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THE SKELETA OF CONVEX BODIES

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Abstract. The connectivity and measure theoretic properties of the skeleta of convex bodies in Euclidean space are discussed, together with some long standing problems and recent results.

In this note, I wish to highlight some unsolved problems relating to the skeleta of convex bodies. Much of the important work in this area was done in the 1960's and early 1970's in response to the interest in polytopes arising from problems in linear programming. So, many of these unsolved problems are approaching their fortieth birthdays.

For a d-polytope P, the k-skeleton (write $\operatorname{skel}_k P$) of P is the union of all faces of P of dimension at most k.

However, for a d-dimensional convex body C, we have two possible definitions for the k-skeleton of C. First recall that an *exposed face* F of C is a subset F of C such that there exists a supporting hyperplane H to C with $H \cap C = F$.

Then, the *exposed k-skeleton* of C is the union of all exposed faces of C of dimension at most k.

This is not an easy concept to work with. For example, an exposed face of an exposed face of C is not necessarily an exposed face of C.

The *extreme k-skeleton* of C is the set of those points of C which are not the centre of some $k+1$-dimensional ball lying entirely in C.

The extreme k-skeleton, which we will call the k-skeleton $\operatorname{skel}_k C$ of C, contains the exposed k-skeleton of C.

How should we measure $\operatorname{skel}_k C$? We can, of course, use the k-dimensional Hausdorff measure $H^k(\operatorname{skel}_k C)$.

Another measure, introduced by Eggleston, Grünbaum and Klee [2], is

$$\eta^k(C) = \liminf_{P \to C} H^k(\operatorname{skel}_k P), \qquad P \text{ a } d\text{-polytope}.$$

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An important unresolved problem is the following:

CONJECTURE (Rolf Schneider [13], [14]).

$$\eta^k(C) = H^k(\operatorname{skel}_k C).$$

Geoffrey Burton [1] proved

$$\eta^k(C) \geq H^k(\operatorname{skel}_k C).$$

I proved [11]

$$\eta^{d-2}(C) = H^{d-2}(\operatorname{skel}_{d-2} C),$$

which, in particular, applies to the 1-skeleton of a 3-dimensional convex body.

For a d-polytope P, Eggleston, Grünbaum and Klee [2] proved that

$$H^{a+b}(\operatorname{skel}_{a+b} P) \leq H^a(\operatorname{skel}_a P).H^b(\operatorname{skel}_b P).$$

So, if $s = mr, s, m, r$ positive integers, $s \leq d$

$$H^s(\operatorname{skel}_s P) \leq (H^r(\operatorname{skel}_r P))^m,$$

i.e.

$$(H^s(\operatorname{skel}_s P))^{\frac{1}{s}} \leq (H^r(\operatorname{skel}_r P))^{\frac{1}{r}}.$$

In the same paper, they posed the general extension of this inequality as a (yet unresolved) problem:

PROBLEM (E. G. K. 1964 [2]). For given r, s , $1 \leq r < s \leq d$, does there exist a constant $\gamma(d, r, s)$ such that

$$(H^s(\operatorname{skel}_s P))^{\frac{1}{s}} \leq \gamma(d, r, s)(H^r(\operatorname{skel}_r P))^{\frac{1}{r}}?$$

It is true for s a multiple of r or if $s = d - 1$ or $s = d$. Are there analogues for convex bodies?

In the theory of linear programming, the simplex algorithm has played a pivotal role. In geometric terms, the simplex algorithm produces, for a given polytope P in $\mathbb{R}^d$, a given linear functional l on $\mathbb{R}^d$, and a given vertex p of P, a simple path Q such that

(i) Q consists only of edges of P.
(ii) On Q, $l(x)$ is strictly increasing from its starting point p to its finishing point q (a vertex of P) where $l(q) = \max_{x \in P} l(x)$.

We can easily show geometrically that such a path Q exists. Suppose we are in $\mathbb{R}^d$ and the linear functional l corresponds to the hyperplane $x_d = 0$. Consider the set U of unit directions, lying in the hyperplane $x_d = 0$ which are parallel to a line segment $[a, b]$ on the boundary ∂C of C, where

$$l(p) < l(a) < \max_{x \in P} l(x).$$

For any particular facet D of P, these directions $D(l)$ have zero $(d-2)$-measure. Consequently U has zero $(d-2)$-measure.

So, choosing any direction u in $\{x_d = 0\}$, but not in U, from which p "can be seen" the result follows by induction by producing the path on the projection of P in direction u then lifting that path onto P.

There are two other simple but important results for d-polytopes:

(i) The 1-skeleton of a d-polytope is d-connected.
(ii) The 1-skeleton of a d-polytope contains a refinement of the complete graph on $d+1$ vertices.

Both the path produced by the simplex method and the d-connectedness of the 1-skeleton can be extended, see [8], to the 1-skeletons of the convex bodies in $\mathbb{R}^d$. However, we can only

Conjecture. The 1-skeleton of a convex body in $\mathbb{R}^d$ contains the refinement of the complete graph on $d+1$ vertices.

This conjecture is true in $\mathbb{R}^3$ but remains an important obstacle to proving other connectivity properties of the 1-skeleton.

An important tool in extending the connectivity results on the 1-skeletons of convex polytopes to the 1-skeletons of convex bodies is the sharpness of shadow boundaries which we now explain:

Shadow boundaries. In $\mathbb{R}^d$, let $\Gamma(l)$ denote the set of l-dimensional subspaces of $\mathbb{R}^d$ equipped with the Haar measure $\gamma(l)$, normalised so that $\gamma(l)(\Gamma(l)) = 1$. For a convex body C in $\mathbb{R}^d$ and X in $\Gamma(l)$, let $S(C, X)$ denote the shadow boundary in direction $X^\perp$, i.e. if π_X denotes the orthogonal projection of $\mathbb{R}^d$ onto X, then $S(C, X)$ is the set $\pi_X^{-1}(\operatorname{relbdy} \pi_X(C)) \cap C$. $S(C, X)$ is also the set of points c on ∂C such that

$$(c + X^\perp) \cap \operatorname{int} C = \emptyset.$$

We will say $S(C, X)$ is *sharp* if for all $c \in S(C, X)$, $(c + X^\perp) \cap \partial C$ is a single point c.

For a d-polytope P, $S(P, X)$ will be sharp provided $X^\perp$ is not parallel to some facet F of P, i.e. a translate of $X^\perp$ is not in $\operatorname{aff} F$.

So, since the set of $X \in \Gamma(l)$ such that $X^\perp$ is parallel to some facet of P has measure zero in $\Gamma(l)$, $S(P, X)$ is almost always sharp, i.e. sharp except on a set of $\gamma(l)$ measure zero.

In 1970 Ewald, Larman and Rogers [3] proved the same result for general convex bodies C, i.e.

$$S(C, X) \text{ is almost always sharp.} \tag{1}$$

Of course, for a d-polytope P, a sharp $S(P, X)$ has finite Hausdorff H^{l-1} measure, and it was conjectured that for general d-convex bodies $S(C, X)$ almost always has finite H^{l-1} measure. (More on this later.)

Using (1) with $l = d-1$, the shadow boundaries $S(C, X)$ are almost always paths on the 1-skeleton of C, and this gives us a tool to create paths in the 1-skeleton of C.

Returning to the d-polytopes it is easy to see that the 1-skeleton of a d-polytope is d-connected in the sense that if $d-1$ vertices V are removed, the remaining vertices can be connected, in pairs, in the 1-skeleton of P, by paths which do not pass through any vertex of V. Consequently, using Menger's theorem, given $2d$ vertices $a_1, \dots, a_d; b_1, \dots, b_d$ in a d-polytope P, there exist d disjoint paths $L_1, \dots, L_d$ joining a_i to some $b_{j(i)}, i = 1, \dots, d$.

We can apply the same methods, which only depend on almost all $d-1$ shadow boundaries being sharp to prove a similar result for $2d$ exposed points $a_1, \ldots, a_d; b_1, \ldots, b_d$ of a d-convex body C, see D. G. Larman and C. A. Rogers [8].

However, it is not always possible to ensure that a_i is joined to b_i, $i = 1, \ldots, d$. This is most easily seen by taking diagonal pairs on a facet of a 3-cube C. Any path joining one diagonal pair intersects any path joining the other pair. A $(d-3)$-fold pyramid over the 3-cube gives a similar example in d dimensions.

These results extend to infinite-dimensional convex bodies C in a Banach space (D. G. Larman [10]). For example, given any two exposed points a, b of C and any positive integer n, there exist n paths $P_1, \ldots, P_n$ in the 1-skeleton of C, each joining a to b and $P_i \cap P_j = a \cup b, i \neq j$.

It would be interesting to investigate whether n can be replaced by ∞.

Returning to polytopes, we can ask for the largest number $g(d)$ such that if we have $g(d)$ pairs of distinct vertices $[b_{2i-1}, b_{2i}]_{i=1}^{g(d)}$ on a d-polytope P then b_{2i-1} can be joined to b_{2i} by a path L_i in the 1-skeleton of P, $i = 1, \ldots, g(d)$ and the L_i's are disjoint.

Of course, the $(d-3)$-fold pyramid over the square yields $g(d) \leq [\frac{d}{2}]$ and Peter Mani and I [6] conjectured that $g(d) = [\frac{d}{2}]$. We proved that $g(d) \geq [\frac{1}{3}(d+1)]$ but S. Gallivan [4] gave an example which showed $g(d) \leq [\frac{2(d+2)}{5}]$. So there is still a considerable gap to be filled.

Our proof of $g(d) \geq [\frac{1}{3}(d+1)]$ used, as a giant 'fly over' system, the existence of the refinement of a complete $d{+}1$ graph in the 1-skeleton of a d-polytope. As previously mentioned, I believe that the 1-skeleton of a d-convex body must contain the refinement of a complete $d{+}1$ graph and, if true we would also have $g(d) \geq [\frac{1}{3}(d+1)]$ for all d-convex bodies.

Increasing paths. If we think of $\mathbb{R}^d$ as $\mathbb{R}^{d-1} \times \mathbb{R}$, with the last coordinate giving a sense of height, there exists in the 1-skeleton of any d-polytope P a strictly increasing path going from the bottom of P to the top of P. This is easily seen via the simplex algorithm or the shadow boundary argument used earlier. The same result is true for all d-convex bodies and follows from the following theorem of Rogers and myself [9].

THEOREM. *The directions of the line segments on the boundary of a d-convex body C, parallel to a fixed hyperplane H and not lying in the two corresponding support hyperplanes to C, parallel to H, have zero $(d-2)$-measure.*

W_v-Paths. Let F be a face of a d-convex body C and let Q be a path in the 1-skeleton of C. Then $F \cap Q$ consists of connected components which we call visits of Q to F.

Klee [5] made the following conjecture, which is true in three dimensions but still unresolved in higher dimensions.

CONJECTURE. If P is a d-polytope, then any two vertices of P can be joined by an edge path which visits every face of P at most once. He called such a path a W_v-path.

Klee's conjecture implies the famous Hirsch Conjecture, i.e. that any two vertices of a d-polytope with n facets can be joined by a wedge path of length at most $n-d$. The implication is simple: every time a $W-v$-path leaves a vertex it leaves at least one facet, never to return. As the end vertex is in at least d facets, the path length is at most $n-d$.

In three dimensions, the shortest Euclidean path along the 1-skeleton of a Schlegel Diagram of a 3-polytope is a W_v-path.

I proved [6] that there exists a path which visits each face at most 2^{d-3} times, which trivially produces the bound $2^{d-3}n$ in the Hirsch conjecture.

PROBLEM. Between any two extreme points of a d-convex body C, is there a path in the 1-skeleton of C which visits each face of C at most 2^{d-3} times? (or even, at most once?).

In all of the above problems/results for convex bodies, the analogues for convex polytopes have paths of finite length. I would conjecture that most of the results for convex bodies are also true with the extra condition that the paths have finite length.

The first step would be to improve the result of Ewald, Larman and Rogers [3] from almost all shadow boundaries being sharp, to almost all shadow boundaries having finite measure.

This was a well known conjecture in the 1970's and recently P. Mani and myself [12] have been able to establish that this is indeed the case.

Whilst the technical details are complicated, the basic idea is quite simple. If $\gamma(l)$ denotes the Haar measure of l-dimensional subspaces of $\mathbb{R}^d$, P a d-polytope in $\mathbb{R}^d$ and X an l-dimensional subspace of $\mathbb{R}^d$, let $S(P,X)$ denote the shadow boundary of P in 'direction' $X^\perp$. It is readily seen that

$$\int_{\Gamma(l)} H^{l-1}(S(P,X))d\gamma(l)(X) = \alpha(l,d)W_{d-l+1}(P)$$

where H^{l-1} is Hausdorff $l-1$ measure and $W_{d-l+1}(P)$ is the $l-d+1$ Quermass integral of P.

Then, for a general d-convex body C we take a sequence of polytopes approaching C and, avoiding a certain set $X \in \gamma(l)$ of zero Haar measure, we use a lower semi-continuity argument to establish

$$\int_{\Gamma(l)} H^{l-1}(S(C,X))d\gamma(l)(X) \le \alpha(l,d)W_{d-l+1}(C).$$

References

[1] G. R. Burton, *The measure of the s-skeleton of a convex body*, Mathematika 17 (1979), 290–301.

[2] H. G. Eggleston, B. Grünbaum and V. Klee, *Some semicontinuity theorems for convex polotopes and cell complexes*, Commentarii Mathematici Helvetici 39 (1964), 165–188.

[3] G. Ewald, D. G. Larman and C. A. Rogers, *The directions of the line segments and of the r-dimensional balls on the boundary of a convex body in Euclidean space*, Mathematika 17 (1970), 1–20.

[4] S. Gallivan, *Disjoint edge paths between given vertices of a convex polytope*, J. Combinatorial Theory Ser. A 39 (1985), 112–115.

[5] V. L. Klee, *Paths on polyhedra I*, J. SIAM (4) 13 (1965), 946–956.

[6] D. G. Larman, *Paths on polytopes*, Proc. London Math. Soc. (3) 20 (1968), 161–178.

[7] D. G. Larman and P. Mani, *On the existence of certain configurations within graphs and the 1-skeletons of polytopes*, Proc. London Math. Soc. (3) 20 (1970), 144–160.

[8] D. G. Larman and C. A. Rogers, *Paths in the one-skeleton of a convex body*, Mathematika 17 (1970), 293–314.

[9] D. G. Larman and C. A. Rogers, *Increasing paths on the one-skeleton of a convex body and the directions of line segments on the boundary of a convex body*, Proc. London Math. Soc. (3) 23 (1971), 683–698.

[10] D. G. Larman, *On the one-skeleton of a compact convex set in a Banach space*, Proc. London Math. Soc. (3) 34 (1977), 137–144.

[11] D. G. Larman, *The $d-2$ skeletons of polytopal approximations to a convex body in E^d*, Mathematika 27 (1980), 122–133.

[12] D. G. Larman and P. Mani, *Almost all shadow boundaries have finite measure*, to appear.

[13] R. Schneider, *Problem 12*, in: Konvexe Körper, Oberwolfach, 1978.

[14] R. Schneider, *Boundary structure and curvature of convex bodies*, in: Contributions to Geometry, Birkhäuser, Basel, 1979, 13–59.