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THE LEVI-CIVITA EQUATION, VECTOR MODULES AND SPECTRAL SYNTHESIS

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Abstract. The purpose of this paper is to give a survey on some recent results concerning spectral analysis and spectral synthesis in the framework of vector modules and in close connection with the Levi-Civita functional equation. Further, we present some open problems in this subject.

1. Basic concepts and example. In this paper $\mathbb{R}$, resp. $\mathbb{C}$, denotes the set of real, resp. complex numbers. All topological spaces in the paper are supposed to be Hausdorff.

Let R be a ring and let X be a topological vector space. We say that X is a *vector module over* R if X, as an Abelian group, is a module over R, and for each r in R the mapping $x \mapsto r \cdot x$ is a continuous linear operator on X. If R has a unit e, then we require that the corresponding linear operator $x \mapsto e \cdot x$ is the identity operator on X. We remark that if no topology is specified on X, then we always consider it with the discrete topology. By a *vector submodule*, or simply a *submodule* of a vector module we mean a linear subspace, which is also a vector module over R, with the same meaning of $r \cdot x$, of course. A closed vector submodule is called a *variety.* The intersection of any nonempty family of submodules, resp. varieties, is a submodule, resp. variety. For any x in X the smallest submodule, resp. variety, is the intersection of all submodules, resp. varieties, including x, which is called the *submodule, resp. variety, generated by* x.

If X is a topological vector space, then $\mathcal{L}(X)$ denotes the algebra of all continuous linear mappings, that is, the linear operators on X. On $\mathcal{L}(X)$ one usually considers the strong operator topology, hence all topological concepts on this space refer to that topology. In this topology a generalized sequence $(A_i)_{i\in I}$ of operators converges to the

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operator A if and only if the generalized sequence $\big(A_i(x)\big)_{i\in I}$ converges to $A(x)$ in X for each x in X.

Let X be a topological vector space and R a ring. By a *representation* of the ring R on X we mean a homomorphism of R into $\mathcal{L}(X)$. If R is a topological ring and this homomorphism is continuous, then we call it a *continuous representation*. If R has a unit, then we require that it is mapped onto the identity operator. Similarly, if an algebra $\mathcal{A}$ is given, then by a *representation* of this algebra on X we mean a representation of the ring on X, which is also a homomorphism of the linear space structure of $\mathcal{A}$. Continuity of an algebra representation is meant in the obvious way.

THEOREM 1.1. *Let X be a topological vector space and suppose that a representation of the ring R is given on X. If at this representation the element r of R is mapped onto the operator A_r in $\mathcal{L}(X)$, then, by defining $r\cdot x = A_r x$ for each r in R and x in X, X is a vector module over R. Every vector module uniquely arises in this way.*

Proof. Let $\Phi : R \to \mathcal{L}(X)$ denote the given representation, that is,

$$\Phi(r) = A_r$$

holds for each r in R. Then we have for each x, y in X and r, s in R

$$r\cdot(x+y) = A_r(x+y) = A_r x + A_r y = r\cdot x + r\cdot y\,,$$
$$(r+s)\cdot x = A_{r+s}x = \Phi(r+s)(x) = \Phi(r)(x) + \Phi(s)(x) = A_r x + A_s x = r\cdot x + s\cdot x\,,$$
$$(rs)\cdot x = A_{rs}x = \big(A_r A_s\big)x = A_r\big(A_s x\big) = r\cdot(s\cdot x)\,,$$

hence X, as an Abelian group, is a module over R. As A_r is continuous, X is a vector module.

Suppose now that X is a vector module over the ring R. Then, by definition, the mapping $A_r : x \mapsto r\cdot x$ is a continuous linear mapping, that is, a linear operator on X, and clearly the mapping $\Phi : r \mapsto A_r$ is a representation of R on X, which induces the given vector module structure on X.

The uniqueness statement is obvious. ∎

This theorem shows that any vector module can be realized as an ordered pair consisting of a topological vector space X and a ring of linear operators on X. This is similar to the situation of ordinary modules: these are pairs consisting of an Abelian group and a ring of endomorphisms of it. In the case of vector modules the submodules are exactly those subspaces, which are invariant under the operators belonging to the given ring. But these are exactly those linear subspaces of X which are invariant under the linear operators belonging to the operator algebra generated by the ring of linear operators in question. Hence, in what follows, we may always suppose that our vector modules over rings are actually vector modules over operator algebras of linear operators on the given topological vector space. More exactly, we shall consider vector modules over operator algebras, by which we mean a topological vector space X together with a unital algebra $\mathcal{A}$ of linear operators on X. Hence submodules of this vector module are exactly the $\mathcal{A}$-invariant subspaces of X. Here we give some simple examples.

1. Let X be a vector space over the field F. Then scalar operators in $\mathcal{L}(X)$, that means, the scalar multiples of the identity, form an operator algebra $\mathcal{A}$ on X and X is a vector module over $\mathcal{A}$. Submodules are exactly the linear subspaces of X.
2. Let X be a topological vector space and A a linear operator on X. Then X is a vector module over the operator algebra $\mathcal{A}_A$ generated by A. We always suppose, unless the contrary is explicitly stated, that this algebra includes the identity operator, too. Submodules are exactly the A-invariant linear subspaces of X.
3. Let $X = \mathbb{C}^{\mathbb{N}}$ denote the vector space of all complex sequences with the product topology and let τ be the shift operator defined by
$$(\tau x)_n = x_{n+1}$$
for each sequence $(x_n)_{n\in\mathbb{N}}$ in X and n in N. Then X is a vector module over the operator algebra $\mathcal{A}_\tau$ generated by the shift operator. Submodules are exactly the shift invariant linear spaces of sequences.

2. Vector modules, group representations and group actions. Let X be a topological vector space and G a group. By a *representation* of G on X we mean a homomorphism of G into $\mathcal{L}(X)$, where the unit element of G is mapped onto the identity operator. If G is a topological group and the homomorphism is continuous, then we call it a *continuous representation.* If T denotes this homomorphism, then let $\mathcal{A}_T$ denote the subalgebra in $\mathcal{L}(X)$ generated by the image of G under T. Obviously, $\mathcal{A}_T$ is the set of all finite linear combinations of operators of the form $T(g)$ with g in G. Then X is a vector module over the operator algebra $\mathcal{A}_T$. We say that this vector module is *induced by the representation* T. Submodules are those linear subspaces which are invariant under all operators $T(g)$ with g in G. We may call them *G-invariant subspaces*, however these depend not just on G, but rather on T. We remark that if T is a representation of G on X, then we may write T_g instead of $T(g)$.

Let E be a topological space and suppose that a topological group G is given which acts continuously on E. This means, that a continuous map $\pi : G \times E \to E$ is given with $\pi\big(g_1, \pi(g_2, a)\big) = \pi(g_1 g_2, a)$ and $\pi(e, a) = a$ for each a in E and g_1, g_2 in G, where e denotes the unit element of G. The map π will be referred to as an *action* of G and the function $x \mapsto \pi(g, x)$ will be denoted by π_g for each g in G.

Let X be an arbitrary topological vector space of complex valued functions on E. Suppose that this function space is *π-invariant*, which means that for any f in X and g in G the function $T_\pi(g)f$, defined by
$$T_\pi(g)f(a) = f\big(\pi(g, a)\big) \tag{1}$$
whenever a is in E, belongs to X. Suppose, moreover, that $f \mapsto T_\pi(g)f$ is a linear operator on X. Then $T_\pi : g \mapsto T_\pi(g)$ is a representation of G on X. In this case we say that this representation is *induced by the action* π.

This means that if a continuous action of G on E is given which induces the representation T_π of G on the π-invariant function space X, then X becomes a vector module over the algebra $\mathcal{A}_{T_\pi}$. The vector submodules of X are exactly those linear subspaces of X which are π-invariant. If an action π of G on E is given and X is a π-invariant function space on E, then we may call X a *π-module.*

3. Spectral analysis and spectral synthesis on vector modules. Let X be a vector module over the algebra $\mathcal{A}$. We say that *$\mathcal{A}$-spectral analysis*, or simply *spectral analysis*, holds on X, if every nonzero submodule in X has a nonzero finite-dimensional submodule. We say that *$\mathcal{A}$-spectral synthesis*, or *spectral synthesis*, holds on X, if for each submodule X_0 in X the sum of all finite-dimensional vector submodules of X_0 is dense in X_0. If $\mathcal{A}$ is of the form $\mathcal{A}_A$, or $\mathcal{A}_T$, resp. $\mathcal{A}_{T_\pi}$, as above, then we speak about A-spectral analysis and A-spectral synthesis, or T-spectral analysis and T-spectral synthesis, resp. π-spectral analysis and π-spectral synthesis. Clearly, if X is nonzero, then $\mathcal{A}$-spectral synthesis implies $\mathcal{A}$-spectral analysis on X. Here we give some simple examples.

1. Obviously spectral synthesis holds for each nonzero finite-dimensional vector module.
2. Let X be a nonzero topological vector space over a field F. Then, as we have seen, X is a vector module over the algebra of scalar operators and submodules are exactly the linear subspaces of X. It follows that spectral synthesis holds on X, as every subspace is the sum of all finite-dimensional subspaces of it.
3. This example shows the connection between spectral analysis and the invariant subspace problem (see e.g. [1], [26]). Let X be a topological vector space and A a linear operator in $\mathcal{L}(X)$. As we have seen, X is a vector module over the algebra $\mathcal{A}_A$ and the submodules are exactly the A-invariant linear subspaces of X. Hence A-spectral analysis for X is equivalent to the existence of a nonzero finite-dimensional invariant subspace of A.
4. Suppose that X is a Banach space and A is a compact operator on X. By the spectral theory of compact operators, X is the sum of A-invariant subspaces and each eigensubspace corresponding to a nonzero element of the spectrum of A is finite-dimensional. It follows that A-spectral analysis holds on X. Moreover, A-spectral synthesis holds on X if and only if the kernel of A is finite-dimensional.

The purpose of this paper is to give a survey on some recent results concerning spectral analysis and spectral synthesis in the framework of vector modules. Further, we present some open problems in this subject.

4. The Levi-Civita functional equation. The following theorem is of fundamental importance.

THEOREM 4.1. *Let X be a vector module over the algebra $\mathcal{A}$. Spectral analysis holds on X if and only if for each submodule X_0 there is a positive integer n and there are linearly independent vectors $x_1, x_2, \dots, x_n$ in X_0 such that*

$$Ax_i = \sum_{j=1}^{n} \lambda_{i,j}(A)x_j \tag{2}$$

holds for $i = 1, 2, \dots, n$ and for each A in $\mathcal{A}$ with some functions $\lambda_{i,j} : \mathcal{A} \to \mathbb{C}$. In particular, they satisfy the system of functional equations

$$\lambda_{i,j}(AB) = \sum_{k=1}^{n} \lambda_{k,j}(A)\lambda_{i,k}(B) \tag{3}$$

for each A, B in $\mathcal{A}$ and for $i, j = 1, 2, \dots, n$.

Proof. Suppose that spectral analysis holds on X and X_0 a nonzero submodule. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of a finite-dimensional submodule of X_0. Then for each $i, j = 1, 2, \ldots, n$ there exist complex numbers $\lambda_{i,j}$ such that (2) holds for $i = 1, 2, \ldots, n$ and for each A in $\mathcal{A}$. Putting AB for A in (2) we get

$$ABx_i = \sum_{j=1}^{n} \lambda_{i,j}(AB)x_j\,. \tag{4}$$

On the other hand, we have

$$A(Bx_i) = A\Big(\sum_{k=1}^{n} \lambda_{i,k}(B)(x_k)\Big) = \sum_{k=1}^{n}\sum_{j=1}^{n} \lambda_{i,k}(B)\lambda_{k,j}(A)x_j\,. \tag{5}$$

By the linear independence of the x_j's we get

$$\lambda_{i,j}(AB) = \sum_{k=1}^{n} \lambda_{k,j}(A)\lambda_{i,k}(B) \tag{6}$$

and the necessity of the conditions of the theorem is proved.

Conversely, suppose that the conditions in the theorem are satisfied with some elements $x_1, x_2, \ldots, x_n$ in X_0 and functions $\lambda_{i,j} : G \to \mathbb{C}$ $(i, j = 1, 2, \ldots, n)$. Then, by (2), the subspace in X_0 spanned by $x_1, x_2, \ldots, x_n$ is nonzero and $\mathcal{A}$-invariant, hence spectral analysis holds on X. ∎

The functional equation in (3) is called *Levi-Civita equation.* Clearly, the functions $\lambda_{i,j}$ define a representation Λ of the algebra $\mathcal{A}$ on the Hilbert space $\mathbb{C}^n$ in the following way: let $\{e_1, e_2, \ldots, e_n\}$ denote the standard orthonormal basis in $\mathbb{C}^n$, where the j-th component of e_i is $\delta_{i,j}$, the Kronecker's symbol and we use the Euclidean inner product $\langle e_i, e_j\rangle = \delta_{i,j}$ for $i, j = 1, 2, \ldots, n$. Then, for each A in $\mathcal{A}$ we let

$$\langle \Lambda(A)e_i, e_j\rangle = \lambda_{i,j}(A)\,,$$

whenever $i, j = 1, 2, \ldots, n$. By (3) it follows

$$\Lambda(AB) = \Lambda(A)\Lambda(B)$$

for each A, B in $\mathcal{A}$, which proves our statement. Obviously, $\Lambda(A)$ can be realized as an $n \times n$ matrix. If $\mathcal{A}$ is unital, then $\Lambda(I)$ is the identity matrix and if $\mathcal{A}$ is closed under taking inverses, then for each invertible operator A in $\mathcal{A}$ the matrix $\Lambda(A)$ is regular:

$$\Lambda(I) = \Lambda(AA^{-1}) = \Lambda(A)\Lambda(A^{-1})\,,$$

which implies

$$\Lambda(A)^{-1} = \Lambda(A^{-1})\,.$$

Hence, in this case $\Lambda : \mathcal{A} \to GL_n(\mathbb{C})$ is a representation of the group of invertible operators in $\mathcal{A}$.

According to our remarks and to the terminology in [19], vectors satisfying a system of equations of the form (2) are called matrix elements. More exactly, an element x in X is called a *matrix element* if it is contained in a finite-dimensional submodule. Clearly, an element of a vector module is a matrix element if and only if it generates a finite-dimensional variety. Using this terminology we can formulate the following theorem.

Theorem 4.2. *Let X be a vector module over the algebra $\mathcal{A}$. Spectral analysis holds on X if and only if each nonzero submodule of X contains a matrix element. Spectral synthesis holds on X if and only if in each submodule X_0 the matrix elements in X_0 span a dense subspace in X_0.*

In the case of commutative $\mathcal{A}$ we have the following theorem.

Theorem 4.3. *Let X be a vector module over the commutative algebra $\mathcal{A}$. Spectral analysis holds on X if and only if in each nonzero submodule there exists a common eigenvector for $\mathcal{A}$.*

Proof. Clearly, if $x \neq 0$ is a common eigenvector for all operators in $\mathcal{A}$, then the one-dimensional subspace spanned by x is an invariant subspace of all operators in $\mathcal{A}$, hence it is a one-dimensional submodule. Conversely, if spectral analysis holds on X and X_0 is a nonzero finite-dimensional submodule, then, by commutativity, there exists a common eigenvector in X_0 of all operators in $\mathcal{A}$. ∎

5. Spectral synthesis and group actions. Concerning spectral analysis and synthesis with respect to group actions we use the concept of matrix elements, as well. Obviously, we have the corresponding theorems 4.2 and 4.3, too.

Here we give a simple example for spectral analysis and synthesis with respect to a group action. Let $E \neq \{0\}$ be a complex vector space, let X denote the set of all complex valued functions on E equipped with the topology of pointwise convergence and with the linear operations. Then X is a locally convex topological vector space. Let G be the multiplicative group $\{1, -1\}$. We define the action π of G on E by $\pi(\epsilon, x) = \epsilon x$ for ϵ in G and x in E. Obviously X is π-invariant. By (1), we have $T_\pi(1)f = f$ and $T_\pi(-1)f(a) = f(-a) = \check{f}(a)$ for each a in E. It follows that a subspace X_0 of X is π-invariant if and only if it contains $\check{f}$ for each f in X_0. On the other hand, common eigenfunctions of $T_\pi(1)$ and $T_\pi(-1)$ are exactly the nonzero even functions:

$$\check{f} = f$$

and the nonzero odd functions:

$$\check{f} = -f.$$

Hence spectral analysis holds for X if and only if every nonzero submodule X_0 contains either an even or an odd function. But if X_0 is nonzero, then for any nonzero f in X_0 we have that the function $\check{f}$, consequently, also the even function $2f_e = f + \check{f}$ and the odd function $2f_o = f - \check{f}$ belong to X_0. As $f = f_e + f_o$, it follows that either f_e or f_o is nonzero, hence spectral analysis holds on X. Finally, by $f = f_e + f_o$, we have that actually spectral synthesis holds on X, too.

6. Decomposition of vector modules. Let X be a vector module over the algebra $\mathcal{A}$. The family $(X_i)_{i\in I}$ of nondense submodules of X is called a *decomposition* of X if the sum of the X_i's is dense in X. Clearly, if the family $(X_i)_{i\in I}$ is a decomposition of X, then so is the family $(X_i^{cl})_{i\in I}$. This implies that we can always suppose that the members of a decomposition are closed, that is, they are varieties. It follows that a decomposition has at least two different nonzero members. We say that X is *decomposable* if it has a decomposition. It is obvious that X is decomposable if and only if it is the closure of the sum of two proper subvarieties. If X is not decomposable, then we call it *indecomposable*. A matrix element is called *indecomposable* if it generates an indecomposable variety, which is—by definition—finite-dimensional.

THEOREM 6.1. *Let X be a vector module over the algebra $\mathcal{A}$. Every finite-dimensional variety can be expressed uniquely as the sum of finitely many indecomposable varieties. Every matrix element can be expressed uniquely as the sum of finitely many indecomposable matrix elements.*

Proof. If X_0 is an indecomposable variety, then we are ready. If not, then X_0 is the closure of the sum of two submodules $A, B \subseteq X_0$, which are not dense in X_0. As finite-dimensional subspaces of X_0 are closed, it follows that X_0 is the sum of the two proper nonzero subspaces A, B. If both are indecomposable, then the proof is finished. Otherwise we can repeat this argument and decompose either A or B, or both into proper subvarieties. Continuing this process the dimensions decrease, hence after finitely many steps we arrive at a finite decomposition of X_0 into indecomposable submodules and the proof of the first statement is complete. The second statement is just a reformulation of the first one. ■

As a consequence we get the following theorem.

THEOREM 6.2. *Let X be a vector module over the algebra $\mathcal{A}$. Spectral analysis holds on X if and only if each nonzero submodule in X contains a nonzero finite-dimensional indecomposable variety, or, equivalently, contains a nonzero indecomposable matrix element. Spectral synthesis holds on X if and only if in each submodule the indecomposable matrix elements span a dense subspace.*

Let G be a topological group. Then G acts continuously on $\mathcal{C}(G)$, the space of continuous complex valued functions on G equipped with the pointwise linear operations and the uniform convergence on compact sets. Two natural actions are induced by the *left translation* τ_y defined by $\tau_y f(x) = f(y^{-1}x)$ and by the *right translation* ρ_y defined by $\rho_y f(x) = f(xy)$ for each f in $\mathcal{C}(G)$, x, y in G. In this paper we call these actions the *left regular action* and the *right regular action* of G, respectively. If G is commutative, then we call them *regular action*. Closed subspaces in $\mathcal{C}(G)$ invariant with respect to both the left and the right regular actions are called two-sided varieties. In the commutative case we may omit the adjective "two-sided". In particular, if G is a topological group and spectral analysis, respectively spectral synthesis, holds on every two-sided variety in $\mathcal{C}(G)$, then we say that *spectral analysis*, respectively *spectral synthesis, holds on G*. Actually, spectral synthesis and spectral analysis in this situation is the classical setting of these spectral problems. Results in this respect can be found in the discrete Abelian case in [11], [3], [20], [21], [22], [2], [9], [23], [10], in the case $G = \mathbb{R}$ in [18], in the case $G = \mathbb{R}^n$ in [4] and in the case of nondiscrete Abelian or compact nonabelian groups in [25].

If G is a commutative topological group, then matrix elements have a more explicit description. We call the continuous function $m : G \to \mathbb{C}$ an *exponential* if it is a homomorphism of G into the multiplicative group of nonzero complex numbers. The continuous function $a : G \to \mathbb{C}$ is called an *additive function* if it is a homomorphism of G into the additive group of complex numbers. The continuous function $p : G \to \mathbb{C}$ is called a *polynomial* if it has the form

$$p(x) = P\big(a_1(x), a_2(x), \ldots, a_n(x)\big) \tag{7}$$

for each x in G, where n is a positive integer, P is a complex polynomial in n variables and $a_1, a_2, \ldots, a_n : G \to \mathbb{C}$ are additive functions. The function $\varphi : G \to \mathbb{C}$ is called an *exponential monomial* if it is the product of an exponential and a polynomial. Finally, linear combinations of exponential monomials are called *exponential polynomials*. For more about polynomials and exponential polynomials on groups see e.g. [20]. By Theorem 3.4.8 on p. 45 in [20] it follows that exponential monomials, hence also exponential polynomials, in particular polynomials are matrix elements with respect to the regular action. Moreover, by Theorem 5.2.1 on p. 76, the converse is also true: every matrix element is an exponential polynomial. The following theorem gives further information on matrix elements.

THEOREM 6.3. *Let G be a commutative topological group. Then the indecomposable matrix elements with respect to the regular action are exactly the exponential monomials.*

Proof. Suppose that $\varphi \neq 0$ is an exponential monomial of the form

$$\varphi(x) = p(x)m(x) = P\big(a_1(x), a_2(x), \ldots, a_n(x)\big)m(x)$$

for each x in G, where m is an exponential and p is a nonzero polynomial. Suppose that $\tau(\varphi)$, the variety generated by φ, is the closure of the sum of two subvarieties, none of them being dense in $\tau(\varphi)$. As $\tau(\varphi)$ is of finite dimension, it follows that

$$\tau(\varphi) = A + B\,,$$

where A, B are proper subvarieties of $\tau(\varphi)$. This means that

$$p(x)m(x) = \varphi_1(x) + \varphi_2(x) \tag{8}$$

holds for each x in G with some exponential polynomials φ_1 in A and φ_2 in B. By Theorem 3.4.3 on p. 42 in [20] the representation of exponential polynomials as a sum of different nonzero exponential monomials is unique, hence in (8) we have $\varphi_1 = 0$ or $\varphi_2 = 0$, which implies either $B = \tau(\varphi)$ or $A = \tau(\varphi)$. This is a contradiction and φ is indecomposable.

Conversely, suppose that φ is an indecomposable exponential monomial. If there appear at least two different exponential monomials in the canonical representation of φ as a sum of exponential monomials, then clearly $\tau(\varphi)$ is a sum of at least two proper subvarieties, which is impossible. Hence φ is an exponential monomial and the theorem is proved. ∎

7. Duals and annihilators. Let X be a vector module over the algebra $\mathcal{A}$. Then $\mathcal{A}^*$, the set of the adjoints of the operators in $\mathcal{A}$, is an algebra of operators on the space X^*, hence we can naturally consider X^* as a vector module over the algebra $\mathcal{A}$. This vector module will be called the *dual module* of the vector module X over $\mathcal{A}$. Clearly

$$A^*\varphi(x) = \varphi(Ax) \tag{9}$$

holds for each φ in X^*, A in $\mathcal{A}$ and x in X.

We recall that if V is a subset of X, then the *annihilator* $V^{\perp}$ of V in X^* is defined by

$$V^{\perp} = \{\varphi : \varphi \in X^*,\ \varphi(x) = 0 \text{ for } x \in V\}$$

and similarly, if I is a subset of X^*, then its *annihilator* $I^{\perp}$ in X is defined by

$$I^{\perp} = \{x : x \in X,\ \varphi(x) = 0 \text{ for } \varphi \in I\}\,.$$

Then $V^{\perp}$ and $I^{\perp}$ are closed subspaces in X^* and in X, respectively. If X is locally convex, then so is X^* and if V, respectively I is a closed subspace in X, respectively in X^*, then we have

$$V^{\perp\perp} = V, \text{ and } I \subseteq I^{\perp\perp}\,.$$

Nevertheless, the second inclusion may be proper (see [10], p. 4). In addition, the closed subspace V is a variety in X if and only if $V^{\perp}$ is an ideal in X^*.

The following statement is well-known.

LEMMA 7.1. *Let X be a locally convex topological vector space and $V \subseteq X$ a closed subspace. Then $X^*/V^{\perp}$ is isomorphic to V^*.*

Actually, the natural homomorphism Φ on X^* mapping each φ in X^* onto its restriction to V is a topological isomorphism between $X^*/V^{\perp}$ and V^*.

Let G be a locally compact topological group and consider the left regular action of G. It is well-known that the dual of $X = \mathcal{C}(G)$ can be identified with the space $X^* = \mathcal{M}_c(G)$ of compactly supported complex Borel measures on G. The space $\mathcal{M}_c(G)$ is an algebra with unit under convolution. Actually, the algebra $\mathcal{A}$ generated by the operators T_y^* in $\mathcal{L}(X^*)$ is a dense subalgebra of $\mathcal{M}_c(G)$. Indeed, for each y in G and μ in $\mathcal{M}(G)$ we have

$$T_y^*\mu(f) = \mu(T_y f) = \int f(yx)\,d\mu(x) = \int\int f(zx)\,d\mu(x)\,d\mu_y(z) = (\mu_y * \mu)(f)\,,$$

where μ_y is the Dirac measure supported by the point y in G. All the convolution operators of the form $\mu \mapsto \mu_y * \mu$ form a weak*-dense subalgebra in $\mathcal{M}_c(G)$. It follows that G-invariant subspaces, resp. G-varieties, in $\mathcal{M}_c(G)$ are exactly the subspaces, resp. closed subspaces, of $\mathcal{M}_c(G)$, which are invariant under all convolution operators corresponding to elements in $\mathcal{M}_c(G)$. In other words, submodules, resp. varieties, in the vector module $\mathcal{M}_c(G)$ over the left regular action of G are exactly the left ideals, resp. the closed left ideals, of the ring $\mathcal{M}_c(G)$.

8. Noetherian modules. We recall that a ring is called a *Noetherian ring*, if the set of its ideals satisfies the *ascending chain condition*, that is, any ascending chain of ideals must stop after finite number of steps. Similarly, a module is called a *Noetherian module*, if the set of its submodules satisfies the ascending chain condition.

THEOREM 8.1. *Suppose that G is a discrete Abelian group and spectral analysis holds on G. Then spectral synthesis holds on G if and only if the dual of every variety in $\mathcal{C}(G)$, which has a unique minimal subvariety, is a Noetherian ring.*

Proof. Suppose first that spectral synthesis fails to hold for G. Then, by the results in [10], the torsion free rank of G is infinite. It follows that G has a subgroup H, which is isomorphic to the direct product of $\aleph_0$ copies of $\mathbb{Z}$, say $H = \prod_{n\in\mathbb{N}} Z_n$, where $Z_n = \mathbb{Z}$ for each $n = 0, 1, \dots$. Let p_n denote the projection of H onto $\mathbb{Z}_n$, that is

$$p_n(x) = x(n)$$

for each $x : \mathbb{N} \to \mathbb{Z}$ in H and n in N. Then $p_n : H \to \mathbb{C}$ is a homomorphism of H into the additive group of $\mathbb{C}$, that is

$$p_n(x + y) = p_n(x) + p_n(y)$$

holds for all x, y in H and n in $\mathbb{N}$. It is well-known that any homomorphism of a subgroup of an Abelian group into a divisible Abelian group can be extended to a homomorphism of the whole group. As the additive group of complex numbers is obviously divisible, the homomorphisms p_n of H can be extended to complex homomorphisms of the whole group G. We shall denote the extensions by p_n, too. Let V_n denote the smallest variety in $\mathcal{C}(G)$ containing p_m for all $m = n, n+1, \dots$. Clearly the constant functions belong to the variety $V = V_0$, and they form a one-dimensional variety, which is minimal. Obviously $V_n \supseteq V_{n+1}$ for all n. We show that p_n is not in V_{n+1}. Clearly, the linear hull W_n of the translates of the set $\{p_m : m = n, n + 1, \dots\}$ is generated by the functions $1, p_m$ with $m = n, n + 1, \dots$. On the other hand, V_n is the closure of W_n. Hence, if p_n is in V_{n+1}, then it is the limit of a net in W_{n+1}, that is

$$p_n(x) = \lim_{\alpha} \Big[\lambda_\alpha + \sum_{m=n+1}^{\infty} \lambda_{\alpha,m} p_m(x)\Big]$$

holds for each x in G. Here $(\lambda_\alpha), (\lambda_{\alpha,m})$ are nets of complex numbers, where $\lambda_{\alpha,m} \neq 0$ holds for at most a finite number of m's. Putting $x = 0$ we get

$$\lim_{\alpha} \lambda_\alpha = 0\,,$$

hence

$$p_n(x) = \lim_{\alpha} \sum_{m=n+1}^{\infty} \lambda_{\alpha,m} p_m(x) \tag{10}$$

holds for each x in G. Let $x^{(m)}$ denote the element of H for which $x^{(m)}(k) = 1$ for $k = m$ and $x^{(m)}(k) = 0$ for $k \neq m$ $(m, k \in \mathbb{N})$. Then putting $x = x^{(n)}$ in (10) we have

$$1 = p_n(x^{(n)}) = \lim_{\alpha} \sum_{m=n+1}^{\infty} \lambda_{\alpha,m} p_m(x^{(n)}) = 0\,,$$

a contradiction. This means that the varieties V_n form a descending chain of cardinality $\aleph_0$ of proper varieties in V, hence, their annihilators in V^* form an ascending chain of proper ideals, which does not terminate after finitely many steps. Now we show that V has no other minimal subvariety than the one formed by the constant functions. Supposing that V has a nonzero subvariety V_0, which does not contain the constant functions, then V_0, by spectral analysis on G, contains an exponential $m \neq 1$. Hence m is in V and, as above, m is the limit of a net in W_0, that is

$$m(x) = \lim_{\alpha} \Big[\lambda_\alpha + \sum_{m=0}^{\infty} \lambda_{\alpha,m} p_m(x)\Big]$$

holds for each x in G. Here $(\lambda_\alpha), (\lambda_{\alpha,m})$ are nets of complex numbers, where $\lambda_{\alpha,m} \neq 0$ holds for at most a finite number of m's. Putting $x = 0$ we get

$$\lim_\alpha \lambda_\alpha = 1\,,$$

hence

$$m(x) = 1 + \lim_\alpha \sum_{m=0}^{\infty} \lambda_{\alpha,m} p_m(x) \tag{11}$$

holds for each x in G. Putting $x + y$ for x in (11) and subtracting (11) from the new equation we get

$$m(x)\big(m(y) - 1\big) = \lim_\alpha \sum_{m=0}^{\infty} \lambda_{\alpha,m} p_m(y) \tag{12}$$

for all x, y in G. Repeating this with (12) instead of (11) we have

$$m(x)\big(m(y) - 1\big)^2 = 0 \tag{13}$$

for all x, y in G. As $m \neq 1$ there is a y in G for which $m(y) \neq 1$, and we have a contradiction. This means that V has a unique minimal subvariety, and an infinite descending chain of proper subvarieties. The annihilators of these subvarieties form an ascending chain of ideals in V^*, which does not terminate after finitely many steps. This contradicts to the fact that V^* is a Noetherian ring.

To prove the converse we suppose that spectral synthesis holds on G. Then, by the results in [23], the torsion free rank of G is finite, say $n \geq 0$. By the results in [24] there are linearly independent complex homomorphisms $\alpha_i : G \to \mathbb{C}$ for $i = 1, 2, \ldots, n$ such that each complex homomorphism of G is a linear combination of these functions. By the linear independence of the α's there exist elements x_j, $j = 1, 2, \ldots, n$, such that the matrix $(\alpha_i(x_j))_{i,j=1}^n$ is regular. Let $e^{(i)}$ denote the element in $\mathbb{C}^n$ whose i-th component is 1, all the others being 0 ($i = 1, 2, \ldots, n$). For each $k = 1, 2, \ldots, n$ we consider the system of linear equations

$$e_i^{(k)} = \sum_{j=1}^{n} \lambda_{k,j} \alpha_j(x_i)$$

where $i = 1, 2, \ldots, n$ for the unknowns $\lambda_{k,1}, \lambda_{k,2}, \ldots, \lambda_{k,n}$. This has a unique solution, and we let

$$a_k = \sum_{j=1}^{n} \lambda_{k,j} \alpha_j\,.$$

Then $a_k : G \to \mathbb{C}$ is a homomorphism and

$$a_k(x_i) = \sum_{j=1}^{n} \lambda_{k,j} \alpha_j(x_i) = e_i^{(k)}\,,$$

in particular, $a_1, a_2, \ldots, a_n$ are linearly independent complex homomorphisms of G. Letting $a = (a_1, a_2, \ldots, a_n)$ we have that $a : G \to \mathbb{C}^n$ is a homomorphism of G onto a subgroup $a(G)$ of $\mathbb{C}^n$. If H denotes the subgroup of G generated by the elements $x_1, x_2, \ldots, x_n$, then $a(H)$ is isomorphic to $\mathbb{Z}^n$. Moreover, the ring of polynomials on G is isomorphic to the ring of polynomials $\mathbb{C}[z_1, z_2, \ldots, z_n]$.

Suppose that V is a variety in $\mathcal{C}(G)$ which has a unique minimal subvariety. As spectral analysis holds on G, V contains an exponential m, hence this minimal subvariety consists of the constant multiples of m. It is easy to see that by multiplying all elements of V by the function $1/m$ we get another variety, which has a minimal subvariety, too, namely, the set of all constant functions. In other words, with no loss of generality we may assume that the unique minimal subvariety in V is the set of constant functions. Then, by spectral synthesis, V is spanned by all polynomials contained in V. We have to show that V^* is Noetherian. Supposing the contrary there is an infinite strictly ascending chain of ideals in V^*, hence their annihilators in V form an infinite strictly descending chain $V = V_0 \supset V_1 \supset \ldots \supset V_n \supset \ldots$ of varieties in V. If P_n denote the set of all polynomials in V_n, then P_n is a translation invariant linear space of polynomials. By spectral synthesis, the closure of P_n is V_n, hence $(P_n)_{n\in\mathbb{N}}$ is a strictly descending chain of translation invariant subspaces of polynomials. But this contradicts Lemma 8 in [10] and our theorem is proved. ∎

THEOREM 8.2. *Suppose that G is a discrete Abelian group and in the ring $\mathcal{M}_c(G)$ the cardinality of every strictly ascending chain of ideals is less than the continuum. Then spectral analysis holds on G.*

Proof. Suppose the contrary, that is, spectral analysis fails to hold for G. Then, by the results in [9], the torsion free rank of G is at least continuum. It follows that G has a subgroup H which is isomorphic to the direct product of continuum copies of $\mathbb{Z}$, say $H = \prod_{t\in\mathbb{R}_+} Z_t$, where $Z_t = \mathbb{Z}$ for each t in $\mathbb{R}_+$. Let p_t denote the projection of H onto $\mathbb{Z}_t$, that is

$$p_t(x) = x(t)$$

for each $x : \mathbb{R}_+ \to \mathbb{Z}$ in H and t in $\mathbb{R}_+$. Then $p_t : H \to \mathbb{C}$ is a homomorphism of H into the additive group of $\mathbb{C}$, that is

$$p_t(x + y) = p_t(x) + p_t(y)$$

holds for all x, y in H and for t in $\mathbb{R}_+$. It is well-known that any homomorphism of a subgroup of an Abelian group into a divisible Abelian group can be extended to a homomorphism of the whole group. As the additive group of complex numbers is obviously divisible, the homomorphisms p_t of H can be extended to complex homomorphisms of the whole group G. We shall denote the extensions by p_t, too. Let V_t denote the smallest variety in $\mathcal{C}(G)$ containing p_s for all $s > t$. Obviously $V_t \supseteq V_s$ for $t < s$. We show that p_t is not in V_t. Clearly, the linear hull W_t of the translates of the set $\{p_s : s > t\}$ is generated by the functions $1, p_s$ with $s > t$. On the other hand, V_t is the closure of W_t. Hence, if p_t is in V_t, then it is the limit of a net in W_t, that is

$$p_t(x) = \lim_{\alpha} \Big[\lambda_\alpha + \sum_{s>t} \lambda_{\alpha,s} p_s(x)\Big]$$

holds for each x in G. Here $(\lambda_\alpha), (\lambda_{\alpha,s})$ are nets of complex numbers, where $\lambda_{\alpha,s} \neq 0$ holds for at most a finite number of s's. Putting $x = 0$ we get

$$\lim_{\alpha} \lambda_\alpha = 0\,,$$

hence

$$p_t(x) = \lim_\alpha \sum_{s>t} \lambda_{\alpha,s} p_s(x) \tag{14}$$

holds for each x in G. Let $x^{(s)}$ denote the element of H for which $x^{(s)}(u) = 1$ for $u = s$ and $x^{(s)}(u) = 0$ for $u \neq s$ $(s, u \in \mathbb{R}_+)$. Then putting $x = x^{(t)}$ in (14) we have

$$1 = p_t(x^{(t)}) = \lim_\alpha \sum_{s>t} \lambda_{\alpha,s} p_s(x^{(t)}) = 0\,,$$

a contradiction. This means that the varieties V_t form a descending chain of continuum cardinality of proper varieties, hence, their annihilators in $\mathcal{M}_c(G)$ form an ascending chain of continuum cardinality of proper ideals, which is impossible. The theorem is proved. ∎

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