FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

ON KOTTMAN'S CONSTANTS IN BANACH SPACES

JESÚS M. F. CASTILLO

Departamento de Matemáticas, Universidad de Extremadura
Avenida de Elvas s/n, 06071 Badajoz, España
E-mail: castillo@unex.es

PIER LUIGI PAPINI

Dipartimento di Matematica, Universitá degli Studi di Bologna
Piazza Porta S. Donato 5, 40126 Bologna, Italia
E-mail: papini@dm.unibo.it

Abstract. This paper deals with a few, not widely known, aspects of Kottman's constant of a Banach space and its symmetric and finite variations. We will consider their behaviour under ultrapowers, relations with other parameters such as Whitley's or James' constant, and connection with the extension of c_0-valued Lipschitz maps.

1. Kottman's constants. This paper deals with a few, not widely known, aspects of Kottman's constant of a Banach space X, with unit ball B_X and unit sphere S_X, defined as follows:

$$K(X) = \sup\{\sigma > 0 : \exists (x_n)_{n\in\mathbb{N}} \in B_X \ \forall n \neq m \ \|x_n - x_m\| \geq \sigma\}.$$

It was introduced and studied by Kottman in [20, 21]. It is clear that $K(X) = 0$ if and only if X is finite-dimensional. A well-known, although highly non-trivial, result of Elton and Odell [12] (see also [10, p. 241]) establishes that $K(X) > 1$ for every infinite-dimensional Banach space. Kottman's constant has been considered in several papers and its exact calculus in different classical Banach spaces has been performed (see e.g. [2, 7, 8, 9, 11, 14, 15, 16, 20, 21, 22, 25, 28, 31, 32, 34]).

2010 *Mathematics Subject Classification*: 46B04, 46B08, 46B20.
Key words and phrases: Kottman's constant, separation constants, packing.
The research of the first author has been supported in part by project MTM2010-20190-C02-01, Junta de Extremadura (Spain) and Fondos FEDER (GR10113).
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-5

Variations of the Kottman constant can be defined: the *finite separation constant*

$$K_f(X) = \sup\{\sigma > 0 : \forall k \in \mathbb{N} \quad \exists\{x_1, \dots, x_k\} \subset B_X \ \forall n \neq m \ \|x_n - x_m\| \geq \sigma\};$$

the *finite symmetric separation constant*, implicitly considered in [26]:

$$K_f^s(X) = \sup\{\sigma > 0 : \forall k \in \mathbb{N} \quad \exists\{x_1, \dots, x_k\} \subset B_X \ \forall n \neq m \ \|x_n \pm x_m\| \geq \sigma\};$$

and the *symmetric separation constant*:

$$K^s(X) = \sup\{\sigma > 0 : \exists(x_n)_{n\in\mathbb{N}} \in B_X \ \forall n \neq m \ \|x_n \pm x_m\| \geq \sigma\}.$$

In the definition of the four above constants, we can substitute S_X to B_X; also, it is clear that $1 \leq K^s \leq K \leq K_f \leq 2$ and $K^s \leq K_f^s \leq K_f$.

The equality between K and K^s holds in several classical spaces such as ℓ_p spaces, where $K(\ell_p) = K^s(\ell_p) = 2^{1/p}$, $1 \leq p < \infty$; or c_0 since $K^s(c_0) = 2$ as the sequence $x_n = e_{n+1} - \sum_{j=1}^n e_j$ shows.

The first question that arises is

Problem 1. Does the Elton-Odell theorem hold for $K^s(\cdot)$? Precisely, is it always $K^s(X) > 1$ for every infinite-dimensional Banach space?

A partial answer will be presented in Corollary 2.3. Recall that every infinite-dimensional Banach space (by the Dvoretzky-Rogers theorem) contains, for every n, almost isometric copies of ℓ_2^n. Since the elements of the canonical basis of ℓ_2 verify $\|e_i \pm e_j\| = \sqrt{2}$ one has $K_f(X) \geq K_f^s(X) \geq \sqrt{2}$ for every infinite-dimensional Banach space X. Since $K^s(\ell_p) = K(\ell_p) = 2^{1/p}$ for $1 < p < +\infty$, it is clear that strict inequalities $K(X) < K_f(X)$ and $K^s(X) < K_f^s(X)$ are possible (see also [20]). Further examples will be exhibited below, near the end of Section 2.

Let us affirmatively prove a conjecture of Kottman ([21, p. 24]) about the stability of K under vector sums. This generalizes [21, Lemma 8] (where only the "ℓ_p" sum is considered), with a much simpler proof. Recall that given a Banach space λ with unconditional basis (e_n), and a sequence (X_n) of Banach spaces, their λ-vector sum is defined as

$$\lambda(X_n) = \Big\{(x_n) \in \ell_\infty(X_n) : \|(x_n)\| = \Big\|\sum \|x_n\| e_n\Big\|_\lambda < +\infty\Big\}.$$

One has

Proposition 1.1.

$$K(\lambda(X_n)) = \sup\{K(\lambda), K(X_n);\ n \in \mathbb{N}\}.$$

Proof. Let $\delta = K(\lambda(X_n)) - \sup\{K(\lambda), K(X_n);\ n \in \mathbb{N}\} > 0$. Let (x^k) be a sequence in $\lambda(X_n)$ for which we assume that $\|x^k - x^l\| > K(\lambda(X_n)) - \delta/3$ for different k, l. There is no loss of generality assuming that each of the sequences x^k is finitely supported and with its support—i.e., the set of non-zero coordinates—contained in $[1, k]$: indeed, $\big\|\sum_n \|x_n^k\| e_n\big\|_\lambda < +\infty$ implies $\big\|\sum_{n\geq N} \|x_n^k\| e_n\big\|_\lambda < \varepsilon$ for large N. From now on we will denote by $[1, n]x$ the sequence with support contained in $[1, n]$ and whose first n coordinates coincide with those of x. Let α_k be an accumulation point for $\|[1, k]x^l\|$. Passing to subsequences, we get a final sequence—relabelled as (x^k)—for which $\lim_k \|[1, n]x^k\| = \alpha_n$ for all n.

Case 1. $\alpha_n = 0$ for almost all n. This means that the sequence is formed by "blocks", hence their mutual distances cannot go beyond $K(\lambda)$.

Case 2. Otherwise, find N such that $\|\sum_{n>N} \alpha_n e_n\|_\lambda < \frac{1}{4}\delta$. Thus, for large k, m one has

$$\begin{aligned} K(\lambda(X_n)) &< \|x^k - x^m\| + \delta/3 \\ &= \Big\|\sum_{n\le N} \|x^k(n) - x^m(n)\| e_n\Big\|_\lambda + \Big\|\sum_{n>N} \|x^k(n) - x^m(n)\| e_n\Big\|_\lambda + \delta/3 \\ &\le \sup\{K(\lambda), K(X_n);\ n \in \mathbb{N}\} + 2\delta/4 + \delta/3 < K(\lambda(X_n)). \end{aligned}$$

Therefore $K(\lambda(X_n)) \le \sup\{K(\lambda), K(X_n);\ n \in \mathbb{N}\}$, and then the equality. ■

The argument in the previous proof implies that given a norm $|\cdot|$ in $\mathbb{R}^2$ and two Banach spaces X, Y, if we set $X \oplus_{|\cdot|} Y$ to mean the product space $X \oplus Y$ endowed with the norm $\|(x, y)\| = \big|(\|x\|, \|y\|)\big|$ then $K(X \oplus_{|\cdot|} Y) = \max\{K(X), K(Y)\}$ and $K^f(X \oplus_{|\cdot|} Y) = \max\{K^f(X), K^f(Y), K^f([\mathbb{R}^2, |\cdot|])\}$. It is not however true that given a norm $\|\cdot\|$ on $X \times Y$ one has $K([X \times Y, \|\cdot\|]) = \max\{K(X), K(Y)\}$: in [26, Ex. 3.1] one can find a hyperplane H of a certain Banach space X in such a way that $K(X) > K(H)$; it is clear however that there is a norm $\|\cdot\|$ on $H \times \mathbb{R}$ so that X is isometric to $[H \times \mathbb{R}, \|\cdot\|]$ and thus $K(X) > K(H) = \max\{K(H), K(\mathbb{R})\}$. One also has (see [9, Claim 4.4] for the case $\lambda = 1$):

LEMMA 1.2. *For $1 \le \theta < 2$, let Y be a θ-complemented finite-codimensional subspace of X. Then $K(X) \le \theta K(Y)$.*

Proof. Let $p : X \to Y$ be a norm θ-projection onto Y with finite-dimensional kernel. Given $\varepsilon > 0$, let (x_n) be a sequence in X such that $\|x_n - x_m\| \ge K(X) - \varepsilon$. Then $(x_n - p(x_n))$ admits a Cauchy subsequence; hence a subsequence—no need to relabel—such that for all n, m one has $\|(x_n - p(x_n)) - (x_m - p(x_m))\| \le \varepsilon$. Consequently $(p(x_n))$ is a sequence in the ball of radius θ of Y such that

$$\|p(x_n) - p(x_m)\| \ge \|x_n - x_m\| - \|(x_n - p(x_n)) - (x_m - p(x_m))\| \ge K(X) - 2\varepsilon.$$

Consequently, $K(Y) \ge \theta^{-1} K(X)$. ■

2. Other packing or covering constants. Kottman's constant of a Banach space X is related to the size of infinite sets of balls which can be packed inside the unit ball B_X. Precisely, let $P(X)$ denote the packing constant of B_X, defined as the supremum of the $r > 0$ such that B_X contains infinitely many non-overlapping balls of radius at least r; then

$$P(X) = \frac{K(X)}{2 + K(X)}.$$

More interesting for us is Whitley's constant $T(X)$ introduced in [30] and called the *thickness* of X (see also [24]):

$$T(X) = \inf\{\sigma > 0 : \exists\{x_1, \dots, x_n\} \subset S : S \subset \bigcup_{i\in\{1,\dots,n\}} B(x_i, \sigma)\}.$$

If $\dim(X) = \infty$, then $T(X) \in [1, 2]$. The value of $T(\cdot)$ in many spaces is known (see for example [27] where connections with other constants were indicated; see also [24] for further results on $T(\cdot)$). The paper [6] contains the connections between $T(X)$ and another parameter concerning coverings by sequences of balls with radii converging to 0.

Moreover, one has

LEMMA 2.1. *$T(X) \le K^s(X)$.*

Proof. Let $\varepsilon > 0$ and set $k = K^s(X) + \epsilon$. There is no infinite k symmetrically separated set in S; so we can take a finite maximal set $A = \{\pm x_i\}_{i \in I}$ such that $\|x_i \pm x_j\| \ge k$ for $i \ne j$; $i, j \in I$. This means that for every $x \in S$, at least one of the distances $\operatorname{dist}(x, A)$ and $\operatorname{dist}(-x, A)$ is less than k: therefore, either $\|x - x_i\| < k$ for some $i \in I$, or $\|x + x_j\| < k$, so $\|x - (-x_j)\| < k$, for some $j \in I$. This shows that A is a finite k-net for S; thus $T(X) \le k$. Since $\epsilon > 0$ is arbitrary, the result is proved. ■

It is a well known result that if finitely many convex closed sets cover the unit sphere of a Banach space then they cover the entire unit ball. A more general version of this result can be seen in [6]. A simple proof goes as follows: observe that if $S_X \subset \bigcup_{i \in \{1,\dots,n\}} C_i$, then taking the weak*-closures in X^{**} we get $B_{X^{**}} \subset \bigcup_{i \in \{1,\dots,n\}} \overline{C_i}^{w^*}$. Now, intersection with X yields $B_X \subset \bigcup_{i \in \{1,\dots,n\}} \overline{C_i}^{w^*} \cap X$: but $\overline{C_i}^{w^*} \cap X = \overline{C_i}^{\text{weak}} = C_i$. From this we get $T(X^{**}) \le T(X)$.

We set the following notation: $M(x, y) = \max\{\|x - y\|, \|x + y\|\}$ and $m(x, y) = \min\{\|x - y\|, \|x + y\|\}$. The following constants have been considered in [27, 5]:

- $g(X) = \inf_{x \in S} \inf_{y \in S} M(x, y)$
- $J(X) = \sup_{x \in S} \sup_{y \in S} m(x, y)$.

It is clear that $1 \le g \le J \le 2$. The constant g has been considered in several papers (see for example [27]) and is the smallest radius of a ball, centred at some $x \in S$, which can contain an antipodal pair $y, -y$ of S. The constant J is often called *James constant* and is the infimum of all radii such that for every $x \in S$ the two balls centred at x and $-x$ with that radius cover S. One has

LEMMA 2.2. *$g \le T \le K^s \le J$.*

Proof. Let $\varepsilon > 0$ and take a finite covering of B_X with balls centred at points of S_X with radius smaller than or equal to $T(X) + \varepsilon$. According to an old result of Ljusternik and Šnirel'man, at least one of the balls must contain an antipodal pair $(y, -y)$. If x is the centre of such a ball, then $\max\{\|x + y\|, \|x - y\|\} \le T(X) + \varepsilon$; and since ε is arbitrary, the inequality $g \le T$ is clear. The inequality $T \le K^s$ has already been proved. To show the inequality $K^s \le J$, we can find points $\pm x_1, \dots, \pm x_n$ such that the union of the balls $B(\pm x_i, r)$, $i = 1, \dots, n$ with r near to $K^s(X)$ does not cover S_X. So, for each of the x_i's, $\min\{\|x_i + y\|, \|x_i - y\|\} > r$ for some y and then $\sup\{m(x_i, y) : y \in S_X\} > r$, which implies $J > r$, and then $J \ge K^s$. ■

The inequalities in the previous lemma can be strict. In fact, one has $g(\ell_1) = 1 < 2 = T(\ell_1)$; $T(\ell_\infty) = 1 < 2 = K^s(\ell_\infty)$. Concerning K^s and J see below.

In [26], the condition $K_f^s < 2$ was called *O-convexity*, while the condition $K_f < 2$ is usually called *P-convexity* (see also [1]).

A Banach space X is called *Uniformly Non-Square* (UNS for short) if $J(X) < 2$. It is clear that (UNS) implies $K^s \le K_f^s < 2$; Example 3.2 in [26] shows that it does not imply $K < 2$. This also shows that K^s and K can be different and that the inequality $K \le J$ does not always hold.

The condition $K_f < 2$ does not imply (UNS): it is enough to consider the product $\ell_2 \oplus_1 \ell_\infty^2$. This also shows that $K^s < J$ is possible (by Lemma 2.2 we know that $K^s \le J$). By adapting Example 1.8 in [20], one can see that $K < 2$ does not imply $K_f^s < 2$. Thus, $J \le K$ is not true in general.

Example 3.3 in [26] shows that a space can be O-convex without being P-convex. Therefore for such a space one has $K_f^s < 2 = K = K_f$. Thus $K_f^s < 2$ does not imply $K < 2$; moreover, $K_f^s < K_f$ is possible.

A partial answer to Problem 1 is the following

COROLLARY 2.3. *For every UNS space X we have $K^s(X) > 1$.*

Proof. Since the functions g and J are connected by the equality $gJ = 2$ [5], one gets that if $J(X) < 2$ then $1 < g(X) \le K^s(X)$. ∎

3. Kottman's constants and ultrapowers. Let us briefly recall the definition and some basic properties of ultraproducts of Banach spaces. For a detailed study of this construction at the elementary level needed here we refer the reader to Heinrich's survey paper [13] or to Sims' notes [29]. Let I be a set, U be an ultrafilter on I, and X_i a family of Banach spaces. Then

$$\ell_\infty(X_i) = \{(x_i) : x_i \in X_i,\ \sup_i \|x_i\| < \infty\},$$

endowed with the supremum norm, is a Banach space, and

$$c_0^U(X_i) = \{(x_i) \in \ell_\infty(X_i) : \lim_{U(i)} \|x_i\| = 0\}$$

is a closed subspace of $\ell_\infty(X_i)$. The ultraproduct of the spaces X_i following U is defined as the quotient

$$[X_i]_U = \ell_\infty(X_i)/c_0^U(X_i).$$

We denote by $[(x_i)]$ the element of $[X_i]_U$ which has the family (x_i) as a representative. It is not difficult to show that

$$\|[(x_i)]\| = \lim_{U(i)} \|x_i\|.$$

In the case $X_i = X$ for all i, we denote the ultraproduct by X_U, and call it the ultrapower of X following U.

The bidual, or any even dual, of a Banach space X is complemented in some ultrapower of X. Indeed, take the set $\mathcal{F}$ of triples (F, F^*, ε) where F is a finite-dimensional subspace of X^{**} and F^* a finite-dimensional subspace of X^*, with the order $(F, F^*\varepsilon) \le (G, G^*, \varepsilon')$ for $F \subset G$, $F^* \subset G^*$ and $\varepsilon' \le \varepsilon$. Let U be a filter refining the Fréchet filter. The principle of local reflexivity yields for each (F, F^*, ε) an operator $T_{F,F^*,\varepsilon} : F \to X$ such that $T_{F,F^*,\varepsilon}(x) = x$ when $x \in X$, $T^*_{F,F^*,\varepsilon} f = f$ for $f \in F^*$ and $\|T_{F,F^*,\varepsilon}\| \le 1 + \varepsilon$.

An embedding $\tau : X^{**} \to X_U$ is thus given by $\tau(x^{**}) = [T_{F,F^*,\varepsilon} x^{**}]$. For this embedding

$$P([x_{F,F^*,\varepsilon}]) = \text{weak}^* - \lim_{U(F,F^*,\varepsilon)} x_{F,F^*,\varepsilon}$$

is a projection.

Definition 3.1. An ultrafilter U on a set I is *countably incomplete* if there is a decreasing sequence (I_n) of subsets of I such that $I_n \in U$ for all n, and $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

It is obvious that any countably incomplete ultrafilter is non-principal (i.e., not formed with all the sets containing a certain element), and also that every non-principal (or free) ultrafilter on $\mathbb{N}$ is countably incomplete. Assuming that all free ultrafilters are countably incomplete is consistent with the Zermelo-Fraenkel axioms of set theory plus the Continuum Hypothesis, since the cardinal of a set supporting a free countably complete ultrafilter should be measurable, hence strongly inaccessible.

In [17] it has been shown that $J(X) = J(X_U)$. From this and the relation $gJ = 2$ it immediately follows that also $g(X) = g(X_U)$. Let us consider the stability of T, K and K^s by ultrapowers.

Proposition 3.2.

1) *There is a free countably incomplete ultrafilter U such that*

$$K(X) \le K(X^{**}) \le K(X_U)$$

and

$$K^s(X) \le K^s(X^{**}) \le K^s(X_U).$$

2) *If U is a countably incomplete ultrafilter then for every $k \in \mathbb{N}$ one has*

$$K_f(X) = K_f(X_U) = K(X_U) = K((X^{2k})_U) = K_f(X^{2k})$$

and

$$K^s_f(X) = K^s_f(X_U) = K^s(X_U) = K^s((X^{2k})_U) = K^s_f(X^{2k}).$$

3) *If U is a countably complete ultrafilter then*

$$K(X) = K(X_U)$$

and

$$K^s(X) = K^s(X_U).$$

Proof. (1) is clear since there are isometric embeddings $X \to X^{**} \to X_U$.

To show (2) one only has to prove that $K_f(X) = K(X_U)$. To show that $K_f(X) \le K(X_U)$, take for each n a finite sequence $x_1^n, \dots, x_n^n$ so that $\|x_j^n - x_k^n\| \ge K_f(X) - \frac{1}{n}$. Since there is a decreasing sequence A_n of elements of U such that $\bigcap_n A_n = \emptyset$, we form the following sequence of elements of X_U: $z_k = [x_k^i]$ for $i \in A_k \setminus A_{k+1}$ (and 0 on the rest). One has

$$\|z_k - z_l\| = \lim_U \|z_k^i - z_l^i\| \ge K_f(X).$$

We show that $K(X_U) \le K_f(X)$. Let (z_k) be an infinite sequence in X_U such that $\|z_k - z_l\| \ge K$. If $z_k = [z_k^n]$, since

$$\|z_k - z_l\| = \lim_{U(i)} \|z_k^i - z_l^i\| \ge K,$$

with the meaning that for every $\varepsilon > 0$

$$A_{k,l} = \{i \in I : \|z_k^i - z_l^i\| \geq K - \varepsilon\} \in U,$$

then, for every $n \in \mathbb{N}$,

$$\bigcap_{k<n} A_{k,n} \in U.$$

Thus, there must exist some $i \in I$ in such a way that the distance between any two elements of the set $\{z_1^i, \dots, z_n^i\}$ is at least $K - \varepsilon$. Now, $K_f(X) \leq K_f(X^{**})$ is clear; while Goldstine's lemma yields $K_f(X^{**}) \leq K_f(X)$. Finally,

$$K((X^{2k})_U) = K_f(X^{2k}) = K_f(X) = K(X_U)$$

follows by iteration.

To prove (3), take a countably complete ultrafilter U and $(r_i)_{i\in I} \in \ell_\infty(I)$; then $\lim_U r_i = r$ implies $\{i \in I : r_i = r\} \in U$. Let $z^n = [x_i^n]$ be a sequence in X_U with $\|z^n - z^m\| > K$. The countable family $A_{n,m} = \{i \in I : \|x_i^n - x_i^m\| > K\} \in U$, as well as its intersection $\bigcap_{n,m} A_{n,m}$; which is therefore non-empty. Let j be an index in that set. Thus $\|x_j^n - x_j^m\| > K$ for all n, m and $K(X) \geq K(X_U)$.

The proofs for K^s are entirely analogous. ■

Observe that since it is known that $K_f(X) < 2$ implies X reflexive, one gets that $K_f(X) < 2$ implies that X is superreflexive (see [1]): indeed, $K_f(X) = K(X_U) < 2$ implies X_U reflexive, hence X is superreflexive. Since $K(X_U) = K_f(X)$ and it is not hard to find spaces X for which $K(X) < K_f(X)$—say $X = \ell_2(\ell_\infty^n)$—it is clear that in general $K(X)$ and $K(X_U)$ are different. This is explicitly remarked in [17] with an example due to Prus.

For the thickness constant one has

Proposition 3.3. *Let U be a free ultrafilter on $\mathbb{N}$. If X_U denotes the corresponding ultrapower of X then $T(X_U) \leq T(X)$.*

Proof. Let $t > T(X)$ and assume, as we can, that $B(X) \subset \bigcup_{i\in\{1,\dots,n\}} B(x_i, t)$. Let us show that

$$B(X)_U \subset \bigcup_{i\in\{1,\dots,n\}} B(x_i, t)_U.$$

To do this, let $[z_n] \in B(X)_U$. For each z_n there is at least one $x_{i(n)}$ such that $\|z_n - x_{i(n)}\| \leq t$. So $\lim_U \|z_n - x_{i(n)}\| \leq t$, which means $\|[z_n] - [x_{i(n)}]\|_U \leq t$. Since for some index $1 \leq m \leq n$ one has $[x_{i(n)}] = [x_m, x_m, \dots, x_m, \dots]$ the proof concludes. ■

These results suggest the following questions

Problem 2.

- Does $T(X^{**}) = T(X)$ hold for every Banach space?
- Does $K(X^{**}) = K(X)$ hold for every Banach space?
- Does $K^s(X^{**}) = K^s(X)$ hold for every Banach space?

Since both $K^s(c_0) = K(c_0) = 2 = K(\ell_1) = K^s(\ell_1)$, and a Banach space containing isomorphic copies of any of those spaces also contains an almost-isometric copy, it is clear that a Banach space such that $K(X) < K(X^{**})$ (resp. $K^s(X) < K^s(X^{**})$), if it exists, cannot contain either c_0 or ℓ_1. Hence it cannot have unconditional basis. It cannot be either an $\mathcal{L}_p$-space: for $1 < p < +\infty$ they are reflexive; for $p = 1$ they contain ℓ_1; and since the bidual of an $\mathcal{L}_\infty$ space contains ℓ_∞, an $\mathcal{L}_\infty$ counterexample should be an $\mathcal{L}_\infty$ space verifying $K(\mathcal{L}_\infty) < 2$ (resp. $K^s(\mathcal{L}_\infty) < 2$). However, one has

PROPOSITION 3.4. *For every $\mathcal{L}_\infty$-space X one has $K^s(X) = K(X) = 2$.*

Proof. It is well known that a separable Banach space X has c_0 as a quotient if (and only if) X^* contains ℓ_1 (see [23, p. 104]). Therefore every separable $\mathcal{L}_\infty$-space has c_0 as a quotient. Since a Banach space isomorphic to c_0 contains an almost isometric copy of c_0 (see [23, p. 97]), one can assume that the norm induced by the quotient map $q : X \to c_0$ is $(1+\varepsilon)$-isometric to the natural sup-norm on c_0. And this immediately yields $K^s(X) = K(X) = 2$. Finally, every $\mathcal{L}_\infty$-space contains a separable $\mathcal{L}_\infty$-space. ∎

In any case, let us remark that in the literature $\mathcal{L}_\infty$-spaces not containing either c_0 or ℓ_1 are few: the Bourgain-Delbaen second type of examples [4], which are isomorphic preduals of ℓ_1 without copies of c_0 or ℓ_1; and the Argyros-Haydon Hereditarily Indecomposable $\mathcal{L}_\infty$ space [3].

4. Kottman's constant and the extension of Lipschitz maps. For a separable Banach space X we define the constant $\lambda_0(X)$ as the infimum of all λ such that for every subspace E of X every Lipschitz map $f : E \to c_0$ admits an extension to X with Lipschitz constant $\mathrm{Lip}(F) \le \lambda \,\mathrm{Lip}(f)$. Kalton proved in [18, Prop. 5.8] the following unexpected result:

PROPOSITION 4.1. $K(X) = \lambda_0(X)$.

The proof given there is rather awkward; we present here a proof, streamlined from papers [19, 18], that $K(X) \le \lambda_0(X)$.

Proof. The key is to characterize the extension property of Lipschitz maps in terms of forbidden sequences. Precisely: Let M be a metric space and let $\lambda \ge 1$. We show that if every c_0-valued Lipschitz map with Lipschitz constant L defined on a subset E of M admits a Lipschitz extension to M with Lipschitz constant at most λL, then given $\varepsilon > 0$ and $a \in M$ it is impossible to find a sequence (x_n) in M such that for $1 \le j \le n-1$ one has

$$\lambda d(a, x_n) + \varepsilon \le d(x_j, x_n).$$

Otherwise, assume that such a sequence (x_n) in M exists; let $E = \{x_n\}_{n\in\mathbb{N}}$ and form the Lipschitz map $f : E \to c_0$ given by

$$f(x) = \big(\min_{1\le j\le k} d(x, x_j)\big)_k.$$

We have $\mathrm{Lip}(f) = 1$; note that the same is true if we apply f to the whole space c_0, or to a bounded subset of c_0, but in this case the range of f is not contained in c_0 (it is contained in ℓ_∞).

If f admits an extension to a Lipschitz map $F : M \to c_0$ with $\mathrm{Lip}(F) \le \lambda \,\mathrm{Lip}(f)$ then set $F(a) = (\xi_k)$. Fixed $k \in \mathbb{N}$, we have

$$\|F(a) - F(x_{k+1})\| \le \lambda d(a, x_{k+1}),$$

so

$$\min_{1 \le j \le k} d(x_{k+1}, x_j) - \xi_k \le \lambda d(a, x_{k+1}) \le \min_{1 \le j \le k} d(x_{k+1}, x_j) - \varepsilon,$$

which means $\xi_k \ge \varepsilon$ for all k, and thus a contradiction.

Now, let X be a Banach space; for $\lambda = K(X) - 2\varepsilon$ and $a = 0$ the choice of a sequence (x_n) such that $\|x_n\| \le 1$, $\|x_n - x_m\| \ge K(X) - \varepsilon$, makes $\lambda + \varepsilon \le \|x_n - x_m\|$ possible. Thus, the assertion $K(X) \le \lambda_0(X)$ holds. ∎

References

[1] D. Amir, C. Franchetti, *The radius ratio and convexity properties in normed linear spaces*, Trans. Amer. Math. Soc. 282 (1984), 275–291.

[2] Yu. Appell, E. M. Semenov, *The packing constant of rearrangement-invariant spaces* (Russian), Funktsional. Anal. i Prilozhen. 32 (1998), no. 4, 69–72; English transl.: Funct. Anal. Appl. 32 (1998), 273–275.

[3] S. A. Argyros, R. G. Haydon, *A hereditarily indecomposable $\mathcal{L}_\infty$-space that solves the scalar-plus-compact problem*, Acta Math. 206 (2011), 1–54.

[4] J. Bourgain, F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. 145 (1980), 155–176.

[5] E. Casini, *About some parameters of normed linear spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 80 (1986), 11–15.

[6] J. M. F. Castillo, P. L. Papini, *Smallness and the covering of a Banach space*, Preprint, 2010.

[7] Y. Cui, H. Hudzik, *On the Banach-Saks and weak Banach-Saks properties of some Banach sequence spaces*, Acta Sci. Math. (Szeged) 65 (1999), 179–187.

[8] Y. Cui, H. Hudzik, *Packing constant for Cesaro sequence spaces*, in: Proceedings of the Third World Congress of Nonlinear Analysts, Part 4 (Catania, 2000), Nonlinear Anal. 47 (2001), 2695–2702.

[9] S. Delpech, *Asymptotic uniform moduli and Kottman constant of Orlicz sequence spaces*, Rev. Mat. Complut. 22 (2009), 455–467.

[10] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, New York, 1984.

[11] J. Dronka, L. Olszowy, *A note on the Kottman constant for Sobolev spaces*, Nonlinear Funct. Anal. Appl. 11 (2006), 37–46.

[12] J. Elton, E. Odell, *The unit ball of every infinite-dimensional normed linear space contains a $(1+\varepsilon)$-separated sequence*, Colloq. Math. 44 (1981), 105-109.

[13] S. Heinrich, *Ultraproducts in Banach space theory*, J. Reine Angew. Math. 313 (1980), 72–104.

[14] H. Hudzik, *Every nonreflexive Banach lattice has the packing constant equal to* 1/2, in: Third International Conference "Function Spaces" (Poznan, 1992), Collect. Math. 44 (1993), 129–134.

[15] H. Hudzik, T. Landes, *Packing constant in Orlicz spaces equipped with the Luxemburg norm*, Boll. Un. Mat. Ital. A (7) 9 (1995), 225–237.

[16] H. Hudzik, C. X. Wu, Y. N. Ye, *Packing constant in Musielak-Orlicz sequence spaces equipped with the Luxemburg norm*, Rev. Mat. Univ. Complut. Madrid 7 (1994), 13–26.

[17] W. Kaczor, A. Stachura, J. Walczuk, M. Zoła, *Measures of non-compactness in ultraproducts*, Bull. Aust. Math. Soc. 80 (2009), 165–172.

[18] N. J. Kalton, *Extending Lipschitz maps into $\mathcal{C}(K)$-spaces*, Israel J. Math. 162 (2007), 275–315.

[19] N. J. Kalton, *Extension of linear operators and Lipschitz maps into $\mathcal{C}(K)$-spaces*, New York J. Math. 13 (2007), 317–381.

[20] C. A. Kottman, *Packing and reflexivity in Banach spaces*, Trans. Amer. Math. Soc. 150 (1970), 565–576.

[21] C. A. Kottman, *Subsets of the unit ball that are separated by more than one*, Studia Math. 53 (1975), 15–27.

[22] A. Kryczka, S. Prus, *Separated sequences in nonreflexive Banach spaces*, Proc. Amer. Math. Soc. 129 (2001), 155–163.

[23] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Ergeb. Math. Grenzgeb. 92, Springer, Berlin, 1977.

[24] E. Maluta, P. L. Papini, *Relative centers and finite nets for the unit ball and its finite subsets*, Boll. Un. Mat. Ital. B (7) 7 (1993), 451–472.

[25] E. Maluta, P. L. Papini, *Estimates for Kottman's separation constant in reflexive Banach spaces*, Colloq. Math. 117 (2009), 105–119.

[26] S. V. R. Naidu, K. P. R. Sastry, *Convexity conditions in normed linear spaces*, J. Reine Angew. Math. 297 (1978), 35–53.

[27] P. L. Papini, *Some parameters of Banach spaces*, Rend. Sem. Mat. Fisico Milano 53 (1983), 131–148.

[28] S. Prus, *Constructing separated sequences in Banach spaces*, Proc. Amer. Math. Soc. 138 (2010), 225–234.

[29] B. Sims, *"Ultra"-techniques in Banach Space Theory*, Queen's Papers in Pure and Appl. Math. 60, Queen's Univ., Kingston, ON, 1982.

[30] R. Whitley, *The size of the unit sphere*, Canad. J. Math. 20 (1968), 450–455.

[31] Y. Q. Yan, *Some results on packing in Orlicz sequence spaces*, Studia Math. 147 (2001), 73–88.

[32] Y. Q. Yan, *Exact values of some geometric constants of Orlicz spaces*, Acta Math. Sin. (Engl. Ser.) 23 (2007), 827–846.

[33] H. S. Yin, *On a problem of Kottman* (Chinese), Chinese Ann. Math. 3 (1982), 617–623.

[34] X. Yu, *The packing spheres constant for a class of separable Orlicz function spaces*, J. Mathematics Research 1 (2009), 21–27.