$$\frac{(M-t+1)(M-t+2)}{2}$$

pairs (k,m) satisfying $r \le k \le m \le M + r/2$, where either $r = 2t$ or $r = 2t-1$. Now we have

$$\sum_{t=1}^{M}(M-t+1)(M-t+2) = \sum_{s=1}^{M} s(s+1) = \frac{M(M+1)(M+2)}{3}$$

which completes the proof. ■

Let $\mathcal{R}$ be the smallest equivalence relation containing $\overrightarrow{\mathcal{R}}$. Now we are going to describe the elements (k,m,n) of Σ and their $\mathcal{R}$-equivalence classes. We assume that $0 < k \le m \le n < k+m$, $m+n-k \le 2M$ and put $r := k+m-n$. We have $1 \le r \le 2M$. If r is even we put $r = 2t$, otherwise $r = 2t-1$, so that $1 \le t \le M$.

Case 0. If $m+n+2-k \le 2M$ then (k,m,n) has no successor, so $[(k,m,n)]_{\mathcal{R}} = \{(k,m,n)\}$. Applying Proposition 2.3 to $M-1$ we see that the number of such triples is $(M-1)M(M+1)/3$.

Case 1. Assume that $m+n+2-k > 2M$ and $k+n+2-m \le 2M$. The former, together with the third inequality in (15) leads to $m = M + \left[\frac{r}{2}\right]$. Hence, we have to choose r and k such that $1 \le r \le 2M$ and $r \le k < M + \left[\frac{r}{2}\right]$, which gives $M(M-1)$ choices. Note that in this case, for $s \ge 0$ the only successor of $(k, m+s, n+s)$ is $(k, m+s+1, n+s+1)$. Therefore

$$[(k,m,n)]_{\mathcal{R}} = \{(k, m+s, n+s) : s \ge 0\}.$$

Case 2. Assume that $k+n+2-m > 2M$ and $k+m+2-n \le 2M$. Then the former, together with (15), leads to $k = m = M + \left[\frac{r}{2}\right]$ (hence $n = 2M-1$ if r is odd and $n = 2M$ if r is even), while the latter means that $r \le 2M-2$. Hence $k = m = M+t-\epsilon$, $n = 2M-\epsilon$, $\epsilon \in \{0,1\}$, $1 \le t \le M-1$. This leads to $2M-2$ classes:

$$\begin{aligned}
&[(M+t-\epsilon, M+t-\epsilon, 2M-\epsilon)]_{\mathcal{R}} \\
&= \{(M+t+p-\epsilon, M+t+q-\epsilon, 2M+p+q-\epsilon) : p,q \ge 0\} \\
&= \{(a, b, a+b-2t+\epsilon) : a,b \ge M+t-\epsilon\}.
\end{aligned}$$

Case 3. Finally, assume that $r = k+m-n > 2M-2$. Then (15) implies that $r = 2M-\epsilon$, $\epsilon \in \{0,1\}$, and $r = k = m = n$, which leads to two classes:

$$\begin{aligned}
&[(2M-\epsilon, 2M-\epsilon, 2M-\epsilon)]_{\mathcal{R}} \\
&= \{(2M-\epsilon+q+s, 2M-\epsilon+p+s, 2M-\epsilon+p+q) : p,q,s \ge 0\} \\
&= \{(a,b,c) : a+b-c, a+c-b, b+c-a \ge 2M-\epsilon, \text{ and } a+b+c-\epsilon \text{ is even}\}.
\end{aligned}$$

We will denote by Σ_i the set of those triples in Σ which fall to Case i.

2.1. The case $M = 2$. Let us apply our results to the case when $M = 2$. Interesting examples of polynomials of this kind and the corresponding measures were studied in [7]. We have

$$\Sigma_0 = \{(1,1,1),(2,2,2)\}, \quad \Sigma_1 = \{(1,2,2),(2,3,3)\},$$
$$\Sigma_2 = \{(2,2,3),(3,3,4)\}, \quad \Sigma_3 = \{(3,3,3),(4,4,4)\}.$$

Assume that

$$\gamma_n = \begin{cases} a & \text{if } n = 0, \\ b & \text{if } n = 1, \\ c & \text{if } n \geq 2, \end{cases} \qquad \beta_n = \begin{cases} u & \text{if } n = 0, \\ v & \text{if } n = 1, \\ w & \text{if } n \geq 2. \end{cases} \tag{16}$$

THEOREM 2.4. *If (16) holds then we have:*

o1) $L(1,1,1) = a(v-u)$,
o2) $L(1,m,m) = \Gamma(m)(w-u)$ *if* $m \geq 2$,
o3) $L(k,m,k+m-1) = \Gamma(k+m-1)(2w-u-v)$ *if* $2 \leq k \leq m$,
o4) $L(k,m,n) = \Gamma(s-1)(2c(2w-u-v) - b(w-u))$ *if* $3 \leq k \leq m \leq n \leq k+m-3$ *and* $k+m+n = 2s+1$,
e1) $L(2,2,2) = ab(c+b-a+(w-u)(w-v))$,
e2) $L(2,m,m) = \Gamma(m)(2c-a+(w-u)(w-v))$ *if* $m \geq 3$,
e3) $L(k,m,k+m-2) = \Gamma(k+m-2)(3c-a-b+(w-u)(w-v))$ *if* $3 \leq k \leq m$,
e4) $L(k,m,n) = \Gamma(s-1)(4c-a-2b+(w-u)(w-v))$ *if* $4 \leq k \leq m \leq n \leq k+m-4$ *and* $k+m+n = 2s$.

Proof. The formulas (o1), (o2), (o3) and (e1), (e2), (e3) are consequences of (10) and (11) respectively. Having them, we can use (9) to compute $L(3,3,3)$ and $L(4,4,4)$. Now we apply Lemma 2.1 to conclude the proof. ∎

COROLLARY 2.5. *Assuming (16), the polynomials $\{P_n\}$ admit nonnegative product linearization if and only if the following inequalities hold:*

$$\begin{aligned} u &\leq v, \\ u &\leq w, \\ u+v &\leq 2w, \\ b(w-u) &\leq 2c(2w-u-v), \\ 0 &\leq c+b-a+(w-u)(w-v), \\ 0 &\leq 4c-a-2b+(w-u)(w-v). \end{aligned}$$

In particular, if in addition $v = w$ then the nonnegative product linearization holds if and only if $u \leq v$, $b(v-u) \leq 2c(v-u)$, $a \leq b+c$ and $2b+a \leq 4c$.

Proof. To avoid the inequalities related to (e2) and (e3) we use the fact that if the first and the last element of a finite arithmetic sequence are nonnegative then all its elements are nonnegative. ∎

2.2. The case $M = 3$. If $M = 3$ then we have

$$\Sigma_0 = \{(1,1,1),(1,2,2),(2,2,3),(3,3,3),(2,2,2),(2,3,3),(3,3,4),(4,4,4)\},$$
$$\Sigma_1 = \{(1,3,3),(2,3,4),(3,4,4),(2,4,4),(3,4,5),(4,5,5)\},$$
$$\Sigma_2 = \{(3,3,5),(4,4,5),(4,4,6),(5,5,6)\},$$
$$\Sigma_3 = \{(5,5,5),(6,6,6)\}.$$

Now we assume that:

$$\gamma_n = \begin{cases} a & \text{if } n = 0, \\ b & \text{if } n = 1, \\ c & \text{if } n = 2, \\ d & \text{if } n \geq 3, \end{cases} \qquad \beta_n = \begin{cases} u & \text{if } n = 0, \\ v & \text{if } n = 1, \\ w & \text{if } n = 2, \\ z & \text{if } n \geq 3. \end{cases} \tag{17}$$

THEOREM 2.6. *Assuming (17) we have*

o1) $L(1,1,1) = a(v-u)$,
o2) $L(1,2,2) = ab(w-u)$,
o3) $L(1,m,m) = \Gamma(m)(z-u)$ *if* $m \geq 3$,
o4) $L(2,2,3) = abc(w+z-u-v)$,
o5) $L(2,m,m+1) = \Gamma(m+1)(2z-u-v)$ *if* $m \geq 3$,
o6) $L(k,m,k+m-1) = \Gamma(k+m-1)(3z-u-v-w)$ *if* $3 \leq k \leq m$,
o7) $L(3,3,3) = abc(a(w-z)+b(u-z)+c(2z-u-v)+d(3z-u-v-w) + (z-u)(z-v)(z-w))$,
o8) $L(3,m,m) = \Gamma(m)(a(w-z)+b(u-z)+2d(3z-u-v-w)+(z-u)(z-v)(z-w))$ *if* $m \geq 4$,
o9) $L(k,m,k+m-3) = \Gamma(k+m-3)(a(w-z)+b(u-z)+c(u+v-2z)+3d(3z-u-v-w)+(z-u)(z-v)(z-w))$ *if* $4 \leq k \leq m$,
o10) $L(k,m,n) = \Gamma(s-2)(a(w-z)+b(u-z)+2c(u+v-2z)+4d(3z-u-v-w)+(z-u)(z-v)(z-w))$ *if* $m \geq 4$, *if* $5 \leq k \leq m \leq n \leq k+m-5$, $k+m+n = 2s-1$,
e1) $L(2,2,2) = ab(c+b-a+(w-u)(w-v))$,
e2) $L(2,3,3) = abc(c+d-a+(z-u)(z-v))$,
e3) $L(2,m,m) = \Gamma(m)(2d-a+(z-u)(z-v))$ *if* $m \geq 4$,
e4) $L(3,3,4) = abcd(c+2d-a-b+(z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w))$,
e5) $L(3,m,m+1) = \Gamma(m+1)(3d-a-b+(z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w))$ *if* $m \geq 4$,
e6) $L(k,m,k+m-2) = \Gamma(k+m-2)(4d-a-b-c+(z-u)(z-v) + (z-u)(z-w)+(z-v)(z-w))$ *if* $4 \leq k \leq m$,
e7) $L(4,4,4) = abcd(ac+d(5d-2a-2b-c)-c(z-u)(z-v)+2d((z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w)))$,
e8) $L(4,m,m) = \Gamma(m)(ac+d(6d-2a-2b-2c)-c(z-u)(z-v)+2d((z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w))$ *if* $m \geq 5$,
e9) $L(k,m,k+m-4) = \Gamma(k+m-4)(ac+d(7d-2a-2b-3c)-c(z-u)(z-v)+2d((z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w))$ *if* $5 \leq k \leq m$,
e10) $L(k,m,n) = \Gamma(s-2)(ac+d(8d-2a-2b-4c)-c(z-u)(z-v)+2d((z-u)(z-v) + (z-u)(z-w)+(z-v)(z-w))$ *if* $6 \leq k \leq m \leq n \leq k+m-6$, $k+m+n = 2s$.

Proof. The formulas (o1)–(o6) and (e1)–(e6) are consequences of (10) and (11). Then, using (9), we compute $L(3,3,3)$, $L(3,4,4)$ and $L(4,4,5)$, which leads to (o7), (o8) and (o9). Using these we get $L(5,5,5)$ and hence (o10). In the same way we prove (e1)–(e10). ∎

In the same way as before we obtain

COROLLARY 2.7. *If we assume (17) then the polynomials $\{P_n\}$ admit nonnegative product linearization if and only if the following inequalities hold:*

$$\begin{aligned}
u &\le v,\\
u &\le w,\\
u &\le z,\\
u+v &\le w+z,\\
u+v &\le 2z,\\
u+v+w &\le 3z,\\
a(z-w)+b(z-u) &\le d(3z-u-v-w)+c(2z-u-v)+(z-u)(z-v)(z-w),\\
a(z-w)+b(z-u) &\le 4d(3z-u-v-w)-2c(2z-u-v)+(z-u)(z-v)(z-w),\\
a &\le c+b+(w-u)(w-v),\\
a &\le c+d+(z-u)(z-v),\\
a &\le 2d+(z-u)(z-v),\\
a+b &\le 2d+c+(z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w),\\
a+b &\le 4d-c+(z-u)(z-v)+(z-u)(z-w)+(z-v)(z-w),\\
d(2a+2b+c-5d) &\le ac+(2d-c)(z-u)(z-v)+2d(2z-u-v)(z-w),\\
d(2a+2b+4c-8d) &\le ac+(2d-c)(z-u)(z-v)+2d(2z-u-v)(z-w). \blacksquare
\end{aligned}$$

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References

[1] R. Askey, *Linearization of the product of orthogonal polynomials*, in: Problems in Analysis, R. Gunning (ed.), Princeton University Press, Princeton, NJ, 1970, 223–228.

[2] W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Studies in Mathematics 20, Berlin, 1995.

[3] M. Bożejko, M. Leinert and R. Speicher, *Convolution and limit theorems for conditionally free random variables,* Pacific J. Math. 175 (1996), 357–388.

[4] M. Bożejko and W. Bryc, *On a class of free Lévy laws related to a regression problem,* J. Funct. Anal. 236 (2006), 59–77.

[5] T. Chihara, *An Introduction to Orthogonal Polynomials*, Mathematics and Its Applications 13, Gordon and Breach, New York 1978.

[6] M. Hinz and W. Młotkowski, *Free cumulants of some probability measures,* Banach Center Publications 78 (2007), 165–170

[7] A. D. Krystek and Ł. J. Wojakowski, *Associative convolutions arising from conditionally free convolution,* Infin. Dimens. Anal. Quantum Probab. Related Top. 8 (2005), 515–545.

[8] W. Młotkowski, *Some class of polynomial hypergroups,* Banach Center Publications 73 (2006), 357–362.

[9] W. Młotkowski and R. Szwarc, *Nonnegative linearization for polynomials orthogonal with respect to discrete measures,* Constructive Approximation 17 (2001), 413–429.

[10] N. Saitoh and H. Yoshida, *The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory,* Probability and Mathematical Statistics 21 (2001), 159–170.

[11] R. Szwarc, *Orthogonal polynomials and a discrete boundary value problem, I, II,* SIAM J. Math. Anal. 23 (1992), 959–969.