Abstract. The aim of this note is to give a clearer and more direct proof of the main result of another paper of the author. Moreover, we give some complementary results related to $R$-complete algebraic foliations with $R$ a rational function of type $\mathbb{C}^\times$.

1. Introduction

1.1. Vector fields \[7\]. A vector field $X$ on $\mathbb{C}^2$ is a section of the tangent bundle of $\mathbb{C}^2$

$$X = R \frac{\partial}{\partial x} + S \frac{\partial}{\partial y}, \quad R, S \in \mathcal{O}_{\mathbb{C}^2}.$$  

Associated to $X$ we have the following system:

$$\begin{cases} 
\dot{x}(t) = R(x,y), \\
\dot{y}(t) = S(x,y).
\end{cases} \quad (1)$$

According to the theorem on existence and uniqueness of local solutions of complex differential equations, for a fixed initial condition $z = (x,y) \in \mathbb{C}^2$, there exist a disk $\mathbb{D}_{r_z}$ of center zero and radius $r_z$ and a holomorphic function $t \in \mathbb{D}_{r_z} \mapsto \varphi_z(t)$ that satisfies $\square$ with $\varphi_z(0) = z$. Given $t \mapsto \varphi_z(t)$, we can extend it by analytic continuation along the paths from zero to the points outside $\mathbb{D}_{r_z}$ to the maximal domain of definition $\Omega_z$ (Riemann domain spread over $\mathbb{C}$). This map $\varphi_z : \Omega_z \to \mathbb{C}^2$ is the solution of $X$ through $z$ and its image $C_z$ defines the trajectory of $X$ through $z$. A trajectory $C_z$ is said to be proper if its topological closure in $\mathbb{C}^2$ defines an analytic curve of pure dimension one. The vector field $X$ is complete if for any $z \in \mathbb{C}^2$ the solution $\varphi_z$ is an entire map. In this case $(t,z) \mapsto \varphi(t,z) = \varphi_z(t)$ defines a holomorphic action of $(\mathbb{C},+)$ on $\mathbb{C}^2$ by global

2010 Mathematics Subject Classification: Primary 32M25; Secondary 32L30, 32S65.

Key words and phrases: complete vector field, complex orbit, holomorphic foliation.

The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc94-0-7 [143] © Instytut Matematyczny PAN, 2011
holomorphic automorphisms. The map $\varphi$ is the (global) flow of $X$. If for any $t \in \mathbb{C}$, $z \mapsto \varphi(t, z)$ is a polynomial automorphism of $\mathbb{C}^2$ the flow $\varphi$ is said to be algebraic. If all the trajectories of $X$ are proper the flow $\varphi$ is said to be proper.

1.2. Algebraic foliations on $\mathbb{C}^2$ [11, Chapter 2]. Let $X$ be a polynomial vector field of degree $m$. Let us consider the atlas $\{(U_i, \phi_i^{-1})\}_{i=0,1,2}$ of $\mathbb{CP}^2$ defined by open sets $U_i := \{(z_0 : z_1 : z_2), z_i \neq 0\}$ and homeomorphisms $\phi_0(z_1, z_2) = [1 : z_1 : z_2], \phi_1(y_1, y_2) = [y_1 : 1 : y_2]$ and $\phi_2(w_1, w_2) = [w_1 : w_2 : 1]$. The vector field $X$ defines a rational vector field on $\mathbb{CP}^2$ given by $(\phi_i^{-1} \circ \phi_0)_*X$ in each chart $(U_i, \phi_i^{-1})$. The pole of $X$ along the line at infinity $L_\infty$ is of order $d = m - 1$ or $m - 2$. If we remove it we obtain on each $(U_i, \phi_i^{-1})$ a polynomial vector field $X_i$ with isolated zeroes. These vector fields $\{X_i\}_{i=0,1,2}$ define a global section $\mathcal{F}_X$ of $\mathcal{O}(d) \otimes T\mathbb{CP}^2$, for $\mathcal{O}(d)$ the line bundle of $\mathbb{CP}^2$ of degree $d$, which is the foliation defined by $X$ (modulo multiplication by a non-zero complex number). The singular set $Sing(\mathcal{F}_X)$ of $\mathcal{F}_X$ is the set of singularities of $X_i$. A singular point $p \in Sing(\mathcal{F}_X)$ is reduced if $\mathcal{F}_X$ around $p$ is generated by a vector field whose first jet at $p$ has eigenvalues $\lambda_1$ and $\lambda_2$ such that either $\lambda_1 \neq 0 \neq \lambda_2$ and $\lambda_1/\lambda_2 \notin \mathbb{Q}^+$, or $\lambda_1 \neq 0 = \lambda_2$.

There is a foliation $\tilde{\mathcal{F}}$ defined on a rational surface $M$ after pulling back $\mathcal{F}_X$ by a birational morphism $\pi : M \to \mathbb{CP}^2$, that is a finite composition of blowing up, with reduced singularities only (Seidenberg’s Theorem). Associated to this resolution one has:

(a) the Zariski open set $U = \pi^{-1}(\mathbb{C}^2)$ of $M$. Note that $X$ can be lifted to it as a holomorphic vector field,
(b) the exceptional divisor $E$ of $U$, and
(c) the divisor at infinity
$$D = M \setminus U = \pi^{-1}(\mathbb{CP}^2 \setminus \mathbb{C}^2) = \pi^{-1}(L_\infty),$$
that is a tree of a smooth rational curves.

1.3. Results of M. Suzuki [10], [11], [12]. Let us recall some important facts about complete vector fields $X$ on $\mathbb{C}^2$:

(I) The trajectories of $X$ are isomorphic to $\mathbb{C}$ or $\mathbb{C}^*$.

(II) There exists a set $E \subset \mathbb{C}^2$ invariant by $X$ of logarithmic capacity zero such that for any $z \in \mathbb{C}^2 \setminus E$, the trajectory $C_z$ is always of the same type. Thus $X$ is either of type $\mathbb{C}$ or $\mathbb{C}^*$, depending on the type of its generic trajectory.

(III) A trajectory of $X$ of type $\mathbb{C}^*$ is proper.

(IV) If $X$ is of type $\mathbb{C}^*$ it defines a proper flow and it has a meromorphic first integral.

Suzuki’s classification. M. Suzuki in [10] classified $\mathbb{C}^2$ algebraic flows and proper flows, modulo holomorphic automorphisms. The vector fields $X$ of the two classifications together are of the form:

1) $[a(x)y + b(x)] \frac{\partial}{\partial y}, \ a(x), b(x) \in \mathbb{C}(x)$
2) $\lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y}, \ \lambda, \mu \in \mathbb{C}$
3) $\lambda x \frac{\partial}{\partial x} + (\lambda my + x^m) \frac{\partial}{\partial y}, \ \lambda \in \mathbb{C}^*, \ m \in \mathbb{N}$
4) $\frac{\alpha(z)}{x^\ell} \cdot \left\{ nx^{\ell+1} \frac{\partial}{\partial x} - [(m + n\ell)x^\ell y + mp(x) + nxp(x)] \frac{\partial}{\partial y} \right\}$,

where $m, n \in \mathbb{N}^*$, $p$ is a polynomial whose degree is not greater than $\ell - 1$ with $p(0) \neq 0$ when $\ell > 0$ or $p \equiv 0$ otherwise, and $\alpha \in \mathbb{C}(z)$ ($z = x^m(x^\ell y + p(x))^n$) with a zero of order $\geq \ell/m$ at $z = 0$.

Proper flows are defined by vector fields of 1), 2) if $\lambda/\mu \in \mathbb{Q}$, 3) with $m = 0$, and 4).

This implies that there is a rational first integral of $X$, modulo holomorphic automorphism, of the form $x, y^p/x^q$ ($p/q = \lambda/\mu \in \mathbb{Q}$) or $x^m(x^\ell y + p(x))^n$.

1.4. Questions. According to Suzuki’s classification a complete holomorphic vector field has a proper flow if and only if it has a rational first integral of one of the above three types, modulo holomorphic automorphism. Therefore, if $X$ is in Suzuki’s list it is of the form $f \cdot Y$, with $Y$ a polynomial vector field and $f \in \mathcal{O}_{\mathbb{C}^2}$, and the foliation generated by $X$ is the algebraic foliation $\mathcal{F}_Y$. It is natural to try to answer the following questions:

- Of what form are the complete vector fields on $\mathbb{C}^2$ that define an algebraic foliation? Or in other words, what can be said about the vector fields of the form $f \cdot Y$ where $Y$ is a polynomial vector field and $f \in \mathcal{O}_{\mathbb{C}^2}$?
- Do they define other complete vector fields different from those in Suzuki’s list?
- Do they define other complete vector fields until now unknown?

We can make a simplification and assume that $f$ is transcendental by Brunella’s classification of complete polynomial vector fields. The result that answers the above questions is [4, Theorem 1.1].

Theorem. Let $X$ be a complete vector field on $\mathbb{C}^2$ of the form $f \cdot Y$, where $Y$ is a polynomial vector field and $f \in \mathcal{O}_{\mathbb{C}^2}$. Then $X$ defines a proper flow and, up to a holomorphic automorphism, $X$ is in Suzuki’s list.

2. Proof

2.1. Assumptions. If $X = f \cdot Y$, we will denote the foliation $\mathcal{F}_Y$ by $\mathcal{F}$. Let $\tilde{\mathcal{F}}$ be its resolution $\pi^* \mathcal{F}$ on $M$, and $E$ and $D$ its divisors.

We may assume that $\tilde{\mathcal{F}}$ has no rational first integrals and that $X$ is of type $\mathbb{C}$ (see (IV) of §1.3). Then $Y$ is of type $\mathbb{C}$ because $\{f = 0\}$ is $\emptyset$ or an invariant set by $Y$. In this situation $E$ and $D$ are $\tilde{\mathcal{F}}$-invariant.

On the other hand, $\tilde{\mathcal{F}}$ admits lots of tangent entire curves; most of them are Zariski dense in $M$ (Darboux’s Theorem). This implies that the Kodaira dimension $\text{kod}(\tilde{\mathcal{F}})$ of $\tilde{\mathcal{F}}$ is either 0 or 1 [8, §IV] (see also [11, p. 131]).

2.2. $\text{kod}(\tilde{\mathcal{F}}) = 1$. According to [8, §IV] the absence of a first integral implies that $\tilde{\mathcal{F}}$ is a Riccati or a turbulent foliation, that is to say, the existence of a fibration

$$g : M \to B$$

whose generic fibre is a rational curve or an elliptic curve transverse to $\tilde{\mathcal{F}}$, respectively.

Remark that $B$ is $\mathbb{CP}^1$ since $M$ is a rational surface.

Lemma 1. $\tilde{\mathcal{F}}$ is a Riccati foliation.
Proof. Let us suppose that $\tilde{F}$ is turbulent. There is a component $D_0 \subset D$ transversal to the generic fibre $G_0$ of $g$. Otherwise we have an elliptic curve contained in $C^2$, which is impossible ($C^2$ is Stein). As $D_0$ is $\tilde{F}$-invariant, one can construct a rational first integral as pointed out in [2, Lemma 1].

**Lemma 2.** $g|_U$ is projected by $\pi$ as a rational function $R$ of type $\mathbb{C}$ or $\mathbb{C}^*$. 

**Proof.** Up to contraction of rational curves inside fibers of $g$, which can produce cyclic quotient singularities of the surface but on which the foliation is always regular, there are five possible models for the fibers of $g$ [3, §7], [2, p. 439]. Let $L_0$ be the leaf of the foliation defined by a trajectory $\tilde{C}_z$ of $X$ transversal to $g$. One can conclude that the orbifold universal covering $\tilde{L}_0$ of $L_0$ is equal to the one of $B_0$, $\tilde{B}_0$, where $B_0$ is defined as $\mathbb{C}P^1$ minus the points over tangent fibres of $g$ with the natural orbifold structure inherited from the orbifold structure on $\mathbb{C}P^1$ induced by (the local models of) $g$. Since $X$ is complete on $C_z$, $\tilde{L}_0$ is biholomorphic to $\mathbb{C}$ and then $L_0$ is parabolic. Then $\text{kod}(\tilde{F}) = 1$ implies by [2, Lemma 2] that there must be at least one fibre $G_0$ tangent to the foliation of one of the following classes:

(d): the fibre is rational with two saddle-nodes of the same multiplicity $m$, with strong separatrices inside the fibre, or

(e): the fibre is rational with two quotient singularities of order 2, and a saddle-node of multiplicity $l$, with strong separatrix inside the fibre.

The components of $D \cup E$ which are not contained in fibers of $g$ define separatrices through singularities of $\tilde{F}$. Then $G_0$ must cut $D \cup E$ in at most one or two points. Therefore $R = g \circ \pi^{-1}$ is of type $\mathbb{C}$ or $\mathbb{C}^*$. ■

Analogously to the polynomial case one can define as in [2] that $F$ is $R$-complete if there exists a finite set $Q \subset \mathbb{C}P^1$ such that for all $t \notin Q$: (i) $R^{-1}(t)$ is transverse to $F$, and (ii) there is a neighbourhood $U_t$ of $t$ in $\mathbb{C}P^1$ such that $R : R^{-1}(U_t) \rightarrow U_t$ induces a holomorphic fibration on $M$ and the restriction of $F$ to $R^{-1}(U_t)$ defines a local trivialization of this fibration on $M$.

**Lemma 3.** $F$ is $R$-complete.

**Proof.** By the proof of Lemma 2 it is enough to note that around $G_0$, removing one or two $\tilde{F}$-invariant disks contained in $D \cup E$, $R$ is a fibration trivialized by the leaves of $\tilde{F}$. ■

2.2.1. $R$ of type $\mathbb{C}$. Up to a polynomial automorphism $R = x$ (see [9]). As $F$ is $x$-complete, $Y$ extends to $\mathbb{C}P^1 \times \mathbb{C}P^1$ as holomorphic vector field leaving $\mathbb{C}P^1 \times \infty$ invariant. In particular

$$Y = a(x) \frac{\partial}{\partial x} + [b(x)y + c(x)] \frac{\partial}{\partial y},$$

with $a$, $b$ and $c \in \mathbb{C}[x]$. As the solutions of $Y$ can only avoid at most one vertical line by Picard’s Theorem, $a(x) = \lambda x^N$, $N \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. Let us take $\varepsilon = 0$ if $N \geq 1$ and $\varepsilon = 1$ otherwise. Then $X$ can be decomposed as

$$X = f \cdot x^{N-1+\varepsilon} \cdot Z = f \cdot x^{N-1+\varepsilon} \cdot 1/x^{N-1+\varepsilon}Y$$  (2)
As $Z$ is of type $\mathbb{C}$ and (rational) complete, the restriction of $f \cdot x^{N-1+\varepsilon}$ to each solution $\varphi_\varepsilon$ of $Z$ is constant. Then $f \cdot x^{N-1+\varepsilon}$ is a meromorphic first integral of $Y$, and then $X$ defines a proper flow.

$\textbf{2.2.2. } R \text{ of type } \mathbb{C}^*$. By Suzuki (see [10]) we may assume that

$$R = x^m(x^\ell y + p(x))^n,$$

where $m \in \mathbb{N}^*$, $n \in \mathbb{Z}^*$, with $(m, n) = 1$, $\ell \in \mathbb{N}$, $p \in \mathbb{C}[x]$ of degree $< \ell$ with $p(0) \neq 0$ if $\ell > 0$ or $p(x) \equiv 0$ if $\ell = 0$, up to a polynomial automorphism.

According to the relations $x = u^n$ and $x^\ell y + p(x) = v u^{-m}$ it is enough to take the rational map $H$ from $u \neq 0$ to $x \neq 0$ defined by

$$(u, v) \mapsto (x, y) = (u^n, u^{-(m+n\ell)}[v - u^mp(u^n)])$$

(3) in order to get $R \circ H(u, v) = v^n$. Although $R$ is not necessarily a polynomial ($n \in \mathbb{Z}$), it is a consequence of the proof of [5, Proposition 3.2] that $H^*F$ is a Riccati foliation adapted to $v^n$ having $u = 0$ as invariant line. Thus

$$H^*Y = u^k \cdot Z = \left\{a(v)u \frac{\partial}{\partial u} + c(v) \frac{\partial}{\partial v}\right\},$$

(4)

where $k \in \mathbb{Z}$, and $a, c \in \mathbb{C}[v]$.

Applying directly the local models of [2], it is proved in [4, Lemma 2] that at least one of the irreducible components of $R$ over 0 must be an $F$-invariant line. Hence the polynomial $c(v)$ of [4] is, in fact, a monomial, and thus of the form $cv^N$ with $c \in \mathbb{C}$ and $N \in \mathbb{N}$. Then

$$H^*X = f \circ H(u, v) \cdot u^k \cdot v^{N-1+\varepsilon} \cdot 1/v^{N-1+\varepsilon} \cdot Z$$

(5)

As $1/v^{N-1+\varepsilon} \cdot Z$ is complete and of type $\mathbb{C}$, the restriction of $f \circ H(u, v) \cdot u^k \cdot v^{N-1+\varepsilon}$ to each solution $\varphi_\varepsilon$ of $1/v^{N-1+\varepsilon} \cdot Z$ through $z \in u \neq 0$ is constant. Projecting by $H$,

$$f^{mn} \cdot x^{mk} \cdot (x^m(x^\ell y + p(x))^n)^{m(N-1+\varepsilon)}$$

is a meromorphic first integral of $Y$, and $X$ defines a proper flow.

**Remark 1.** In §3, we will obtain an explicit first integral of $Y$ which does not depend on $f$ but that nevertheless can be multivalued. Moreover, using that integral we will give an alternative proof of the existence of an invariant line for $F$ [4, Lemma 2].

$\textbf{2.3. } \text{kod}(\tilde{F}) = 0$. According to [8, §III and §IV], [2, p. 443] we can contract $\tilde{F}$-invariant rational curves on $M$ via a contraction $s$ to obtain a new surface $\tilde{M}$ (maybe singular with cyclic quotient singularities), a reduced foliation $\tilde{F}$ on this surface, and a finite covering map $r$ from a smooth compact projective surface $S$ to $\tilde{M}$ such that: 1) $r$ ramifies only over cyclic (quotient) singularities of $\tilde{M}$ and 2) the foliation $r^*(\tilde{F})$ is generated by a complete holomorphic vector field $Z_0$ on $S$ with isolated zeroes.
It follows from [2, p. 443] that the covering $r$ can be lifted to $M$. That is, there are a surface $T$, a birational morphism $g : T \to S$ and a ramified covering $h : T \to M$ such that $s \circ h = r \circ g$

\[
\begin{array}{c}
\text{M} \\ \downarrow s \circ h \\
\text{T} \\
\downarrow r \circ g \\
\text{S} \\
\end{array}
\]

Let $\bar{Z}_0$ be the lift $g^*Z_0$ of $Z_0$ on $T$ via $g$. Then $\bar{Z}_0$ must be a rational vector field on $T$ generating the foliation $\bar{F}$ given by $g^*(r^*(\bar{F})) = h^*\bar{F}$. On the other hand, $\bar{F}$ is also generated by the rational vector field $\bar{Y}$ on $T$ given by $h^*\bar{Y}$, with $\bar{Y} = \pi^*Y$. Hence there is a rational function $\bar{F}$ on $T$ such that

$$\bar{Y} = \bar{F} \cdot \bar{Z}_0. \quad (6)$$

**Remark 2.** From the above construction we notice that:

- The map $g$ is a composition of blowing-ups at a finite set $\Theta = \{\theta_i\}_{i=1}^s \subset S$ of regular points of $Z_0$. In fact $\Theta = r^{-1}(\text{Sing}(\bar{M}))$. The poles of $\bar{Z}_0$ are in $g^{-1}(\Theta)$ and they define a divisor $P \subset T$ invariant by $\bar{F}$. Hence $\bar{Z}_0$ is holomorphic on $T \setminus P$. Note that in $T \setminus P$, $\bar{Z}_0$ has only isolated zeroes.
- $P$ is the exceptional divisor of $g$, $h(P)$ is the exceptional divisor of $s$ and is $\bar{F}$-invariant. Then $h_{|T \setminus P} : T \setminus P \to M \setminus h(P)$ is a regular covering map.
- Let $C_{\theta_i}$ be the trajectory of $Z_0$ through $\theta_i$. $\bar{Z}_0$ is a complete holomorphic vector field on $W \setminus \{g^{-1}(C_{\theta_i})\}_{i=1}^s$.

**Lemma 4.** $h$ is a birational map.

**Proof.** The set of components of the divisor $h(P)$ of the contraction $s$, that define curves in $\mathbb{C}^2$ after projection via $\pi_{|U}$ is or empty or an affine line $L$ ([2, Lemma 6]). In the latter case $Y$ is always of type $\mathbb{C}^*$ ([2, p. 445]). Thus there is Zariski open $W \subset T$ such that $\pi \circ h : W \to \mathbb{C}^2 \setminus \pi(E)$ is a regular covering. Therefore $h$ is birational.

We can project (6) by $\pi \circ h$ to obtain a decomposition $X = f \cdot Y = f \cdot F \cdot Z$, where $Z$ is a rational vector field of type $\mathbb{C}$ which is complete outside a finite set of trajectories. Hence the restriction of $f \cdot F$ to each solution of $Z$ must be constant, and $f \cdot F$ is a meromorphic first integral for $X$.

### 3. $R$-complete foliations with $R$ of type $\mathbb{C}^*$.

Let us assume that $\mathcal{F}$ is $R$-complete with $R$ a rational map of type $\mathbb{C}^*$. One can assume that $\mathcal{F}$ is defined after the rational change $H$, [3], by the rational vector field $H^*Y$ given in (4).

**Proposition 1.** $\mathcal{F}$ has a multivaluated meromorphic first integral.

**Proof.** If one takes $v_0$ with $c(v_0) \neq 0$, the trajectories of $H^*Y$ except the horizontal ones and $\{u = 0\}$ are parameterized by maps $\sigma(w_0, t)$, where $w_0$ is a fixed point and $\sigma$ is a multivaluated holomorphic map defined on $\mathbb{C}^* \times (\mathbb{C} \setminus \{c = 0\})$ of the form

$$\sigma(w, t) = (u(w, t), v(w, t)) = (we^{\int_{v_0}^{v(w,t)} \frac{dz}{c(z)}}, t). \quad (7)$$
It is enough to extend the local solution through \((w_0,v_0)\), with \(w_0 \in \mathbb{C}^*\), of \(1/c(v) \cdot Z\) by analytic continuation along paths in \(\mathbb{C} \setminus \{c(v) = 0\}\). This map is defined as \(\sigma(w,t)\) with \(\sigma\) equals \([7]\) (see \([6]\) Section 2)).

Let us take the one-form \(\omega = [a(z)/c(z)]\,dz\) that appears in \([7]\). It has a fraction expansion of the form

\[
\omega = \left\{ s(z) + \sum_{j=0}^{r} \frac{A^j_1}{(z - \xi_j)} + \frac{A^j_2}{(z - \xi_j)^2} + \cdots + \frac{A^j_r}{(z - \xi_j)^r} \right\} \, dz,
\]

where \(s(z) \in \mathbb{C}[z]\), \(\xi_j\) are those roots of multiplicity \(r_j\) of the denominator of \(a(z)/c(z)\) after simplifying \(a(z)\) and \(c(z)\), and \(A^j_i \in \mathbb{C}\), for \(1 \leq i \leq r_j\). If \(\xi_j\) is zero we assume that it is \(\xi_0\). Otherwise \(A^0_i = 0\) and the sum of \(\text{(8)}\) begins from \(j = 1\). Let us fix

\[
\Gamma(z) = e^{s(z)} \prod_{j=0}^{r} \Gamma_j(z) = e^{s(z)} \prod_{j=0}^{r} e^{\lambda^j_1 \log(z-\xi_j) + \frac{\lambda^j_2}{(z-\xi_j)} + \cdots + \frac{\lambda^j_r}{(z-\xi_j)^r}}
\]

where \(s(z) = \int^z s(t) \, dt\), and \(\lambda^j_1 = A^j_i\) and \(\lambda^j_2 = A^j_i / (-i + 1)\) for \(2 \leq i \leq r_j\). If we introduce \([8]\) in \([7]\), after explicit integration of \(\omega\), one has that \(\sigma(w,t)\) is of the form \((w \cdot \Gamma(t))/\Gamma(v_0), t)\). Then

\[
F(u,v) = \frac{u}{\Gamma(v)}
\]

is a first integral of \(H^*Y\). Finally, we can express \([10]\) in terms of \(x\) and \(y\) according to \([3]\),

\[
G(x,y) = \frac{x^{1/n}}{\Gamma(x^{m/n} \cdot (x^ny + p(x)))},
\]

and thus obtain a (multivalued) first integral of \(Y\), and then of \(\mathcal{F}\).

**Proposition 2.** The line \(x = 0\) is invariant by \(\mathcal{F}\).

**Proof.** Let us suppose that \(x = 0\) is not invariant. Each trajectory of \(Y\) through a non-singular point \((0, y_0)\), \(y_0 \neq 0\), can be then locally parametrized by a map \(t \mapsto \gamma(t) = (t, y(t))\), with \(t\) in a sufficiently small disk \(\mathbb{D}\) and \(y(0) = y_0\). In order to study the restriction of \(G\) to each of them we will consider the non-reduced parametrization \(\gamma(t^{m_i})\).

**a) Case** \(n > 0\). Let us take the function

\[
q(t) = t^{m_i} (t^{n_i} y(t^{n_i}) + p(t^{n_i})),
\]

It follows from \([9]\) and \([11]\) that

\[
G \circ \gamma(t^{n_i}) = \Omega(t) \cdot \Delta(t),
\]

where

\[
\Omega(t) = t^{(1-m\lambda_0)} \cdot e^{-\frac{\lambda_0}{\gamma^0(t)}} \cdots e^{-\frac{\lambda_0}{\gamma^0(t)^{n_i-1}}}
\]

is in general a *multivalued* holomorphic function in \(\mathbb{D}^*\), and

\[
\Delta(t) = \frac{e^{-\bar{s}(q(t)) - \lambda_0 \log(t^{n_i} y(t) + p(t^{n_i}))}}{\prod_{j \geq 1} \Gamma_j(q(t))}
\]

is a holomorphic in \(\mathbb{D}\) with \(\Delta(0) \neq 0\).
On the other hand, as $\gamma(\mathbb{D}^*)$ is contained in a trajectory of $Y$, we can determine $\delta(y_0) \in \mathbb{C}$ such that $\gamma(\mathbb{D}^*) \subset \{G = \delta(y_0)\}$. This implies that (12) must be constantly equal to $\delta(y_0)$, and hence $r_0 = 1$ and $1 - m\lambda_1^0 = 0$ in (13). Thus we can assume that $\Omega(t) \equiv 1$ and $G \circ \gamma(0) = \Delta(0) = \delta(y_0)$.

a.1) If $\ell > 0$, we know from (14) that the value

$$\Delta(0) = \frac{e^{-s(0) - \lambda_1^0\log(p(0))}}{\prod_{j=1}^{r} e^{\lambda_j^0 \log(-\xi_j) + \frac{\lambda_j^1}{(-\xi_j)^{j-1}}}}$$

does not depend on $y_0$. Therefore (fixed the logarithmic branch) we may assume that $\delta(y_0) \equiv \delta = \Delta(0)$ for any $y_0$. In particular, there is an open set $\mathcal{N}$ of analytic dimension 2 containing $\{x = 0\} \cap \{Y \neq 0\}$ and such that $\mathcal{N} \setminus \{x = 0\} \subset \{G = \delta\}$, which gives us a contradiction.

a.2) If $\ell = 0$, by a simple inspection in (11), using $r_0 = 1$ and $1 - m\lambda_1^0 = 0$, we see that $y = 0$ must be invariant by $Y$. But this line can be assumed to be $x = 0$ after a symmetry $(x, y) \mapsto (y, x)$, contradicting again our assumptions.

b) Case $n < 0$. Let us take the function

$$\tilde{q}(t) = \frac{t^{-n}y(t^n) + p(t^n)}{t^m}.$$  

It follows from (9) and (11) that

$$G \circ \gamma(t^{-n}) = \tilde{\Omega}(t) \cdot \tilde{\Delta}(t),$$

where

$$\tilde{\Omega}(t) = t^{-(1-m)\sum_{j=0}^{r} \lambda_j^1} \cdot e^{-\bar{s}(\tilde{q}(t))}$$

is in general a multivalued holomorphic function in $\mathbb{D}^*$, and

$$\tilde{\Delta}(t) = \frac{e^{-\sum_{j=0}^{r} \lambda_j^0 \log(t^m(\tilde{q}(t)-\xi_j))}}{\prod_{j=0}^{r} e^{\lambda_j^1/(\tilde{q}(t)-\xi_j)} \cdots e^{\lambda_j^r/(\tilde{q}(t)-\xi_j)^{r-j}}}$$

is holomorphic in $\mathbb{D}$ with $\tilde{\Delta}(0) \neq 0$. As $\gamma(\mathbb{D}^*)$ is contained in a trajectory of $Y$, we can determine $\delta(y_0) \in \mathbb{C}$ such that $\gamma(\mathbb{D}^*) \subset \{G = \delta(y_0)\}$. Then (15) must be constantly equal to $\delta(y_0)$, and hence $\bar{s}(z) \equiv 0$ (note that $\bar{s}(z)$ is zero when it is constant) and $1 - m \sum_{j=0}^{r} \lambda_j^1 = 0$ in (16). Thus $\tilde{\Omega}(t) \equiv 1$ and $G \circ \gamma(0) = \tilde{\Delta}(0) = \delta_0$.

b.1) If $\ell > 0$, we know from (17) that the value

$$\tilde{\Delta}(0) = e^{-\sum_{j=0}^{r} \lambda_j^0 \log(p(0))}$$

does not depend of $y_0$. So analogously to a.1), fixed the logarithmic branch, we may assume that $\delta(y_0) \equiv \delta = \tilde{\Delta}(0)$ for any $y_0$. In particular, there is an open set $\mathcal{N}$ of analytic dimension 2 containing $\{x = 0\} \cap \{Y \neq 0\}$ and such that $\mathcal{N} \setminus \{x = 0\} \subset \{G = \delta\}$, which gives us a contradiction.

b.2) If $\ell = 0$ we can proceed as in a.1).
References


