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LIPSCHITZ CONTINUITY IN MUCKENHOUPT $\mathcal{A}_1$ WEIGHTED FUNCTION SPACES

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Abstract. We study continuity envelopes of function spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ where the weight belongs to the Muckenhoupt class $\mathcal{A}_1$. This essentially extends partial forerunners in [13, 14]. We also indicate some applications of these results.

1. Introduction. The purpose of this paper is to use the recently introduced concept of continuity envelopes in function spaces in order to characterise weighted spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ where w belongs to the Muckenhoupt class $\mathcal{A}_1$. This can be understood as some counterpart of a parallel consideration for growth envelopes in [15].

It is well-known that weights from the Muckenhoupt class $\mathcal{A}_1 \subset \mathcal{A}_\infty$ may have local singularities, which can influence properties of the corresponding function spaces. One may take the weight function

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha, & |x| < 1, \\ |x|^\beta, & |x| \geq 1, \end{cases} \qquad x \in \mathbb{R}^n, \tag{1}$$

as typical example, where $\alpha, \beta > -n$. Weighted Besov and Triebel-Lizorkin spaces with Muckenhoupt weights are meanwhile well known concepts, cf. [3, 4] for a first systematic approach; we refer to subsequent papers and some history in further detail below. In contrast to this, the study of continuity envelopes has a rather short history; this new tool was developed in [12, 13, 35], initially intended for a more precise characterisation of function spaces. It turned out, however, that it leads not only to surprisingly sharp results based on classical concepts, but allows a lot of applications, too, e.g. to the study of compact embeddings. We return to this point later. Roughly speaking, a continuity

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envelope $\mathfrak{E}_{\mathsf{C}}(X)$ of a function space X consists of a so-called continuity envelope function

$$\mathcal{E}_{\mathsf{C}}^X(t) \sim \sup_{\|f|X\|\leq 1} \frac{\omega(f,t)}{t}, \quad t>0,$$

together with some *'fine index'* $u_{\mathsf{C}}^X \in (0,\infty]$; here $\omega(f,t)$ stands for the modulus of continuity, as usual.

By dealing with continuity envelopes of weighted spaces of type $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ first (special) results were obtained in [13, 14], essentially concentrating on the model weight (1). Our main intention now is to extend this idea, that is, to study the interplay between the weight $w\in\mathcal{A}_1$ and the Lipschitz continuity of functions from spaces $B^s_{p,q}(\mathbb{R}^n,w)$ and $F^s_{p,q}(\mathbb{R}^n,w)$ characterised by their continuity envelopes. Moreover, we restrict ourselves to the smallest weight class in this context, i.e., to $\mathcal{A}_1$. Here we have a complete result which essentially coincides with the unweighted situation if we assume, in addition, that

$$\inf_{m\in\mathbb{Z}^n} w(Q_{0,m}) \geq c_w > 0, \tag{2}$$

where $Q_{0,m}$ are unit cubes in $\mathbb{R}^n$ centred at $m\in\mathbb{Z}^n$, and $w(\Omega)=\int_\Omega w(x)\,\mathrm{d}x$. Our main outcome, Theorem 4.5 below, establishes that for $0<p<\infty$, $0<q\leq\infty$, $s>\frac{n}{p}$, and $w\in\mathcal{A}_1$ with (2),

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(\mathbb{R}^n,w)) = \begin{cases} (t^{-n/p+s-1},q), & \frac{n}{p}<s<\frac{n}{p}+1, \\ (|\log t|^{1/q'},q), & s=\frac{n}{p}+1 \quad\text{and}\quad 1<q\leq\infty, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(\mathbb{R}^n,w)) = \begin{cases} (t^{-n/p+s-1},p), & \frac{n}{p}<s<\frac{n}{p}+1, \\ (|\log t|^{1/p'},p), & s=\frac{n}{p}+1 \quad\text{and}\quad 1<p<\infty. \end{cases}$$

All this will be explicated by our model weight (1) and another one, $w_{\varkappa,\Gamma}$, related to a fractal d-set Γ. Moreover, we briefly indicate some applications of our results to Hardy type inequalities, criteria for sharp embeddings, and compact embeddings. In particular, we prove that under the above assumptions,

$$B^s_{p,q}(\mathbb{R}^n,w)\hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n) \quad\text{if and only if}\quad \begin{cases} s>\frac{n}{p}+1, & \text{or} \\ s=\frac{n}{p}+1 \quad\text{and}\quad 0<q\leq 1, \end{cases}$$

and

$$F^s_{p,q}(\mathbb{R}^n,w)\hookrightarrow \mathrm{Lip}^1(\mathbb{R}^n) \quad\text{if and only if}\quad \begin{cases} s>\frac{n}{p}+1, & \text{or} \\ s=\frac{n}{p}+1 \quad\text{and}\quad 0<p\leq 1. \end{cases}$$

Here $\mathrm{Lip}^1(\mathbb{R}^n)$ stands for the classical Lipschitz space defined as the space of all functions $f\in C(\mathbb{R}^n)$ such that

$$\big\|f|\mathrm{Lip}^1(\mathbb{R}^n)\big\| = \|f|C(\mathbb{R}^n)\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t}$$

is finite.

The main tools to prove such results are unweighted counterparts, sharp embeddings and atomic decompositions of corresponding spaces. We also benefit from related observations on embeddings and local singularities $\mathbf{S}_{\mathrm{sing}}(w)$ of the weight $w \in \mathcal{A}_1$ contained in [17, 18].

The paper is organised as follows. In Section 2 we collect all the material on Muckenhoupt weights, weighted spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$, $F^s_{p,q}(\mathbb{R}^n, w)$, and embeddings that will be needed below. This is followed by a short introduction to the concept of continuity envelopes in Section 3, before we deal exclusively with $w \in \mathcal{A}_1$ in Section 4 and determine the corresponding continuity envelopes of $B^s_{p,q}(\mathbb{R}^n, w)$, $F^s_{p,q}(\mathbb{R}^n, w)$. Finally we present a number of applications in Section 5.

2. Weighted function spaces. We fix some notation. By $\mathbb{N}$ we mean the set of natural numbers, by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$, and by $\mathbb{Z}^n$ the set of all lattice points in $\mathbb{R}^n$ having integer components. The positive part of a real function f is denoted by $f_+(x) = \max(f(x), 0)$, the integer part of $a \in \mathbb{R}$ by $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \le a\}$. If $0 < u \le \infty$, the number u' is given by $\frac{1}{u'} = (1 - \frac{1}{u})_+$. For two positive real sequences $\{\alpha_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ we mean by $\alpha_k \sim \beta_k$ that there exist constants $c_1, c_2 > 0$ such that $c_1\alpha_k \le \beta_k \le c_2\alpha_k$ for all $k \in \mathbb{N}$; similarly for positive functions. Given two (quasi-) Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

All unimportant positive constants will be denoted by c, occasionally with subscripts. For convenience, let both $\mathrm{d}x$ and $|\cdot|$ stand for the (n-dimensional) Lebesgue measure in the sequel. If not otherwise indicated, log is always taken with respect to base 2.

As we shall mainly deal with function spaces on $\mathbb{R}^n$, we may often omit the '$\mathbb{R}^n$' from their notation for convenience.

2.1. Muckenhoupt weights. We briefly recall some fundamentals on the Muckenhoupt class $\mathcal{A}_1$. By a weight w we shall always mean a locally integrable function $w \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$, positive a.e. in the sequel. Let M stand for the Hardy-Littlewood maximal operator given by

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n, \tag{3}$$

where $\mathcal{B}$ is the collection of all open balls

$$B(x,r) = \{y \in \mathbb{R}^n : |y - x| < r\}, \quad r > 0.$$

Definition 2.1. Let w be a weight on $\mathbb{R}^n$. Then w *belongs to the Muckenhoupt class* $\mathcal{A}_1$ if there exists a constant $0 < A < \infty$ such that the inequality

$$Mw(x) \le Aw(x) \tag{4}$$

holds for almost all $x \in \mathbb{R}^n$.

The class $\mathcal{A}_1$ is a special 'extremal' case of the larger scale of so-called $\mathcal{A}_\infty$ weights, see the pioneering work of Muckenhoupt in [22–24], and the monographs [11], [29, Ch. V], [30], and [31, Ch. IX] for a complete account on the theory of Muckenhoupt weights. We shall concentrate on the special class $\mathcal{A}_1$ only in this paper. As usual, we use the abbreviation

$$w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^n$ is some bounded, measurable set. Then a weight w on $\mathbb{R}^n$ belongs to $\mathcal{A}_1$ if and only if

$$\frac{1}{|B|}\int_B f(y)\,\mathrm{d}y \le \frac{c}{w(B)}\int_B f(x)w(x)\,\mathrm{d}x$$

for all nonnegative f and all balls B. In particular, with $E \subset B$ and $f = \chi_E$, this implies that

$$\frac{|E|}{|B|} \le c'\,\frac{w(E)}{w(B)}\,, \qquad E \subset B,\ w \in \mathcal{A}_1. \tag{5}$$

EXAMPLES 2.2.

(i) One of the most prominent examples of a Muckenhoupt weight $w \in \mathcal{A}_1$ is given by $w(x) = |x|^{\varrho}$, where $-n < \varrho \le 0$.

(ii) We modified this example in [17, 19] by

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha}, & |x| < 1,\\ |x|^{\beta}, & |x| \ge 1,\end{cases} \tag{6}$$

and

$$w_{\log}(x) = \begin{cases} |x|^{\alpha}(1-\log|x|)^{\gamma}, & |x| < 1,\\ |x|^{\beta}(1+\log|x|)^{\delta}, & |x| \ge 1,\end{cases}$$

where $\alpha, \beta > -n$, and $\gamma, \delta \in \mathbb{R}$. Plainly, $w_{\alpha,\beta} = w_{\log}$ when $\gamma = \delta = 0$. For simplicity we shall only regard $w_{\alpha,\beta}$ in this paper. A straightforward calculation shows that $w_{\alpha,\beta} \in \mathcal{A}_1$ when

$$-n < \min(\alpha,\beta) \le \max(\alpha,\beta) \le 0. \tag{7}$$

(iii) Finally we recall a 'fractal' example studied in [16]. Let $\Gamma \subset \mathbb{R}^n$ be a *d-set*, $0 < d < n$, in the sense of [34, Def. 3.1], [20] (which is different from [8]), i.e., there exists a Borel measure μ in $\mathbb{R}^n$ such that $\operatorname{supp}\mu = \Gamma$ compact, and there are constants $c_1, c_2 > 0$ such that for arbitrary $\gamma \in \Gamma$ and all $0 < r < 1$

$$c_1 r^d \le \mu(B(\gamma, r) \cap \Gamma) \le c_2 r^d.$$

We proved in [16] that the weight $w_{\varkappa,\Gamma}$ given by

$$w_{\varkappa,\Gamma}(x) = \begin{cases} \operatorname{dist}(x,\Gamma)^{\varkappa}, & \text{if} \quad \operatorname{dist}(x,\Gamma) \le 1,\\ 1, & \text{if} \quad \operatorname{dist}(x,\Gamma) \ge 1,\end{cases} \tag{8}$$

satisfies $w_{\varkappa,\Gamma} \in \mathcal{A}_1$ if $-(n-d) < \varkappa \le 0$.

In a slight abuse of notation one may incorporate in this approach the distance weight to some hyperplane in $\mathbb{R}^n$, $\Gamma = \{(x', 0) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}\} \sim \mathbb{R}^{n-1}$, with $d = n-1$, $-1 < \varkappa \le 0$, and

$$w_{\varkappa,n-1}(x) = \begin{cases} |x_n|^{\varkappa}, & \text{if} \quad |x_n| \le 1,\\ 1, & \text{if} \quad |x_n| \ge 1.\end{cases}$$

For further examples we refer to [9, 17, 18].

We need some refined study of the singularity behaviour of Muckenhoupt $\mathcal{A}_1$ weights. Let for $m \in \mathbb{Z}^n$ and $\nu \in \mathbb{N}_0$, $Q_{\nu,m}$ denote the n-dimensional cube with sides parallel to the axes of coordinates, centred at $2^{-\nu}m$ and with side length $2^{-\nu}$. In [18] we introduced the following notion of their *set of singularities* $\mathbf{S}_{\text{sing}}(w)$.

DEFINITION 2.3. For $w \in \mathcal{A}_1$ we define the *set of singularities* $\mathbf{S}_{\text{sing}}(w)$ by

$$\mathbf{S}_{\text{sing}}(w) = \mathbf{S}_\infty(w) = \Big\{x_0 \in \mathbb{R}^n : \sup_{Q_{\nu,m} \ni x_0} \frac{w(Q_{\nu,m})}{|Q_{\nu,m}|} = \infty\Big\}.$$

REMARK 2.4. This is a special case of $\mathbf{S}_{\text{sing}}(w_1, w_2)$ defined in [18] with $w_2 \equiv 1$, $w_1 \equiv w$. Moreover, for $w \notin \mathcal{A}_1$ another set $\mathbf{S}_0(w)$ may appear (in obvious analogy to $\mathbf{S}_\infty(w)$ and thus explaining the notation), but we proved in [18] that $\mathbf{S}_0(w) = \emptyset$ for $w \in \mathcal{A}_1$.

EXAMPLES 2.5. Let $w_{\alpha,\beta}$ be given by (6) with (7) such that

$$\frac{w_{\alpha,\beta}(Q_{\nu,m})}{|Q_{\nu,m}|} \sim \begin{cases} 2^{-\nu\alpha} & \text{if } m = 0, \\ |2^{-\nu}m|^\alpha & \text{if } 1 \le |m| < 2^\nu, \\ |2^{-\nu}m|^\beta & \text{if } |m| \ge 2^\nu. \end{cases} \tag{9}$$

Hence

$$\mathbf{S}_{\text{sing}}(w_{\alpha,\beta}) = \begin{cases} \{0\}, & \text{if} \quad \alpha < 0, \\ \emptyset, & \text{if} \quad \alpha = 0. \end{cases}$$

In case of the weight $w_{\varkappa,\Gamma}$ introduced in (8) with $0 < d < n$ and $-(n-d) < \varkappa \le 0$, one obtains that

$$\mathbf{S}_{\text{sing}}(w_{\varkappa,\Gamma}) = \begin{cases} \emptyset, & \text{if } \varkappa = 0, \\ \Gamma, & \text{if } \varkappa < 0, \end{cases}$$

based on the estimate

$$\frac{w_{\varkappa,\Gamma}(Q_{\nu,m})}{|Q_{\nu,m}|} \sim \begin{cases} 1, & \text{if} \quad 2Q_{\nu,m} \cap \Gamma = \emptyset, \\ 2^{-\nu\varkappa}, & \text{otherwise}, \end{cases}$$

see [16].

REMARK 2.6. Note that we always have $|\mathbf{S}_{\text{sing}}(w)| = 0$ for $w \in \mathcal{A}_1$, cf. [18].

2.2. Function spaces of type $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ with $w \in \mathcal{A}_1$. Let $w \in \mathcal{A}_1$ be a Muckenhoupt weight and $0 < p < \infty$. Then the weighted Lebesgue space $L_p(\mathbb{R}^n, w)$ contains all measurable functions such that

$$\|f|L_p(\mathbb{R}^n, w)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\Big)^{1/p}$$

is finite. For $p = \infty$ one obtains the classical (unweighted) Lebesgue space,

$$L_\infty(\mathbb{R}^n, w) = L_\infty(\mathbb{R}^n), \quad w \in \mathcal{A}_1;$$

we thus mainly restrict ourselves to $p < \infty$ in what follows.

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and its dual $\mathcal{S}'(\mathbb{R}^n)$ of all complex-valued tempered distributions have their usual meaning here. Let $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp}\varphi \subset \{y \in \mathbb{R}^n : |y| < 2\} \quad \text{and} \quad \varphi(x) = 1 \quad \text{if} \quad |x| \le 1,$$

and for each $j \in \mathbb{N}$ let $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. Then $\{\varphi_j\}_{j=0}^\infty$ forms a *smooth dyadic resolution of unity*. Given any $f \in \mathcal{S}'(\mathbb{R}^n)$, we denote by $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ its Fourier transform and its inverse Fourier transform, respectively. Let $f \in \mathcal{S}'(\mathbb{R}^n)$, then the Paley-Wiener-Schwartz theorem implies that $\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)$ is an entire analytic function on $\mathbb{R}^n$.

Definition 2.7. Let $w \in \mathcal{A}_1$, $0 < q \le \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $\{\varphi_j\}_{j\in\mathbb{N}_0}$ a smooth dyadic resolution of unity.

(i) *The weighted Besov space* $B^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f|B^s_{p,q}(\mathbb{R}^n,w)\big\| = \Big\|\big\{2^{js}\big\|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)|L_p(\mathbb{R}^n,w)\big\|\big\}_{j\in\mathbb{N}_0}|\ell_q\Big\|$$

is finite.

(ii) *The weighted Triebel-Lizorkin space* $F^s_{p,q}(\mathbb{R}^n, w)$ is the set of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\big\|f|F^s_{p,q}(\mathbb{R}^n,w)\big\| = \Big\|\big\|\{2^{js}|\mathcal{F}^{-1}(\varphi_j\mathcal{F}f)(\cdot)|\}_{j\in\mathbb{N}_0}|\ell_q\big\|\ \Big|\ L_p(\mathbb{R}^n,w)\Big\|$$

is finite.

Remark 2.8. The spaces $B^s_{p,q}(\mathbb{R}^n, w)$ and $F^s_{p,q}(\mathbb{R}^n, w)$ are independent of the particular choice of the smooth dyadic resolution of unity $\{\varphi_j\}_j$ appearing in their definitions. They are quasi-Banach spaces (Banach spaces for $p, q \ge 1$), and $\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$, where the first embedding is dense if $q < \infty$, similarly for the F-case; cf. [3]. Moreover, for $w_0 \equiv 1 \in \mathcal{A}_1$ these are the usual (unweighted) Besov and Triebel-Lizorkin spaces; we refer, in particular, to the series of monographs [32–36] for a comprehensive treatment of the unweighted spaces.

The above spaces with weights of type $w \in \mathcal{A}_\infty$ have been studied systematically in [3, 4], with subsequent papers [5, 6]. It turned out that many of the results from the unweighted situation have weighted counterparts: e.g., we have $F^0_{p,2}(\mathbb{R}^n, w) = h_p(\mathbb{R}^n, w)$, $0 < p < \infty$, where the latter are Hardy spaces, see [3, Thm. 1.4]. Further details can be found in [1–4, 10, 11, 25, 26]. In [27] the above class of weights was extended in order to incorporate locally regular weights, too, creating in that way the class $\mathcal{A}^{\ell oc}_p$. We partly rely on our approaches in [16–19].

We briefly recall the definition of atoms.

Definition 2.9. Let $K \in \mathbb{N}_0$ and $b > 1$.

(i) The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an 1_K*-atom* if $\operatorname{supp} a \subset bQ_{0,m}$ for some $m \in \mathbb{Z}^n$, and $|\mathrm{D}^\alpha a(x)| \le 1$ for $|\alpha| \le K$, $x \in \mathbb{R}^n$.

(ii) Let $s \in \mathbb{R}$, $0 < p \le \infty$, and $L + 1 \in \mathbb{N}_0$. The complex-valued function $a \in C^K(\mathbb{R}^n)$ is said to be an $(s, p)_{K,L}$*-atom* if for some $\nu \in \mathbb{N}_0$,

$$\begin{aligned} &\operatorname{supp} a \subset bQ_{\nu,m} && \text{for some } m \in \mathbb{Z}^n,\\ &|\mathrm{D}^\alpha a(x)| \le 2^{-\nu(s-n/p)+|\alpha|\nu} && \text{for } |\alpha| \le K,\ x \in \mathbb{R}^n,\\ &\int_{\mathbb{R}^n} x^\beta a(x)\,\mathrm{d}x = 0 && \text{for } |\beta| \le L. \end{aligned}$$

We shall denote an atom $a(x)$ supported in some $Q_{\nu,m}$ by $a_{\nu,m}$ in the sequel. Choosing $L = -1$ in (ii) means that no moment conditions are required. For $0 < p < \infty$, $0 < q \le \infty$,

$w \in \mathcal{A}_1$, we introduce suitable sequence spaces $b_{pq}(w)$ by

$$b_{pq}(w) = \Big\{ \lambda = \{\lambda_{\nu,m}\}_{\nu,m} : \lambda_{\nu,m} \in \mathbb{C}, \\ \big\|\lambda|b_{pq}(w)\big\| \sim \Big\| \Big\{ \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^p 2^{\nu n} w(Q_{\nu,m}) \Big)^{1/p} \Big\}_{\nu\in\mathbb{N}_0} |\ell_q \Big\| < \infty \Big\}.$$

The atomic decomposition result used below reads as follows.

PROPOSITION 2.10. *Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, and $w \in \mathcal{A}_1$. Let $K, L+1 \in \mathbb{N}_0$ with*

$$K \ge (1 + \lfloor s \rfloor)_+ \quad \textit{and} \quad L \ge \max\Big(-1, \Big\lfloor n\Big(\frac{1}{p} - 1\Big)_+ - s \Big\rfloor \Big). \tag{10}$$

Then $f \in \mathcal{S}(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n, w)$ if and only if it can be written as a series

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m} a_{\nu,m}(x), \quad \textit{converging in } \mathcal{S}'(\mathbb{R}^n), \tag{11}$$

where $a_{\nu,m}(x)$ are 1_K-atoms ($\nu = 0$) or $(s,p)_{K,L}$-atoms ($\nu \in \mathbb{N}$) and $\lambda \in b_{pq}(w)$. Furthermore,

$$\inf \|\lambda|b_{pq}(w)\| \tag{12}$$

is an equivalent quasi-norm in $B^s_{pq}(\mathbb{R}^n, w)$, where the infimum ranges over all admissible representations (11).

REMARK 2.11. The above result is a special case of [16, Thm. 3.10], cf. also [1, Theorem 5.10]. There are parallel F-results, too.

Notational agreement. We adopt the nowadays usual custom to write $A^s_{p,q}$ instead of $B^s_{p,q}$ or $F^s_{p,q}$, respectively, when both scales of spaces are meant simultaneously in some context.

2.3. Continuous embeddings. We collect some embedding results for weighted spaces that will be used later. We immediately specify the general criterion obtained in [17] to one-weight situations with $w \in \mathcal{A}_1$, cf. [17, 18]. Recall that we deal with function spaces on $\mathbb{R}^n$ exclusively, and will thus omit the '$\mathbb{R}^n$' from their notation.

PROPOSITION 2.12. *Let $w \in \mathcal{A}_1$ and*

$$-\infty < s_2 \le s_1 < \infty, \quad 0 < p_1 < \infty, \quad 0 < p_2 \le \infty, \quad 0 < q_1, q_2 \le \infty. \tag{13}$$

Then

$$\mathrm{id}_w : B^{s_1}_{p_1,q_1}(w) \hookrightarrow B^{s_2}_{p_2,q_2}$$

is continuous if and only if

$$\begin{cases} \inf_m w(Q_{0,m}) \ge c > 0, & \textit{and} \\ \big\{2^{-j\boldsymbol{\delta}_*}\big\}_{j\in\mathbb{N}_0} \in \ell_{q^*}, & \textit{and} \\ p_1 \le p_2, \end{cases} \tag{14}$$

where

$$\boldsymbol{\delta}_* = s_1 - \frac{n}{p_1} - s_2 + \frac{n}{p_2} \quad \textit{and} \quad \frac{1}{q^*} = \Big(\frac{1}{q_2} - \frac{1}{q_1}\Big)_+.$$

In particular,

$$B^s_{p,q}(w) \hookrightarrow B^s_{p,q} \quad \textit{if and only if} \quad \inf_m w(Q_{0,m}) \geq c > 0. \tag{15}$$

REMARK 2.13. If $w \equiv 1 \in \mathcal{A}_1$, then (14) reduces to $p_1 \leq p_2$ and $\boldsymbol{\delta}_* > 0$ with the extension to $\boldsymbol{\delta}_* \geq 0$ if $q_1 \leq q_2$. This recovers the well-known unweighted result.

In case of our special weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ this reads as follows, cf. [19].

COROLLARY 2.14. *Let the parameters satisfy* (13).

(i) *Let $w_{\alpha,\beta}$ be given by* (6) *with* (7). *The embedding*

$$\mathrm{id}_{\alpha,\beta} : B^{s_1}_{p_1,q_1}(w_{\alpha,\beta}) \hookrightarrow B^{s_2}_{p_2,q_2}$$

is continuous if and only if $p_1 \leq p_2$, $\beta = 0$, and $\left\{2^{-j\boldsymbol{\delta}_*}\right\}_{j\in\mathbb{N}_0} \in \ell_{q^*}$. In particular,

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^s_{p,q} \quad \textit{if and only if} \quad \beta = 0.$$

(ii) *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ be given by* (8) *with $-(n-d) < \varkappa \leq 0$. The embedding*

$$\mathrm{id}_{\varkappa,\Gamma} : B^{s_1}_{p_1,q_1}(w_{\varkappa,\Gamma}) \hookrightarrow B^{s_2}_{p_2,q_2} \tag{16}$$

is continuous if and only if $p_1 \leq p_2$ and $\left\{2^{-\nu\boldsymbol{\delta}_*}\right\}_{\nu\in\mathbb{N}_0} \in \ell_{q^*}$, in particular,

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow B^s_{p,q}.$$

REMARK 2.15. Though we deal in this paper with $\mathcal{A}_1$ weights mainly, we recall the following extension of Corollary 2.14 to values $\alpha \geq 0$, $\beta \geq 0$, $\varkappa \geq 0$. In [15,17] we proved that for $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\alpha > -n$, $\beta \geq 0$, $\varkappa > -(n-d)$,

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow B^{s-\max(\alpha,0)/p}_{p,q} \qquad \text{if and only if} \qquad \beta \geq 0, \tag{17}$$

and

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow B^{s-\max(\varkappa,0)/p}_{p,q}. \tag{18}$$

In [17, 18] we also considered situations where both source and target spaces are weighted with the same $w \in \mathcal{A}_\infty$. Here we shall only need the following basic observation.

PROPOSITION 2.16. *Let $0 < q \leq \infty$, $0 < p < \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_1$.*

(i) *Let $-\infty < s_1 \leq s_0 < \infty$ and $0 < q_0 \leq q_1 \leq \infty$, then*

$$A^{s_0}_{p,q}(w) \hookrightarrow A^{s_1}_{p,q}(w) \quad \textit{and} \quad A^s_{p,q_0}(w) \hookrightarrow A^s_{p,q_1}(w).$$

(ii) *We have*

$$B^s_{p,\min(p,q)}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^s_{p,\max(p,q)}(w). \tag{19}$$

(iii) *Assume that*

$$\inf_m w(Q_{0,m}) \geq c_w > 0. \tag{20}$$

Let $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ satisfy

$$s_0 - \frac{n}{p_0} = s - \frac{n}{p} = s_1 - \frac{n}{p_1}\,. \tag{21}$$

Then

$$B^{s_0}_{p_0,p}(w) \hookrightarrow F^{s}_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,p}(w). \tag{22}$$

REMARK 2.17. These embeddings are natural extensions from the unweighted case $w \equiv 1$, see [32, Prop. 2.3.2/2, Thm. 2.7.1] and [28, Thm. 3.2.1]. The above result essentially coincides with [3, Thm. 2.6] and can be found in [15, 17]. For later use, let us recall the following extension of (iii) to values $\alpha \ge 0$, $\beta \ge 0$, $\varkappa \ge 0$ for our weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$, cf. [15]: Let $\beta \ge 0$, $\alpha > -n$, and $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ satisfy

$$s_0 - \frac{\max(\alpha,0)+n}{p_0} = s - \frac{\max(\alpha,0)+n}{p} = s_1 - \frac{\max(\alpha,0)+n}{p_1}\,. \tag{23}$$

Then (22) holds for $w = w_{\alpha,\beta}$. In case of

$$s_0 - \frac{\max(\varkappa,0)+n}{p_0} = s - \frac{\max(\varkappa,0)+n}{p} = s_1 - \frac{\max(\varkappa,0)+n}{p_1}\,, \tag{24}$$

(22) is true for $w = w_{\varkappa,\Gamma}$, where we assume $\varkappa > -(n-d)$, $0 < d < n$.

3. Envelopes

3.1. Definition and basic properties. Let C be the space of all complex-valued bounded uniformly continuous functions on $\mathbb{R}^n$, equipped with the sup-norm as usual. Recall that the classical Lipschitz space Lip^1 is defined as the space of all functions $f \in C$ such that

$$\left\| f|\mathrm{Lip}^1 \right\| = \|f|C\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t} \tag{25}$$

is finite, the expression (25) defining its norm, where $\omega(f,t)$ stands for the modulus of continuity,

$$\omega(f,t) = \sup_{|h|\le t} \sup_{x\in\mathbb{R}^n} |f(x+h) - f(x)|, \quad t > 0.$$

DEFINITION 3.1. Let $X \hookrightarrow C$ be some quasi-normed function space on $\mathbb{R}^n$.

(i) The *continuity envelope function* $\mathcal{E}_{\mathsf{C}}^X : (0,\infty) \to [0,\infty]$ of X is defined by

$$\mathcal{E}_{\mathsf{C}}^X(t) = \sup_{\|f|X\|\le 1} \frac{\omega(f,t)}{t}\,, \quad t > 0.$$

(ii) Assume $X \not\hookrightarrow \mathrm{Lip}^1$. Let $\varepsilon \in (0,1)$, $H(t) = -\log \mathcal{E}_{\mathsf{C}}^X(t)$, $t \in (0,\varepsilon]$, and let μ_H be the associated Borel measure. The number u_{C}^X, $0 < u_{\mathsf{C}}^X \le \infty$, is defined as the infimum of all numbers v, $0 < v \le \infty$, such that

$$\left(\int_0^\varepsilon \Big(\frac{\omega(f,t)}{t\,\mathcal{E}_{\mathsf{C}}^X(t)}\Big)^v \mu_H(\mathrm{d}t)\right)^{1/v} \le c\,\|f|X\| \tag{26}$$

(with the usual modification if $v = \infty$) holds for some $c > 0$ and all $f \in X$. The couple

$$\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^X(\cdot), u_{\mathsf{C}}^X\right)$$

is called *continuity envelope* for the function space X.

This concept was introduced and first studied in [35, Ch. 2], [12], see also [13]. For convenience we recall some properties. In view of (i) we obtain—strictly speaking—equivalence classes of continuity envelope functions when working with equivalent quasi-norms in X as we shall usually do. But we do not want to distinguish between representative and equivalence class in what follows and thus stick at the notation introduced in (i). Concerning (ii) we shall assume that we can choose a continuous representative in the equivalence class $[\mathcal{E}_{\mathsf{C}}^X]$, for convenience (but in a slight abuse of notation) denoted by $\mathcal{E}_{\mathsf{C}}^X$ again. It is obvious that (26) holds for $v = \infty$ and any X. Moreover, one verifies that

$$\sup_{0<t\le\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \le c_1 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{1/v_1} \le c_2 \left(\int_0^\varepsilon \left(\frac{g(t)}{\mathcal{E}_{\mathsf{C}}^X(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{1/v_0}$$

for $0 < v_0 < v_1 < \infty$ and all non-negative monotonically decreasing functions g on $(0, \varepsilon]$; cf. [35, Prop. 12.2]. So since $\mathcal{E}_{\mathsf{C}}^X$ is equivalent to some monotonically decreasing function, we observe that the left-hand sides in (26) are monotonically ordered in v and it is natural to look for the smallest possible one.

Proposition 3.2.

(i) *Let $X_i \hookrightarrow C$, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$. Then $X_1 \hookrightarrow X_2$ implies that there is some positive constant c such that for all $t > 0$,*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \le c\,\mathcal{E}_{\mathsf{C}}^{X_2}(t). \tag{27}$$

(ii) *We have $X \hookrightarrow \mathrm{Lip}^1$ if and only if $\mathcal{E}_{\mathsf{C}}^X$ is bounded.*

(iii) *Let X_i, $i = 1, 2$, be some function spaces on $\mathbb{R}^n$ with $X_1 \hookrightarrow X_2$. Assume for their continuity envelope functions*

$$\mathcal{E}_{\mathsf{C}}^{X_1}(t) \sim \mathcal{E}_{\mathsf{C}}^{X_2}(t), \qquad t \in (0, \varepsilon),$$

for some $\varepsilon > 0$. Then we get for the corresponding indices $u_{\mathsf{C}}^{X_i}$, $i = 1, 2$, that

$$u_{\mathsf{C}}^{X_1} \le u_{\mathsf{C}}^{X_2}. \tag{28}$$

This result coincides with [13, Props. 5.3, 6.4].

3.2. Continuity envelopes in unweighted spaces. We briefly summarise some results for unweighted spaces, in particular, for Besov and Triebel-Lizorkin spaces and Lipschitz spaces of type Lip^a, $0 < a < 1$, and $\mathrm{Lip}^{(1,-b)}$, $b \ge 0$. The latter represent the natural extensions of (25) and collect all $f \in C$ such that

$$\|f|\mathrm{Lip}^a\| = \|f|C\| + \sup_{t\in(0,1)} \frac{\omega(f,t)}{t^a},$$

and

$$\left\|f|\mathrm{Lip}^{(1,-b)}\right\| = \|f|C\| + \sup_{t\in(0,1/2)} \frac{\omega(f,t)}{t\,|\log t|^b},$$

respectively, are finite.

Proposition 3.3.

(i) *Let* $0 < a < 1$, $b \geq 0$. *Then*

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^a\right) = \left(t^{-(1-a)}, \infty\right),$$
$$\mathfrak{E}_{\mathsf{C}}(C) = \left(t^{-1}, \infty\right),$$

and

$$\mathfrak{E}_{\mathsf{C}}\left(\mathrm{Lip}^{(1,-b)}\right) = \left(|\log t|^b, \infty\right).$$

(ii) *Let* $0 < p \leq \infty$ (*with* $p < \infty$ *in* F*-case*), $0 < q \leq \infty$, $s \geq \frac{n}{p}$. *Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}) = \begin{cases} \left(t^{-n/p+s-1}, q\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ \left(|\log t|^{1/q'}, q\right), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < q \leq \infty, \\ \left(t^{-1}, \infty\right), & s = \frac{n}{p} \quad \text{and} \quad 0 < q \leq 1, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}) = \begin{cases} \left(t^{-n/p+s-1}, p\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ \left(|\log t|^{1/p'}, p\right), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < p < \infty, \\ \left(t^{-1}, \infty\right), & s = \frac{n}{p} \quad \text{and} \quad 0 < p \leq 1. \end{cases}$$

Remark 3.4. Case (i) can be found in [13, Sect. 5.3]. Note that this explains the saying that continuity envelopes 'measure' the *lack* or *deviation* from Lipschitz continuity of function spaces. For proofs and further discussion in (ii) we refer to [13, Thms. 9.2, 9.4, 9.10], [35, Sect. 14], and to [21].

There is a number of partial results in the weighted setting: in [13,14] we studied the situation of $A^s_{p,q}(w_{\alpha,\beta})$ in some cases and obtained,

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(t) \sim \mathcal{E}_{\mathsf{C}}^{F^s_{p,q}(w_{\alpha,\beta})}(t) \sim t^{-n/p-\max(\alpha,0)/p+s-1}, \quad 0 < t < 1, \tag{29}$$

if $\beta \geq 0$, $-n < \alpha \leq \beta$, $\frac{\max(\alpha,0)}{p} < s - \frac{n}{p} < \frac{\max(\alpha,0)}{p} + 1$.

4. Continuity envelope for $w \in \mathcal{A}_1$. In this section we characterise the deviation from Lipschitz continuity of spaces $A^s_{p,q}(\mathbb{R}^n, w) = A^s_{p,q}(w)$, $w \in \mathcal{A}_1$, in terms of their continuity envelopes. As a preparation we recall our result from [15] about the embedding $A^s_{p,q}(w) \hookrightarrow C$, i.e., where the concept of continuity envelopes makes sense. Afterwards we give a sufficient condition for the embedding $A^s_{p,q}(w) \hookrightarrow \mathrm{Lip}^1$, i.e., where no deviation from Lipschitz continuity in the sense of continuity envelopes appears. Later it will turn out that this condition is also necessary. Borderline situations $s = \frac{n}{p}$ are mainly out of the scope of the present approach.

Lemma 4.1. *Let $0<p<\infty$, $0<q\leq\infty$, $w\in\mathcal{A}_1$ with* (20).

(i) *Then*

$$B^s_{p,q}(w)\hookrightarrow C \quad \text{if and only if} \quad \begin{cases} s>\frac{n}{p}, & \text{or} \\ s=\frac{n}{p} & \text{and} \quad 0<q\leq 1, \end{cases}$$

and

$$F^s_{p,q}(w)\hookrightarrow C \quad \text{if and only if} \quad \begin{cases} s>\frac{n}{p}, & \text{or} \\ s=\frac{n}{p} & \text{and} \quad 0<p\leq 1. \end{cases}$$

(ii) *Let $s>\frac{n}{p}+1$ or $s=\frac{n}{p}+1$ and $0<q\leq 1$. Then*

$$B^s_{p,q}(w)\hookrightarrow \mathrm{Lip}^1. \tag{30}$$

Proof. As mentioned, (i) is covered by [15] (where the target space L_∞ there can be replaced by C here). As for (ii) we use embedding (15) with (20), thus the well-known embedding $B^s_{p,q}\hookrightarrow \mathrm{Lip}^1$ for $s>\frac{n}{p}+1$ or $s=\frac{n}{p}+1$ and $0<q\leq 1$ implies $B^s_{p,q}(w)\hookrightarrow \mathrm{Lip}^1$. ■

Remark 4.2. In Corollary 5.2 below we shall prove that for $w\in\mathcal{A}_1$ we have (30) if and only if $s>\frac{n}{p}+1$ or $s=\frac{n}{p}+1$ and $0<q\leq 1$ (as in the unweighted case), and a counterpart for F-spaces. For the moment we conclude from the above result and general facts about continuity envelopes that their study makes sense for parameters $\frac{n}{p}\leq s\leq\frac{n}{p}+1$.

4.1. Continuity envelope function. We show that whenever $s\geq\frac{n}{p}$ and $w\in\mathcal{A}_1$ satisfies (20), then

$$\mathcal{E}_{\mathsf{C}}^{A^s_{p,q}(w)}(t)\sim\mathcal{E}_{\mathsf{C}}^{A^s_{p,q}}(t) \qquad \text{for} \quad t\to 0.$$

Proposition 4.3. *Let $0<p<\infty$, $0<q\leq\infty$, $s\geq\frac{n}{p}$, $w\in\mathcal{A}_1$ with* (20).

(i) *Let $\frac{n}{p}<s<\frac{n}{p}+1$. Then*

$$\mathcal{E}_{\mathsf{C}}^{A^s_{p,q}(w)}(t)\sim t^{-n/p+s-1}, \qquad t\to 0. \tag{31}$$

(ii) *Let $s=\frac{n}{p}+1$. Then*

$$\mathcal{E}_{\mathsf{C}}^{A^{1+n/p}_{p,q}(w)}(t)\sim\begin{cases} |\log t|^{1/q'}, & \text{if} \quad A^{1+n/p}_{p,q}=B^{1+n/p}_{p,q} \quad \text{and} \quad 1<q\leq\infty, \\ |\log t|^{1/p'}, & \text{if} \quad A^{1+n/p}_{p,q}=F^{1+n/p}_{p,q} \quad \text{and} \quad 1<p<\infty, \end{cases} \qquad t\to 0.$$

(iii) *Let $s=\frac{n}{p}$ and assume, in addition, that $0<q\leq 1$ if $A^{n/p}_{p,q}=B^{n/p}_{p,q}$, and $0<p\leq 1$ if $A^{n/p}_{p,q}=F^{n/p}_{p,q}$. Then*

$$\mathcal{E}_{\mathsf{C}}^{A^{n/p}_{p,q}(w)}(t)\sim t^{-1}, \qquad t\to 0. \tag{32}$$

Proof. Step 1. Note first that it is sufficient to deal with B-spaces only: Assume that we have already proved (i) with $A^s_{p,q}=B^s_{p,q}$; then (19) together with (27) complete the argument in case of $A^s_{p,q}=F^s_{p,q}$; concerning (ii) and (iii) we apply Proposition 2.16(iii).

The estimates from above immediately follow from (15) together with (27) and Proposition 3.3(ii). For the converse assertions we use special atoms (where we do not need any moment conditions, recall (10)). We adapt the unweighted arguments appropriately and construct functions $f_{j,x_0} \in B^s_{p,q}(w)$ with $\|f_{j,x_0}|B^s_{p,q}(w)\| \sim 1$, such that

$$\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w)}(2^{-j}) \geq c \sup_{x_0} \frac{\omega(f_{j,x_0}, 2^{-j})}{2^{-j}}, \quad j \in \mathbb{N}.$$

Step 2. We begin with (i) and (iii) and assume that $\frac{n}{p} \leq s < \frac{n}{p}+1$. Let for $x_0 \in \mathbb{R}^n$, $j \in \mathbb{N}$,

$$f_{j,x_0}(x) = 2^{-js} \varphi\left(2^j(x-x_0)\right) w\left(B(x_0, 2^{-j})\right)^{-1/p}, \tag{33}$$

where φ is a mollified version of

$$\widetilde{\varphi}(x) = \begin{cases} 0, & |x| \geq 1, \\ 1-|x|, & |x| \leq 1, \end{cases} \qquad x \in \mathbb{R}^n,$$

such that $\operatorname{supp} \varphi(2^j \cdot) \subset \{y \in \mathbb{R}^n : |y| \leq c2^{-j}\}$, $j \in \mathbb{N}$, and

$$\frac{\omega\left(\varphi(2^j \cdot), t\right)}{t} \sim 2^j, \qquad t \sim 2^{-j}, \quad j \in \mathbb{N}.$$

Thus

$$\frac{\omega(f_j, t)}{t} \sim 2^{-j(s-1)} w\left(B(x_0, 2^{-j})\right)^{-1/p}, \qquad t \sim 2^{-j}, \quad j \in \mathbb{N}.$$

We put

$$a_j(x) = 2^{-j(s-n/p)} \varphi\left(2^j(x-x_0)\right)$$

and observe that these are special atoms according to Definition 2.9, since $\operatorname{supp} a_j \subset \operatorname{supp} \varphi(2^j(\cdot - x_0)) \subset B(x_0, 2^{-j})$,

$$|\mathrm{D}^\alpha a_j(x)| \leq c_{\alpha,\varphi} 2^{-j(s-n/p)+j|\alpha|}, \quad |\alpha| \leq K,$$

and our assumption on s implies that we do not need to impose moment conditions, see (10). Now let $\lambda_j = 2^{-jn/p} w\left(B(x_0, 2^{-j})\right)^{-1/p}$, then

$$f_{j,x_0}(x) = \lambda_j a_j(x)$$

is a special atomic decomposition (11) and we obtain

$$\left\|f_{j,x_0}|B^s_{p,q}(w)\right\| \leq \|\lambda|b_{pq}(w)\| \sim \lambda_j 2^{jn/p} w\left(B(x_0, 2^{-j})\right)^{1/p} = 1.$$

This leads to

$$\begin{aligned} \mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w)}(2^{-k}) &\geq c \sup_{j,x_0} \frac{\omega(f_{j,x_0}, 2^{-k})}{2^{-k}} \\ &\geq c \sup_{x_0} \frac{\omega(f_{k,x_0}, 2^{-k})}{2^{-k}} \\ &\geq c' \sup_{x_0} 2^{-k(s-1)} w\left(B(x_0, 2^{-k})\right)^{-1/p} \\ &\geq c'' 2^{-k(s-n/p-1)} \sup_{x_0} \Big(\frac{|B(x_0, 2^{-k})|}{w\left(B(x_0, 2^{-k})\right)} \Big)^{1/p}. \end{aligned}$$

In view of (31) and (32) it is sufficient to prove that there exists some $x_0 \in \mathbb{R}^n$ such that

$$\frac{w\left(B(x_0, 2^{-k})\right)}{|B(x_0, 2^{-k})|} \leq c \tag{34}$$

independent of $k \in \mathbb{N}_0$; but since $|\mathbf{S}_\infty(w)| = |\mathbf{S}_{\rm sing}(w)| = 0$, recall Remark 2.6, we can always find some $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ and this completes the argument for (i) and (iii).

Step 3. We modify the above approach in order to prove (ii). Let again $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ and put

$$f_m(x) = m^{-1/q} \sum_{j=1}^{m} 2^{-j} \varphi\left(2^j(x-x_0)\right), \quad m \in \mathbb{N}. \tag{35}$$

Then

$$\omega\left(f_m, 2^{-k}\right) \geq m^{-1/q} k\, 2^{-k}, \quad k = 1, \ldots, m,$$

in particular,

$$\frac{\omega\left(f_m, 2^{-m}\right)}{2^{-m}} \geq m^{1-1/q}, \quad m \in \mathbb{N}. \tag{36}$$

Regarding (35) as an atomic decomposition of f_m (with $a_j(x) = 2^{-j}\varphi\left(2^j(x-x_0)\right)$, $\lambda_j = m^{-1/q}$, $j = 1, \ldots, m$), we conclude that

$$\left\| f_m | B^{1+n/p}_{p,q}(w) \right\| \leq \|\lambda | b_{pq}(w)\| \sim m^{-1/q} \Big(\sum_{j=1}^{m} 2^{jqn/p} w\big(B(x_0, 2^{-j})\big)^{q/p} \Big)^{1/q} \leq c, \tag{37}$$

where we applied (34) with $j = k$ and $x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$. The rest is similar to Step 2, since (36) implies

$$\mathcal{E}_{\mathsf{C}}^{B^{1+n/p}_{p,q}(w)}(2^{-k}) \geq c \sup_m \frac{\omega\left(f_m, 2^{-k}\right)}{2^{-m}} \geq c' \frac{\omega\left(f_k, 2^{-k}\right)}{2^{-k}} \geq c'' k^{1/q'}, \quad k \in \mathbb{N}.$$

This concludes the proof. ■

Remark 4.4. Note that we did not use the assumptions $w \in \mathcal{A}_1$ and (20) in Step 2 of the above proof. Hence we always obtain

$$\mathcal{E}_{\mathsf{C}}^{B^{s}_{p,q}(w)}(2^{-k}) \geq c\, 2^{-k(s-n/p-1)} \sup_{x_0 \in \mathbb{R}^n} \Big(\frac{|B(x_0, 2^{-k})|}{w(B(x_0, 2^{-k}))} \Big)^{1/p}, \tag{38}$$

leading to

$$\mathcal{E}_{\mathsf{C}}^{A^{s}_{p,q}(w)}(t) \geq c t^{-n/p+s-1} \quad \text{for} \quad t \to 0,$$

where $w \in \mathcal{A}_\infty$ and $s \geq \frac{n}{p}$.

4.2. Continuity envelopes. We complete the characterisation of $A^s_{p,q}(w)$, $w \in \mathcal{A}_1$, in terms of their continuity envelopes.

Theorem 4.5. *Let* $0 < p < \infty$, $0 < q \leq \infty$, $w \in \mathcal{A}_1$ *with* (20).

(i) *Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(w)) = \begin{cases} (t^{-n/p+s-1}, q), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/q'}, q), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < q \leq \infty. \end{cases}$$

(ii) *Then*

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(w)) = \begin{cases} (t^{-n/p+s-1}, p), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/p'}, p), & s = \frac{n}{p}+1 \quad \textit{and} \quad 1 < p < \infty. \end{cases}$$

Proof. In view of Proposition 4.3, (28), and Proposition 3.3(ii) it remains to prove that $u_{\mathsf{C}}^{B^s_{p,q}(w)} \geq q$ and $u_{\mathsf{C}}^{F^s_{p,q}(w)} \geq p$. By Proposition 2.16(iii) and another application of (28) we may restrict ourselves to the B-case.

Step 1. Let first $\frac{n}{p} < s < \frac{n}{p}+1$ and $\varepsilon > 0$. We have to verify that

$$\left(\int_0^\varepsilon \left[\frac{\omega(f,t)}{t^{s-n/p}}\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \leq c \left\|f|B^s_{p,q}(w)\right\| \tag{39}$$

for all $f \in B^s_{p,q}(w)$ implies $v \geq q$. For this purpose we construct extremal functions based on a combination of the functions f_{j,x_0} given by (33). We choose $\{x_j\}_j \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$ with, say, $|x_j - x_r| \geq 4$, $j \neq r$, such that $\operatorname{supp} \varphi(2^j(\cdot - x_j)) \cap \operatorname{supp} \varphi(2^r(\cdot - x_r)) = \emptyset$ for $j \neq r$, $j, r \in \mathbb{N}_0$, and φ behaves as described in Step 2 of the proof of Proposition 4.3 above. Let $\{b_j\}_{j\in\mathbb{N}}$ be a sequence of non-negative numbers where we may assume, in addition, that $b_1 = \ldots = b_{J-1} = 0$, and J is suitably chosen such that $2^{-J} \sim \varepsilon$. Let

$$f_b(x) = \sum_{j=1}^\infty 2^{-js} b_j \varphi(2^j(x-x_j)) w(B(x_j, 2^{-j}))^{-1/p}. \tag{40}$$

Seen as atomic decomposition of f_b with atoms $a_j = 2^{-j(s-n/p)}\varphi(2^j(\cdot - x_j))$ and coefficients $\lambda_j = 2^{-jn/p} b_j w(B(x_j, 2^{-j}))^{-1/p}$, this implies

$$\begin{aligned} \left\|f_b|B^s_{p,q}(w)\right\| &\leq \|\lambda|b_{pq}(w)\| \\ &\leq c\Big(\sum_{j=J}^\infty 2^{-jqn/p} w(B(x_j,2^{-j}))^{-q/p} b_j^q 2^{jqn/p} w(B(x_j,2^{-j}))^{q/p}\Big)^{1/q} \sim \|b|\ell_q\|. \end{aligned}$$

Since

$$\omega(f_b, 2^{-j}) \geq c b_j 2^{-js} w(B(x_j,2^{-j}))^{-1/p} \geq c' b_j 2^{-j(s-n/p)}, \quad j \in \mathbb{N}, \tag{41}$$

inequality (39) can be extended on both sides to

$$\|b|\ell_v\| = \Big(\sum_{j=J}^\infty b_j^v\Big)^{1/v} \leq c_1 \left(\int_0^\varepsilon \left[\frac{\omega(f_b,t)}{t^{s-n/p}}\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \leq c_2 \|f_b|B^s_{p,q}(w)\| \leq c_3 \|b|\ell_q\|$$

for arbitrary sequences of non-negative numbers. This obviously requires $v \geq q$.

Step 2. It remains to consider the situation $s = \frac{n}{p}+1$. We have to show the sharpness of $v = q$ in

$$\left(\int_0^\varepsilon \left[\frac{\omega(f,t)}{t\,|\log t|^{1/q'+1/v}}\right]^v \frac{\mathrm{d}t}{t}\right)^{1/v} \leq c\left\|f|B^{1+n/p}_{p,q}(w)\right\|, \quad f \in B^{1+n/p}_{p,q}(w), \tag{42}$$

but this works exactly as before, that is, we proceed by contradiction and assume $v < q$. We refine the approach presented in Step 3 of the proof of Proposition 4.3. Let

$x_0 \in \mathbb{R}^n \setminus \mathbf{S}_\infty(w)$, $m \in \mathbb{N}$, and

$$f_{m,b}(x) = \sum_{j=1}^{m} b_j 2^{-j} \varphi\left(2^j(x - x_0)\right), \quad x \in \mathbb{R}^n, \tag{43}$$

where

$$b_j = j^{-1/q}(1 + \log j)^{-1/v}, \quad j = 1, \ldots, m,$$

in (40). Then similarly to (37),

$$\left\| f_{m,b} | B^{1+n/p}_{p,q}(w) \right\| \le c \, \|b|\ell_q\| = c\Big(\sum_{j=1}^{m} \frac{1}{j(1 + \log j)^{q/v}}\Big)^{1/q} \le c_2$$

since $v < q$, where c_2 does not depend on $m \in \mathbb{N}$. On the other hand, by our choice of $\{b_j\}_j$,

$$\omega(f_{m,b}, 2^{-k}) \ge c 2^{-k} \sum_{j=1}^{k} b_j \ge c 2^{-k} k b_k, \quad k = 1, \ldots, m,$$

that is,

$$\frac{\omega(f_{m,b}, 2^{-k})}{2^{-k}} \ge c k^{1-1/q}(1 + \log k)^{-1/v}, \quad k = 1, \ldots, m, \quad m \in \mathbb{N},$$

such that for $m \ge J$,

$$\begin{aligned}\Big(\int_0^\varepsilon \left[\frac{\omega(f_{m,b}, t)}{t\, |\log t|^{1/q'+1/v}}\right]^v \frac{\mathrm{d}t}{t}\Big)^{1/v} &\ge c_1 \Big(\sum_{k=1}^{m} \left[\frac{\omega(f_{m,b}, 2^{-k})}{2^{-k} k^{1/q'+1/v}}\right]^v\Big)^{1/v} \\ &\ge c_2 \Big(\sum_{k=1}^{m} \frac{1}{k(1 + \log k)}\Big)^{1/v}.\end{aligned}$$

Obviously the expression on the right-hand side diverges for $m \to \infty$, such that there are functions $f_{m,b} \in B^{1+n/p}_{p,q}(w)$, not satisfying (42). This completes the proof. ∎

Remark 4.6. It is natural to expect that

$$\mathfrak{E}_{\mathsf{C}}\left(A^{n/p}_{p,q}(w)\right) = \left(t^{-1}, \infty\right), \quad w \in \mathcal{A}_1,$$

where we have, in addition, to assume that $0 < p \le 1$ in case of $F^{n/p}_{p,q}(w)$ and $0 < q \le 1$ in case of $B^{n/p}_{p,q}(w)$. However, the celebrated result in [21] for the unweighted case is rather tricky to prove, so its weighted counterpart is postponed.

Remark 4.7. Let $w \in \mathcal{A}_1$ with (20). Then Proposition 4.3 and Theorem 4.5 describe exactly the counterparts of the unweighted situations with $w \equiv 1$, see Proposition 3.3 (apart from borderline cases). In other words, though we only have the embedding (15) in this setting, the spaces are so close together that their Lipschitz continuity behaviour (measured in continuity envelopes) cannot be distinguished. This phenomenon is already known from similar studies concerning singularity behaviour [15] and questions of compactness, cf. [18].

We separately formulate Theorem 4.5 for our example weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$. Note that (20) requires $\beta \geq 0$. Hence to apply Theorem 4.5 to $w_{\alpha,\beta}$ requires $\beta = 0$ in view of (7), thus only the local behaviour can differ from the unweighted setting.

Corollary 4.8. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s > \frac{n}{p}$.*

(i) *Let $w_{\alpha,\beta}$ be given by (6) with $-n < \alpha \leq 0$, and $\beta = 0$. Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(w_{\alpha,0})) = \begin{cases} \left(t^{-n/p+s-1}, q\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/q'}, q), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < q \leq \infty, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(w_{\alpha,0})) = \begin{cases} \left(t^{-n/p+s-1}, p\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/p'}, p), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < p < \infty. \end{cases}$$

(ii) *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ given by (8) with $-(n-d) < \varkappa \leq 0$. Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-n/p+s-1}, q\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/q'}, q), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < q \leq \infty, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-n/p+s-1}, p\right), & \frac{n}{p} < s < \frac{n}{p}+1, \\ (|\log t|^{1/p'}, p), & s = \frac{n}{p}+1 \quad \text{and} \quad 1 < p < \infty. \end{cases}$$

Remark 4.9. Plainly, the case $-n < \beta < 0$ in the above example, referring to weights $w \in \mathcal{A}_1$ which do not satisfy (20) is of some interest, too, but not yet covered by our above techniques, apart from lower estimates, see Remark 4.4.

Though we dealt in this paper with $\mathcal{A}_1$ weights only, we may extend the above Corollary 4.8 for our examples $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ to values $\alpha \geq 0$, $\beta \geq 0$, $\varkappa \geq 0$. This is also based on the embeddings recalled in Remark 2.15.

Corollary 4.10. *Let $0 < p < \infty$, $0 < q \leq \infty$, and $s > \frac{n}{p}$.*

(i) *Let $\alpha > -n$, $\beta \geq 0$, and $w_{\alpha,\beta}$ given by (6). Assume $s - \frac{\max(\alpha,0)}{p} > \frac{n}{p}$. Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(w_{\alpha,\beta})) = \begin{cases} \left(t^{-n/p+s-1-\max(\alpha,0)/p}, q\right), & s < \frac{n}{p}+1+\frac{\max(\alpha,0)}{p}, \\ (|\log t|^{1/q'}, q), & s = \frac{n}{p}+1+\frac{\max(\alpha,0)}{p},\ 1 < q \leq \infty, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(w_{\alpha,\beta})) = \begin{cases} \left(t^{-n/p+s-1-\max(\alpha,0)/p}, p\right), & s < \frac{n}{p}+1+\frac{\max(\alpha,0)}{p}, \\ (|\log t|^{1/p'}, p), & s = \frac{n}{p}+1+\frac{\max(\alpha,0)}{p},\ 1 < p < \infty. \end{cases}$$

(ii) *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ be given by* (8) *with $\varkappa > -(n-d)$. Assume $s - \frac{\max(\varkappa,0)}{p} > \frac{n}{p}$. Then*

$$\mathfrak{E}_{\mathsf{C}}(B^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-n/p+s-1-\max(\varkappa,0)/p}, q\right), & s < \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \\ (|\log t|^{1/q'}, q), & s = \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \ 1 < q \leq \infty, \end{cases}$$

and

$$\mathfrak{E}_{\mathsf{C}}(F^s_{p,q}(w_{\varkappa,\Gamma})) = \begin{cases} \left(t^{-n/p+s-1-\max(\varkappa,0)/p}, p\right), & s < \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \\ (|\log t|^{1/p'}, p), & s = \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \ 1 < p < \infty. \end{cases}$$

Proof. Step 1. We begin with (i) and may concentrate on the B-case again in view of the extended version of Proposition 2.16(iii) according to Remark 2.17. Thus the upper estimates are a consequence of embedding (17) together with Proposition 3.3(ii). The lower estimate for $\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(t)$ in case of $s < \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}$ immediately follows from (38) and $x_0 = 0$, recall (9). So all what is left to show is the lower bound for $\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}(t)$ in case of $s = \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}$, as well as the lower bound for the index $u_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})}$. We modify the corresponding arguments in the above proofs appropriately. Let $\alpha > 0$ and $s = \frac{n+\alpha}{p} + 1$. We use f_m given by (35) with $x_0 = 0$ and observe that again this can be understood as an atomic decomposition of f_m with atoms $a_j(x) = 2^{-j(1+\alpha/p)}\varphi(2^j x)$ and coefficients $\lambda_j = m^{-1/q}2^{j\alpha/p}$, $j = 1, \ldots, m$. We conclude that

$$\left\| f_m | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leq \| \lambda | b_{pq}(w_{\alpha,\beta}) \| \sim m^{-1/q} \Big(\sum_{j=1}^{m} 2^{jq(n+\alpha)/p} w(B(0, 2^{-j}))^{q/p} \Big)^{1/q} \leq c,$$

where we applied (9). The rest works as above. It remains to show that $u_{\mathsf{C}}^{B^s_{p,q}(w_{\alpha,\beta})} \geq q$ if $\frac{n+\alpha}{p} < s \leq \frac{n+\alpha}{p} + 1$. We proceed similarly to the approach in the proof of Theorem 4.5 and consider the function

$$f_b(x) = \sum_{j=1}^{\infty} 2^{-j(s-(n+\alpha)/p)} b_j \varphi(2^j x),$$

where $\{b_j\}_{j\in\mathbb{N}}$ is a sequence of non-negative numbers. Since $2^{-j(s-n/p)}\varphi(2^j x)$ are atoms according to Definition 2.9 (no moment conditions needed), we obtain by Proposition 2.10 that

$$\left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leq c \Big(\sum_{j=1}^{\infty} 2^{jq(n+\alpha)/p} b_j^q w(B(0, 2^{-j}))^{q/p} \Big)^{1/q}, \tag{44}$$

(with obvious modification if $q = \infty$). Since $w(B(0, 2^{-j})) \sim 2^{-j(\alpha+n)}$, (44) implies that

$$\left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \leq c \, \| b | \ell_q \| \,. \tag{45}$$

We follow the same line of arguments as in the proof of Theorem 4.5. The counterpart of (39) reads for $s < \frac{n+\alpha}{p} + 1$ as

$$\left(\int_0^{\varepsilon} \left[t^{(n+\alpha)/p-s} \, \omega(f_b, t) \right]^v \frac{\mathrm{d}t}{t} \right)^{1/v} \leq c \left\| f_b | B^s_{p,q}(w_{\alpha,\beta}) \right\| \tag{46}$$

and can thus be extended on both sides to

$$\Big(\sum_{j=J}^{\infty} b_j^v\Big)^{1/v} \le c_1\bigg(\int_0^{\varepsilon} \Big[t^{(n+\alpha)/p-s}\,\omega(f_b,t)\Big]^v \frac{\mathrm{d}t}{t}\bigg)^{1/v} \le c_2 \left\|f_b|B^s_{p,q}(w_{\alpha,\beta})\right\| \le c_3\|b|\ell_q\|$$

for arbitrary sequences of non-negative numbers with, say, $b_1 = \ldots = b_J = 0$ for some $J \in \mathbb{N}$ with $2^{-J} \sim \varepsilon$. This follows by (45), (46) and the counterpart of (41),

$$\omega(f_b, 2^{-j}) \ge c b_j 2^{-j(s-(n+\alpha)/p)},$$

and leads to $v \ge q$. In case of $s = \frac{n+\alpha}{p} + 1$ we take $b_j = 0$, $j > m$, such that f_b obtains the special form

$$f_b(x) = \sum_{j=1}^{m} b_j 2^{-j}\varphi(2^j x), \quad m \in \mathbb{N}.$$

The rest is now completely parallel to the end of the proof of Theorem 4.5.

Step 2. The argument to verify part (ii) works completely parallel. Since we now have $\mathbf{S}_{\rm sing}(w_{\varkappa,\Gamma}) = \Gamma$, $\varkappa \ne 0$, instead of $\mathbf{S}_{\rm sing}(w_{\alpha,\beta}) = \{0\}$, $\alpha \ne 0$, one has to choose $x_0 \in \Gamma$ for $\varkappa > 0$ accordingly. ∎

REMARK 4.11. The above result does not only extend Corollary 4.8 above, but (i) also completes partial forerunners in [13, 14], recall also (29). In case of singularity behaviour characterised by growth envelopes the complete counterpart can be found in [15].

5. Applications. We briefly present some typical applications of the preceding envelope results: Hardy type inequalities, sharp embedding criteria, and estimates for approximation numbers of related compact embeddings.

5.1. Hardy type inequalities

COROLLARY 5.1. *Let $0 < p < \infty$, $0 < q \le \infty$, $s > \frac{n}{p}$, $w \in \mathcal{A}_1$ with (20), and $\varepsilon > 0$ be small.*

(i) *Let $\frac{n}{p} < s < \frac{n}{p} + 1$, $0 < u \le \infty$ and let $\varkappa$ be a positive monotonically decreasing function on $(0, \varepsilon]$. Then*

$$\bigg(\int_0^{\varepsilon} \Big[\varkappa(t) t^{n/p-s}\omega(f,t)\Big]^u \frac{\mathrm{d}t}{t}\bigg)^{1/u} \le c\,\|f|A^s_{p,q}(w)\|$$

for some $c > 0$ and all $f \in A^s_{p,q}(w)$ if and only if $\varkappa$ is bounded and

$$\begin{cases} q \le u \le \infty, & \text{if} \quad A^s_{p,q} = B^s_{p,q}, \\ p \le u \le \infty, & \text{if} \quad A^s_{p,q} = F^s_{p,q}, \end{cases}$$

with the modification

$$\sup_{t\in(0,\varepsilon)} \varkappa(t) t^{n/p-s}\omega(f,t) \le c\,\|f|A^s_{p,q}(w)\| \tag{47}$$

if $u = \infty$. In particular, if $\varkappa$ is an arbitrary non-negative function on $(0, \varepsilon]$, then (47) holds if and only if $\varkappa$ is bounded.

(ii) *Let* $s = \frac{n}{p} + 1$, $1 < q \le \infty$, $0 < u \le \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$. *Then*

$$\left(\int_0^{\varepsilon} \left[\varkappa(t) \, \frac{\omega(f,t)}{t \, |\log t|^{1/q'}} \right]^u \frac{\mathrm{d}t}{t\,|\log t|} \right)^{1/u} \le c \left\| f | B^{1+n/p}_{p,q}(w) \right\|$$

for some $c > 0$ *and all* $f \in B^{1+n/p}_{p,q}(w)$ *if and only if* $\varkappa$ *is bounded and* $q \le u \le \infty$, *with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t) \, \frac{\omega(f,t)}{t \, |\log t|^{1/q'}} \le c \left\| f | B^{1+n/p}_{p,q}(w) \right\| \tag{48}$$

if $u = \infty$. *In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$, *then* (48) *holds if and only if* $\varkappa$ *is bounded.*

(iii) *Let* $s = \frac{n}{p} + 1$, $1 < p < \infty$, $0 < u \le \infty$ *and let* $\varkappa$ *be a positive monotonically decreasing function on* $(0, \varepsilon]$. *Then*

$$\left(\int_0^{\varepsilon} \left[\varkappa(t) \, \frac{\omega(f,t)}{t \, |\log t|^{1/p'}} \right]^u \frac{\mathrm{d}t}{t\,|\log t|} \right)^{1/u} \le c \left\| f | F^{1+n/p}_{p,q}(w) \right\|$$

for some $c > 0$ *and all* $f \in F^{1+n/p}_{p,q}(w)$ *if and only if* $\varkappa$ *is bounded and* $p \le u \le \infty$, *with the modification*

$$\sup_{t \in (0,\varepsilon)} \varkappa(t) \, \frac{\omega(f,t)}{t \, |\log t|^{1/p'}} \le c \left\| f | F^{1+n/p}_{p,q}(w) \right\| \tag{49}$$

if $u = \infty$. *In particular, if* $\varkappa$ *is an arbitrary non-negative function on* $(0, \varepsilon]$, *then* (49) *holds if and only if* $\varkappa$ *is bounded.*

This follows immediately from Definition 3.1 and Theorem 4.5. Of course, the above Hardy type inequalities can be explicated for the particular weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ considered in Corollary 4.10, but this is left to the reader.

5.2. Sharp embedding criteria. Another type of application concerns sharp (or limiting) embeddings which naturally can be understood as sharp inequalities, too. We begin with a general result and restrict ourselves afterwards to a few model cases only to demonstrate the method.

Corollary 5.2. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s > \frac{n}{p}$, $w \in \mathcal{A}_1$ *with* (20). *Then*

$$B^s_{p,q}(w) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \quad \begin{cases} s > \frac{n}{p} + 1, & \textit{or} \\ s = \frac{n}{p} + 1 \quad \textit{and} & 0 < q \le 1. \end{cases}$$

Similarly,

$$F^s_{p,q}(w) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \quad \begin{cases} s > \frac{n}{p} + 1, & \textit{or} \\ s = \frac{n}{p} + 1 \quad \textit{and} & 0 < p \le 1. \end{cases}$$

Proof. Again the F-result follows from the B-assertion, embeddings (19) and Proposition 2.16(iii). The sufficiency is covered by Lemma 4.1(ii), so it remains to disprove $B^s_{p,q}(w) \hookrightarrow \mathrm{Lip}^1$ when $s < \frac{n}{p} + 1$ or $s = \frac{n}{p} + 1$ and $1 < q \le \infty$. However, in these situations we have the unboundedness of $\mathcal{E}_{\mathsf{C}}^{B^s_{p,q}(w)}(t)$ when $t \to 0$ in view of Proposition 4.3(i),(ii) which is by Proposition 3.2(ii) equivalent to $B^s_{p,q}(w) \not\hookrightarrow \mathrm{Lip}^1$. ∎

REMARK 5.3. The above criterion completes Proposition 4.1(ii) and can be seen as the counterpart of Proposition 4.1(i). In the unweighted setting $w \equiv 1$ this is well-known, cf. [35, Thm. 11.4] and [7, Thm. 2.1].

In case of our special weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ this can be extended based on Corollary 4.10.

COROLLARY 5.4. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$.*

(i) *Assume that $\alpha > -n$, $\beta \geq 0$. Then*

$$B^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \begin{cases} \textit{either} & s > \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}, \quad 0 < q \leq 1, \end{cases}$$

and

$$F^s_{p,q}(w_{\alpha,\beta}) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \begin{cases} \textit{either} & s > \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + 1 + \frac{\max(\alpha,0)}{p}, \quad 0 < p \leq 1. \end{cases}$$

(ii) *Let $\Gamma \subset \mathbb{R}^n$ be a d-set, $0 < d < n$, and $w_{\varkappa,\Gamma}$ be given by (8) with $\varkappa > -(n-d)$. Then*

$$B^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \begin{cases} \textit{either} & s > \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \quad 0 < q \leq 1, \end{cases}$$

and

$$F^s_{p,q}(w_{\varkappa,\Gamma}) \hookrightarrow \mathrm{Lip}^1 \quad \textit{if and only if} \begin{cases} \textit{either} & s > \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \\ \textit{or} & s = \frac{n}{p} + 1 + \frac{\max(\varkappa,0)}{p}, \quad 0 < p \leq 1. \end{cases}$$

REMARK 5.5. In the same spirit one can prove criteria for limiting embeddings between spaces of type $A^s_{p,q}(w)$; for instance, we have shown in [15] with similar arguments that for $w \in \mathcal{A}_1$ with (20), $0 < p_0 < p < p_1 < \infty$, $s_1 < s < s_0$ which satisfy (21), and $0 < u, v \leq \infty$,

$$B^{s_0}_{p_0,u}(w) \hookrightarrow F^s_{p,q}(w) \hookrightarrow B^{s_1}_{p_1,v}(w) \quad \text{if and only if} \quad u \leq p \leq v. \tag{50}$$

This obviously refines Proposition 2.16(iii).

5.3. Compact embeddings. Finally we want to focus on the relation between continuity envelopes and approximation numbers of compact embeddings. We briefly recall this concept. Let A_1 and A_2 be two complex (quasi-) Banach spaces and let $T \in \mathcal{L}(A_1, A_2)$ be a linear and continuous operator from A_1 into A_2. The *k-th approximation number* of T is given by

$$a_k(T) = \inf\{\|T - S\| : S \in \mathcal{L}(A_1, A_2),\ \mathrm{rank}\, S < k\}.$$

Now let $\Omega \subset \mathbb{R}^n$ be some bounded domain, X some function space on $\mathbb{R}^n$, and $X(\Omega)$ be defined by restriction. Assume that $X(\Omega) \hookrightarrow C(\Omega)$. We proved in [13, Cor. 11.18] that there exists $c > 0$ such that for all $k \in \mathbb{N}$,

$$a_{k+1}(\mathrm{id} : X(\Omega) \longrightarrow C(\Omega)) \leq c k^{-1/n} \mathcal{E}_{\mathsf{C}}^X(k^{-1/n}). \tag{51}$$

This implies the following result.

Corollary 5.6. *Let $w \in \mathcal{A}_1$ and assume that $\Omega \subset \mathbb{R}^n$ is bounded and sufficiently large such that $\overline{\mathbf{S}_{\mathrm{sing}}(w)} \subsetneq \mathring{\Omega}$. Let $0 < p < \infty$, $0 < q \leq \infty$, $\frac{n}{p} < s < \frac{n}{p} + 1$. Then*

$$a_k \left(\mathrm{id} : A^s_{p,q}(\Omega, w) \hookrightarrow C(\Omega)\right) \leq c k^{-s/n+1/p}, \quad k \in \mathbb{N}.$$

Proof. This is an immediate consequence of (51) together with (the counterpart of) Theorem 4.5. Note that the assumption $\overline{\mathbf{S}_{\mathrm{sing}}(w)} \subsetneq \mathring{\Omega}$ admits that the construction of extremal functions in our proofs can be adapted to this situation immediately, whereas the upper estimates are due to the introduction of these spaces by restriction. Hence we have the counterpart of Theorem 4.5 for spaces $A^s_{p,q}(\Omega, w)$ in this situation. ∎

Remark 5.7. In [18] we could prove that this upper estimate is indeed an equivalence in case of $p > 2$. Plainly, Corollary 5.6 can be explicated for our weights $w_{\alpha,\beta}$ and $w_{\varkappa,\Gamma}$ in the slightly more general situation covered by Corollary 4.10. But this is left to the reader, since the method seems obvious.

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