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MURPHY'S *POSITIVE DEFINITE KERNELS AND HILBERT C*-MODULES* REORGANIZED

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Abstract. The paper the title refers to is that in *Proceedings of the Edinburgh Mathematical Society*, 40 (1997), 367–374. Taking it as an excuse we intend to realize a twofold purpose:

1° to atomize that important result showing by the way connections which are out of favour,
2° to rectify a tiny piece of history.

The objective 1° is going to be achieved by adopting means adequate to goals; it is of great gravity and this is just Mathematics. The other, 2°, comes from the author's internal need of showing how ethical values in Mathematics are getting depreciated. The latter has nothing to do with the previous issue; the coincidence is totally accidental.

Reproducing kernel Hilbert C*-modules

Rudiments of the theory of Hilbert C*-modules. Let $\mathfrak{A}$ be a C*-algebra with its norm denoted[1] by $\|\cdot\|_{\mathfrak{A}}$. An *inner product* $\mathfrak{A}$*-module* is a right $\mathfrak{A}$-module $\mathcal{E}$ (with scalar multiplication compatible with this in $\mathcal{E}$ as well as that in $\mathfrak{A}$) with a mapping

$$\mathcal{E}\times\mathcal{E}\ni(x,y)\mapsto\langle x,y\rangle\in\mathfrak{A}$$

such that

(a) it is $\mathbb{C}$-linear in the second variable;
(b) $\langle x,y\mathfrak{a}\rangle=\langle x,y\rangle\mathfrak{a}$, $x,y\in\mathcal{E}$, $\mathfrak{a}\in\mathfrak{A}$;
(c) $\langle y,x\rangle=\langle x,y\rangle^*$, $x,y\in\mathcal{E}$;

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The paper is in final form and no version of it will be published elsewhere.

[1] Warning: we do not copy slavishly the notation of [17] though this is a standard monograph of the subject. In order to protect the reader from being in a whirl we put a subscript in notation of norms; inner products are less dangerous.

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(d) $\langle x, x\rangle \geqslant 0$; if $\langle x, x\rangle = 0$ then $x = 0$.

If the second condition in (d) is dropped we call $\mathcal{E}$ a *semi-inner product* $\mathfrak{A}$*-module*.

Proposition 1 (Proposition 2.3 in [22]). *Conditions* (a)–(d) *above imply that* $\|x\|_{\mathcal{E}} \stackrel{\text{def}}{=} \sqrt{\|\langle x, x\rangle\|_{\mathfrak{A}}}$ *is a norm on* $\mathcal{E}$ *and*[2]

(i) $\|x\mathfrak{a}\|_{\mathcal{E}} \leqslant \|x\|_{\mathcal{E}}\|\mathfrak{a}\|_{\mathfrak{A}}$, $x \in \mathcal{E}$, $\mathfrak{a} \in \mathfrak{A}$;
(ii) $\langle y, x\rangle\langle x, y\rangle \leqslant \|y\|^2_{\mathcal{E}}\langle x, x\rangle$, $x, y \in \mathcal{E}$;
(iii) $\|\langle x, y\rangle\|_{\mathfrak{A}} \leqslant \|x\|_{\mathcal{E}}\, \|y\|_{\mathcal{E}}$, $x, y \in \mathcal{E}$.

If $\mathcal{E}$ is complete with respect to the norm $\|\cdot\|_{\mathcal{E}}$, it is called a *Hilbert* $\mathfrak{A}$*-module* (it belongs to the category of C*-modules if one wants to hide $\mathfrak{A}$).

A Hilbert $\mathfrak{A}$-module is a Banach space and for that reason one can consider bounded linear operators between two such spaces, $\mathcal{E}$ and $\mathcal{E}_1$ say; denote the totality of those by $\mathbf{B}(\mathcal{E}, \mathcal{E}_1)$. Call a map $T\colon \mathcal{E} \to \mathcal{E}_1$ *adjointable* if there is another map $T^*\colon \mathcal{E}_1 \to \mathcal{E}$ such that

$$\langle Tx, x_1\rangle = \langle x, T^* x_1\rangle, \quad x \in \mathcal{E},\ x_1 \in \mathcal{E}_1.$$

If T is adjointable then it must necessarily be $\mathbb{C}$-linear[3] as well as $\mathfrak{A}$-linear, and, due to Banach–Steinhaus theorem also bounded. Denote by $\mathbf{B}^*(\mathcal{E}, \mathcal{E}_1)$ the set of all adjointable operators; apparently $\mathbf{B}^*(\mathcal{E}, \mathcal{E}_1) \subset \mathbf{B}(\mathcal{E}, \mathcal{E}_1)$. For further use we make a shorthand notation $\mathbf{B}^*(\mathcal{E}) \stackrel{\text{def}}{=} \mathbf{B}^*(\mathcal{E}, \mathcal{E})$; this, with the involution *, is a C*-algebra. Sometimes we may have a need to get the C*-algebra involved in the notation; we just put the C*-algebra in the subscript like in $\mathbf{B}^*_{\mathfrak{A}}(\mathcal{E})$.

For $x \in \mathcal{E}$ and $x_1 \in \mathcal{E}_1$ define the operators $T_{x,y}$ by and T_x by

$$T_{x,x_1} y \stackrel{\text{def}}{=} x_1\langle x, y\rangle, \qquad T_x y \stackrel{\text{def}}{=} \langle x, y\rangle, \qquad y \in \mathcal{E}. \tag{1}$$

Then T_{x,x_1}'s belong to $\mathbf{B}^*(\mathcal{E}, \mathcal{E}_1)$ and T_x's belong to $\mathbf{B}^*(\mathcal{E}, \mathfrak{A})$. Moreover, $T_{x,\mathfrak{a}} = T_{x\mathfrak{a}^*}$. Let $\mathbf{K}(\mathcal{E}, \mathcal{E}_1)$ stand for the closed (in $\mathbf{B}(\mathcal{E}, \mathcal{E}_1)$) linear span of all the T_{x,x_1}'s. It is clear that T_{x,x_1} and T_x are $\mathfrak{A}$-linear.

Notice that $\mathfrak{A}$ itself is a Hilbert $\mathfrak{A}$-module with the inner product

$$\langle a, b\rangle_{\mathfrak{A}} \stackrel{\text{def}}{=} a^* b, \quad a, b \in \mathfrak{A}$$

and the $\mathfrak{A}$-module norm coincides with that of the C*-algebra $\mathfrak{A}$. Another thing which is worthy of mention is that $\mathbf{B}^*(\mathfrak{A})$ is isomorphic to $\mathfrak{A}$ itself, cf. [17] p.10.

Comments. If $\mathfrak{A}$ has a unit $\mathfrak{e}$ then $\mathbf{K}(\mathcal{E}, \mathfrak{A}) = \mathbf{B}^*(\mathcal{E}, \mathfrak{A})$, cf. [17] p.13. This is so because $T \in \mathbf{B}^*(\mathcal{E}, \mathfrak{A})$ is of the form $T = T_{T^*\mathfrak{e}}$, which as such belongs to $\mathbf{K}(\mathcal{E}, \mathfrak{A})$. Furthermore, $\mathbf{K}(\mathcal{E}, \mathfrak{A})$ is precisely the set of all those bounded $\mathfrak{A}$-linear mappings which are Riesz representable through an $\mathfrak{A}$-inner product.

On the other hand, the Riesz representation theorem does not always apply to the members of $\mathbf{B}(\mathcal{E}, \mathfrak{A})$; if it does $\mathcal{E}$ is called *self-dual* (more on self-dual Hilbert C*-modules is in [9]).

[2] The proof in [2] caries over to the case of semi-inner product as well, for another look at [17], p. 3.

[3] 'Linear' or in abbreviation 'lin' always refers to $\mathbb{C}$-linearity. If $\mathfrak{A}$-has a unit, it is needless to think separately of linearity when $\mathfrak{A}$-linearity is around.

There is a substantial difference in behaviour of Hilbert C*-modules compared to Hilbert spaces: the orthocomplemention does not perform as involution. However, one useful tool remains: if $\langle x, y\rangle = 0$ for fixed x and y ranging over a dense subset of $\mathcal{E}$ then $x = 0$. It is so because the inner product is continuous according to (iii) of Proposition 1.

C*-positive definite kernels. Let S be a set. Call a mapping $K\colon S\times S \to \mathfrak{A}$ a $\mathfrak{A}$-*kernel on* S or briefly a kernel if no confusion arises. An $\mathfrak{A}$-kernel on S is said to be $\mathfrak{A}$-*positive definite* (again, positive definite if no confusion arises) if

$$\sum_{k,l} \mathfrak{a}_k^* K(s_k, s_l)\mathfrak{a}_l \geqslant 0 \text{ for any finite choice of } (s_n)_n \subset S \text{ and } (\mathfrak{a}_n)_n \subset \mathfrak{A}. \tag{2}$$

Using the standard quadratic form (in two complex variables) argument we immediately get Hermitian symmetry

$$K(s,t) = K(t,s)^*, \quad s,t \in S. \tag{3}$$

Two typical Schwarz inequalities can be derived from (ii) and (iii) of Proposition 1 later.

Reproducing kernel Hilbert $\mathfrak{A}$-module: the construction. Let K be a $\mathfrak{A}$-positive definite kernel on S. Set $K_s \overset{\text{def}}{=} K(s,\,\cdot\,)$ for the sections and

$$\mathcal{D}_K \overset{\text{def}}{=} \operatorname{lin}\{K_s\mathfrak{a}\colon\ s \in S,\ \mathfrak{a} \in \mathfrak{A}\}.$$

Therefore the members of $\mathcal{D}_K$ are of the form $\sum_i K_{s_i}\mathfrak{a}_i$, which means they are mappings [4] from S to $\mathfrak{A}$. Let us try to define an $\mathfrak{A}$-inner product on $\mathcal{D}_K$ as follows:

$$\Big\langle \sum_k K_{t_k}\mathfrak{b}_k, \sum_i K_{s_i}\mathfrak{a}_i \Big\rangle_K \overset{\text{def}}{=} \sum_{i,k} \mathfrak{b}_k^* K(t_k, s_i)\mathfrak{a}_i, \qquad (s_m)_m, (t_n)_n \subset S,\ (\mathfrak{a}_m)_m, (\mathfrak{b}_n)_n \subset \mathfrak{A}. \tag{4}$$

To see the inner product is well defined notice first that $\sum_i K_{s_i}\mathfrak{a}_i{=}0$ forces $\sum_{i,k}\mathfrak{b}_k^* K(t_k, s_i)\mathfrak{a}_i = 0$ regardless what $\sum_k K_{t_k}\mathfrak{b}_k$ is. Then we get in a standard way that

$$\sum_{i,k} \mathfrak{b}_k^* K(t_k, s_i)\mathfrak{a}_i = \sum_{j,l} \mathfrak{b}_l'^* K(t_l', s_j')\mathfrak{a}_i' \quad \text{if } \sum_i K_{s_i}\mathfrak{a}_i = \sum_j K_{s'_j}\mathfrak{a}_j \text{ and } \sum_k K_{t_k}\mathfrak{b}_k = \sum_l K_{t'_l}\mathfrak{b}_l,$$

which proves the claim [5].

The defining formula (4) turns into the reproducing kernel property

$$F(s) = \langle F, K_s\rangle_K, \quad s \in S,\ F \in \mathcal{D}_K. \tag{5}$$

It is clear that $\mathcal{D}_K$ is an $\mathfrak{A}$-inner product module. Now we want to complete it still having the completion to be a Hilbert $\mathfrak{A}$-module composed of $\mathfrak{A}$-functions on S [6]. For

[4] It seems to be suggestive to call mappings from S to $\mathfrak{A}$ just $\mathfrak{A}$-*functions* on S.

[5] Let us notice that this simple observation is the key to the RKHS approach being so exciting. Usually people, even if they decide to follow the construction up to the very end, at this point make the argument rather enigmatic if any at all. It is apparently needless to say most of the RKHS-like constructions bear hallmarks of schism.

[6] At this point the reproducing kernel space idea was abandoned in [20].

this let $\widehat{\mathcal{E}}$ be an arbitrary Hilbert $\mathfrak{A}$-module in which $\mathcal{D}_K$ is densely imbedded via the isometry[7] $\mathcal{D} \ni F \mapsto \tilde{F} \in \widehat{\mathcal{E}}$. For every $G \in \widehat{\mathcal{E}}$ the formula

$$F_G(s) \stackrel{\text{def}}{=} \langle G, K_s \rangle_{\widehat{\mathcal{E}}}$$

determines, by density of $\mathcal{D}_K$ in $\widehat{\mathcal{E}}$, a unique $\mathfrak{A}$-function F_G on S. It is a matter of straightforward calculation to check that $\mathcal{E}_K \stackrel{\text{def}}{=} \{F_G\colon\ G \in \widehat{\mathcal{E}}\}$ is a Hilbert $\mathfrak{A}$-module of $\mathfrak{A}$-functions on S with the inner product being an 'inverse image' of that in $\widehat{\mathcal{E}}$. Needless to say that $\mathcal{E}_K$ does not depend on a particular choice of $\widehat{\mathcal{E}}$[8].

Reproducing kernel Hilbert $\mathfrak{A}$-module: the properties. The first feature is the reproducing kernel property (5) extends as

$$F(s) = \langle F, K_s \rangle_K, \quad s \in S,\ F \in \mathcal{E}_K. \tag{6}$$

What is important when one wants to think of any kind of minimality is that the inner product $\mathfrak{A}$-module $\mathcal{D}_K$ is already dense in $\mathcal{E}_K$ by the construction.

The *evaluation mapping* φ_s at $s \in S$ given by

$$\varphi_s\colon \mathcal{E}_K \ni F \mapsto F(s) \in \mathfrak{A}$$

is $\mathfrak{A}$-linear and, due to (iii) of Proposition 1, bounded. Moreover,

$$\langle \varphi_s(K_t), \mathfrak{a} \rangle_{\mathfrak{A}} = K_t(s)^* \mathfrak{a} = K(t,s)\mathfrak{a} = \langle K_t, K_s \mathfrak{a} \rangle_K,$$

which means $(\varphi_s)^* \mathfrak{a} = K_s \mathfrak{a}$. All this leads to what some people (including the author) may see as the archetype of the Kolmogorov decomposition.

THEOREM 2. *For every $s \in S$, $\varphi_s \in \mathbf{B}^*(\mathcal{E}_K, \mathfrak{A})$ and $(\varphi_s)^*$ acts as $(\varphi_s)^* \mathfrak{a} = K_s \mathfrak{a}$. Moreover,*

$$K(s,t) = \varphi_s (\varphi_t)^*, \quad s,t \in S. \tag{7}$$

REMARK 3. Every Hilbert $\mathfrak{A}$-module $\mathcal{E}$ is a reproducing kernel Hilbert $\mathfrak{A}$-module over itself with the kernel K defined as

$$K(x,y) \stackrel{\text{def}}{=} \langle x, y \rangle_K, \quad x, y \in \mathcal{E}.$$

NOTE. The amazing grace of the reproducing kernel Hilbert spaces, when they are constructed according to the rules contained in [1], is in the space being composed of functions. The same happens also to the Hilbert C*-modules. However, the latter lack the RKHS test unless the module is self-dual. This is so because the Riesz representation theorem, which is the only reason for the test to work, fails to hold.

There is an occurrence when the RKHS test is certainly valid[9] too. It allows to determine precisely which $\mathfrak{A}$-functions constitute the space $\mathcal{E}_K$. This highlights the extraordinary features of the reproducing kernel spaces structure.

[7] This is in fact an $\mathfrak{A}$-isometry, that is it preserves $\mathfrak{A}$-inner products, as a more careful look at the argument presented on p. 10 of [17] may ensure us.

[8] Though elements of the RKHS approach can be traced on many occasions we would like to advertise here [38], at least for those who can read it.

[9] For the Hilbert space case cf. [37], property (y).

PROPOSITION 4. *Suppose the $\mathfrak{A}$-Hilbert module $\mathcal{E}_K$ is self-dual. For an $\mathfrak{A}$-function F on S the following conditions are equivalent:*

① *F belongs to $\mathcal{E}_K$;*
② *there is $c \geqslant 0$ such that for any finite choice of $(s_m)_m \subset S$ and $(\mathfrak{a}_n)_n \subset \mathfrak{A}$*

$$\sum_{k,l} \mathfrak{a}_k^* F(s_k)^* F(s_l)\mathfrak{a}_l \leqslant c \sum_{k,l} \mathfrak{a}_k^* K(s_k, s_l)\mathfrak{a}_l.$$

Proof. Use the same argument as that in [37]. ∎

COMMENTS. An important representative of kernels with separated variables, as opposed to what is going to follow, is that allied to the cosine function as well as to those alike, see for instance [21].

Semigroups in action. Theorem 2 is a ground floor version of Murphy's Theorem 2.3. Specifying $\mathfrak{A} = \mathbf{B}^*(\mathcal{E})$, where $\mathcal{E}$ is already a Hilbert C*-module, we may go upstairs to get precisely that Theorem. However, we prefer still to keep moving on the ground floor and pass to Theorem 2.4 of [20] this route.

Let $\mathfrak{S}$ be a multiplicative semigroup of left actions on S. Let us define two operators in $\mathcal{E}_K$ related to a given $\mathfrak{s}$. For $F \in \mathcal{E}_K$ and $\mathfrak{s} \in \mathfrak{S}$ define first the $\mathfrak{s}$'s translate $F_{[\mathfrak{s}]}$ of F as $F_{[\mathfrak{s}]} \overset{\text{def}}{=} F(\mathfrak{s}s)$, $s \in S$. The operator $\Psi_K(\mathfrak{s})$ is well defined by

$$\mathcal{D}(\Psi_K(\mathfrak{s})) \overset{\text{def}}{=} \{F \in \mathcal{E}_K\colon\ F_{[\mathfrak{s}]} \in \mathcal{E}_K\}, \quad \Psi_K(\mathfrak{s})F \overset{\text{def}}{=} F_{[\mathfrak{s}]}.$$

$\Psi_K(\mathfrak{s})$ may be an unbounded operator with domain $\mathcal{D}(\Psi_K(\mathfrak{s}))$ different from $\mathcal{E}_K$. The other operator, $\Phi_K(\mathfrak{s})$ may not be well defined; if it is, it is always densely defined

$$\mathcal{D}(\Phi_K(\mathfrak{s})) \overset{\text{def}}{=} \mathcal{D}_K, \quad \Phi_K(\mathfrak{s}) \sum_i K_{s_i}\mathfrak{a}_i = \sum_i K_{\mathfrak{s}s_i}\mathfrak{a}_i, \quad (s_i)_i \subset S. \tag{8}$$

The reproducing kernel property (5) implies

$$\langle F, K_{\mathfrak{s}s}\rangle_K = \langle \Psi_K(\mathfrak{s})F, K_s\rangle_K, \quad F \in \mathcal{D}(\Psi_K(\mathfrak{s})),\ s \in S$$

and this in turn helps to prove the following

PROPOSITION 5. *$\Psi_K(\mathfrak{s})$ is a closed operator. If $\Psi_K(\mathfrak{s})$ is densely defined, then $\Phi_K(\mathfrak{s})$ is well defined and $\Phi_K(\mathfrak{s})^* = \Psi_K(\mathfrak{s})$, and* vice versa.

In principle, $\Psi(\mathfrak{s})$ may not be densely defined while $\Phi(\mathfrak{s})$ may not be well defined as an operator. In this paper we are interested exclusively in the case when these two operators are bounded. This case is described as follows.

PROPOSITION 6. *$\Phi_K(\mathfrak{s})$ is a well defined bounded operator if and only if there is a number $c(\mathfrak{s}) \geqslant 0$ such that*

$$\Big\| \sum_{i,j} \mathfrak{a}_i^* K(\mathfrak{s}s_i, \mathfrak{s}s_j)\mathfrak{a}_j \Big\|_{\mathfrak{A}} \leqslant c(\mathfrak{s}) \Big\| \sum_{i,j} \mathfrak{a}_i^* K(s_i, s_j)\mathfrak{a}_j \Big\|_{\mathfrak{A}}, \quad (s_m)_m \subset S,\ (\mathfrak{a}_m)_m \subset \mathfrak{A}. \tag{9}$$

If this happens then $\Psi_K(\mathfrak{s})$ is a densely defined bounded operator and $\|\Psi_K(\mathfrak{s})\| = \|\Phi_K(\mathfrak{s})\| \leqslant c(\mathfrak{s})$; keeping the same notation for the extensions of these operators we have

$$\Phi_K(\mathfrak{s}) = \Psi_K(\mathfrak{s})^* \text{ and } \Psi_K(\mathfrak{s}) \text{ as well as } \Phi_K(\mathfrak{s}) \text{ are in } \mathbf{B}^*(\mathcal{E}_K).$$

Moreover, the mapping

$$\Phi_K\colon \mathfrak{S} \ni \mathfrak{s} \mapsto \Phi_K(\mathfrak{s}) \in \mathbf{B}^*(\mathcal{E}_K) \tag{10}$$

is multiplicative while the mapping

$$\Psi_K\colon \mathfrak{S} \ni \mathfrak{s} \mapsto \Psi_K(\mathfrak{s}) \in \mathbf{B}^*(\mathcal{E}_K) \tag{11}$$

is antimultiplicative. If $\mathfrak{S}$ *is unital with unit* 1, $\Phi(1) = \Psi(1) = 1_{\mathcal{E}_K}$ = *the identity operator in* $\mathcal{E}_K$.

COMMENTS. The important notice is that one of the responsibilities of the boundedness condition (9) is to ensure the operator $\Phi(\mathfrak{s})$ to be well defined; it goes together unnoticeably with boundedness of $\Phi(\mathfrak{s})$. These two matters merge.

Now the Kolmogorov factorization (7) resembles more than some people still would like to have.

COROLLARY 7. *Suppose* (9) *holds. Then for* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}$, $s, t \in S$

$$K(\mathfrak{s}\, s, \mathfrak{t}\, t) = \langle \Phi_K(\mathfrak{s}) K_s, \Phi_K(\mathfrak{t}) K_t \rangle_K, \tag{12}$$
$$K(s, \mathfrak{t}\, t) = \langle K_s, \Phi_K(\mathfrak{t}) K_t \rangle_K.$$

The appearance of the homomorphism Φ_K makes the resulting $\mathfrak{A}$-module a kind of C^**-correspondence* in the sense of [19]. We come to this notion closer as we specify more S and $\mathfrak{S}$.

Involution in $\mathfrak{S}$ added. Suppose $\mathfrak{S}$ has an involution and the action of $\mathfrak{S}$ is *transitive transitive transitive* with respect to the kernel K or the kernel K is $\mathfrak{S}$-*invariant*, which means anyway that

$$K(s, \mathfrak{s}\, t) = K(\mathfrak{s}^* s, t), \quad \mathfrak{s} \in \mathfrak{S},\ s, t \in S.$$

COROLLARY 8. *Suppose* (9) *holds. Then* $\Psi_K(s^*) = \Phi_K(s)$ *and the homomorphism* Φ *as in* (10) *becomes a* $*$*-homomorphism while the other* Ψ*, that in* (11)*, is anti-*$*$*-homomorphism.*

Now we have to change the meaning

$$S = \mathfrak{S} \text{ a } *\text{-semigroup},$$
$$K(\mathfrak{s}, \mathfrak{t}) = \omega(\mathfrak{s}^* \mathfrak{t}) \text{ with } \omega\colon \mathfrak{S} \to \mathfrak{A}$$

and the notation

$$\mathcal{D}_\omega,\ \mathcal{E}_\omega,\ \omega_{\mathfrak{s}},\ \langle\,\cdot\,, -\rangle_\omega,\ \Phi_\omega \text{ and } \Psi_\omega$$
instead of
$$\mathcal{D}_K,\ \mathcal{E}_K,\ K_{\mathfrak{s}},\ \langle\,\cdot\,, -\rangle_K,\ \Phi_K \text{ and } \Psi_K\ .$$

According to (2), $\mathfrak{A}$-positive definiteness of ω means,

$$\sum_{k,l} \mathfrak{a}_k^* \omega(\mathfrak{s}_k^* \mathfrak{s}_l) \mathfrak{a}_l \geqslant 0 \text{ for any finite choice of } (\mathfrak{s}_n)_n \subset \mathfrak{S} \text{ and } (\mathfrak{a}_n)_n \subset \mathfrak{A}$$

and the decomposition (12) (together with the defining formula (8)) takes the form

$$\omega(\mathfrak{s}_1^* \mathfrak{s}_2^* \mathfrak{t}_2 \mathfrak{t}_1) = \langle \Phi_\omega(\mathfrak{s}_2) \omega_{\mathfrak{s}_1}, \Phi_\omega(\mathfrak{t}_2) \omega_{\mathfrak{t}_1} \rangle_\omega = \langle \omega_{\mathfrak{s}_2 \mathfrak{s}_1}, \omega_{\mathfrak{t}_2 \mathfrak{t}_1} \rangle_\omega, \quad \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1, \mathfrak{t}_2 \in \mathfrak{S}.$$

Via the reproducing kernel Hilbert $\mathfrak{A}$-module construction, the inequalities (ii) and (iii) of Proposition 1, which we are going to use, now take the form

$$\Big(\sum_{i,k}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{t}_k)\mathfrak{b}_k\Big)^*\sum_{i,k}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{t}_k)\mathfrak{b}_k \leqslant \Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}\sum_{k,l}\mathfrak{b}_k^*\omega(\mathfrak{t}_k^*\mathfrak{t}_l)\mathfrak{b}_l, \quad (\mathfrak{s}_m)_m,(\mathfrak{t}_n)_n\subset\mathfrak{S},\ (\mathfrak{a}_m)_m,(\mathfrak{b}_n)_n\subset\mathfrak{A} \tag{13}$$

and

$$\Big\|\sum_{i,k}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{t}_k)\mathfrak{b}_k\Big\|_{\mathfrak{A}}^2 \leqslant \Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}\Big\|\sum_{k,l}\mathfrak{b}_k^*\omega(\mathfrak{t}_k^*\mathfrak{t}_l)\mathfrak{b}_l\Big\|_{\mathfrak{A}}, \quad (\mathfrak{s}_m)_m,(\mathfrak{t}_n)_n\subset\mathfrak{S},\ (\mathfrak{a}_m)_m,(\mathfrak{b}_n)_n\subset\mathfrak{A}. \tag{14}$$

Notice that

$$\Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}} = \Big\|\Big\langle\sum_i\omega_{\mathfrak{s}_i}\mathfrak{a}_i,\sum_i\omega_{\mathfrak{s}_i}\mathfrak{a}_i\Big\rangle\Big\|_{\mathfrak{A}} = \Big\|\sum_i\omega_{\mathfrak{s}_i}\mathfrak{a}_i\Big\|_{\omega}^2. \tag{15}$$

Remark 9. If $\omega\in\mathcal{E}_\omega$ then $\Phi(\mathfrak{s})\omega=\omega_{\mathfrak{s}}$ for all $\mathfrak{s}$ regardless $\mathfrak{S}$ or $\mathfrak{A}$ is unital or not. Indeed, for any $\mathfrak{t}\in\mathfrak{S}$

$$\langle\Psi(\mathfrak{s})\omega,\omega_{\mathfrak{t}}\rangle_\omega = \langle\omega_{[s]},\omega_{\mathfrak{t}}\rangle = \omega(\mathfrak{s}\mathfrak{t}) = \omega_{\mathfrak{s}^*}(\mathfrak{t}) = \langle\omega_{\mathfrak{s}^*},\omega_{\mathfrak{t}}\rangle.$$

Because $\Phi(\mathfrak{s})=\Psi(\mathfrak{s}^*)$ and $\omega_{\mathfrak{t}}$'s span $\mathcal{E}_\omega$, we get it.

Towards the KSGNS theorem

The boundedness condition, several versions. We turn to the *boundedness condition* (9) which under the current circumstances takes the form (a) below.

Proposition 10. *The following conditions are equivalent:*

(a) *for every* $\mathfrak{s}\in\mathfrak{S}$ *there is a constant* $c(\mathfrak{s})\geqslant 0$ *such that*

$$\Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}^*\mathfrak{s}\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}} \leqslant c(\mathfrak{s})\Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}, \quad (\mathfrak{s}_m)_m\subset\mathfrak{S},\ (\mathfrak{a}_m)_m\subset\mathfrak{A};$$

(b) *for every* $\mathfrak{s}\in\mathfrak{S}$ *there is a constant* $c(\mathfrak{s})\geqslant 0$ *such that*

$$\|\mathfrak{a}^*\omega(\mathfrak{t}^*\mathfrak{s}^*\mathfrak{s}\mathfrak{t})\mathfrak{a}\|_{\mathfrak{A}} \leqslant c(\mathfrak{s})\|\mathfrak{a}^*\omega(\mathfrak{t}^*\mathfrak{t})\mathfrak{a}\|_{\mathfrak{A}}, \quad \mathfrak{t}\in\mathfrak{S},\ \mathfrak{a}\in\mathfrak{A};$$

(c) *there is a submultiplicative function* $c\colon\mathfrak{S}\to[0,+\infty)$ *such that*

$$\|\mathfrak{a}^*\omega(\mathfrak{t}^*\mathfrak{s}^*\mathfrak{s}\mathfrak{t})\mathfrak{a}\|_{\mathfrak{A}} \leqslant d(\mathfrak{t},\mathfrak{a})c(\mathfrak{s}), \quad \mathfrak{s},\mathfrak{t}\in\mathfrak{S},\ \mathfrak{a}\in\mathfrak{A};$$

(d) $\liminf_n\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*(\mathfrak{s}^*\mathfrak{s})^{2^n}\mathfrak{s}_j)\mathfrak{a}_j\|_{\mathfrak{A}}^{2^{-n}}$ *is finite and does not depend on the choice of* $(\mathfrak{s}_n)_m$ *and* $(\mathfrak{a}_m)_m$.

Proof. (a) ⇒ (b) trivially. To get (c) from (b) notice that requiring $c(\mathfrak{s})$ to be minimal in (b) uniformly in $\mathfrak{t}$ and $\mathfrak{a}$ implies submultiplicativity of the function $c\colon\mathfrak{S}\to[0,+\infty)$. Applying (14) we get

$$\|\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}^*\mathfrak{s}\mathfrak{s}_j)\mathfrak{a}_j\|_{\mathfrak{A}}^2 \leqslant \|\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}^*\mathfrak{s}\mathfrak{s}_i)\mathfrak{a}_i\|_{\mathfrak{A}}\|\mathfrak{a}_j^*\omega(\mathfrak{s}_j^*\mathfrak{s}^*\mathfrak{s}\mathfrak{s}_j)\mathfrak{a}_j\|_{\mathfrak{A}}.$$

Therefore,

$$\Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*(\mathfrak{s}^*\mathfrak{s})^{2^n}\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}^{2^{-n}} \leqslant \Big(\sum_{i,j}\|\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*(\mathfrak{s}^*\mathfrak{s})^{2^n}\mathfrak{s}_i)\mathfrak{a}_i\|_{\mathfrak{A}}^{1/2}\|\mathfrak{a}_j^*\omega(\mathfrak{s}_j^*(\mathfrak{s}^*\mathfrak{s})^{2^n}\mathfrak{s}_j)\mathfrak{a}_j\|_{\mathfrak{A}}^{1/2}\Big)^{2^{-n}}$$
$$\leqslant d(c((\mathfrak{s}^*\mathfrak{s})^{2^{n-1}})^{1/2}c((\mathfrak{s}^*\mathfrak{s})^{2^{n-1}})^{1/2})^{2^{-n}} \leqslant dc(\mathfrak{s}^*\mathfrak{s})^{-1/2},$$

with $d \stackrel{\mathsf{def}}{=} \max_i\{d(\mathfrak{s}_i,\mathfrak{a}_i)\}$. Thus (d) follows.

Now repeated use of (14) with $\mathfrak{b}_i = \mathfrak{a}_i$ and $\mathfrak{t}_i = \mathfrak{s}^*\mathfrak{s}\mathfrak{s}_i$ gives us

$$\Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}^*\mathfrak{s}\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}^2 \leqslant \Big\|\sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*(\mathfrak{s}^*\mathfrak{s})^{2^k}\mathfrak{s}_j)\mathfrak{a}_j\Big\|_{\mathfrak{A}}^{2^{-k}}\Big\|\sum_{k,l}\mathfrak{a}_k^*\omega(\mathfrak{s}_k^*\mathfrak{s}_l)\mathfrak{a}_l\Big\|_{\mathfrak{A}}^{1-2^{-k}}.$$

The limit passage, after taking into account (d), leads to (a). ∎

Condition (b) may be viewed as a diagonalization of (a).

REMARK 11. The above versions of the boundedness condition have been discussed by the present author on different occasions, always for positive definite operator valued kernels. Condition (a) is Sz.-Nagy's boundedness condition in his general dilation theorem in [39]. Condition (b) is in [33], condition (c) is singled out in [31] and condition (d), the forerunner of the whole case, is already in [29].

COMMENTS. Notice either $\mathfrak{S}$ or $\mathfrak{A}$ need not be unital for the conclusion of Proposition 10 to hold. Anyway, now any of (a)–(d) may be viewed as a boundedness condition for ω. Moreover, the conditions (c) extends from $\mathfrak{S}$ to $\mathfrak{S}^+$ with ease.

Unitization of $\mathfrak{S}$. If $\mathfrak{S}$ has no unit define its *unitization* $\mathfrak{S}^+$ as $\mathfrak{S}^+ \stackrel{\mathsf{def}}{=} \mathfrak{S}\cup\{\imath\}$ with semigroup multiplication and involution as $\imath\mathfrak{s} = \mathfrak{s}\imath = \mathfrak{s}$ and $\imath^* = \imath$. If $\mathfrak{S}$ is already unital, that is it has a unit, for homogenization purpose set $\mathfrak{S}^+ \stackrel{\mathsf{def}}{=} \mathfrak{S}$. The following is important enough to be particularized

$$\boxed{\omega \text{ belongs to } \mathcal{E}_\omega \text{ and } \omega(\mathfrak{s}^*) = \omega(\mathfrak{s})^*,\ \mathfrak{s}\in\mathfrak{S}.} \qquad (*)$$

PROPOSITION 12. *Suppose the C*-algebra $\mathfrak{A}$ is* unital *with the unit denoted by $\mathfrak{e}$. Consider the following conditions:*

(α) *with some $c > 0$*

$$\omega(\mathfrak{s}^*) = \omega(\mathfrak{s})^*, \quad \mathfrak{s}\in\mathfrak{S},$$
$$c\sum_{i,,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i)^*\omega(\mathfrak{s}_j)\mathfrak{a}_j \leqslant \sum_{i,j}\mathfrak{a}_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)\mathfrak{a}_j, \quad (\mathfrak{s}_m)_m\subset\mathfrak{S},\ (\mathfrak{a}_m)_m\subset\mathfrak{A};$$

(β) *$\omega\colon\mathfrak{S}\to\mathfrak{A}$ extends to a positive definite function $\omega^+\colon\mathfrak{S}^+\to\mathfrak{A}$;*

(γ) *condition $(*)$ holds.*

Then $(\gamma)\Rightarrow(\alpha)\Leftrightarrow(\beta)$. If $\mathcal{E}_\omega$ is self-dual, then $(\alpha)\Rightarrow(\gamma)$.

An $\mathfrak{A}$-function ω on $\mathfrak{S}$ satisfying any of the equivalent conditions (α) or (β) of Proposition 12 is said to have the *extension property.* If $\mathfrak{S}$ is unital this is nothing but $\mathfrak{A}$-positive definiteness (actually, the second condition in (α) guarantees this at once).

The equivalence $(\alpha)\Leftrightarrow(\beta)$ for Hilbert space operator valued functions is in [32], for Hilbert C*-module valued ones in [16].

Proof. Suppose (α) holds. Extending ω to ω^+ by $\omega^+(\mathfrak{1}) \stackrel{\text{def}}{=} c^{-1/2}\mathfrak{e}$ we have

$$\begin{aligned}\sum_{i,j} \mathfrak{a}_i^* \omega^+({\mathfrak{s}_i}^* \mathfrak{s}_j)\mathfrak{a}_j &= \sum_{\mathfrak{s}_i \neq 1 \neq \mathfrak{s}_j} \mathfrak{a}_i^* \omega({\mathfrak{s}_i}^* \mathfrak{s}_j)\mathfrak{a}_j + \sum_{\mathfrak{s}_i \neq 1} \mathfrak{a}_i^* \omega({\mathfrak{s}_i}^*)\mathfrak{a}_1 \\ &+ \sum_{\mathfrak{s}_j \neq 1} \mathfrak{a}_1^* \omega(\mathfrak{s}_j)\mathfrak{a}_j + \mathfrak{a}_1^* c^{-1} \mathfrak{a}_1 \\ &\geqslant c\Big(\sum_{\mathfrak{s}_i \neq 1 \neq \mathfrak{s}_j} \mathfrak{a}_i^* \omega(\mathfrak{s}_i)^* \omega(\mathfrak{s}_j)\mathfrak{a}_j + \sum_{\mathfrak{s}_i \neq 1} c^{-1} \mathfrak{a}_i^* \omega(\mathfrak{s}_i)^* \mathfrak{a}_1 \\ &+ \sum_{\mathfrak{s}_j \neq 1} c^{-1} \mathfrak{a}_1^* \omega(\mathfrak{s}_j)\mathfrak{a}_j + c^{-2} \mathfrak{a}_1^* \mathfrak{a}_1 \Big) \\ &= c\Big(\sum_{\mathfrak{s}_i \neq 1} \omega(\mathfrak{s}_i)\mathfrak{a}_{\mathfrak{s}} + c^{-1}\mathfrak{a}_1 \Big)^* \Big(\sum_{\mathfrak{s}_i \neq 1} \omega(\mathfrak{s}_i)\mathfrak{a}_{\mathfrak{s}} + c^{-1}\mathfrak{a}_1 \Big) \geqslant 0.\end{aligned}$$

Thus ω^+ is positive definite on $\mathfrak{S}^+$.

Back to the proof suppose ω is extendible, that is (β) holds. Writing (13) for ω^+ with $\mathfrak{s}_i = \mathfrak{1}$ and $\mathfrak{a}_k = \mathfrak{e}$ and then restricting the resulting inequality to $\mathfrak{S}$ (remember, $\omega^+(\mathfrak{s}) = \omega(\mathfrak{s})$ for $\mathfrak{s} \in \mathfrak{S}$) we get the second of ($\alpha$). The first comes from (3).

Suppose now (γ) holds. Then the reproducing kernel property

$$\omega(\mathfrak{s}) = \langle \omega, \omega_{\mathfrak{s}} \rangle_\omega$$

when combined with condition (ii) of Proposition 1 gives

$$\begin{aligned}\sum_{i,,j} \mathfrak{a}_i^* \omega(\mathfrak{s}_i)^* \omega(\mathfrak{s}_i)\mathfrak{a}_i &= \Big\langle \omega, \sum_i \omega_{\mathfrak{s}} \mathfrak{a}_i \Big\rangle_\omega^* \Big\langle \omega, \sum_j \omega_{\mathfrak{s}} \mathfrak{a}_j \Big\rangle_\omega \\ &\leqslant \|\omega\|_\omega^2 \Big\langle \sum_i \omega_{\mathfrak{s}} \mathfrak{a}_i, \sum_j \omega_{\mathfrak{s}} \mathfrak{a}_j \Big\rangle_\omega \\ &\leqslant \|\omega\|_\omega^2 \sum_{i,j} \mathfrak{a}_i^* \omega(\mathfrak{s}_i^* \mathfrak{s}_j)\mathfrak{a}_j. \end{aligned} \tag{16}$$

This is the second of (α), the first has just to be copied.

Suppose now the Hilbert $\mathfrak{A}$-module $\mathcal{E}_\omega$ is self-dual. Then the inequality in (α) fits in the condition ② of Proposition 4 with $F = \omega$ and the last conclusion follows ■

Call a net $(1_\lambda)_\lambda \subset \mathfrak{S}$ an *approximate unit* for ω if there is a net $(\mathfrak{a}_\lambda)_\lambda \subset \mathfrak{A}$ such that

$$\mathfrak{a}_\lambda^* \omega(\mathfrak{1}_\lambda \mathfrak{s}) \xrightarrow{\mathfrak{A}} \omega(\mathfrak{s}) \text{ and } \omega(\mathfrak{s} 1_\lambda^*)\mathfrak{a}_\lambda \xrightarrow{\mathfrak{A}} \omega(\mathfrak{s}), \tag{17}$$

$$\|\mathfrak{a}_\lambda^* \omega(\mathfrak{1}_\lambda \mathfrak{1}_\lambda^*)\mathfrak{a}_\lambda\|_{\mathfrak{A}} \text{ is bounded in } \lambda; \tag{18}$$

it is called a *strong approximate unit* for ω if (17) holds and, instead of (18),

$$(\mathfrak{a}_\lambda^* \omega(\mathfrak{1}_\lambda \mathfrak{1}_\lambda^*)\mathfrak{a}_\lambda)_\lambda \text{ is a Cauchy net.} \tag{19}$$

Notice that (19) implies (18), thus strong approximate unit seems to be really stronger. Is it?

PROPOSITION 13. *Suppose $\mathfrak{A}$ is unital. If ω has an approximate unit $(\mathfrak{1}_\lambda)_\lambda$ then it has an extension property. If ω has a strong approximate unit $(\mathfrak{1}_\lambda)_\lambda$ then $(*)$ holds. Conversely, if $(*)$ holds then there are two arrays $((\mathfrak{s}_i^n)_{i \in \{\text{finite}\}})_{n=0}^\infty \subset \mathfrak{S}$ and $((\mathfrak{a}_i^n)_{i \in \{\text{finite}\}})_{n=0}^\infty \subset \mathfrak{A}$*

such that

$$\sum_i (\mathfrak{a}_i^n)^*\omega(\mathfrak{s}_i^n\mathfrak{s}) \xrightarrow{\mathfrak{A}} \omega(\mathfrak{s}) \text{ and } \sum_i \omega(\mathfrak{s}(\mathfrak{s}_i^n)^*)\mathfrak{a}_i^n \xrightarrow{\mathfrak{A}} \omega(\mathfrak{s}), \tag{20}$$

$$\sum_{i,j} (\mathfrak{a}_i^n)^*\omega(\mathfrak{s}_i^n(\mathfrak{s}_j^n)^*)\mathfrak{a}_j^n \text{ is a Cauchy sequence in } n. \tag{21}$$

Proof. Insert the approximate unit into (3) and (13) and proceed in an appropriate way so as get both parts of (α).

Suppose ω has a strong approximate unit $(1_\lambda)_\lambda$. Then, (19) with a little help of (15) implies $(\omega_{1_\lambda^*}\mathfrak{a}_\lambda)_\lambda$ is a Cauchy net in $\mathcal{E}_\omega$. Therefore, there is $F \in \mathcal{E}_\omega$ such that $\omega_{1_\lambda^*}\mathfrak{a}_\lambda \xrightarrow{\mathcal{E}_\omega} F$ and, consequently,

$$\omega(\mathfrak{s}) \xleftarrow{\mathfrak{A}} \omega(1_\lambda\mathfrak{s})\mathfrak{a}_\lambda = \langle \omega_{1_\lambda^*}\mathfrak{a}_\lambda, \omega_\mathfrak{s}\rangle \xrightarrow{\mathfrak{A}} \langle F, \omega_\mathfrak{s}\rangle = F(\mathfrak{s}), \quad \mathfrak{s} \in \mathfrak{S}.$$

Therefore, $\omega = F \in \mathcal{E}_\omega$. The Hermitian symmetry of ω goes straightforwardly from (3).

If $\mathfrak{S}$ is unital then (20) and (21) hold with the trivial choice of the required data. If not, then there is always a sequence $(\sum_i \omega_{(\mathfrak{s}_i^n)^*}\mathfrak{a}_i^n)_n$ converges to ω in $\mathcal{E}_\omega$ (density of $\mathcal{D}_\omega$!) and so does $\Phi_\omega(\mathfrak{s})(\sum_i \omega_{(\mathfrak{s}_i^n)^*}\mathfrak{a}_i^n)_n$ to $\omega(\mathfrak{s})$, use Remark 9 on a way. This gives (21). Because, due to the reproducing property norm convergence implies pointwise, (20) follows as well. ∎

REMARK 14. If ω satisfies the extension property, then the mapping $W\colon \omega_\mathfrak{s} \to \omega_\mathfrak{s}^+$, $\mathfrak{s} \in \mathfrak{S}$, extends to a linear operator of $\mathcal{E}_\omega$ into $\mathcal{E}_{\omega^+}$. The operator W is adjointable with adjoint given by $W^*\omega_\mathfrak{t}^+ = \omega_\mathfrak{t}$ if $t \neq 1$ and 0 otherwise. Furthermore, W is an $\mathfrak{A}$ isometry onto

$$\mathcal{E}_{\omega^+}^0 \overset{\text{def}}{=} \operatorname{clolin}\{\omega_\mathfrak{s}^+\mathfrak{a}\colon\ \mathfrak{s} \in \mathfrak{S},\ \mathfrak{a} \in \mathfrak{A}\}.$$

It is a matter of direct verification that the basic RKHS operators are related by

$$W\Phi_\omega(\mathfrak{s})W^* = \Phi_{\omega^+}(\mathfrak{s}), \quad \mathfrak{s} \in \mathfrak{S}.$$

The basic dilation theorem. Given an $\mathfrak{A}$-function ω on a $*$-semigroup $\mathfrak{S}$, it is clear that if there exists a Hilbert $\mathfrak{A}$-module $\mathcal{E}$, a multiplicative $*$-homomorphism $\Phi\colon \mathfrak{S} \to \mathbf{B}^*(\mathcal{E})$ and $V \in \mathbf{B}^*(\mathfrak{A}, \mathcal{E})$ such that

$$\omega(\mathfrak{s}) = V^*\Phi(\mathfrak{s})V, \quad \mathfrak{s} \in \mathfrak{S}, \tag{22}$$

then ω is $\mathfrak{A}$-positive definite. Moreover, (22) implies

$$\|\mathfrak{a}^*\omega(\mathfrak{t}^*\mathfrak{s}^*\mathfrak{s}\mathfrak{t})\mathfrak{a}\|_\mathfrak{A} = \|\Phi_\omega(\mathfrak{s})V\mathfrak{a}\|_\mathcal{E} \leqslant \|V\mathfrak{a}\|_\mathcal{E}\,\|\Phi(\mathfrak{s})\|_\mathcal{E} \tag{23}$$

with $\|\Phi_\omega(\mathfrak{s})\|_\omega$ submultiplicative. Thus (c) of Proposition 10 holds. To show ω has an extension property use (22) like in (16); the Hermitian symmetry of ω follows from (22) immediately. This solves in the rather trivial way most of the one side of the dilation story. The other is the masterpiece for those who may appreciate it. We are aware of the fact that the category theory fans are going to be disappointed; RKHS is too rich in information it carries to be a categorical object on call.

Another issue which we are not going to touch is uniqueness of minimal dilations whatever the latter means. This can be done in a standard way anytime a need appears. The notion of minimality in (23) has to be introduced anyway. We say that the triple

$(\mathcal{E}, \Phi, V)$ is *minimal*[10] for ω if $\mathcal{E} = \operatorname{clolin}\{\Phi(\mathfrak{s})V\mathfrak{a}\colon\ \mathfrak{s} \in \mathfrak{S},\ \mathfrak{a} \in \mathfrak{A}\}$. Minimality is always done when $\mathfrak{S}$ is unital.

THEOREM 15. *Let $\mathfrak{A}$ be a unital C^*-algebra. For an $\mathfrak{A}$-positive definite function ω on $\mathfrak{S}$ the following conclusions hold.*

1° *There is $V^+ \in \mathbf{B}^*(\mathfrak{A}, \mathcal{E}_{\omega^+})$ such that*

$$\mathfrak{a}^*\omega(\mathfrak{s})\mathfrak{b} = \langle V^+\mathfrak{a}, \Phi_\omega^+(\mathfrak{s})V^+\mathfrak{b}\rangle_{\omega^+}, \quad \mathfrak{s} \in \mathfrak{S},\ \mathfrak{a}, \mathfrak{b} \in \mathfrak{A} \tag{24}$$

if and only if ω satisfies the boundedness condition, that is any of the conditions (a)–(d) *of Proposition 10 as well as it has the extension property, that is any of the conditions* (α)–(β) *of Proposition 12 holds.*

2° *Suppose condition* ($*$) *holds*[11]*. Then there is $V \in \mathbf{B}^*(\mathfrak{A}, \mathcal{E}_\omega)$ such that*

$$\mathfrak{a}^*\omega(\mathfrak{s})\mathfrak{b} = \langle V\mathfrak{a}, \Phi_\omega(\mathfrak{s})V\mathfrak{b}\rangle_\omega, \quad \mathfrak{s} \in \mathfrak{S}\ \ \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}. \tag{25}$$

The operator V is defined by (30) *and its adjoint by* (31)*. Therefore,* (25) *can be written as*

$$\mathfrak{a}^*\omega(\mathfrak{s})\mathfrak{b} = \langle \omega\mathfrak{a}, \Phi_\omega(\mathfrak{s})\omega\mathfrak{b}\rangle_\omega, \quad \mathfrak{s} \in \mathfrak{S}\ \ \mathfrak{a}, \mathfrak{b} \in \mathfrak{A}. \tag{26}$$

3° *The triple $(\mathcal{E}_\omega, \Phi_\omega, V)$ generated in* 2° *is minimal. Consequently,* (25) *can be written as*

$$\omega(\mathfrak{s}) = V^*\Phi(\mathfrak{s})V, \quad \mathfrak{s} \in \mathfrak{S}. \tag{27}$$

Notice that in the case $\mathfrak{S}$ is unital the extendibility procedure is needless, $\mathfrak{A}$-positive definiteness and the boundedness condition are enough to play the game. The whole embarrassment is caused by a possible lack of unit in $\mathfrak{S}$; this happens in C*-algebras but that case is always weaponed with a substitute, approximate unit. On the other hand, the reproducing kernel construction we have carried out here in full allows to put forward condition ($*$). Its simplicity is an effective counterbalance for more sophisticated technology. At least it reduces the number of new objects involved from two to one: just the Φ_ω. If $\mathfrak{A}$ is unital, the dilation formula (26) takes the shortest possible form

$$\omega(\mathfrak{s}) = \langle \omega, \Phi_\omega(\mathfrak{s})\omega\rangle_\omega, \quad \mathfrak{s} \in \mathfrak{S} \tag{28}$$

which resembles the reproducing kernel property by the way. This observation justifies the rough statement:

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Proof. The kernel $(\mathfrak{s}, \mathfrak{t}) \to \omega(\mathfrak{s}^*\mathfrak{t})$ is positive definite in the sense of (2) therefore the outcomes of the reproducing kernel construction are at our disposal.

Let us prove first conclusion 2°. Due to ($*$), $\omega \in \mathcal{E}_\omega$ and the reproducing kernel property (6) gives us

$$\omega(\mathfrak{s}) = \langle \omega, \omega_\mathfrak{s}\rangle_\omega, \quad \mathfrak{s} \in \mathfrak{S}$$

and this, in turn, allows us to write

$$\langle \omega(\mathfrak{s})\mathfrak{a}, \mathfrak{b}\rangle_\mathfrak{A} = \mathfrak{a}^*\omega(\mathfrak{s})^*\mathfrak{b} = \mathfrak{a}^*\langle \omega_\mathfrak{s}, \omega\rangle_\omega\mathfrak{b} = \langle \omega_\mathfrak{s}\mathfrak{a}, \omega\mathfrak{b}\rangle_\omega. \tag{29}$$

[10] Other names which appear on this occasion are *nondegenerate* or *essential.*

[11] This happens when ω has a strong approximate unit, cf. Remark 9.

Reading (29) in a proper way (remember $\mathfrak{A}$ is unital) we infer that $\omega(\mathfrak{s})\mathfrak{a}$ is the only c such that $\langle c,\mathfrak{b}\rangle_{\mathfrak{A}} = \langle\omega_{\mathfrak{s}}\mathfrak{a},\omega\mathfrak{b}\rangle_\omega$. This means that

$$V\colon \mathfrak{A}\ni\mathfrak{a}\mapsto\omega\mathfrak{a}\in\mathcal{E}_\omega \tag{30}$$

is adjointable with

$$V^*\colon \mathcal{E}_\omega\ni\omega_{\mathfrak{s}}\mathfrak{a}\mapsto\omega(\mathfrak{s})\mathfrak{a}\in\mathfrak{A}. \tag{31}$$

By Remark 9, we have

$$\langle V\mathfrak{a},\Phi_\omega(\mathfrak{s})V\mathfrak{b}\rangle_\omega = \langle\omega\mathfrak{a},\Phi_\omega(\mathfrak{s})\omega\mathfrak{b}\rangle_\omega = \langle\omega\mathfrak{a},\omega_{\mathfrak{s}}\mathfrak{b}\rangle_\omega = \mathfrak{a}^*\omega(\mathfrak{s})\mathfrak{b}$$

and this establishes (24).

The essential direction in 1° goes as follows (the reverse has already been discussed before the theorem). If $\mathfrak{S}$ is unital, we are in a position of 2°. If not, we continue the play between $\mathfrak{S}$ and $\mathfrak{S}^+$. Keeping the notation up for the operator V in the unital case of ω^+, as done in 2° for this case, we have it in $\mathbf{B}^*(\mathfrak{A},\mathcal{E}_{\omega^+})$. Setting now $V^+\overset{\mathrm{def}}{=}VW$ we use the merits of Remark 14 to come to (24).

Putting to use again Remark 9 we get

$$\sum_i\Phi_\omega(\mathfrak{s}_i)Va_i = \sum_i\Phi_\omega(\mathfrak{s}_i)\omega a_i = \sum_i\omega_{\mathfrak{s}_i}a_i$$

which makes all the arrangements for minimality of the triple $(\mathcal{E}_\omega,\Psi_\omega,V)$ as $\sum_i\omega_{\mathfrak{s}_i}a_i$ span linearly $\mathcal{D}_\omega$, the dense subspace of $\mathcal{E}_\omega$. ∎

The part 1° of above Theorem for a Hilbert space operator valued ω is in [30]. It was long ago!

COMMENTS. The presence of ω in the reproducing kernel Hilbert C*-module, expressed in (∗), not only refreshes the dilation formula (24) by giving it the simply looking form (28) but also works efficiently for minimality.

Closer to the originals

KSGNS now. The door has been opened for consequences. First KSGNS, for those who are eager for seeing it again.

Suppose $\mathfrak{S}$ and $\mathfrak{A}$ are <u>both</u> C*-algebras; the latter to be unital. The map ω is now <u>linear</u> and <u>completely</u> <u>positive</u>. It is well known that this is the pleasant case when complete positivity is <u>equivalent</u> to positive definiteness or rather to $\mathfrak{A}$-positive definiteness according to the present circumstances. In other words, positivity of an $\mathfrak{A}$-matrix, at the very beginning of the exploration, is replaced by factorization; this seems to be the only case, compare the items (A) and (B) below.

The boundedness condition (b) of Proposition 10 follows immediately from the rudimentary inequality $\mathfrak{t}^*\mathfrak{s}^*\mathfrak{s}\mathfrak{t}\leqslant\|\mathfrak{s}\|^2_{\mathfrak{S}}\,\mathfrak{t}^*\mathfrak{t}$.

Suppose $\{1_\lambda\}_\lambda$ is an approximate unit in $\mathfrak{S}$. Insert in the Schwarz inequality (13) $\mathfrak{s}_i = 1_\lambda$, $i\in\{\text{singleton}\}$ and then perform the limit passage (boundedness of ω in use) taking advantage of all the attributes of $\{1_\lambda\}_\lambda$ so as to come to the second condition (λ) of Proposition 12. The first condition comes from (3) with help of the approximative unit too.

Therefore both the boundedness condition and the extension property are inherited from the structural properties of C*-algebras.

SUMMING-UP. If ω is a completely positive map between two C*-algebras $\mathfrak{S}$ and $\mathfrak{A}$, with $\mathfrak{A}$ unital, then (24) holds. This is the (ground level again) KSGNS, or rather its existence part which is settled here in a more elementary environment than usually. Its uniqueness is a matter of further conditions unless $\mathfrak{S}$ is unital. Now we have two possibilities: (a) either to assume there is an approximating unit in $\mathfrak{S}$ which is a strong approximate unit for ω, or (b) to assume $(*)$ right away. The mutual relationship is in Proposition 13; an immediate one is in the case $\mathfrak{S} = \mathfrak{A}$ and ω to be $\mathfrak{A}$-linear they coincide.

Notice that linearity of the dilated map Φ is of secondary importance, it comes for free from its minimality.

What is meant by KSGNS refers to the case when $\mathfrak{A}$ is just $\mathbf{B}^*(\mathfrak{A}, \mathcal{E}_\omega)$ which is C*-algebra for itself. Therefore, our pre-KSGNS, so to speak, can be cultivated for this instance.

Is the acronym KSGNS long enough? Now we are going to provide arguments for our claim, or rather insistence, of extending the acronym by the well-deserving initials. It was in 1955 when two essential events happened: the PAMS paper [26] of Stinespring (included) and Szőkefalvi-Nagy's Appendix [39] (forgotten); the English version [40] appeared five years later. Stinespring's paper stimulated an explosion of new ideas, getting more and more abstract, Sz.-Nagy's has been left out of favour.

The first thing we want to stress on is these two 1955 results are (logically) equivalent, see [34]. This is so as long as bounded operators are involved in a way. An attempt at extending them to the 'unbounded' circumstances causes a splitting into two, no longer equivalent:

(A) the direction of [24] in which complete positivity further on provides with abstract characterizations of dilatability [12] – less useful in concrete cases;
(B) positive definiteness which alone hardly becomes a sufficient condition for dilatability – much more handy if available, look at [33] or [36] to catch some flavour.

The difference between (A) and (B) can be even seen in the case $\mathfrak{A} = \mathbb{C}$, that is when dealing with moment problems.

The main object in [39] is an involution semigroup [13] (or, a $*$-semigroup) and positive definiteness is defined on them. The positive definite mappings are bounded Hilbert space operator valued which positions the Appendix before 'K' in the acronym; the mappings go to Hilbert $\mathbb{C}$-modules yet are defined on simpler algebraic structures than C*-algebras. The theorem says [14] roughly that positive definiteness and the boundedness condition (a) of Proposition 10 are necessary and sufficient for an operator function to have a dilation provided the $*$-semigroup $\mathfrak{S}$ has a unit. Additional conclusions concern continuity and linearity properly understood in the context of semigroups.

[12] Dilatability also means extendibility, according to §5 of [39].
[13] This notion seems to be originated there.
[14] This is in §6 of [39]. Other versions are in [29], [30], [31] and [33].

Let us exhibit the diversity of $*$-semigroups to which the Sz.-Nagy general dilation theorem applies.

- ▷ Groups (commutative or not) with involution $\mathfrak{s}^* \stackrel{\text{def}}{=} \mathfrak{s}^{-1}$. The boundedness condition (a) of Proposition 10 turns into equality with $c(\mathfrak{s}) = 1$. Therefore the dilations are unitary representations of groups, which happen in Harmonic Analysis. The case $\mathfrak{S} = \mathbb{Z}$ is that of the 1953 frequently quoted Sz.-Nagy dilation theorem for contractions, cf. [39], §4.
- ▷ Inverse semigroups. In this case the boundedness condition (d) of Proposition 10 trivializes [15], which gives more prominence to Proposition 10.
- ▷ σ-algebras of sets with set intersection as the semigroup operation and the identity map as an involution. Again the boundedness condition (a) of Proposition 10 turns into equality with $c(\mathfrak{s}) = 1$. This leads to Naimark's dilation of semispectral measures to spectral ones, cf. [39], §2 and [18].
- ▷ Subnormality. This is a 1950 invention of Halmos [13] who characterized it in terms of positive definiteness plus some boundedness condition which, due to Bram [6], turned out to be needless (see also [31] for another argument presented also in [41]). It was Sz.-Nagy who put Halmos result into more general framework, his general dilation theorem of [39], cf. §5; another, quite different, kind of sentiment is exposed in [2]. Nevertheless, subnormality can be considered also from the C*-algebra point of view, see [7] and [35]. What is worthy to rescue is appearance of the unital $*$-semigroup $\mathbb{N} \times \mathbb{N}$ with involution $(m, n)^* \stackrel{\text{def}}{=} (n, m)$ which makes all this possible.
- ▷ $*$-algebras, in particular C*-algebras. This case has been already discussed on occasion of KSGSN.
- ▷ Moment problems. This is an extremely spectacular area rooted in Classical Analysis. Here are three interesting cases
  - $\mathfrak{S} \stackrel{\text{def}}{=} \mathbb{N}$ with identical involution. This is an environment of the classical (one variable) moment problems, both scalar and operator valued. Here positive definiteness is enough for integral representation (read: dilatability). Any of the boundedness conditions localizes the measure on a compact set.
  - $\mathfrak{S} \stackrel{\text{def}}{=} \mathbb{N}^d$ with coordinate addition and identical involution again. This is the very sensitive case, cf. [11], positive definiteness is only a necessary condition of being a moment multisequence. Therefore, the boundedness condition not only guarantees dilatability but also localizes the measure on a compact set, again.
  - The Sz.-Nagy semigroup mentioned on occasion of subnormality is related to the complex moment problem, for more consult [28].

  It seems to be needless to say that the above can be extended to the Hilbert C*-module context as well.

The careful reader has noticed that author's ambition here is to confront complete positivity with positive definiteness and to heighten awareness the latter suits more situations in

[15] Recall, to $\mathfrak{s} \in \mathfrak{S}$ there is a unique $\mathfrak{s}^*$ such that $\mathfrak{s} = \mathfrak{s}\mathfrak{s}^*\mathfrak{s}$ and $\mathfrak{s}^* = \mathfrak{s}^*\mathfrak{s}\mathfrak{s}^*$

a rather elementary manner. So, the question turns up again: is the acronym KSGNS long enough? Maybe KS Sz.-N GNS? The only thing against might be that, as a pictograph, it violates someone's aesthetical habits. Nevertheless, notice 'Sz.' is from Sz.-Nagy and seems to have nothing in common with the present author.

Sz.-Nagy and Stinespring side by side. These two 1955 theorems we have already mentioned are logically equivalent in the sense one of them can be derived by means of the other without any appeal to their proofs. This was in fact done already in [34]. Let us point out the main arguments here just to strengthen integrity of this paper.

Positive definiteness versus complete positivity. The dilation theorems in question involve two different kind of positivity. Positive definiteness is the principal theme of this paper. The other, complete positivity can be defined as follows. Let $\omega\colon \mathfrak{A} \to \mathfrak{B}$ be a mapping between two C*-algebras $\mathfrak{A}$ and $\mathfrak{B}$; call it *positive* if $\omega(\mathfrak{a}) \geqslant 0$ whenever $\mathfrak{a} \geqslant 0$. Furthermore, for a C*-algebra $\mathfrak{C}$ denote by $\mathbf{M}_n(\mathfrak{C})$ the unique C*-algebra of $n \times n$ matrices with entries in $\mathfrak{C}$. Then ω induced a mapping $\omega_n\colon \mathbf{M}_n(\mathfrak{A}) \to \mathbf{M}_n(\mathfrak{B})$ acting entrywise. The mapping ω is said to be *completely positive*[16] if ω_n is positive for any positive integer n.

If $\omega\colon \mathfrak{A} \to \mathfrak{B}$ is a linear mapping between C*-algebras then complete positivity and positive definiteness coincide. This is so because positive elements in $\mathbf{M}_n(\mathfrak{A})$ turn out to be of the form $(\sum_k \mathfrak{a}_{k,i}^*\mathfrak{a}_{k,j})_{i,j=1}^n$ (square root in $\mathbf{M}_n(\mathfrak{A})$ involved!).

Let us state explicitly our points of reference. Let $\mathcal{H}$ be a Hilbert space, $\mathfrak{S}$ is an involution semigroup and $\mathfrak{A}$ a C*-algebra, both with unit.

THEOREM (Sz.-Nagy [39], Théorème Principal). *A mapping $\omega\colon \mathfrak{S} \to \mathbf{B}(\mathcal{H})$ is of the form*

$$\omega(\mathfrak{s}) = V^*\Phi(\mathfrak{s})V, \quad \mathfrak{s} \in \mathfrak{S}, \tag{32}$$

where $\Phi\colon \mathfrak{S} \to \mathbf{B}(\mathcal{K})$ is a $*$homomorphism and $V \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, if and only if ω is positive definite on $\mathfrak{S}$ and satisfies the boundedness condition (a) *of* (10).

The choice of $\mathcal{K}$ can always be arranged as being minimal. If this happens then[17]

$$\begin{gathered}\omega(\mathfrak{x}\mathfrak{s}\mathfrak{y}) = \omega(\mathfrak{x}\mathfrak{t}\mathfrak{y}) + \omega(\mathfrak{x}\mathfrak{u}\mathfrak{y}) \text{ for some } \mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in \mathfrak{S} \\ \text{and all } \mathfrak{x}, \mathfrak{y} \in \mathfrak{S} \text{ implies } \Phi(\mathfrak{s}) = \Phi(\mathfrak{t}) + \Phi(\mathfrak{u}).\end{gathered} \tag{33}$$

THEOREM (Stinespring [26]). *A linear mapping $\widehat{\omega}\colon \mathfrak{A} \to \mathbf{B}(\mathcal{H})$ is of the form*

$$\widehat{\omega}(\mathfrak{a}) = V^*\Phi(\mathfrak{a})V, \quad \mathfrak{a} \in \mathfrak{A}, \tag{34}$$

where $\Phi\colon \mathfrak{A} \to \mathbf{B}(\mathcal{K})$ is a $*$homomorphism and $V \in \mathbf{B}(\mathcal{H}, \mathcal{K})$, if and only if $\widehat{\omega}$ is completely positive on $\mathfrak{A}$.

From Sz.-Nagy's to Stinespring's. Suppose $\widehat{\omega}$ is a completely positive mapping on a C*-algebra $\mathfrak{A}$. As we already know it is positive definite, satisfies the boundedness condition (in its version (c) as in Proposition 10 therefore (a) as well). Now we are in a position to apply Sz.-Nagy's theorem resulting in the $*$-semigroup homomorphism. Applying the

[16] The definition comes from [26]. It has become a standard topic in Operator Algebras since then, see for instance [43]; let us mention the monograph [23] settles it well in the context of the similarity problem including the famous Kadison problem.

[17] This may be viewed as a substitute of additivity missing in a semigroup.

part (33) of that theorem makes it a $*$-algebra homomorphism. The Stinespring theorem has been established as a result.

From Stinespring's to Sz.-Nagy's. This is more involved because the underlying algebraic structure is not so fertile. Suppose now ω is a positive definite mapping on a $*$-semigroup $\mathfrak{S}$ satisfying the boundedness condition (a) of Proposition 10. With c as in (c) of Proposition 10 we construct the weighted Banach $*$-algebra $\ell^1(\mathfrak{S}, c)$ with standard multiplication $\xi * \eta(\mathfrak{s}) \stackrel{\text{def}}{=} \sum_{\mathfrak{tu}=\mathfrak{s}} \xi(\mathfrak{t})\eta(\mathfrak{u})$, $\xi, \eta \in \ell^1(\mathfrak{S}, c)$, or $\stackrel{\text{def}}{=} 0$ if there is no such $\mathfrak{t}$ and $\mathfrak{u}$, and typical involution $\xi^*(\mathfrak{s}) \stackrel{\text{def}}{=} \overline{\xi(\mathfrak{s}^*)}$. In the next step one has to pass to the C*-envelop [18] of $\ell^1(\mathfrak{S}, c)$, call it $\mathfrak{A}$. The boundedness condition (c) allows us to define a linear mapping $\widehat{\omega}\colon \ell^1(\mathfrak{S}, c) \to \mathrm{B}(\mathcal{H})$ via $\widehat{\omega}(\xi) \stackrel{\text{def}}{=} \sum_{\mathfrak{s}} \xi(\mathfrak{s})\omega(\mathfrak{s})$ which extends to $\mathfrak{A}$ with the same notation in use. Positive definiteness of ω yields that of $\widehat{\omega}$, hence complete positivity of the latter. Now Stinespring's theorem leads directly to Sz.-Nagy's.

Comments. Both Sz.-Nagy and Stinespring theorems in their original versions require the algebraic objects to have a unit. There is no problem in relaxing this by using one of the (equivalent) extension conditions in Proposition 12 on one side and keeping in mind the fact that a C*-algebra always has an approximate unit (now Proposition 9 can be in use) on the other.

Notice in Stinespring's paper minimality, and therefore uniqueness, is not touched. On the other hand, in Sz.-Nagy theorem we have normalization $\omega(1) = 1_{\mathcal{H}}$ resulting in $V =$ projection. These two matters can be rearranged by a two line argument.

It should not be any problem with performing the equivalence in question in the case of Hilbert C*-modules.

Conclusions. Summing up:

- Sz.-Nagy – elementary, simple though more laborious, pointing out the essentials, more suitable for individual cases;
- Stinespring and the followers – technologically more advanced, fast tracking, usually engaging more than needed, rather a categorical treatment.

Warning:

- none of them is universal, no *panacea* appear even in Mathematics (very optimistic).

Making it more spatial. Let us notice that for both Stinespring and Sz.-Nagy the common target space for ω is the C*-algebra of bounded operators on a Hilbert space which is a $\mathbb{C}$-module. The difference is in the initial set $\mathfrak{S}$ for the kernel; Sz.-Nagy's is the simplest possible, it does not bear any unnecessary at the moment arrangement. On the other hand, for GNS our choice of the target space in Theorem 15 fits in. We may try to mend this incompleteness.

Suppose $\mathfrak{S}$ is a $*$-semigroup and $\mathfrak{A}$ is a unital C*-algebra; moreover, suppose ω satisfies $(*)$.

[18] Cf. Proposition 2.7.1 in [8]. Notice that under our circumstances ω provides us with quite a number of positive (read: positive definite) forms on $\ell^1(\mathfrak{S}, c)$ the existence of which is indispensable for the definition to work.

Procedure # 1. Fix a faithful $*$-representation π of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. Then (27) can be written as an equality for bounded operators on the Hilbert space $\mathcal{H}$

$$\pi(\omega(\mathfrak{s})) = \pi(V)^*\pi(\Phi_\omega(\mathfrak{s}))\pi(V), \quad \mathfrak{s} \in \mathfrak{S}$$

with $\mathfrak{s} \mapsto \pi(\Phi_\omega(\mathfrak{s}))$ being a $*$-representation of $\mathfrak{S}$ on the Hilbert space $\mathcal{H}$.

Procedure # 2. Now $\mathcal{E}$ is an arbitrary $\mathfrak{A}$-module. For a $\mathbf{B}^*(\mathcal{E})$-function ω on $\mathfrak{S}$ we have a kernel K on $\mathfrak{S} \times \mathcal{E}$ defined as

$$K(\mathfrak{s}, x, \mathfrak{t}, y) \stackrel{\text{def}}{=} \langle \omega(\mathfrak{s}^*\mathfrak{t})x, y\rangle_{\mathcal{E}}, \quad \mathfrak{s}, \mathfrak{t} \in \mathfrak{S}, \ x, y \in \mathcal{E}.$$

Then K becomes an $\mathfrak{A}$-valued kernel which is $\mathfrak{S}$-invariant. Recall $\mathbf{B}^*(\mathcal{E})$-positive definiteness of ω means

$$\sum_{i,j} T_i^*\omega(\mathfrak{s}_i^*\mathfrak{s}_j)T_j \geqslant 0, \quad (\mathfrak{s}_m)_M \subset \mathfrak{S}, \ (T_n)_N \subset \mathbf{B}^*(\mathcal{E}). \tag{35}$$

To show K is $\mathfrak{A}$-positive definite choose[19] $x \in \mathcal{E}$ such that $\langle x, x\rangle_{\mathcal{E}} = e$ and define $T_i \stackrel{\text{def}}{=} T_{x, x_i\mathfrak{a}_i}$ as in (1). Then $T_i \in \mathbf{B}^*(\mathcal{E})$ and $T_i x = x_i\mathfrak{a}_i$

$$\begin{aligned}\sum_{i,j} \mathfrak{a}_i^* K(\mathfrak{s}_i, x_i, \mathfrak{s}_j, x_j)\mathfrak{a}_j &= \sum_{i,j} \mathfrak{a}_i^*\langle \omega(\mathfrak{s}_i^*\mathfrak{s}_j)x_i, x_j\rangle_{\mathcal{E}}\mathfrak{a}_j \\ &= \sum_{i,j} \langle \omega(\mathfrak{s}_i^*\mathfrak{s}_j)x_i\mathfrak{a}_i, x_j\mathfrak{a}_j\rangle_{\mathcal{E}} \\ &= \sum_{i,j} \langle \omega(\mathfrak{s}_i^*\mathfrak{s}_j)T_i x, T_j x\rangle_{\mathcal{E}} \stackrel{(35)}{\geqslant} 0\end{aligned}$$

Therefore K is $\mathfrak{A}$-positive definite provided ω is $\mathbf{B}^*(\mathcal{E})$-positive definite. We can now develop the reproducing kernel routine for K distinguishing the resulting elements of the construction by the subscript $_\omega$.

Notice $\mathfrak{S}$ acts on $S = \mathfrak{S} \times \mathcal{E}$ exclusively through its first variable, the second remains untouched.

To avoid getting involved in nonunital dispute, which would not be a disaster due to our already worked out tools like Proposition 13, assume $\mathfrak{S}$ is unital. Assume also $K(1, x, 1, y) = \langle x, y\rangle_{\mathcal{E}}$ which defines immediately an isometry $V \colon x \to K_{1,x}$. Moreover, $V \in \mathbf{B}^*(\mathcal{E}, \mathcal{E}_\omega)$ with $V^* \colon K_{\mathfrak{s},y} \to \omega(\mathfrak{s}^*)y$. Indeed,

$$\langle Vx, K_{\mathfrak{s},y}\rangle_{\mathcal{E}_\omega} = \langle K_{1,x}, K_{\mathfrak{s},y}\rangle_{\mathcal{E}_\omega} = \langle \omega(\mathfrak{s})x, y\rangle_{\mathcal{E}} = \langle x, \omega(\mathfrak{s}^*)y\rangle_{\mathcal{E}}.$$

Putting together most of what we have experienced so far, especially Corollary 7, we come to the yet another dilation result, one which reminds more those of Sz.-Nagy and Stinespring.

THEOREM 16. *Let the $*$-semigroup $\mathfrak{S}$ and the C^*-algebra $\mathfrak{A}$ be unital. Suppose $\mathcal{E}$ is an inner product $\mathfrak{A}$-module and ω is an $\mathbf{B}^*(\mathcal{E})$-function on $\mathfrak{S}$ such that*

$$\omega(1) = 1_{\mathcal{E}}.$$

[19] It looks like we have to accept existence of such an x as what is often called a technical assumption.

There exist an isometry $V \in \mathbf{B}^*(\mathcal{E}, \mathcal{E}_\omega)$ and a $*$-representation Φ_ω of $\mathfrak{S}$ on $\mathcal{E}_\omega$ such that

$$\langle x, \omega(\mathfrak{s})y\rangle_{\mathcal{E}} = \langle Vx, \Phi_\omega(\mathfrak{s})Vy\rangle_{\mathcal{E}_\omega}, \quad \mathfrak{s} \in \mathfrak{S}, \; x, y \in \mathcal{E}$$

if and only if ω is $\mathbf{B}^*(\mathcal{E})$-positive definite and satisfies the boundedness condition. The triple $(\mathcal{E}_\omega, \Phi_\omega, V)$ remains minimal.

Theorem 16 is in [14] exposed in a different way.

Advice. Combine Procedure # 1 with Procedure # 2 to get whatever is possible. In particular, one may come closer to what is Section 3 of [20]; let us mention that the reproducing kernel construction in case of Hilbert C*-module valued kernels was developed in [15].

Revitalizing a question moved aside *or* a painful case. We have declared in 2° of Abstract to say couple of words on whether a mathematical result survives or not depends on an author rather than the result itself. This kind of situation is in sharp contrast to unquestionable integrity of Mathematics. The pretext for undertaking this 'metamathematical' topic, say, is what we have called the boundedness condition; consult Proposition 10 for it and Remark 11 for the source references. As we have had already pointed out its importance is in the fact that it implies, via boundedness of the dilation operators, some further properties like integrability, so to speak, in the commutative case. In the multivariable moment problem it guarantees its solvability (and also as a kind of *bonus*: compactness of the support of the representing measure). The story concerns mainly condition (c) of Proposition 10 and goes as follows. The condition (c) appeared in the 1977 paper [31] with c (α there) to be submultiplicative. Then, in 1984 two things happened: the paper [4] and the monograph [3] with c, called an absolute value there. In [4] c satisfies a kind of 'sub–C*' condition, in [3], p. 89, it is submultiplicative plus some minor extra requirements. Both sources quote [31], however, in [4] the authors say on p. 167 '*Definition 1.2 is weaker than that given by Szafraniec ...*' while in [3], p. 141, one may find '*but somehow similar conditions are implicit in Szafraniec (1977).*' Why 'weaker' or why 'implicit' is not clear, only the authors know. In two notes quoted here as [27] there is a thorough discussion of 'weaker' put into a pretty much wider context, interesting for itself too. The conclusion therein is this battle is for nothing. On the other hand, in Proposition of [31] the condition in question is explicitly stated as condition (ii) [20]. Does 'implicit' mean 'explicit' or the other way? Nevertheless, so far it looks like it is nothing to quarrel with but once the seeds have been already sowed all the rest has been developing drastically. Among 14 papers reported in MathSciNet as quoting [4] only one, that of [12], does justice. The last of those fourteen [25] makes even the title 'politically correct'. How has it gone this way? Maybe because already the 1987 survey paper [5] was quit of the reference [31]?

So why those bitter words are here? Certainly, because the author has been waiting long enough for the people to get aware of their sins. Also because these things happen more frequently than one might think of and there is no forum for them to be weighed in

[20] The paper [33] makes the moment problem meaning of all the versions of the boundedness condition even more literal.

though sometimes, not too often, one may find a journal note entitled *Acknowledgment of priority ...*

Dear PT Reader, please forgive us that if you find it inappropriate. However this is a proper occasion to quote the dictum *amicus Plato, sed magis amicas veritas.*

Some more author's personal remarks. In the vast literature of the subject there certainly are better, further going and mountainously more involved results than these here. Being an outsider, under the pressure of time, the author has been unable to penetrate all the writings; the owners of those results are asked to forgive him any pain caused by not to be quoted [21]. However he does hope to resume the search, always with simplicity and temperance as a priority.

Getting older the author more and more understands and appreciates what his Master in Mathematics, Professor Tadeusz Ważewski [42], used to mean saying *parasite associations.* Unfortunately, they have been spreading over and over pretty often driving the natural beauty out of Mathematics. This essay's intension is to demonstrate an attempt at slowing down that overwhelming drift. As already mentioned the author is determined to continue elsewhere his efforts of clarifying the topic with emphasis on *what is responsible for what.* As *organic* food or wine is approaching everyday life the time comes for bringing this environmental idea into Mathematics too.

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