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FUNCTION SPACES ON THE SNOWFLAKE

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Abstract. We consider two types of Besov spaces on the closed snowflake, defined by traces and with the help of the homeomorphic map from the interval $[0,3]$. We compare these spaces and characterize them in terms of Daubechies wavelets.

1. Introduction. In [2] we introduced two types of Besov spaces on the Koch curve. In the same manner we can define Besov spaces on the closed snowflake SF, which is a d-set with $d = \frac{\log 4}{\log 3}$. The first possibility is to define Besov spaces $B^s_{pq}(\mathrm{SF}, \mu)$ by traces

$$B^s_{pq}(\mathrm{SF}, \mu) = \mathrm{tr}_\mu\, B^{s+(2-d)/p}_{pq}(\mathbb{R}^2), \quad 1 < p < \infty,\ 0 < q < \infty,\ 0 < s < \infty.$$

The second way is to use the homeomorphic map $\widetilde{H}$ between interval $[0,3] = 3\mathbb{T}$ and SF and define $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ by

$$\mathbb{B}^s_{pq}(\mathrm{SF}, \mu) = \big\{ f \circ \widetilde{H}^{-1} : f \in B^s_{pq}(3\mathbb{T}) \big\} = B^s_{pq}(3\mathbb{T}) \circ \widetilde{H}^{-1},$$

where $B^s_{pq}(3\mathbb{T})$ are 3-periodic Besov spaces.

In the present paper we consider how these types of spaces are interrelated. First we concentrate on the case when $1 < p = q < \infty$, $0 < s < 1$ and then extend our result to the case when $p \neq q$.

This paper is organized as follows. In Section 2 we collect the definitions and preliminaries. For our purposes we slightly modify the definitions and theorems concerning 2π-periodic Besov spaces defined in [3]. We describe the trace method of defining Besov spaces and their characterization in terms of atoms. In Section 3 we present the wavelet characterization of the 3-periodic Besov spaces $B^s_{pq}(3\mathbb{T})$ and then shift it to SF. Then we compare $B^s_{pq}(\mathrm{SF}, \mu)$ and $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$. The main result is contained in Theorem 3.8.

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2. Preliminaries

2.1. Notation and basic definitions. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbb{Z}$ is the set of all integers. Let $\mathbb{R}^n$ be Euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. $L_p(\mathbb{R}^n)$ with $0 < p \leq \infty$, is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}, \quad 0 < p < \infty,$$
$$\|f|L_\infty(\mathbb{R}^n)\| = \operatorname*{ess-sup}_{x\in\mathbb{R}^n} |f(x)|\,.$$

If $\varphi \in S(\mathbb{R}^n)$ then

$$\widehat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(x) e^{-ix\xi}\, dx, \qquad \xi \in \mathbb{R}^n,$$

denotes the Fourier transform of φ. The inverse Fourier transform is given by

$$\varphi^\vee(x) = \mathcal{F}^{-1}\varphi(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \varphi(\xi) e^{ix\xi}\, d\xi, \qquad x \in \mathbb{R}^n.$$

One extends $\mathcal{F}$ and $\mathcal{F}^{-1}$ in the usual way from S to S'. Namely, for $f \in S'(\mathbb{R}^n)$,

$$\mathcal{F}f(\varphi) = f(\mathcal{F}\varphi), \qquad \varphi \in S(\mathbb{R}^n).$$

Let $\varphi_0 \in S(\mathbb{R}^n)$ with

$$\varphi_0(x) = 1, \quad |x| \leq 1 \quad \text{and} \quad \varphi_0(x) = 0, \quad |x| \geq \frac{3}{2}, \tag{1}$$

and let

$$\varphi_k(x) = \varphi_0(2^{-k}x) - \varphi_0(2^{-k+1}x), \quad x \in \mathbb{R}^n, \quad k \in \mathbb{N}. \tag{2}$$

Then, since

$$1 = \sum_{j=0}^{\infty} \varphi_j(x) \qquad \text{for all } x \in \mathbb{R}^n, \tag{3}$$

the φ_j form a dyadic resolution of unity in $\mathbb{R}^n$. According to the Paley-Wiener-Schwartz theorem, $\left(\varphi_k \widehat{f}\right)^\vee$ is an entire analytic function on $\mathbb{R}^n$ for any $f \in S'(\mathbb{R}^n)$. In particular, $\left(\varphi_k \widehat{f}\right)^\vee(x)$ makes sense pointwise.

Definition 2.1. Let $\varphi = \{\varphi_j\}_{j=0}^\infty$ be the dyadic resolution of unity according to (1)–(3), $s \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$ and

$$\|f|B^s_{pq}(\mathbb{R}^n)\|_\varphi = \left(\sum_{j=0}^{\infty} 2^{jsq} \|\left(\varphi_k \widehat{f}\right)^\vee |L_p(\mathbb{R}^n)\|^q\right)^{1/q}$$

(with the usual modification if $q = \infty$). Then the *Besov space* $B^s_{pq}(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that $\|f|B^s_{pq}(\mathbb{R}^n)\|_\varphi < \infty$, [4].

2.2. Trace spaces $B^s_{pq}(\Gamma, \mu)$

Definition 2.2. A measure μ in $\mathbb{R}^n$ is called *Radon* if all Borel sets are μ-measurable and

- $\mu(K) < \infty$ for compact sets $K \subset \mathbb{R}^n$,
- $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}$ for open sets $V \subset \mathbb{R}^n$,
- $\mu(A) = \inf\{\mu(V) : A \subset V,\ V \text{ is open}\}$ for $A \subset \mathbb{R}^n$.

Definition 2.3. A compact set Γ in $\mathbb{R}^n$ is called a *d-set* with $0 < d < n$ if there is a Radon measure μ in $\mathbb{R}^n$ with support Γ such that for some positive constants c_1 and c_2,

$$c_1 r^d \le \mu(B(\gamma, r)) \le c_2 r^d, \quad \gamma \in \Gamma,\ 0 < r < 1,\ 0 < d < n. \tag{4}$$

where $B(x, r)$ is a ball in $\mathbb{R}^n$ centred at $x \in \mathbb{R}^n$ and of radius $r > 0$.

If Γ is a d-set, then the restriction to Γ of the d-dimensional Hausdorff measure H^d satisfies (4) and any measure μ satisfying (4) is equivalent to $\mathrm{H}^d|_\Gamma$. A consequence of this is that the Hausdorff dimension of Γ is d.

$L_p(\Gamma, \mu)$ with $0 < p \le \infty$, is the standard quasi-Banach space (Banach when $p \ge 1$) with respect to measure μ, quasi-normed by

$$\|f|L_p(\Gamma, \mu)\| = \left(\int_\Gamma |f(\gamma)|^p \ \mu(d\gamma)\right)^{1/p}, \qquad 0 < p \le \infty,$$

with usual modification when $p = \infty$.

Definition 2.4. Let μ be a Radon measure in $\mathbb{R}^n$. Let

$$s > 0, \quad 1 < p < \infty, \quad 0 < q < \infty. \tag{5}$$

Let for some $c > 0$,

$$\int_\Gamma |\varphi(\gamma)| \ \mu(d\gamma) \le c\|\varphi|B^s_{pq}(\mathbb{R}^n)\| \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{6}$$

Then the *trace operator* tr_μ,

$$\mathrm{tr}_\mu : B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_1(\Gamma, \mu),$$

is the completion of the pointwise trace $(\mathrm{tr}_\mu\, \varphi)(\gamma) = \varphi(\gamma)$, $\varphi \in S(\mathbb{R}^n)$. Furthermore, the image of tr_μ is denoted by $\mathrm{tr}_\mu\, B^s_{pq}(\mathbb{R}^n)$ and is quasi-normed by

$$\|g|\,\mathrm{tr}_\mu\, B^s_{pq}(\mathbb{R}^n)\| = \inf\left\{\|f|B^s_{pq}(\mathbb{R}^n)\| : \mathrm{tr}_\mu f = g\right\}.$$

Remark 2.5. The above definition is justified since $S(\mathbb{R}^n)$ is dense in $B^s_{pq}(\mathbb{R}^n)$ with (5). We refer to [4], Theorem 2.3.3, p. 48. Due to (6), the trace of f is independent of the approximation of f in $B^s_{pq}(\mathbb{R}^n)$ by $S(\mathbb{R}^n)$-functions.

Definition 2.6. Let Γ be a d-set in $\mathbb{R}^n$. Let $s > 0$, $1 < p < \infty$, $0 < q < \infty$. Then

$$B^s_{pq}(\Gamma, \mu) = \mathrm{tr}_\mu\, B^{s+(n-d)/p}_{pq}(\mathbb{R}^n).$$

The following assertion is covered by Theorem 3, p. 155 in [1], we also refer to [5, Section 1.17.2].

THEOREM 2.7. *Let Γ be a compact d-set in $\mathbb{R}^n$ with $0 < d < n$ and let μ be a corresponding Radon measure. Let $0 < s < 1$, $1 < p < \infty$, $1 \le q \le \infty$, and let tr_μ be the trace operator. Then there is a common linear and bounded extension operator ext_μ with*

$$\mathrm{ext}_\mu : B^s_{pq}(\Gamma, \mu) \hookrightarrow B^{s+(n-d)/p}_{pq}(\mathbb{R}^n) \tag{7}$$

and

$$\mathrm{tr}_\mu \circ \mathrm{ext}_\mu = \mathrm{id} \qquad (\textit{identity in } B^s_{pq}(\Gamma, \mu)). \tag{8}$$

2.3. Atomic characterizations of $B^s_{pq}(\Gamma, \mu)$. Besov spaces on d-sets $B^s_{pq}(\Gamma, \mu)$ with $0 < s < 1$ and $1 < p = q < \infty$ can be characterized in terms of intrinsic building blocks, namely atoms.

Let

$$\Gamma_\delta = \bigcup_{\gamma \in \Gamma} B(\gamma, \delta), \quad \delta > 0,$$

where

$$B(\gamma, \delta) = \{x \in \mathbb{R}^n : |x - \gamma| \le \delta\}, \tag{9}$$

be a δ-neighbourhood of Γ. Let $\varepsilon > 0$ be fixed. Let for $j \in \mathbb{N}_0$,

$$\{\gamma_{j,k}\}_{k=1}^{M_j} \subset \Gamma \tag{10}$$

be a lattice of points with the following properties:

- For some $c_1 > 0$

$$|\gamma_{j,k_1} - \gamma_{j,k_2}| \ge c_1 2^{-\varepsilon j}, \quad j \in \mathbb{N}_0, \quad k_1 \ne k_2. \tag{11}$$

- For some $j_0 \in \mathbb{N}$, some $c_2 > 0$ and $\delta_j = c_2 2^{-\varepsilon j}$,

$$\Gamma_{\delta_j} \subset \bigcup_{k=1}^{M_j} B(\gamma_{j,k}, 2^{-\varepsilon(j+2j_0)}), \quad j \in \mathbb{N}_0, \tag{12}$$

  where $B(\gamma_{j,k}, 2^{-\varepsilon(j+2j_0)})$ are given by (9).

DEFINITION 2.8. Let Γ be a d-set in $\mathbb{R}^n$. Let

$$\varepsilon > 0, \quad 1 < p < \infty, \quad 0 < s < 1.$$

Let

$$B^\Gamma_{j,k} = \{\gamma \in \Gamma : |\gamma - \gamma_{j,k}| \le 2^{-\varepsilon j}\}, \quad j \in \mathbb{N}_0,\ k = 1, \ldots, M_j, \tag{13}$$

be the intersection of a ball in $\mathbb{R}^n$ with Γ, where the lattices $\{\gamma_{j,k}\}_{k=1}^{M_j}$ have the same meaning as in (10)–(12). Then a Lipschitz-continuous function a_{jk} on Γ is called an $(s,p)^*$-*atom*, more precisely an $(s,p)^*$-ε-*atom*, if for $j \in \mathbb{N}_0$ and $k = 1, \ldots, M_j$,

$$\mathrm{supp}\, a_{jk} \subset B^\Gamma_{j,k},$$
$$|a_{jk}(\gamma)| \le c\, \mathrm{H}^d(B^\Gamma_{j,k})^{s/d-1/p}, \qquad \gamma \in \Gamma,$$

and

$$|a_{jk}(\gamma) - a_{jk}(\delta)| \le c\, \mathrm{H}^d(B^\Gamma_{j,k})^{(s-1)/d-1/p}\, |\gamma - \delta|$$

with $\gamma, \delta \in \Gamma$.

Now we can formulate an intrinsic atomic decomposition of the trace spaces $B^s_{pp}(\Gamma,\mu)$.

THEOREM 2.9. *Let* $1 < p < \infty$, *and* $0 < s < 1$. *Let* $\varepsilon > 0$. *Then* $B^s_{pp}(\Gamma,\mu)$ *is the collection of all* $f \in L_1(\Gamma,\mu)$ *which can be represented as*

$$f(\gamma) = \sum_{j=0}^{\infty}\sum_{k=1}^{M_j} \lambda_j^k a_{jk}(\gamma), \quad \gamma \in \Gamma, \tag{14}$$

where

$$\|\lambda\| = \left(\sum_{j=0}^{\infty}\sum_{k=1}^{M_j} \left|\lambda_j^k\right|^p\right)^{1/p} < \infty,$$

a_{jk} *are* (s,p)-ε-*atoms and* (14) *converges absolutely in* $L_1(\Gamma,\mu)$. *Furthermore,*

$$\|f|B^s_{pp}(\Gamma,\mu)\| \sim \inf \|\lambda\|$$

where infimum is taken over all admissible representations (14), [5, Chapter 8.1.3].

2.4. Periodic Besov spaces. The theory of periodic Besov spaces may be found in [3]. We slightly modify the definitions and theorems given there to consider 3-periodic functions.

Let

$$\mathbb{T} = \{x \in \mathbb{R} : 0 \le x \le 1\},$$

where the points 0 and 1 are identified. Let

$$3\mathbb{T} = \{x \in \mathbb{R} : 0 \le x \le 3\}$$

with the points 0 and 3 being identified. We can interpret $3\mathbb{T}$ as a circle of radius $\frac{3}{2\pi}$ with the centre at the origin. We define the distance $\rho(x,y)$ between two points $x, y \in 3\mathbb{T}$ as the length of the shortest arc on the circle connecting them, i.e.

$$\rho(x,y) = \min\{|x-y|, 3 - |x-y|\}. \tag{15}$$

By $D(3\mathbb{T})$ we denote the collection of all complex-valued infinitely differentiable functions on $3\mathbb{T}$. The topology in $D(3\mathbb{T})$ is generated by the family of semi-norms

$$\|\varphi\|_\alpha = \sup_{x \in 3\mathbb{T}} |D^\alpha \varphi(x)|, \quad \alpha \in \mathbb{N}_0.$$

$D'(3\mathbb{T})$ is the class of all continuous linear functionals on $D(3\mathbb{T})$. The continuity of a linear functional f on $D(3\mathbb{T})$ means that there exist $N \in \mathbb{N}$ and $c_N > 0$ such that

$$|f(\varphi)| \le c_N \sum_{\alpha \le N} \|\varphi\|_\alpha,$$

for all $\varphi \in D(3\mathbb{T})$.

Let $0 < p \le \infty$. $L_p(3\mathbb{T})$ is the standard quasi-Banach space with respect to Lebesgue measure, quasi-normed by

$$\|f|L_p(3\mathbb{T})\| = \left(\int_0^3 |f(x)|^p \, dx\right)^{1/p},$$

with the usual modification if $p=\infty$. If $1\le p\le\infty$ then $f\in L_p(3\mathbb{T})$ can be interpreted in a unique way as an element of $D'(3\mathbb{T})$ by

$$f(\varphi)=\int_0^3 f(x)\varphi(x)\,dx,\quad \varphi\in D(3\mathbb{T}). \tag{16}$$

Consequently, for $1\le p\le\infty$ we have

$$D(3\mathbb{T})\subset L_p(3\mathbb{T})\subset D'(3\mathbb{T}), \tag{17}$$

where "$\subset$" here and further on means the topological embedding.

Let $f\in D'(3\mathbb{T})$. Then the numbers

$$\widehat{f}(k)=\frac{1}{3}f(e^{-2\pi ikx/3}),\qquad k\in\mathbb{Z},$$

are said to be the Fourier coefficients of f. If $f\in L_p(3\mathbb{T})$, $1\le p\le\infty$, then (16), (17) imply that

$$\widehat{f}(k)=\frac{1}{3}\int_0^3 f(x)e^{-2\pi ikx/3}\,dx,\qquad k\in\mathbb{Z}.$$

Any $f\in D'(3\mathbb{T})$ can be represented as

$$f=\sum_{k\in\mathbb{Z}}a_k e^{2\pi ikx/3},\quad x\in 3\mathbb{T}\quad (\text{convergence in } D'(3\mathbb{T})), \tag{18}$$

where the Fourier coefficients $\{a_k\}\subset\mathbb{C}$ are of at most polynomial growth,

$$|a_k|\le c\,(1+|k|)^{\kappa},\quad \text{for some } c>0,\quad \kappa>0 \text{ and all } k\in\mathbb{Z}.$$

Definition 2.10. Let $\varphi=\{\varphi_j\}_{j=0}^{\infty}$ be a dyadic resolution of unity in $\mathbb{R}$ according to (1)–(3), $s\in\mathbb{R}$, $0<p\le\infty$, $0<q\le\infty$ and

$$\|f|B^s_{pq}(3\mathbb{T})\|=\Big(\sum_{j=0}^{\infty}2^{jsq}\Big\|\sum_{k\in\mathbb{Z}}\varphi_j\Big(\frac{2\pi k}{3}\Big)a_k e^{2\pi ikx/3}|L_p(3\mathbb{T})\Big\|^q\Big)^{1/q}$$

(with the usual modification if $q=\infty$). Then the Besov space $B^s_{pq}(3\mathbb{T})$ consists of all $f\in D'(3\mathbb{T})$ such that $\|f|B^s_{pq}(3\mathbb{T})\|<\infty$, [3, Chapter 3].

3. Besov spaces on the snowflake. Three Koch curves clipped together form the snowflake curve SF, see Figure 1. Due to the isomorphism H between $[0,1]$ and the Koch curve Γ, described in [2], we may establish isomorphism $\widetilde{H}$ between $[0,3]$ and SF. The snowflake is a d-set with $d=\frac{\log 4}{\log 3}$. Let μ be chosen in such a way that it is the image of the Lebesgue measure under $\widetilde{H}$.

Our approach to defining Besov spaces on the snowflake is the same as in [2]. We start with the same restrictions on the parameters

$$0<s<1,\quad 1<p=q<\infty$$

and then extend our result to the case when $p\ne q$.

3.1. New periodic wavelets on $\mathbb{T}$ and $\mathbb{R}$. Let $C^u(\mathbb{R})$, $u\in\mathbb{N}$, denote the collection of all complex-valued continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u inclusively. Let $\psi_F\in C^u(\mathbb{R})$ and $\psi_M\in C^u(\mathbb{R})$ be a father and a

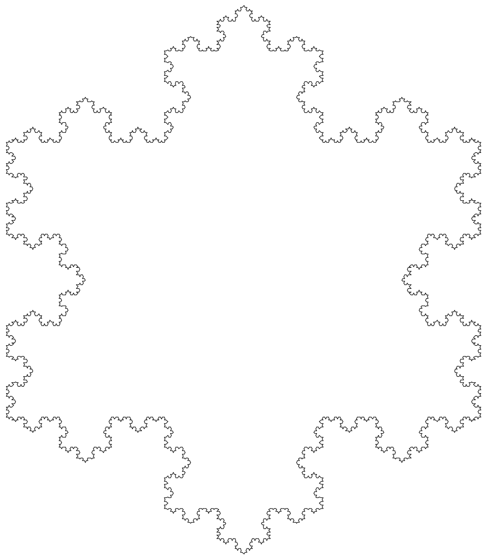

Fig. 1. The snowflake

mother Daubechies wavelet on $\mathbb{R}$, respectively. Since $0 < s < 1$ it is enough to consider $\psi_F \in C^1(\mathbb{R})$ and $\psi_M \in C^1(\mathbb{R})$. Define ψ_j^k by

$$\psi_j^k(x) = \begin{cases} \psi_F(x-k), & j=0,\ k \in \mathbb{Z}, \\ 2^{(j-1)/2}\psi_M(2^{j-1}x-k), & j \in \mathbb{N},\ k \in \mathbb{Z}. \end{cases} \tag{19}$$

Then $\{\psi_j^k\}_{j\in\mathbb{N}_0,k\in\mathbb{Z}}$ is an orthonormal basis in $L_2(\mathbb{R})$. We transform the wavelet basis of $L_2(\mathbb{R})$ into a wavelet basis of $L_2(3\mathbb{T})$ by periodizing each member of the basis.

Let $L \in \mathbb{N}$. One can replace ψ_F and ψ_M by

$$\psi_F^L(\cdot) = \psi_F(2^L\cdot), \qquad \psi_M^L(\cdot) = \psi_M(2^L\cdot),$$

ψ_j^k by

$$\psi_j^{L,k}(\cdot) = 2^{L/2}\psi_j^k(2^L\cdot). \tag{20}$$

We choose and fix L such that

$$\operatorname{supp}\psi_F^L \subset \Big\{x : |x| < \frac{1}{2}\Big\}, \quad \operatorname{supp}\psi_M^L \subset \Big\{x : |x| < \frac{1}{2}\Big\}. \tag{21}$$

Then

$$\operatorname{supp}\psi_j^{L,0} \subset \{x : |x| < 2^{-j}\}, \quad j \in \mathbb{N}.$$

Let

$$N = \sup_{x\in\mathbb{R}} |\psi'_F(x)|\,, \quad M = \sup_{x\in\mathbb{R}} |\psi'_M(x)|\,.$$

ψ_F and ψ_M are Lipschitz-continuous functions. For the functions $\psi_j^{L,k}$ defined by (19) and (20) we have

$$\left|\psi_0^{L,k}(x) - \psi_0^{L,k}(y)\right| \le 2^{3L/2} N\,|x-y|\,, \qquad x, y \in \mathbb{R},$$
$$\left|\psi_j^{L,k}(x) - \psi_j^{L,k}(y)\right| \le 2^{3(j+L-1)/2} M\,|x-y|\,, \quad j \in \mathbb{N}, \quad x, y \in \mathbb{R}.$$

We construct 3-periodic counterparts of $\psi_j^{L,k}$ by the procedure

$$\psi_{j,3\mathrm{per}}^{L,k}(x) = \sum_{l=-\infty}^{\infty} \psi_j^{L,k}(x+3l). \tag{22}$$

Define $\psi_j^{L,k,3\mathrm{per}}$ on $3\mathbb{T}$ by

$$\psi_j^{L,k,3\mathrm{per}}(x) = \psi_{j,3\mathrm{per}}^{L,k}(x), \qquad x \in 3\mathbb{T}.$$

Let

$$\mathbb{P}_0^3 = \left\{k \in \mathbb{Z} : 0 \le k \le 3 \cdot 2^L - 1\right\}$$
$$\mathbb{P}_j^3 = \left\{k \in \mathbb{Z} : 0 \le k \le 3 \cdot 2^{j+L-1} - 1\right\}, \quad j \in \mathbb{N}.$$

Then for $j \in \mathbb{N}_0$ there exists a set of points $\{x_{j,k}\}_{k \in \mathbb{P}_j^3} \subset 3\mathbb{T}$ such that

$$\operatorname{supp} \psi_0^{L,k,3\mathrm{per}} \subset \{x \in 3\mathbb{T} : \rho(x, x_{0,k}) < \tfrac{1}{2}\} = B_{0,k}^{3\mathbb{T}},$$
$$\operatorname{supp} \psi_j^{L,k,3\mathrm{per}} \subset \{x \in 3\mathbb{T} : \rho(x, x_{j,k}) < 2^{-j}\} = B_{j,k}^{3\mathbb{T}}.$$

Recall that $\rho(\cdot,\cdot)$ is the metric on $3\mathbb{T}$ given by (15). For the points $x, y \in B_{j,k}^{3\mathbb{T}}$, $j \in \mathbb{N}_0$, $k \in \mathbb{P}_j^3$,

$$\left|\widetilde{H}(x) - \widetilde{H}(y)\right| \sim \rho(x,y)^{1/d}.$$

Similarly to Proposition 1.34 in [6] one gets that

$$\left\{\psi_j^{L,k,3\mathrm{per}} : j \in \mathbb{N}_0,\ k \in \mathbb{P}_j^3\right\}$$

is an orthonormal basis in $L_2(3\mathbb{T})$. We simplify the notation and omit L in $\psi_j^{L,k,3\mathrm{per}}$.

To characterize periodic Besov spaces in terms of wavelets we first introduce the corresponding sequence spaces.

Definition 3.1. Let $0 < p \le \infty$, $0 < q \le \infty$ and $s \in \mathbb{R}$. Then $b_{pq}^{s,3\mathrm{per}}$ is the collection of all sequences

$$\mu = \left\{\mu_j^k \in \mathbb{C} : j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\right\}$$

such that

$$\|\mu | b_{pq}^{s,3\mathrm{per}}\| = \left(\sum_{j=0}^{\infty} 2^{j(s-1/p)q} \Big(\sum_{k \in \mathbb{P}_j^3} |\mu_j^k|^p\Big)^{q/p}\right)^{1/q} < \infty.$$

Theorem 3.2. *Let $\{\psi_j^{k,3\mathrm{per}}\}$ be the orthonormal basis in $L_2(3\mathbb{T})$. Let $0 < p \le \infty$, $0 < q \le \infty$ and $0 < s < 1$. Let $f \in D'(3\mathbb{T})$. Then $f \in B_{pq}^s(3\mathbb{T})$ if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j^3} \mu_j^k 2^{-(j+L)/2} \psi_j^{k,3\mathrm{per}}, \qquad \mu \in b_{pq}^{s,3\mathrm{per}},$$

unconditional convergence being in $D'(3\mathbb{T})$ *and in any space* $B^{\sigma}_{pq}(3\mathbb{T})$ *with* $\sigma < s$. *Furthermore, this representation is unique,*

$$\mu_j^k = 2^{(j+L)/2} \int_0^3 f(x)\psi_j^{k,3\mathrm{per}}(x)\,dx,$$

and

$$I : f \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\}$$

is an isomorphic map of $B^s_{pq}(3\mathbb{T})$ *onto the sequence space* $b^{s,3\mathrm{per}}_{pq}$. *If, in addition,* $p < \infty$, $q < \infty$, *then* $\{\psi_j^{k,\mathrm{per}}\}$ *is an unconditional basis in* $B^s_{pq}(3\mathbb{T})$.

Remark 3.3. This assertion is the counterpart of Theorem 1.37 in [6] for $B^s_{pq}(3\mathbb{T})$.

Since

$$B^s_{pq}(3\mathbb{T}) \hookrightarrow L_p(3\mathbb{T})$$

with s, p and q satisfying (5) (see [3, Chapter 3.5.1]), we reformulate Theorem 3.2 with additional restrictions on the parameters.

Theorem 3.4. *Let* $\{\psi_j^{k,3\mathrm{per}}\}$ *be the above orthonormal basis in* $L_2(3\mathbb{T})$. *Let* $1 < p < \infty$, $0 < q < \infty$ *and* $0 < s < 1$. *Let* $f \in L_p(3\mathbb{T})$. *Then* $f \in B^s_{pq}(3\mathbb{T})$ *if and only if it can be represented as*

$$f = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j^3} \mu_j^k 2^{-(j+L)/2} \psi_j^{k,\mathrm{per}}, \qquad \mu \in b^{s,3\mathrm{per}}_{pq},$$

unconditional convergence being in $L_p(3\mathbb{T})$. *Furthermore this representation is unique,*

$$\mu_j^k = 2^{(j+L)/2} \int_0^3 f(x)\psi_j^{k,3\mathrm{per}}(x)\,dx,$$

and

$$I : f \to \{\mu_j^k, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\}$$

is an isomorphic map of $B^s_{pq}(3\mathbb{T})$ *onto the sequence space* $b^{s,3\mathrm{per}}_{pq}$.

3.2. Besov spaces $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$. Let

$$\mathbb{B}^s_{pq}(\mathrm{SF}, \mu) = \left\{ f \circ \widetilde{H}^{-1} : f \in B^s_{pq}(3\mathbb{T}) \right\} = B^s_{pq}(3\mathbb{T}) \circ \widetilde{H}^{-1}$$

with

$$\| f \circ \widetilde{H}^{-1} | \mathbb{B}^s_{pq}(\mathrm{SF}, \mu) \| = \| f | B^s_{pq}(3\mathbb{T}) \|.$$

Define $\widetilde{\psi}_{jk}$ by

$$\widetilde{\psi}_{jk}(\gamma) = \psi_j^{k,3\mathrm{per}} \circ \widetilde{H}^{-1}(\gamma).$$

From the corresponding properties of functions $\psi_j^{k,3\mathrm{per}}$ and transform $\widetilde{H}$ the properties of $\widetilde{\psi}_{jk}$ follow, namely:

- The system $\{\widetilde{\psi}_{jk}, j \in \mathbb{N}_0, k \in \mathbb{P}_j^3\}$ is an orthonormal basis in $L_2(\mathrm{SF}, \mu)$.
- For $j \in \mathbb{N}_0$ there is a set of points $\{\gamma_{j,k}\}_{k \in \mathbb{P}_j^3} \subset \mathrm{SF}$ such that

$$\operatorname{supp} \widetilde{\psi}_{0k} \subset \{\gamma \in \mathrm{SF} : |\gamma - \gamma_{0,k}| \le c2^{-1/d}\} = B^{\mathrm{SF}}_{0,k}, \quad k \in \mathbb{P}_0^3,$$
$$\operatorname{supp} \widetilde{\psi}_{jk} \subset \{\gamma \in \mathrm{SF} : |\gamma - \gamma_{j,k}| \le c2^{-j/d}\} = B^{\mathrm{SF}}_{j,k}, \quad k \in \mathbb{P}_j^3.$$

- For $\gamma, \delta \in \operatorname{supp} \widetilde{\psi}_{jk}$

$$\left|\widetilde{\psi}_{jk}(\gamma) - \widetilde{\psi}_{jk}(\delta)\right| \leq c2^{3j/2} \left|\gamma - \delta\right|^{d}$$
$$= c2^{3j/2} \left|\gamma - \delta\right|^{d-1} \left|\gamma - \delta\right| \leq c2^{-j(-1/d-1/2)} \left|\gamma - \delta\right|.$$

The last inequality is due to the fact that for $\gamma, \delta \in B^{\mathrm{SF}}_{j,k}$

$$|\gamma - \delta| \leq |\gamma - \gamma_{j,k}| + |\gamma_{j,k} - \delta| \leq c2^{-j/d}.$$

Define $\widetilde{a}_{jk}$ by

$$\widetilde{a}_{jk} = \begin{cases} 2^{-L/2}\widetilde{\psi}_{jk}, & j = 0,\ k \in \mathbb{P}^3_j, \\ 2^{-j(s-1/p)}2^{-(j+L-1)/2}\widetilde{\psi}_{jk}, & j \in \mathbb{N},\ k \in \mathbb{P}^3_j. \end{cases}$$

Then

$$\operatorname{supp} \widetilde{a}_{jk} \subset B^{\mathrm{SF}}_{j,k},$$
$$|\widetilde{a}_{jk}(\gamma)| \leq c2^{-j(s-1/p)} \leq c\,\mathrm{H}^d(B^{\mathrm{SF}}_{j,k})^{s-1/p}, \quad \text{for any } \gamma \in \mathrm{SF},$$

and for any $\gamma, \delta \in \operatorname{supp} \widetilde{a}_{jk}$

$$|\widetilde{a}_{jk}(\gamma) - \widetilde{a}_{jk}(\delta)| \leq c2^{-j(s-1/d-1/p)} \left|\gamma - \delta\right| \leq c\,\mathrm{H}^d(B^{\mathrm{SF}}_{j,k})^{s-1/d-1/p} \left|\gamma - \delta\right|.$$

According to Definition 2.8 $\widetilde{a}_{jk}$ are (sd, p)-atoms.

Theorem 3.5. *Let $1 < p < \infty$, $0 < q < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(\mathrm{SF}, \mu)$. Then $\widetilde{f} \in \mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ if and only if it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}^3_j} \mu^k_j 2^{-(j+L)/2} \widetilde{\psi}_{jk}, \tag{23}$$

unconditional convergence being in $L_p(\mathrm{SF}, \mu)$. Furthermore this representation is unique,

$$\mu^k_j = 2^{(j+L)/2}(\widetilde{f}, \widetilde{\psi}_{jk})_{\mathrm{SF}} = 2^{(j+L)/2} \int_{\mathrm{SF}} \widetilde{f}(\gamma)\widetilde{\psi}_{jk}(\gamma)\,\mu(d\gamma),$$

and

$$I : \widetilde{f} \to \left\{\mu^k_j, j \in \mathbb{N}_0, k \in \mathbb{P}^3_j\right\} \tag{24}$$

is an isomorphic map of $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ onto the sequence space $b^{s,3\mathrm{per}}_{pq}$.

3.3. Comparison of $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ and $B^s_{pq}(\mathrm{SF}, \mu)$. We have

$$\mathbb{B}^{s/d}_{pp}(\mathrm{SF}, \mu) = B^s_{pp}(\mathrm{SF}, \mu). \tag{25}$$

The inclusion from left to the right follows from Theorem 2.9 and Theorem 3.5. To get the opposite one, we need the characterization of periodic Besov spaces in terms of first differences, we refer to [3, Section 3.5]. The idea is the same as in [2].

To compare $\mathbb{B}^s_{pq}(\mathrm{SF}, \mu)$ and $B^s_{pq}(\mathrm{SF}, \mu)$ with $0 < s < 1$ and $p \neq q$ we use the real interpolation.

Let $0 < \theta < 1$, $1 < p < \infty$, $0 < q < \infty$, $0 < s_0 < 1$, $0 < s_1 < 1$, $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then from Theorem 1 in [3, Ch. 3.6.1] it follows that

$$\left(B^{s_0}_{pp}(3\mathbb{T}), B^{s_1}_{pp}(3\mathbb{T})\right)_{\theta,q} = B^s_{pq}(3\mathbb{T}).$$

Since spaces $B^s_{pq}(3\mathbb{T})$ are isomorphic to sequence spaces $b^{s,3\mathrm{per}}_{pq}$, we have

$$\big(b^{s_0,3\mathrm{per}}_{pp}, b^{s_1,3\mathrm{per}}_{pp}\big)_{\theta,q} = b^{s,3\mathrm{per}}_{pq}.$$

Using the isomorphic map in (24) one gets

$$\big(\mathbb{B}^{s_0}_{pp}(\mathrm{SF},\mu), \mathbb{B}^{s_1}_{pp}(\mathrm{SF},\mu)\big)_{\theta,q} = \mathbb{B}^s_{pq}(\mathrm{SF},\mu). \tag{26}$$

For any d-set the following theorem holds.

THEOREM 3.6. *Let Γ be a d-set in $\mathbb{R}^n$ with $0 < d < n$. Let $0 < \theta < 1$, $1 < p < \infty$, $1 \le q < \infty$, $0 < s_0 < 1$, $0 < s_1 < 1$, $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$. Then*

$$\big(B^{s_0}_{pq_0}(\Gamma,\mu), B^{s_1}_{pq_1}(\Gamma,\mu)\big)_{\theta,q} = B^s_{pq}(\Gamma,\mu). \tag{27}$$

Proof. We put

$$P = \mathrm{ext}_\mu \circ \mathrm{tr}_\mu : B^{s+(n-d)/p}_{pq}(\mathbb{R}^n) \hookrightarrow B^{s+(n-d)/p}_{pq}(\mathbb{R}^n).$$

Then P is a linear and bounded map. From (8) it follows that

$$P^2 = \mathrm{ext}_\mu \circ \mathrm{tr}_\mu \circ \mathrm{ext}_\mu \circ \mathrm{tr}_\mu = P.$$

Hence P is a projection of $B^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$ onto $PB^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$. By $P \circ \mathrm{ext}_\mu = \mathrm{ext}_\mu$, one gets that ext_μ maps $B^s_{pq}(\Gamma,\mu)$ into $PB^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$. On the other hand, if $f \in PB^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$, then $f = \mathrm{ext}_\mu(\mathrm{tr}_\mu(f))$, $\mathrm{tr}_\mu f \in B^s_{pq}(\Gamma)$. Hence ext_μ maps $B^s_{pq}(\Gamma,\mu)$ onto $PB^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$. Since tr_μ and ext_μ are linear bounded operators, one has

$$\|f|B^s_{pq}(\Gamma,\mu)\| \sim \|\mathrm{ext}_\mu f|B^{s+(n-d)/p}_{pq}(\mathbb{R}^n)\| \tag{28}$$

and it follows that

$$\mathrm{ext}_\mu : B^s_{pq}(\Gamma,\mu) \to PB^{s+(n-d)/p}_{pq}(\mathbb{R}^n)$$

is an isomorphic map.

Let

$$\big(B^{s_0}_{pq_0}(\Gamma,\mu), B^{s_1}_{pq_1}(\Gamma,\mu)\big)_{\theta,q} = B_\theta(\Gamma).$$

It is known that

$$\Big(B^{s_0+(n-d)/p}_{pq_0}(\mathbb{R}^n), B^{s_1+(n-d)/p}_{pq_1}(\mathbb{R}^n)\Big)_{\theta,q} = B^{s+(n-d)/p}_{pq}(\mathbb{R}^n). \tag{29}$$

We denote the right-hand side of (29) by $B_\theta(\mathbb{R}^n)$.

By the interpolation property for the spaces on $\mathbb{R}^n$ and Γ

$$\|f|B_\theta(\Gamma)\| = \|\mathrm{tr}_\mu \circ \mathrm{ext}_\mu f|B_\theta(\Gamma)\| \le c\|\mathrm{ext}_\mu f|B_\theta(\mathbb{R}^n)\| \le c'\|f|B_\theta(\Gamma)\|. \tag{30}$$

Hence

$$\|f|B_\theta(\Gamma)\| \sim \|\mathrm{ext}_\mu f|B^{s+(n-d)/p}_{pq}(\mathbb{R}^n)\|. \tag{31}$$

Together with (28) this leads to

$$\|f|B_\theta(\Gamma)\| \sim \|f|B^s_{pq}(\Gamma,\mu)\|.$$

This completes the proof. ∎

REMARK 3.7. The proof essentially uses the way of reasoning in [5, Ch. 1.11.8].

Using (25), (26) and (27) one gets that for $0 < s < 1$, $1 < p < \infty$, $1 \le q < \infty$

$$B^s_{pq}(\mathrm{SF}, \mu) = \mathbb{B}^{s/d}_{pq}(\mathrm{SF}, \mu).$$

Thus we may conclude that the following theorem holds.

THEOREM 3.8. *Let $1 < p < \infty$, $1 \le q < \infty$ and $0 < s < 1$. Let $\widetilde{f} \in L_p(\mathrm{SF}, \mu)$. Then $\widetilde{f} \in B^s_{pq}(\mathrm{SF}, \mu)$ if and only if it can be represented as*

$$\widetilde{f} = \sum_{j=0}^{\infty} \sum_{k \in \mathbb{P}_j} \mu^k_j 2^{-(j+L)/2} \widetilde{\psi}^k_j,$$

unconditional convergence being in $L_p(\mathrm{SF}, \mu)$. Furthermore this representation is unique,

$$\mu^k_j = 2^{(j+L)/2} (\widetilde{f}, \widetilde{\psi}^k_j)_{\mathrm{SF}},$$

and

$$I : \widetilde{f} \to \left\{ \mu^k_j, j \in \mathbb{N}_0, k \in \mathbb{P}_j \right\} \tag{32}$$

is an isomorphic map of $B^s_{pq}(\mathrm{SF}, \mu)$ onto the sequence space $b^{s/d,\mathrm{per}}_{pq}$.

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