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GELFAND AND KOLMOGOROV NUMBERS OF EMBEDDING OF RADIAL BESOV AND SOBOLEV SPACES

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Abstract. We prove asymptotic formulas for the behavior of Gelfand and Kolmogorov numbers of Sobolev embeddings between Besov and Triebel-Lizorkin spaces of radial distributions. Our method works also for Weyl numbers.

1. Introduction. Today we have good knowledge about asymptotic behavior of entropy and approximation numbers of compact embedding between spaces of Sobolev-Besov-Hardy type. The attention has been focused on spaces defined on bounded domains and weighted spaces. Spaces defined on bounded domains were thoroughly investigated by D. E. Edmunds and H. Triebel [ET1] and [ET2], cf. also [ET]. Embeddings of weighted function spaces where investigated by D. Haroske and H. Triebel [HT1] and D. Haroske [DH1], [DH2]. Further development can be found in the series of papers [KLSS2], [KLSS3], [KLSS4].

Compactness of Sobolev embeddings of spaces of radial functions was noticed by W. Strauss [WS] and S. Coleman, V. Glazer, A. Martin [CGM]. Asymptotic behavior of entropy numbers and approximation numbers of the embeddings were studied in the paper by Th. Kühn, H. G. Leopold, W. Sickel and L. Skrzypczak [KLSS1] and by L. Skrzypczak, B. Tomasz [ST]. However nobody has studied Gelfand and Kolmogorov numbers of the embeddings of spaces of radial functions, so in this paper we investigate asymptotic behavior of the corresponding numbers. In a similar way we can estimate the asymptotic behavior of Weyl numbers of the embeddings.

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2. Function spaces of radial distributions. We assume that the reader is acquainted with the definition and basic properties of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, $1 \le p, q \le \infty$, $s \in \mathbb{R}$, and Triebel-Lizorkin spaces $F^s_{p,q}(\mathbb{R}^n)$, $1 \le p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$. We will denote the continuous embeddings between Banach spaces by $\hookrightarrow$.

Let $p_1 \le p_2$ and $s_1 \ge s_2$. It is well known that if $\delta = s_1 - s_2 - n(\frac{1}{p_1} - \frac{1}{p_2}) \ge 0$ and $q_1, q_2 \ge 1$ ($q_1 \le q_2$ if $\delta = 0$) then

$$B^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R}^n).$$

The above embeddings are never compact. However if we take subspaces of radial functions or distributions then some of the above embeddings are compact, cf. [SS].

Definition 2.1. A tempered distribution f is called *radial* if $f^g = f$ for $g \in O(\mathbb{R}^n)$ where $f^g(\varphi) = f(\varphi^{g^{-1}})$ and $\varphi^{g^{-1}}(x) = \varphi(g^{-1}x)$, $\varphi \in S(\mathbb{R}^n)$. For $s \in \mathbb{R}$, $1 \le p, q \le \infty$, we define

$$RB^s_{p,q}(\mathbb{R}^n) = \{f \in B^s_{p,q}(\mathbb{R}^n) : f \text{ is radial}\},$$
$$RF^s_{p,q}(\mathbb{R}^n) = \{f \in F^s_{p,q}(\mathbb{R}^n) : f \text{ is radial}\}.$$

Remark. The spaces $RB^s_{p,q}(\mathbb{R}^n)$ and $RF^s_{p,q}(\mathbb{R}^n)$ are complemented subspaces of $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$ respectively, cf. [LS]. The restriction to the subspaces of radial distributions allows us to overcome the problem of compactness. We get the following theorem, cf. [SS].

Theorem 2.2. *The embedding* $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *is compact if and only if* $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$ *and* $p_1 < p_2$.

The behavior of radial functions from Besov spaces at infinity is strictly related to the behavior of functions belonging to some weighted Besov spaces defined on the real line. Before we formulate the known result we need some definitions. For $\alpha \in \mathbb{R}$ we introduce a weight function $\omega_\alpha : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$\omega_\alpha(x) = (1 + |x|^2)^{\alpha/2}. \tag{1}$$

Definition 2.3. Let $0 < u < \infty$, $1 \le p_1 \le p_2 \le \infty$ and $s, \alpha \in \mathbb{R}$. Then

$$B^s_{p,q}([u,\infty), \omega_\alpha) = \big\{f \in B^s_{p,q}(\mathbb{R}, \omega_\alpha) : \operatorname{supp} f \subset \{x \in \mathbb{R} : x \ge u\}\big\},$$

where

$$B^s_{p,q}(\mathbb{R}^n, \omega_\alpha) = \{f \in S'(\mathbb{R}^n) : \omega_\alpha f \in B^s_{p,q}(\mathbb{R}^n)\},$$

with

$$\|f|B^s_{p,q}(\mathbb{R}^n, \omega_\alpha)\| = \|\omega_\alpha f|B^s_{p,q}(\mathbb{R}^n)\|.$$

Moreover we define

$$RB^s_{p,q}(\mathbb{R}^n \setminus B(0,u)) = \big\{f \in RB^s_{p,q}(\mathbb{R}^n) : \operatorname{supp} f \subset \{x \in \mathbb{R}^n : |x| \ge u\}\big\}.$$

The above function spaces are related to each other by the trace operator tr^* defined by

$$\mathrm{tr}^* : f(x_1, \ldots, x_n) \mapsto f(u, 0, \ldots, 0), \qquad \text{where } u = |x_1|.$$

THEOREM 2.4. *Suppose $n \geq 2$, $0 < u < \infty$, $1 \leq p, q \leq \infty$ and $s > 0$. Then:*

1. *The operator* tr^* *maps* $RB^s_{p,q}(\mathbb{R}^n \setminus B(0,u))$ *continuously onto* $B^s_{p,q}([u,\infty), \omega_\alpha)$.
2. *There is a linear and continuous extension operator* ext *which maps* $B^s_{p,q}([u,\infty), \omega_\alpha)$ *into* $RB^s_{p,q}(\mathbb{R}^n \setminus B(0,u))$ *and such that* $\mathrm{tr}^* \circ \mathrm{ext} = \mathrm{id}$.
3. *The operator* tr^* *is an isomorphism.*

The above theorem is proved in [KLSS1].

The next interesting dependence is a relationship between the radial Besov space and some weighted sequence space. We will work with the following sequence spaces.

DEFINITION 2.5. Let $1 \leq p, q \leq \infty$ and $\delta \geq 0$. In the sequel, ω_α stands for (1). We define

$$l_q(2^{j\delta} l_p(\omega_\alpha)) = \Big\{ (s_{j,k})_{j,k} : \|s_{j,k}|l_q(2^{j\delta} l_p(\omega_\alpha))\| = \Big(\sum_{j=0}^{\infty} \Big(\sum_{k=0}^{\infty} 2^{j\delta} \omega_\alpha(k) |s_{j,k}|^p \Big)^{q/p} \Big)^{1/q} < \infty \Big\}$$

(with the usual modification if $p = \infty$ or $q = \infty$). If $\delta = 0$ we will write $l_q(l_p(\omega_\alpha))$.

Now we have the following proposition.

PROPOSITION 2.6 ([KLSS1]). *Let $p_1 < p_2$, $1 \leq q_1, q_2 \leq \infty$ and $s_1 - \frac{n}{p_1} > s_2 - \frac{n}{p_2}$. We put $\delta = s_1 - s_2 + n(\frac{1}{p_2} - \frac{1}{p_1})$. There exist bounded operators S, T such that the following diagram is commutative*

$$\begin{array}{ccc} RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) & \xrightarrow{\;S\;} & l_{q_1}(2^{j\delta} l_{p_1}(\omega_{n-1})) \\ \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{Id}} \\ RB^{s_2}_{p_2,q_2}(\mathbb{R}^n) & \xleftarrow{\;T\;} & l_{q_2}(l_{p_2}(\omega_{n-1})), \end{array}$$

i.e. $T \circ S = \mathrm{id}$.

The details can be found in [KLSS1].

3. Gelfand and Kolmogorov numbers. In this section we prove asymptotic estimates of Gelfand and Kolmogorov numbers. We need two well known results concerning this numbers of embeddings of sequence spaces.

First we consider diagonal operators. We put

DEFINITION 3.1. A *diagonal operator* $D_\sigma : l_{p_1} \mapsto l_{p_2}$ for a nonincreasing sequence $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$ is defined by

$$(\varsigma_k)_k \longmapsto (\sigma_k \varsigma_k)_k.$$

We will need compact diagonal operators defined by sequences of the form $\sigma_k = k^{-\alpha}$, $\alpha > 0$. We will write D_α if $\sigma_k = k^{-\alpha}$. It is very easy to show the following lemma.

LEMMA 3.2. *Let $D_\alpha : l_{p_1} \mapsto l_{p_2}$.*

(i) *Let $1 \leq p_1 \leq p_2 \leq \infty$. D_α is compact if and only if $\alpha > 0$.*
(ii) *Let $1 \leq p_2 < p_1 \leq \infty$ and $\frac{1}{t} = \frac{1}{p_2} - \frac{1}{p_1}$. D_α is compact if and only if $\alpha > \frac{1}{t}$.*

Now we will formulate the key definitions.

DEFINITION 3.3. Let X and Y be Banach spaces and $T \in L(X,Y)$.

(i) For $k \in \mathbb{N}$, we define the *k-th approximation number* by

$$a_k(T) := \inf\{\|T - A\| : A \in L(X,Y),\ \operatorname{rank}(A) < k\},$$

where $\operatorname{rank}(A)$ denotes the dimension of the range $A(X) = \{A(x), x \in X\}$.

(ii) For $k \in \mathbb{N}$, we define the *k-th Gelfand number* by

$$c_k(T) := \inf\{\|TJ_M^X\| : M \subset X,\ \operatorname{codim}(M) < k\},$$

where J_M^X is the natural injection of M into X and M is a closed subspace of the Banach space X.

(iii) For $k \in \mathbb{N}$, we define the *k-th Kolmogorov number* by

$$d_k(T) := \inf\{\|Q_N^Y T\| : N \subset Y,\ \dim(N) < k\},$$

where Q_N^Y is the natural surjection of Y onto the quotient space

$$Y/N = \{y + N : \|y + N\| = \inf_{z \in N} \|y + z\|\}$$

and N is a closed subspace of the Banach space Y.

The approximation numbers, the Gelfand numbers and the Kolmogorov numbers are examples of s-numbers, that were introduced by A. Pietsch, cf. [AP]. In the next proposition we recall the well known properties of the above s-numbers.

PROPOSITION 3.4. *Let $s_k \in \{a_k, c_k, d_k\}$. Then*

1. $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq \ldots$
2. $s_{n+k-1}(T_1 + T_2) \leq s_k(T_1) + s_n(T_2)$ (additivity),
3. $s_{n+k-1}(T_1T_2) \leq s_k(T_1)s_n(T_2)$ (multiplicativity),
4. $s_k(T) = 0$ *when* $\operatorname{rank}(T) < k$ (rank property),
5. $c_k(T) \leq a_k(T)$, $d_k(T) \leq a_k(T)$,
6. *if* $a_k(T) \to 0$ *then T is compact,*
   $c_k(T) \to 0$ *if and only if T is compact,*
   $d_k(T) \to 0$ *if and only if T is compact.*

For a real number $r > 0$ and any operator $T \in L(X,Y)$ we put

$$L_{r,\infty}^{(s)}(T) := \sup_{k \in \mathbb{N}} k^{1/r} s_k(T).$$

The expression $L_{r,\infty}^{(s)}(T)$ is an example of an operator ideal quasi-norm. This means in particular that there exists a number $0 < \sigma \leq 1$ such that

$$L_{r,\infty}^{(s)}\Big(\sum_j T_j\Big)^{\sigma} \leq \sum_j L_{r,\infty}^{(s)}(T_j)^{\sigma},$$

for any sequence of bounded linear operators $T_j : X \to Y$ [HK, 1.c.5].

Directly from the definition we see that the approximation numbers of the diagonal operator $D_\sigma : l_p \mapsto l_p$ defined by the nonincreasing sequence $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \ldots \geq 0$ are given by

$$a_k(D_\sigma : l_p \to l_p) = \sigma_k. \tag{2}$$

This fact can be found in [CS, p. 46] or in [AP, p. 108]. By the boundedness of D_σ and the monotonicity of the scale l_p-spaces we get

$$c_k(D_\sigma : l_{p_1} \to l_{p_2}) \le a_k(D_\sigma : l_{p_1} \to l_{p_2}) \le \sigma_k, \tag{3}$$

if $p_1 \le p_2$, cf. Proposition 3.4. The above and the following facts can be found in [AP]. If X and Y are Banach spaces, then

$$c_k(T^*) = d_k(T) \tag{4}$$

for all compact operators $T \in L(X,Y)$ and

$$c_k(T) = d_k(T^*) \tag{5}$$

for all $T \in L(X,Y)$.

It is well known that

$$D^*_\sigma : l^*_{p_2} \to l^*_{p_1} = D_\sigma : l_{q_2} \to l_{q_1}, \tag{6}$$

where $q_2 = \frac{p_2}{p_2-1}$, $q_1 = \frac{p_1}{p_1-1}$ and $1 \le p_1, p_2 < \infty$ (so $1 < q_1, q_2 \le \infty$). Hence, from (4) and (6), for $1 \le p_1, p_2 < \infty$ we get

$$d_k(D_\sigma : l_{p_1} \to l_{p_2}) = c_k(D_\sigma : l_{q_2} \to l_{q_1}), \tag{7}$$

if $D_\sigma : l_{p_1} \to l_{p_2}$ is a compact operator. In addition from (5) and (6), for $1 \le p_1, p_2 < \infty$ (so $1 < q_1, q_2 \le \infty$) we obtain

$$c_k(D_\sigma : l_{p_1} \to l_{p_2}) = d_k(D_\sigma : l_{q_2} \to l_{q_1}), \tag{8}$$

if $D_\sigma : l_{p_1} \to l_{p_2}$ is continuous.

Now we consider the asymptotic behavior of Gelfand numbers of embedding of finite-dimensional spaces. In [EDG] or [JV] we can find the following lemma.

LEMMA 3.5. *For* $1 \le k \le N < \infty$ *and* $1 \le p_1, p_2 \le \infty$ *define*

$$\Phi(N,k,p_1,p_2) := \begin{cases} (N-k+1)^{\frac{1}{p_2}-\frac{1}{p_1}}, & \text{if } 1 \le p_2 \le p_1 \le \infty, \\ \left(\min\{1, N^{1-\frac{1}{p_1}} k^{-\frac{1}{2}}\}\right)^{(\frac{1}{p_1}-\frac{1}{p_2})/(\frac{1}{p_1}-\frac{1}{2})}, & \text{if } 1 < p_1 < p_2 \le 2, \\ \max\Big\{N^{\frac{1}{p_2}-\frac{1}{p_1}}, \sqrt{1-\frac{k}{N}}^{(\frac{1}{p_1}-\frac{1}{p_2})/(\frac{1}{2}-\frac{1}{p_2})}\Big\}, & \text{if } 2 \le p_1 < p_2 \le \infty, \\ \max\Big\{N^{\frac{1}{p_2}-\frac{1}{p_1}}, \min\{1, N^{1-\frac{1}{p_1}} k^{-\frac{1}{2}}\}\sqrt{1-\frac{k}{N}}\Big\}, & \\ & \text{if } 1 < p_1 \le 2 < p_2 \le \infty. \end{cases}$$

Then

$$c_k(\mathrm{id} : l^N_{p_1} \to l^N_{p_2}) \approx \Phi(N,k,p_1,p_2), \qquad 1 \le k \le N < \infty,$$

if $p_1 > 1$.

In [EDG] we can also find an analogous lemma for Kolmogorov numbers. One can switch from Kolmogorov numbers to Gelfand numbers and vice versa using (4) and (5).

We can use Lemma 3.5 to prove the behavior of Gelfand numbers of diagonal operators acting between infinite-dimensional spaces.

Lemma 3.6. *Let $1 \le p_1, p_2 \le \infty$ and D_α be a diagonal compact operator generated by $\sigma_k = k^{-\alpha}$. Let $\frac{1}{t} = \frac{1}{p_2} - \frac{1}{p_1}$, $\theta = \left(\frac{1}{p_1} - \frac{1}{p_2}\right)/\left(\frac{1}{p_1} - \frac{1}{2}\right)$. There exist positive constants c and C such that for all $k \in \mathbb{N}$*

$$ck^{-\beta} \le c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le Ck^{-\beta},$$

where

$$\beta = \begin{cases} \alpha, & \text{if } 2 \le p_1 < p_2 \le \infty, \\ -\frac{1}{2} + \frac{1}{p_1} + \alpha, & \text{if } 1 < p_1 \le 2 < p_2 \le \infty \text{ and } \alpha > \frac{1}{p_1'}, \\ \frac{\alpha}{2}\frac{p_1}{p_1 - 1}, & \text{if } 1 < p_1 \le 2 < p_2 \le \infty \text{ and } \alpha < \frac{1}{p_1'} \\ & \text{or } 1 < p_1 < p_2 \le 2 \text{ and } \alpha < \frac{\theta}{p_1'}, \\ \alpha - \frac{1}{t}, & \text{if } 1 < p_1 < p_2 \le 2 \text{ and } \alpha > \frac{\theta}{p_1'} \\ & \text{or } 1 \le p_2 \le p_1 \le \infty,\ p_1 > 1 \text{ and } \alpha > \frac{1}{t}. \end{cases}$$

Remark. Similar estimates without proof are given in [RL]. Since we have no good reference for the proof we present our proof below.

Proof.

1) *Estimation from above.* The upper estimates in the case $2 \le p_1 < p_2 \le \infty$ follow from general estimates (3) and (2). Now we regard the case $1 \le p_2 \le p_1 \le \infty$, $p_1 > 1$ and $\frac{1}{t} < \alpha$. Let

$$\Lambda := \{\lambda = (\lambda_l)_l : \lambda_l \in \mathbb{C},\ 1 \le l < \infty\}$$

and let $P_i : \Lambda \to \Lambda$ be a projection defined in the following way

$$(P_i\lambda)_l := \begin{cases} \lambda_l, & \text{if } 2^{i-1} \le l < 2^i, \\ 0, & \text{otherwise}, \end{cases} \qquad l \in \mathbb{N},$$

for $\lambda = (\lambda_l)$, $i = 1, 2, \ldots$. Then

$$D_\alpha = \sum_i D_\alpha \circ P_i.$$

Elementary properties of the Gelfand numbers yield that for any $i \in \mathbb{N}$

$$c_k(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \le 2^{-(i-1)\alpha} c_k(\mathrm{id} : l_{p_1}^{2^{i-1}} \to l_{p_2}^{2^{i-1}}). \tag{9}$$

From (9) and Lemma 3.5 we get

$$L_{s,\infty}^{(c)}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \le C 2^{(i-1)(1/s + 1/t - \alpha)}. \tag{10}$$

Taking $M \in \mathbb{N}_0$ we put

$$P^M := \sum_{i=1}^{M+1} D_\alpha \circ P_i \qquad \text{and} \qquad Q_M := \sum_{i=M+2}^{\infty} D_\alpha \circ P_i.$$

We choose $s > 0$ such that $\frac{1}{s} + \frac{1}{t} - \alpha > 0$. Then, from (10)

$$L_{s,\infty}^{(c)}(P^M)^\sigma \le \sum_{i=1}^{M+1} L_{s,\infty}^{(c)}(D_\alpha \circ P_i)^\sigma \le C_1 \sum_{i=0}^{M} 2^{\sigma i(1/s + 1/t - \alpha)} \le C_2 2^{\sigma M(1/s + 1/t - \alpha)}. \tag{11}$$

So from (11) we get for all $k \in \mathbb{N}$

$$c_k(P^M : l_{p_1} \to l_{p_2}) \le ck^{-1/s} 2^{M(1/s + 1/t - \alpha)}.$$

Let now $k = 2^M$. Then

$$c_{2^M}(P^M : l_{p_1} \to l_{p_2}) \le c2^{M(1/t-\alpha)}. \tag{12}$$

Since $\frac{1}{t} < \alpha$ we can choose $s > 0$ such that $\frac{1}{s} + \frac{1}{t} - \alpha < 0$. Hence, from (10)

$$L^{(c)}_{s,\infty}(Q_M)^\sigma \le C_1 \sum_{i=M+1}^{\infty} 2^{\sigma i(1/s+1/t-\alpha)} \le C_2 2^{\sigma M(1/s+1/t-\alpha)}. \tag{13}$$

Thus taking $k = 2^M$ we get from (13) that

$$c_{2^M}(Q_M : l_{p_1} \to l_{p_2}) \le c2^{M(1/t-\alpha)}. \tag{14}$$

From (12) and (14) we get

$$c_{2^M}(D_\alpha : l_{p_1} \to l_{p_2}) \le c2^{M(1/t-\alpha)},$$

therefore by the properties of Gelfand numbers

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le ck^{1/t-\alpha}, \tag{15}$$

for all $k \in \mathbb{N}$.

Let now $1 < p_1 \le 2 < p_2 \le \infty$. Then by Lemma 3.5

$$c_k(\mathrm{id} : l^N_{p_1} \to l^N_{p_2}) \approx \begin{cases} \sqrt{1 - \frac{k}{N}}, & \text{if } 1 \le k \le N^{2-2/p_1}, \\ k^{-\frac{1}{2}} N^{1-\frac{1}{p_1}} \sqrt{1 - \frac{k}{N}}, & \text{if } N^{2-2/p_1} \le k \le \frac{N}{N^{2/p_2-1}+1}, \\ N^{\frac{1}{p_2}-\frac{1}{p_1}}, & \text{if } \frac{N}{N^{2/p_2-1}+1} \le k \le N. \end{cases} \tag{16}$$

As in the previous case in view of (9) and (16) we conclude that

$$L^{(c)}_{s,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \le C2^{(i-1)(1/s+1/2-1/p_1-\alpha)}, \quad \text{if } s < 2, \tag{17}$$

$$L^{(c)}_{s,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \le C2^{(i-1)((2-2/p_1)/s-\alpha)}, \quad \text{if } s \ge 2. \tag{18}$$

Let $\alpha > \frac{1}{p_1'}$. Take $s < 2$ such that $\frac{1}{s} + \frac{1}{2} - \frac{1}{p_1} - \alpha > 0$. Then formula (17) yields

$$L^{(c)}_{s,\infty}(P^M)^\sigma \le C_1 \sum_{i=0}^{M} 2^{\sigma i(1/s+1/2-1/p_1-\alpha)} \le C_2 2^{\sigma M(1/s+1/2-1/p_1-\alpha)}. \tag{19}$$

Taking $k = 2^M$ we get from (19) that

$$c_{2^M}(P^M : l_{p_1} \to l_{p_2}) \le c2^{M(1/2-1/p_1-\alpha)}. \tag{20}$$

We choose $s < 2$ such that $\frac{1}{s} + \frac{1}{2} - \frac{1}{p_1} - \alpha < 0$. Then from (17)

$$L^{(c)}_{s,\infty}(Q_M)^\sigma \le C_1 \sum_{i=M+1}^{\infty} 2^{\sigma i(1/s+1/2-1/p_1-\alpha)} \le C_2 2^{\sigma M(1/s+1/2-1/p_1-\alpha)}. \tag{21}$$

Hence taking $k = 2^M$ we obtain from (21) that

$$c_{2^M}(Q_M : l_{p_1} \to l_{p_2}) \le c2^{M(1/2-1/p_1-\alpha)}. \tag{22}$$

In view of (20) and (22) we get

$$c_{2^M}(D_\alpha : l_{p_1} \to l_{p_2}) \le c2^{M(1/2-1/p_1-\alpha)},$$

hence

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le ck^{1/2-1/p_1-\alpha}, \tag{23}$$

for all $k \in \mathbb{N}$.

Let $\alpha < \frac{1}{p_1'}$. We take $s \ge 2$ such that $(2 - \frac{2}{p_1})\frac{1}{s} - \alpha > 0$. Therefore, from (18)

$$L^{(c)}_{s,\infty}(P^M)^\sigma \le C_1 \sum_{i=0}^{M} 2^{\sigma i((2-2/p_1)/s-\alpha)} \le C_2 2^{\sigma M((2-2/p_1)/s-\alpha)}. \tag{24}$$

Then for all $k \in \mathbb{N}$ we get from (24) that

$$c_k(P^M : l_{p_1} \to l_{p_2}) \le c k^{-1/s} 2^{M((2-2/p_1)/s-\alpha)}.$$

Taking $k = [2^{M(2-2/p_1)}]$ we get

$$c_k(P^M : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}. \tag{25}$$

Take $s \ge 2$ such that $(2 - \frac{2}{p_1})\frac{1}{s} - \alpha < 0$. From (18) we have

$$\begin{aligned} L^{(c)}_{s,\infty}(Q_M)^\sigma &\le \sum_{i=M+2}^{\infty} L^{(c)}_{s,\infty}(D_\alpha \circ P_i)^\sigma \\ &\le C_1 \sum_{i=M+1}^{\infty} 2^{\sigma i((2-2/p_1)/s-\alpha)} \le C_2 2^{\sigma M((2-2/p_1)/s-\alpha)}. \end{aligned} \tag{26}$$

Thus, for all $k \in \mathbb{N}$ we obtain from (26) that

$$c_k(Q_M : l_{p_1} \to l_{p_2}) \le c k^{-1/s} 2^{M((2-2/p_1)/s-\alpha)}.$$

Taking $k = [2^{M(2-2/p_1)}]$ we get

$$c_k(Q_M : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}. \tag{27}$$

In view of (25) and (27)

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))},$$

if $k = [2^{M(2-2/p_1)}]$, hence

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}, \tag{28}$$

for all $k \in \mathbb{N}$.

Let now $1 < p_1 < p_2 \le 2$. Then by Lemma 3.5

$$c_k(\mathrm{id} : l^N_{p_1} \to l^N_{p_2}) \approx \begin{cases} 1, & \text{if } 1 \le k \le N^{2-2/p_1}, \\ \left(N^{1-1/p_1} k^{-1/2}\right)^\theta, & \text{if } N^{2-2/p_1} \le k \le N. \end{cases} \tag{29}$$

As in the previous case in view of (9) and (29) we conclude that

$$\begin{aligned} L^{(c)}_{s,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) &\le C 2^{(i-1)(1/s+(1/2-1/p_1)\theta-\alpha)} \\ &= C 2^{(i-1)(1/s+1/p_2-1/p_1-\alpha)}, \quad \text{if } s < 2/\theta, \end{aligned} \tag{30}$$

$$L^{(c)}_{s,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \le C 2^{(i-1)((2-2/p_1)/s-\alpha)}, \quad \text{if } s \ge 2/\theta. \tag{31}$$

Let $\alpha > \frac{\theta}{p_1'}$. Take $s > 0$ such that $\frac{1}{s} > \max(\frac{\theta}{2}, \frac{1}{p_1} - \frac{1}{p_2} + \alpha)$. Then formula (30) yields

$$L^{(c)}_{s,\infty}(P^M)^\sigma \le C_1 \sum_{i=0}^{M} 2^{\sigma i(1/s+1/p_2-1/p_1-\alpha)} \le C_2 2^{\sigma M(1/s+1/p_2-1/p_1-\alpha)}. \tag{32}$$

Taking $k = 2^M$ we get from (32) that

$$c_{2^M}(P^M : l_{p_1} \to l_{p_2}) \le c 2^{M(1/p_2-1/p_1-\alpha)}. \tag{33}$$

Since $\alpha > \frac{\theta}{p_1'}$ we can choose $s > 0$ such that $\frac{\theta}{2} < \frac{1}{s} < \frac{1}{p_1} - \frac{1}{p_2} + \alpha$. Then from (30)

$$L_{s,\infty}^{(c)}(Q_M)^\sigma \le C_1 \sum_{i=M+1}^{\infty} 2^{\sigma i(1/s+1/p_2-1/p_1-\alpha)} \le C_2 2^{\sigma M(1/s+1/p_2-1/p_1-\alpha)}. \tag{34}$$

Hence taking $k = 2^M$ we obtain from (34) that

$$c_{2^M}(Q_M : l_{p_1} \to l_{p_2}) \le c 2^{M(1/p_2-1/p_1-\alpha)}. \tag{35}$$

In view of (33) and (35) we get

$$c_{2^M}(D_\alpha : l_{p_1} \to l_{p_2}) \le c 2^{M(1/p_2-1/p_1-\alpha)},$$

hence

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le c k^{1/p_2-1/p_1-\alpha}, \tag{36}$$

for all $k \in \mathbb{N}$.

Let $\alpha < \frac{\theta}{p_1'}$. Then we can choose $s > 0$ such that $\frac{\alpha}{2} p_1' < \frac{1}{s} \le \frac{\theta}{2}$. Therefore, from (31)

$$L_{s,\infty}^{(c)}(P^M)^\sigma \le C_1 \sum_{i=0}^{M} 2^{\sigma i((2-2/p_1)/s-\alpha)} \le C_2 2^{\sigma M((2-2/p_1)/s-\alpha)}. \tag{37}$$

Then for all $k \in \mathbb{N}$ we get from (37) that

$$c_k(P^M : l_{p_1} \to l_{p_2}) \le c k^{-1/s} 2^{M((2-2/p_1)/s-\alpha)}.$$

Taking $k = [2^{M(2-2/p_1)}]$ we get

$$c_k(P^M : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}. \tag{38}$$

Take $s \ge 2/\theta$ such that $\frac{1}{s}(2 - \frac{2}{p_1}) - \alpha < 0$. From (31) we have

$$\begin{aligned} L_{s,\infty}^{(c)}(Q_M)^\sigma &\le \sum_{i=M+2}^{\infty} L_{s,\infty}^{(c)}(D_\alpha \circ P_i)^\sigma \\ &\le C_1 \sum_{i=M+1}^{\infty} 2^{\sigma i((2-2/p_1)/s-\alpha)} \le C_2 2^{\sigma M((2-2/p_1)/s-\alpha)}. \end{aligned} \tag{39}$$

Thus, for all $k \in \mathbb{N}$ we obtain from (39) that

$$c_k(Q_M : l_{p_1} \to l_{p_2}) \le c k^{-1/s} 2^{M((2-2/p_1)/s-\alpha)}.$$

Taking $k = [2^{M(2-2/p_1)}]$ we get

$$c_k(Q_M : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}. \tag{40}$$

In view of (38) and (40)

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))},$$

if $k = [2^{M(2-2/p_1)}]$, hence

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \le c k^{-(\alpha/2)(p_1/(p_1-1))}, \tag{41}$$

for all $k \in \mathbb{N}$.

2) *Estimation from below.* First we note that

$$c_k(D_\alpha : l_{p_1} \to l_{p_2}) \approx c_k(\mathrm{Id} : l_{p_1}(\omega) \to l_{p_2}),$$

where

$$l_{p_1}(\omega)=\Big\{(\lambda_l)_l : \|\lambda|l_{p_1}(\omega)\|=\Big(\sum_{l=1}^{\infty}|\lambda_l l^{\alpha}|^{p_1}\Big)^{1/p_1}<\infty\Big\}.$$

Thus we can regard the following commutative diagram

$$\begin{array}{ccc} l_{p_1}^N & \xrightarrow{\;S\;} & l_{p_1}(\omega) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}} \\ l_{p_2}^N & \xleftarrow{\;T\;} & l_{p_2}. \end{array}$$

We have

$$c_k(\mathrm{id})\le \|S\|\,\|T\|c_k(\mathrm{Id})\quad \text{for } k=1,2,3,\ldots, \tag{42}$$

and the norm of S depends on N and α. Let $v=(v_1,\ldots,v_N)$, $\lambda=(\lambda_i)$ and

$$(S(v))_i=\begin{cases} v_i, & \text{if } N\le i<2N,\\ 0, & \text{otherwise,}\end{cases}$$

then

$$(T(\lambda))_i=\lambda_{i+N-1},\quad \text{if } 1\le i\le N.$$

Hence $\|T\|=1$ and $\|S\|\le CN^{\alpha}$. In consequence we get by (42) that

$$c_k(\mathrm{id})\le CN^{\alpha}c_k(\mathrm{Id})\quad \text{for } k=1,2,3,\ldots. \tag{43}$$

Let $1<p_1<p_2\le 2$. By (43) and Lemma 3.5 we get

$$\big(\min\{1,N^{1-1/p_1}k^{-1/2}\}\big)^{(\frac{1}{p_1}-\frac{1}{p_2})/(\frac{1}{p_1}-\frac{1}{2})}\le CN^{\alpha}c_k(\mathrm{Id})\quad \text{for } k=1,2,3,\ldots. \tag{44}$$

If $\alpha>\frac{\theta}{p_1'}$ and $N=k$, $k\in\mathbb{N}$, then (44) yields

$$Ck^{-\alpha+1/p_2-1/p_1}\le Ck^{-\alpha}\big(\min\{1,k^{1-1/p_1}k^{-1/2}\}\big)^{\theta}\le c_k(\mathrm{Id}).$$

Let $\alpha<\frac{\theta}{p_1'}$ and $N=[k^{p_1/(2(p_1-1))}]$, $k\in\mathbb{N}$. From (44) we get

$$Ck^{-(\alpha/2)(p_1/(p_1-1))}\le Ck^{-(\alpha/2)(p_1/(p_1-1))}\big(\min\{1,k^{p_1(1-1/p_1)/(2(p_1-1))}k^{-1/2}\}\big)^{\theta}\le c_k(\mathrm{Id}).$$

Let us regard the case $2\le p_1<p_2\le\infty$. Then in view of (43) and Lemma 3.5, taking $N=2k$, $k\in\mathbb{N}$, we obtain

$$Ck^{-\alpha}\le c(2k)^{-\alpha}\sqrt{\tfrac{1}{2}}^{(\frac{1}{p_1}-\frac{1}{p_2})/(\frac{1}{2}-\frac{1}{p_2})}\le c_k(\mathrm{Id}).$$

Consider the case $1\le p_2\le p_1\le\infty$, $p_1>1$ and $\frac{1}{t}<\alpha$. Let $N=2k$, $k\in\mathbb{N}$. From Lemma 3.5 and (43) we get

$$Ck^{1/t-\alpha}\le c(2k)^{-\alpha}(k+1)^{1/t}\le c_k(\mathrm{Id}).$$

Take the last case $1<p_1\le 2<p_2\le\infty$. In view of Lemma 3.5 and (43) we get

$$\min\{1,N^{1-1/p_1}k^{-1/2}\}\sqrt{1-\tfrac{k}{N}}\le CN^{\alpha}c_k(\mathrm{Id}). \tag{45}$$

If $\alpha>\frac{1}{p_1'}$ and $N=2k$, $k\in\mathbb{N}$, then (45) yields

$$Ck^{-\alpha+1/2-1/p_1}\le c(2k)^{-\alpha}\min\{1,(2k)^{1-1/p_1}k^{-1/2}\}\sqrt{\tfrac{1}{2}}\le c_k(\mathrm{Id}).$$

Let $\alpha < \frac{1}{p_1'}$ and $N = [(2^k)^{p_1/(2(p_1-1))}]$, $k \in \mathbb{N}$, $k \geq 2$. From (45) we get

$$c(2^k)^{-(\alpha/2)(p_1/(p_1-1))} \min\{1, (2^k)^{p_1(1-1/p_1)/(2(p_1-1))}(2^{k-2})^{-1/2}\} \times \sqrt{1-2^{k(1-p_1/(2(p_1-1)))-2}} \leq c_{2^{k-2}}(\mathrm{Id}).$$

Therefore

$$C(2^k)^{-(\alpha/2)(p_1/(p_1-1))} \leq c(2^k)^{-(\alpha/2)(p_1/(p_1-1))} \min\{1, 2\} \leq c_{2^{k-2}}(\mathrm{Id}). \ \blacksquare$$

REMARK. If $\alpha = \frac{1}{p_1'}$ or $\alpha = \frac{\theta}{p_1'}$, then our method does not work since we cannot use estimates (17), (18) and (30), (31). Moreover in both cases

$$L^{(c)}_{2,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \leq C2^{(i-1)(1-1/p_1-\alpha)},$$

and

$$L^{(c)}_{2/\theta,\infty}(D_\alpha \circ P_i : l_{p_1} \to l_{p_2}) \leq C2^{(i-1)(\theta(1-1/p_1)-\alpha)}.$$

From the above lemma and equations (7), (8) we get

LEMMA 3.7. *Let $1 \leq p_1, p_2 \leq \infty$ and D_α be a diagonal compact operator generated by $\sigma_k = k^{-\alpha}$. Let $\frac{1}{t} = \frac{1}{p_2} - \frac{1}{p_1}$, $\theta' = \left(\frac{1}{p_1} - \frac{1}{p_2}\right)/\left(\frac{1}{2} - \frac{1}{p_2}\right)$. There exist positive constants c and C such that for all $k \in \mathbb{N}$*

$$ck^{-\beta} \leq d_k(D_\alpha : l_{p_1} \to l_{p_2}) \leq Ck^{-\beta},$$

where

$$\beta = \begin{cases} \alpha, & \text{if } 1 \leq p_1 < p_2 \leq 2, \\ \frac{1}{2} - \frac{1}{p_2} + \alpha, & \text{if } 1 \leq p_1 < 2 < p_2 < \infty \text{ and } \alpha > \frac{1}{p_2}, \\ \alpha\frac{p_2}{2}, & \text{if } 1 \leq p_1 < 2 < p_2 < \infty \text{ and } \alpha < \frac{1}{p_2} \\ & \text{or } 2 \leq p_1 < p_2 < \infty \text{ and } \alpha < \frac{\theta'}{p_2}, \\ \alpha - \frac{1}{t}, & \text{if } 2 \leq p_1 < p_2 < \infty \text{ and } \alpha > \frac{\theta'}{p_2} \\ & \text{or } 1 \leq p_2 \leq p_1 \leq \infty,\ p_2 < \infty \text{ and } \alpha > \frac{1}{t}. \end{cases}$$

4. Gelfand and Kolmogorov numbers of embedding of spaces of radial functions.

First we regard the following lemma.

LEMMA 4.1. *Let $1 \leq q_1, q_2 \leq \infty$. Let $\alpha > 0$ if $1 < p_1 < p_2 \leq \infty$. Let $\delta > 0$ and $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$, $\theta = \left(\frac{1}{p_1} - \frac{1}{p_2}\right) / \left(\frac{1}{p_1} - \frac{1}{2}\right)$. There is a positive constant C such that for all $k \in \mathbb{N}$*

$$c_k\big(\mathrm{Id} : l_{q_1}(2^{j\delta} l_{p_1}(\omega_\alpha)) \to l_{q_2}(l_{p_2}(\omega_\alpha))\big) \leq Ck^{-\beta},$$

where

$$\beta = \begin{cases} \frac{\alpha}{p}, & \text{if } 2 \leq p_1 < p_2 \leq \infty, \\ \frac{\alpha}{p} - \frac{1}{2} + \frac{1}{p_1}, & \text{if } 1 < p_1 \leq 2 < p_2 \leq \infty \text{ and } \frac{\alpha}{p} > \frac{1}{p_1'}, \\ \frac{\alpha}{2p}\frac{p_1}{p_1-1}, & \text{if } 1 < p_1 \leq 2 < p_2 \leq \infty \text{ and } \frac{\alpha}{p} < \frac{1}{p_1'} \\ & \text{or } 1 < p_1 < p_2 \leq 2 \text{ and } \frac{\alpha}{p} < \frac{\theta}{p_1'}, \\ \frac{\alpha}{p} + \frac{1}{p}, & \text{if } 1 < p_1 < p_2 \leq 2 \text{ and } \frac{\alpha}{p} > \frac{\theta}{p_1'}. \end{cases}$$

Proof. We decompose Id into the sum $\mathrm{Id} = \sum_{j=0}^{\infty} \mathrm{id}_j$ with $\mathrm{id}_j = E_j \circ \mathrm{id} \circ P_j$, where $P_j(x = (x_{l,i})) = (x_{j,i})_i$ denotes the projection onto j-level and $E_j(y = (y_i)) = (0, \ldots, (y_i), \ldots, 0)$ is the identity operator putting the sequence y into j-level and $\mathrm{id} : l_{p_1}(\omega_\alpha) \to l_{p_2}(\omega_\alpha)$. Thus for any given $j \in \mathbb{N}_0$ we have the following commutative diagram (see [KLSS1])

$$\begin{array}{ccccc} l_{q_1}(2^{j\delta} l_{p_1}(\omega_\alpha)) & \xrightarrow{P_j} & l_{p_1}(\omega_\alpha) & \xrightarrow{D_\mu} & l_{p_1} \\ \Big\downarrow{\scriptstyle \mathrm{id}_j} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle D_\sigma} \\ l_{q_2}(l_{p_2}(\omega_\alpha)) & \xleftarrow{E_j} & l_{p_2}(\omega_\alpha) & \xleftarrow{D_\upsilon} & l_{p_2}, \end{array}$$

where D_μ, D_υ, D_σ are the diagonal operators generated by the sequences $\mu_k = (1+k)^{\alpha/p_1}$, $\upsilon_k = (1+k)^{-\alpha/p_2}$ and $\sigma_k = (1+k)^{-\alpha/p}$, respectively. Since D_μ and D_υ are isometries, $\|P_j\| \le 2^{-j\delta}$ and $\|E_j\| \le 1$, from Lemma 3.6 we get

$$c_k\big(\mathrm{id}_j : l_{q_1}(2^{j\delta} l_{p_1}(\omega_\alpha)) \to l_{q_2}(l_{p_2}(\omega_\alpha))\big) \le c_k(D_\sigma : l_{p_1} \to l_{p_2}) \|P_j\| \le C 2^{-j\delta} k^{-\beta},$$

where β is defined as in the lemma. Taking $r > 0$ such that $\frac{1}{r} = \beta$ we have

$$L^{(c)}_{r,\infty}(\mathrm{Id})^\rho \le c \sum_{j=0}^{\infty} L^{(c)}_{r,\infty}(\mathrm{id}_j)^\rho \le c \sum_{j=0}^{\infty} 2^{-j\delta\rho} < \infty,$$

for some $\rho \in (0, 1]$. ∎

With the above lemma we get upper estimates of Gelfand numbers of embeddings of the radial Besov spaces, whereas using Theorem 2.4 we get an estimation from below.

THEOREM 4.2. *Let the embeddings $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ be compact and $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$. Then*

$$c_k(\mathrm{id} : RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)) \approx k^{-\beta},$$

$$\beta = \begin{cases} \frac{n-1}{p}, & \textit{if } 2 \le p_1 < p_2 \le \infty, \\ -\frac{1}{2} + \frac{1}{p_1} + \frac{n-1}{p}, & \textit{if } 1 < p_1 \le 2 < p_2 \le \infty \textit{ and } n(\frac{1}{p_1} - \frac{1}{p_2}) > \frac{1}{p_2'}, \\ \frac{n-1}{2p} \frac{p_1}{p_1 - 1}, & \textit{if } 1 < p_1 \le 2 < p_2 \le \infty \textit{ and } n(\frac{1}{p_1} - \frac{1}{p_2}) < \frac{1}{p_2'} \\ & \textit{or } 1 < p_1 < p_2 \le 2 \textit{ and } n(\frac{1}{p_1} - \frac{1}{2}) < \frac{1}{2}, \\ \frac{n}{p}, & \textit{if } 1 < p_1 < p_2 \le 2 \textit{ and } n(\frac{1}{p_1} - \frac{1}{2}) > \frac{1}{2}. \end{cases}$$

Proof.

1) *Upper estimates.* Due to Proposition 2.6 we transfer the problem from the level of function spaces to the level of sequence spaces. Now applying Lemma 4.1 with $\alpha = n - 1$ we get the needed estimates.

2) *Lower estimates.* Lower estimates are a consequence of Theorem 2.4 and Proposition 1 in [KLSS2]. Let $s_1 > s_2 > 0$. It follows from Theorem 2.4 that the following diagram is commutative

$$\begin{array}{ccc} B^{s_1}_{p_1,q_1}([1,\infty), w_{(n-1)/p_1}) & \xrightarrow{\mathrm{ext}} & RB^{s_1}_{p_1,q_1}(\mathbb{R}^n \setminus B(0,1)) \\ \Big\downarrow{\scriptstyle \mathrm{Id}_1} & & \Big\downarrow{\scriptstyle \mathrm{Id}} \\ B^{s_2}_{p_2,q_2}([1/2,\infty), \omega_{(n-1)/p_2}) & \xleftarrow{\mathrm{tr}^*} & RB^{s_2}_{p_2,q_2}(\mathbb{R}^n \setminus B(0,1/2)). \end{array} \tag{46}$$

Furthermore

$$c_k(B^{s_1}_{p_1,q_1}([1,\infty),\omega_{(n-1)/p_1}) \hookrightarrow B^{s_2}_{p_2,q_2}([1/2,\infty),\omega_{(n-1)/p_2})) \\ \approx c_k(B^{s_1}_{p_1,q_1}([1,\infty),\omega_{(n-1)/p}) \hookrightarrow B^{s_2}_{p_2,q_2}([1/2,\infty)) \quad (47)$$

and the Gelfand numbers of the embeddings $B^{s_1}_{p_1,q_1}([1,\infty),\omega_{(n-1)/p}) \hookrightarrow B^{s_2}_{p_2,q_2}([1/2,\infty))$ and $B^{s_1}_{p_1,q_1}(\mathbb{R},\omega_{(n-1)/p}) \hookrightarrow B^{s_2}_{p_2,q_2}(\mathbb{R})$ are asymptotically the same. We repeat the simple argumentation taken from [KLSS1]. We take $\psi_1,\psi_2 \in C_\infty(\mathbb{R})$ such that $\operatorname{supp}\psi_1 \subset (-\infty,1/2)$, $\operatorname{supp}\psi_2 \subset (-1/2,\infty)$, where $\psi_1+\psi_2=1$. We get the commutative diagram

$$\begin{array}{ccc} B^{s_1}_{p_1,q_1}(\mathbb{R},\omega_{(n-1)/p}) & \xrightarrow{\psi_1\oplus\psi_2} & B^{s_1}_{p_1,q_1}((-\infty,1],\omega_{(n-1)/p}) \oplus B^{s_1}_{p_1,q_1}([-1,\infty),\omega_{(n-1)/p}) \\ \Big\downarrow \mathrm{Id}_2 & & \Big\downarrow \widetilde{\mathrm{id}} \\ B^{s_2}_{p_2,q_2}(\mathbb{R}) & \xleftarrow{\ \pi\ } & B^{s_2}_{p_2,q_2}((-\infty,3/2]) \oplus B^{s_2}_{p_2,q_2}([-3/2,\infty)), \end{array}$$

where $\psi_i : f \to \psi_i f$, $i=1,2$, and $\pi : (f_1,f_2) \to f_1+f_2$. Furthermore $\widetilde{\mathrm{id}} = \mathrm{id}_1 + \mathrm{id}_2$, where $\mathrm{id}_1 : (f_1,f_2) \to (f_1,0)$ and $\mathrm{id}_2 : (f_1,f_2) \to (0,f_2)$. So

$$c_k(\mathrm{Id}_2) \le c_k(\mathrm{id}_1) + c_k(\mathrm{id}_2) \le 2Cc_k(B^{s_1}_{p_1,q_1}([1,\infty),w_{(n-1)/p}) \hookrightarrow B^{s_2}_{p_2,q_2}([1/2,\infty))). \quad (48)$$

By Proposition 1 in [KLSS2] it is enough to analyze the diagram

$$\begin{array}{ccc} l_{p_1}(\omega_{(n-1)/p}) & \xrightarrow{S_0} & l_{q_1}(2^{j\delta}\tilde{l}_{p_1}(\omega_{(n-1)/p})) \\ \Big\downarrow \mathrm{Id}_4 & & \Big\downarrow \mathrm{Id}_2 \\ l_{p_2} & \xleftarrow{\ T_0\ } & l_{q_2}(l_{p_2}), \end{array} \quad (49)$$

where for $v=(v_l)$ and $\lambda=(\lambda_{j,l})$ we put

$$(S_0(v))_{j,l} = \begin{cases} v_{|l|}, & \text{if } j=0 \text{ and } l \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

$$(T_0(\lambda))_l = \lambda_{0,l}.$$

Hence $\|T_0\|=1$, $\|S_0\| \le 2$ and the sequence spaces $l_{q_1}(2^{j\delta}\tilde{l}_{p_1}(\omega_{(n-1)/p}))$ are defined by

$$l_q(2^{j\delta}\tilde{l}_p(\omega_{(n-1)/p})) = \Big\{(s_{j,k})_{j,k} : \|s_{j,k}|l_q(2^{j\delta}\tilde{l}_p(\omega_\alpha))\| \\ = \Big(\sum_{j=0}^\infty 2^{j\delta q}\Big(\sum_{k\in\mathbb{Z}}^\infty |\omega_{(n-1)/p}(2^{-j}k)s_{j,k}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

(with the usual modification if $p=\infty$ or $q=\infty$). Thus from the estimate

$$c_k(\mathrm{Id}_4 : l_{p_1}(\omega_{(n-1)/p}) \to l_{p_2}) \approx c_k(D_{(n-1)/p} : l_{p_1} \to l_{p_2})$$

we get $Ck^{-\beta} \le c_k(\mathrm{Id}_2)$. On the other hand we have

$$\begin{array}{ccc} RB^{s_1}_{p_1,q_1}(\mathbb{R}^n \setminus B(0,1)) & \longrightarrow & RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \\ \Big\downarrow \mathrm{Id} & & \Big\downarrow \mathrm{id} \\ RB^{s_2}_{p_2,q_2}(\mathbb{R}^n \setminus B(0,1/2)) & \xleftarrow{\ \Phi\ } & RB^{s_2}_{p_2,q_2}(\mathbb{R}^n), \end{array} \quad (50)$$

where $\Phi : f \to \varphi f$ and φ is a radial smooth function such that $\operatorname{supp}\varphi \subset \{x : |x| > \frac{1}{2}\}$, $\varphi(x) = 1$ if $|x| \geq 1$. Hence

$$c_k(RB^{s_1}_{p_1,q_1}(\mathbb{R}^n \setminus B(0,1)) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n \setminus B(0,1/2))) \leq c_k(RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)). \quad (51)$$

This finishes the proof for $s_2 > 0$. The case $s_2 \leq 0$ follows from the lift property for Besov spaces. ■

Now we will show that

$$[RB^s_{p,q}(\mathbb{R}^n)]^* = R(B^s_{p,q})^*(\mathbb{R}^n) = RB^{-s}_{p',q'}(\mathbb{R}^n), \quad (52)$$

where $p' = 1 - \frac{1}{p}$ and $q' = 1 - \frac{1}{q}$. We know that $(B^s_{p,q})^*(\mathbb{R}^n) = B^{-s}_{p',q'}(\mathbb{R}^n)$. Since $RB^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n)$ we get $R(B^s_{p,q})^*(\mathbb{R}^n) \hookrightarrow (B^s_{p,q})^*(\mathbb{R}^n) \hookrightarrow [RB^s_{p,q}(\mathbb{R}^n)]^*$. On the other hand since $RB^s_{p,q}(\mathbb{R}^n)$ is complemented in $B^s_{p,q}(\mathbb{R}^n)$ there exists a projection $P : B^s_{p,q}(\mathbb{R}^n) \to RB^s_{p,q}(\mathbb{R}^n)$. This was proved in [LS]. Hence for $f \in [RB^s_{p,q}(\mathbb{R}^n)]^*$ we put $f \circ P \in (B^s_{p,q})^*(\mathbb{R}^n) = B^{-s}_{p',q'}(\mathbb{R}^n)$. But $f \circ P$ is a radial distribution. This is our claim.

From (52), (4) and (5) we get the following theorem.

Theorem 4.3. *Let the embeddings* $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *be compact and* $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$. *Then*

$$d_k\big(\mathrm{id} : RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)\big) \approx k^{-\beta},$$

$$\beta = \begin{cases} \frac{n-1}{p}, & \text{if } 1 \leq p_1 < p_2 \leq 2, \\ \frac{1}{2} - \frac{1}{p_2} + \frac{n-1}{p}, & \text{if } 1 \leq p_1 < 2 < p_2 < \infty \text{ and } n(\frac{1}{p_1} - \frac{1}{p_2}) > \frac{1}{p_1}, \\ \frac{(n-1)p_2}{2p}, & \text{if } 1 \leq p_1 < 2 < p_2 < \infty \text{ and } n(\frac{1}{p_1} - \frac{1}{p_2}) < \frac{1}{p_1} \\ & \text{or } 2 \leq p_1 < p_2 < \infty \text{ and } n(\frac{1}{2} - \frac{1}{p_2}) < \frac{1}{2}, \\ \frac{n}{p}, & \text{if } 2 \leq p_1 < p_2 < \infty \text{ and } n(\frac{1}{2} - \frac{1}{p_2}) > \frac{1}{2}. \end{cases}$$

The corresponding result about Gelfand and Kolmogorov numbers of embeddings for the Triebel-Lizorkin space follows from elementary embeddings

$$B^s_{p,q_1}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,q_2}(\mathbb{R}^n),$$

if $q_1 \leq \min(p,q)$, $q_2 \geq \max(p,q)$. We get

Corollary 4.4. *Let the embeddings* $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *be compact and* $p_2 < \infty$. *Then*

$$c_k(\mathrm{id} : RF^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RF^{s_2}_{p_2,q_2}(\mathbb{R}^n)) \approx c_k(\mathrm{id} : RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)),$$
$$d_k(\mathrm{id} : RF^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RF^{s_2}_{p_2,q_2}(\mathbb{R}^n)) \approx d_k(\mathrm{id} : RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)).$$

Taking into account that $F^s_{p,2} = H^s_p$, $1 < p < \infty$, as a special case we obtain the estimates for Gelfand and Kolmogorov numbers of Sobolev embeddings of radial subspaces $RH^s_p(\mathbb{R}^n)$ of fractional Sobolev spaces $H^s_p(\mathbb{R}^n)$.

Remark. Let id denote the above compact embedding of function spaces. Let $1 < p_1, p_2 < \infty$, $n(\frac{1}{p_1} - \frac{1}{p_2}) \neq \frac{1}{p_2'}$, $n(\frac{1}{p_1} - \frac{1}{2}) \neq \frac{1}{2}$ and $n(\frac{1}{p_1} - \frac{1}{p_2}) \neq \frac{1}{p_1}$, $n(\frac{1}{2} - \frac{1}{p_2}) \neq \frac{1}{2}$. If we compare the above estimate of Gelfand and Kolmogorov numbers with the related estimates of approximation numbers of [ST], we get easily the following relations:

- if $1 < p_1 \le p_2 \le 2$ then
$$d_k(\mathrm{id}) \approx a_k(\mathrm{id}) \text{ but } c_k(\mathrm{id}) = o(a_k(\mathrm{id})) \text{ if } k \to \infty,$$
- if $2 \le p_1 \le p_2 < \infty$ then
$$c_k(\mathrm{id}) \approx a_k(\mathrm{id}) \text{ but } d_k(\mathrm{id}) = o(a_k(\mathrm{id})) \text{ if } k \to \infty,$$
- if $1 < p_1 < 2 < p_2 < \infty$ and $p_2 = p_1'$ then
$$c_k(\mathrm{id}) \approx d_k(\mathrm{id}) \approx a_k(\mathrm{id}),$$
- if $1 < p_1 < 2 < p_2 < \infty$ and $\min(p_1', p_2) = p_1'$ then
$$c_k(\mathrm{id}) \approx a_k(\mathrm{id}) \text{ but } d_k(\mathrm{id}) = o(a_k(\mathrm{id})) \text{ if } k \to \infty,$$
- if $1 < p_1 < 2 < p_2 < \infty$ and $\min(p_1', p_2) = p_2$ then
$$d_k(\mathrm{id}) \approx a_k(\mathrm{id}) \text{ but } c_k(\mathrm{id}) = o(a_k(\mathrm{id})) \text{ if } k \to \infty.$$

REMARK. Another example of s-numbers are Weyl numbers. For $k \in \mathbb{N}$, we define the k-th Weyl number by
$$x_k(T) := \sup\{a_k(TS) : S \in L(l_2, X) \text{ with } \|S\| \le 1\},$$
where X and Y are Banach spaces and $T \in L(X, Y)$. Using the same methods as above we can also prove the following theorem.

THEOREM 4.5. *Let the embeddings* $RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \hookrightarrow RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *be compact and* $\frac{1}{p} = \frac{1}{p_1} - \frac{1}{p_2}$. *Then*
$$x_k(\mathrm{id} : RB^{s_1}_{p_1,q_1}(\mathbb{R}^n) \to RB^{s_2}_{p_2,q_2}(\mathbb{R}^n)) \approx k^{-\beta}$$
$$\beta = \begin{cases} \frac{n}{p}, & \text{if } 1 \le p_1 \le p_2 \le 2, \\ \frac{1}{p_1} - \frac{1}{2} + \frac{n-1}{p}, & \text{if } 1 \le p_1 < 2 < p_2 < \infty, \\ \frac{(n-1)}{p}, & \text{if } 2 \le p_1 \le p_2 < \infty. \end{cases}$$

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