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REMARKS ON THE SPACES OF DIFFERENTIABLE MULTIFUNCTIONS

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Abstract. In this paper we consider some spaces of differentiable multifunctions, in particular the generalized Orlicz-Sobolev spaces of multifunctions, we study completeness of them, and give some theorems.

1. Introduction. The notion of differential of multifunction was introduced in many papers (see [H, Chapter 6, Section 7]). In this paper we apply the De Blasi definition of differential of multifunction from [DB], and the Martelli–Vignoli definition from [M]. The differential of multifunction in [D] is a Gateaux differential, however we apply the easier extension of the definition of differential of multifunction from [G] and [Hu]. Also we apply the ideas from [K1, K2, K3]. We introduce some multiderivatives and we give the definition of spaces of differentiable multifunctions and some theorems, in particular a generalization of the generalized Orlicz-Sobolev spaces to multifunctions. The aim of this note is to obtain handy space of differentiable multifunctions. We use the one-dimensional Lebesgue measure space on $\mathbb{R}$ only. Let Y be a real Banach space and θ be the zero in Y. Let $T \subset \mathbb{R}$ and

$$X = \{F : T \to 2^Y : F(t) \text{ is nonempty for every } t \in T,\ \text{compact for a.e. } t \in T\}.$$

If $Y = \mathbb{R}$, $F \in X$, we define

$$\underline{f}(F)(t) = \inf_{x \in F(t)} x, \qquad \overline{f}(F)(t) = \sup_{x \in F(t)} x \ \text{ for every } t \in T.$$

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Let $[a, b]$ denote a closed interval for all $a, b \in \overline{\mathbb{R}}$, $a \leq b$. Let $\mathbb{N}$ be the set of all positive integers. For each nonempty and compact $A, B \subset Y$ we define

$$\operatorname{dist}(A, B) = \max\bigl(\max_{x\in A}\min_{y\in B}\|x-y\|, \max_{y\in B}\min_{x\in A}\|x-y\|\bigr).$$

Define:

$$C(Y) = \{A \subset Y : A \text{ is nonempty and compact}\},$$
$$X_c = \{F \in X : F(t) \in C(Y) \text{ for every } t \in T\},$$
$$kC(Y) = \{A \in C(Y) : A \text{ is convex}\},$$
$$X_{kc} = \{F \in X_c : F(t) \in kC(Y) \text{ for every } t \in T\}.$$

Let $B \in C(Y)$ and $|B| = \operatorname{dist}(B, \{\theta\})$. Let $F, G \in X$, $a \in \mathbb{R}$. We define $F + G$ and aF by the formulas

$$(F+G)(t) = \{x + y : x \in F(t), y \in G(t)\}, \qquad (aF)(t) = \{ax : x \in F(t)\}.$$

for every $t \in T$.

We use Lebesgue integral only.

2. Spaces of differentiable multifunctions. Now we assume that $T = [a, b]$, where $a < b$ and $a, b \in \mathbb{R}$.

Definition 2.1. We say that $\mathcal{A} \subset X$ is *X-linear* if $F + G \in \mathcal{A}$ and $aF \in \mathcal{A}$ for all $F, G \in \mathcal{A}$, $a > 0$.

Definition 2.2. Let $\mathcal{A} \subset X$ be X-linear. Let $M : \mathcal{A} \to X$. We say that M is *X-linear on* $\mathcal{A}$ if $M(F + G) = M(F) + M(G)$, $M(aF) = aM(F)$ for all $F, G \in \mathcal{A}$, $a > 0$.

Let $Z(T) = \mathcal{C}(T)$ or $Z(T) = \mathcal{C}^1(T)$ (the spaces of continuous and continuously differentiable functions, respectively). Let $Y = \mathbb{R}$. Define

$$X_{\mathbb{R},Z(T)} = \{F \in X_c : \underline{f}(F), \overline{f}(F) \in Z(T)\}, \tag{1}$$

$$X_{1,\mathbb{R},Z(T)} = \{F \in X_{\mathbb{R},Z(T)} : F(t) \in kC(\mathbb{R}) \text{ for every } t \in T\}, \tag{2}$$

$$X_{1,\mathbb{R},\mathcal{C}^1(T),+} = \{F \in X_{1,\mathbb{R},\mathcal{C}^1(T)} : (\overline{f}(F))'(t) \geq (\underline{f}(F))'(t), \text{ for every } t \in T\}. \tag{3}$$

It is easy to see that $X_{1,\mathbb{R},\mathcal{C}^1(T),+}$ is X-linear in $X_{1,\mathbb{R},\mathcal{C}^1(T)}$.

Definition 2.3. Let $F, F_n \in X_{\mathbb{R},\mathcal{C}(T)}$ (see (1)) for every $n \in \mathbb{N}$. We write $F_n \xrightarrow{\mathcal{C}(T)} F$ iff $\underline{f}(F_n) - \underline{f}(F) \to 0$, $\overline{f}(F_n) - \overline{f}(F) \to 0$ in $\mathcal{C}(T)$ and $\lim_{n\to\infty} \operatorname{dist}(F_n(t), F(t)) = 0$ for every $t \in T$.

Definition 2.4. Let $F_n \in X_{\mathbb{R},\mathcal{C}(T)}$ for every $n \in \mathbb{N}$. We say that $\{F_n\}$ is a Cauchy sequence in $X_{\mathbb{R},\mathcal{C}(T)}$ iff $\{\underline{f}(F_n)\}$, $\{\overline{f}(F_n)\}$ are Cauchy sequences in $\mathcal{C}(T)$ and $\{F_n(t)\}$ is a Cauchy sequence in $B(\mathbb{R})$ for every $t \in T$, where $B(\mathbb{R})$ is a metric space of all compact and nonempty subsets of $\mathbb{R}$ with Hausdorff metric.

We easily obtain

Theorem 2.5. *If $\{F_n\}$ is a Cauchy sequence in $X_{\mathbb{R},\mathcal{C}(T)}$, then there is an $F \in X_{\mathbb{R},\mathcal{C}(T)}$ such that $F_n \xrightarrow{\mathcal{C}(T)} F$.*

If $F \in X_{1,\mathbb{R},\mathcal{C}(T)}$ (see (2)), then $F(t) = \underline{f}(F)(t) + (\overline{f}(F)(t) - \underline{f}(F)(t))[0,1]$.

If $F \in X_{1,\mathbb{R},\mathcal{C}^1(T)}$, we set $\partial F(t) = (\underline{f}(F))'(t) + ((\overline{f}(F))'(t) - (\underline{f}(F))'(t))[0,1]$.

We easily obtain the following:

THEOREM 2.6. *The operator ∂F is X-linear on $X_{1,\mathbb{R},\mathcal{C}^1(T),+}$ (see (3)) but it is not X-linear on $X_{1,\mathbb{R},\mathcal{C}^1(T)}$.*

If $F \in X_{1,\mathbb{R},\mathcal{C}^1(T)}$, then we say that ∂F is the multiderivatives of F.

Now we change the definition of the fundamental spaces of multifunctions. Define

$$(4) \qquad X_{n,Z(T)} = \Big\{F \in X_c : F(t) = f(t) + \sum_{k=1}^{n} f_k(t)[0,1] \ \text{ for every } t \in T\Big\},$$

where $f, f_k \in Z(T)$, for $k = 1, \ldots, n$, n is any natural number, and f_k are such that

$(*)$ for $i \neq j$ $(f_i(t) + f_j(t))[0,1] \neq f_i(t)[0,1] + f_j(t)[0,1]$ on the set of positive measure.

Next,

$$(5) \quad X_{\infty,\mathcal{C}(T)} = \Big\{F \in X_c : F(t) = f(t) + \sum_{k=1}^{\infty} f_k(t)[0,1] \text{ and } \sum_{k=1}^{\infty} |f_k(t)| < \infty \\ \text{for every } t \in T\Big\},$$

where $f, f_k \in \mathcal{C}(T)$ for $k \in \mathbb{N}$, f_k satisfy $(*)$, and

$$\sum_{k=1}^{\infty} f_k(t)[0,1] = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(t)[0,1]$$

for every $t \in T$ (in the Hausdorff metric). Finally,

$$(6) \quad X_{\infty,\mathcal{C}^1(T)} = \Big\{F \in X_{\infty,\mathcal{C}(T)} : f, f_k \in \mathcal{C}^1(T) \text{ for every } k \in \mathbb{N} \\ \text{and } \sum_{k=1}^{\infty} |f_k'(t)| < +\infty \text{ for every } t \in T\Big\}.$$

If $a \in \mathbb{R}$, $F, G \in X_{\infty,\mathcal{C}(T)}$,

$$F(t) = f(t) + \sum_{k=1}^{\infty} f_k(t)[0,1], \qquad G(t) = g(t) + \sum_{k=1}^{\infty} g_k(t)[0,1],$$

define

$$(G + F)(t) = f(t) + g(t) + \sum_{k=1}^{\infty}(f_k(t)[0,1] + g_k(t)[0,1]),$$

$$(aF)(t) = af(t) + \sum_{k=1}^{\infty} af_k(t)[0,1],$$

for every $t \in T$.

It is easy to check that

- $X_{\infty,\mathcal{C}(T)}$ (see (5)) is a linear subset of X.
- if $F \in X_{1,\mathcal{C}(T)}$ (see (4)) and $f_1(t) \geq 0$ for every $t \in T$, then $F(t) = \underline{f}(F)(t) + (\overline{f}(F)(t) - \underline{f}(F)(t))[0,1]$ for every $t \in T$.

- that $X_{n,\mathcal{C}(T)} \subset X_{1,\mathbb{R},\mathcal{C}(T)}$ for every $n \in \mathbb{N}$.
- that $X_{1,\mathcal{C}^1(T)}$ and $X_{1,\mathbb{R},\mathcal{C}^1(T)}$ are different.

Let $F \in X_{n,\mathcal{C}^1(T)}$ and $F(t) = f(t) + \sum_{k=1}^n f_k(t)[0,1]$, where f_k satisfy $(*)$. Define

$$\partial F(t) = f'(t) + \sum_{k=1}^{n} f_k'(t)[0,1],$$

for every $t \in T$. So if $F \in X_{1,\mathcal{C}^1(T)} \cap X_{1,\mathbb{R},\mathcal{C}^1(T)}$, then $(\partial F)(t) = (\underline{f}(F))'(t) + ((\overline{f}(F))'(t) - (\underline{f}(F))'(t))[0,1]$ for every $t \in T$. There are $F \in X_{1,\mathcal{C}^1(T)}$ such that $(\partial F) \notin X_{1,\mathbb{R},\mathcal{C}^1(T)}$, for example $F(t) = t + t[0,1]$ for every $t \in \mathbb{R}$.

If $F \in X_{n,\mathcal{C}^1(T)}$ (see (4)), then we say that ∂F is a multiderivative of F.

THEOREM 2.7. *If $F \in X_{n,\mathcal{C}^1(T)}$, then $\partial F \in X_{n,\mathcal{C}(T)}$.*

Let $F \in X_{\infty,\mathcal{C}^1(T)}$ and

$$F(t) = f(t) + \sum_{k=1}^{\infty} f_k(t)[0,1]$$

for every $t \in T$, then we define

$$\partial F(t) = f'(t) + \sum_{k=1}^{\infty} f_k'(t)[0,1]$$

for every $t \in T$, and we say that ∂F is a multiderivative of F.

DEFINITION 2.8. Let $F, F_n \in X_{\infty,\mathcal{C}(T)}$ for $n \in \mathbb{N}$ and

$$F(t) = f(t) + \sum_{k=1}^{\infty} f_k(t)[0,1], \qquad F_n(t) = f^n(t) + \sum_{k=1}^{\infty} f_k^n(t)[0,1]$$

for every $t \in T$ and every $n \in \mathbb{N}$. We write $F_n \longrightarrow F$ iff

$$f^n - f \to 0, \quad f_k^n - f_k \to 0 \text{ in } \mathcal{C}(T) \text{ for } k \in \mathbb{N} \text{ and}$$
$$\operatorname{dist}(F_n(t), F(t)) \to 0 \text{ as } n \to \infty, \text{ for every } t \in T.$$

DEFINITION 2.9. Let $F_n \in X_{\infty,\mathcal{C}(T)}$ for every $n \in \mathbb{N}$ and

$$F_n(t) = f^n(t) + \sum_{k=1}^{\infty} f_k^n(t)[0,1]$$

for every $t \in T$ and every $n \in \mathbb{N}$.

We say that $\{F_n\}$ is a *Cauchy sequence in $X_{\infty,\mathcal{C}(T)}$* iff $\{f^n\}$, $\{f_k^n\}$ are Cauchy sequences in $\mathcal{C}(T)$, $\{g_n(t)\} = \{\{f_k^n(t)\}\}$ is a Cauchy sequence in l^1 for every $t \in T$,

THEOREM 2.10. *If $\{F_n\}$ is a Cauchy sequence in $X_{\infty,\mathcal{C}(T)}$, then there is an $F \in X_{\infty,\mathcal{C}(T)}$ such that $F_n \longrightarrow F$.*

Proof. By the first assumption there are f, f_k, $k \in \mathbb{N}$ such that $f^n \to f$, $f_k^n \to f_k$ in $\mathcal{C}(T)$ for every $k \in \mathbb{N}$.

By the assumptions for sufficiently large n_0 and every $t \in T$,

$$\sum_{k=1}^{\infty} |f_k(t)| \le \sum_{k=1}^{\infty} |f_k^{n_0}(t)| + \sum_{k=1}^{\infty} |f_k^{n_0}(t) - f_k(t)| < \infty.$$

Let for $t \in T$

$$F(t) = f(t) + \sum_{k=1}^{\infty} f_k(t)[0,1].$$

We have for every $t \in T$

$$\operatorname{dist}(F_n(t), F(t)) \le |f^n(t) - f(t)| + \sum_{k=1}^{\infty} |f_k^n(t) - f_k(t)| \to 0 \text{ as } n \to \infty,$$

so $F \in X_{\infty,\mathcal{C}(T)}$ and $F_n \to F$. ∎

DEFINITION 2.11. Let $F_n \in X_{\infty,\mathcal{C}^1(T)}$ (see (6)) for every $n \in \mathbb{N}$. We say that $\{F_n\}$ is a *Cauchy sequence in* $X_{\infty,\mathcal{C}^1(T)}$ if $\{F_n\}$ and $\{\partial F_n\}$ are Cauchy sequences in $X_{\infty,\mathcal{C}(T)}$.

We easily obtain

THEOREM 2.12. *If $\{F_n\}$ is a Cauchy sequence in $X_{\infty,\mathcal{C}^1(T)}$, then there is an $F \in X_{\infty,\mathcal{C}^1(T)}$ such that $F_n \to F$.*

3. Generalization. In this section $Y = \mathbb{R}^n$. Define

$$X_{Y,\mathcal{C}(T)} = \Big\{F \in X_c : F(t) = \sum_{k=1}^{\infty} f_k(t)A_k \text{ and } \sum_{k=1}^{\infty} |f_k(t)| < \infty \text{ for every } t \in T\Big\}, \tag{7}$$

where $f_k \in \mathcal{C}(T)$, $A_k \in kC(Y)$, $|A_k| \le 1$ for $k \in \mathbb{N}$, and if $i \ne j$, $A_i = A_j$, then

$$(f_i(t) + f_j(t))A_i \ne f_i(t)A_i + f_j(t)A_i$$

on the set of positive measure. Let

$$X_{Y,\mathcal{C}^1(T)} = \Big\{F \in X_{Y,\mathcal{C}(T)} : f_k \in \mathcal{C}^1(T) \text{ for every } k \in \mathbb{N} \text{ and } \sum_{k=1}^{\infty} |f_k'(t)| < \infty \text{ for every } t \in T\Big\}. \tag{8}$$

Let $F \in X_{Y,\mathcal{C}^1(T)}$ (see (8)). It is easy to see that if we define

$$F_n(t) = \sum_{k=1}^{n} f_k(t)A_k, \quad (\partial F_n)(t) = \sum_{k=1}^{n} f_k'(t)A_k, \quad (\partial F)(t) = \sum_{k=1}^{\infty} f_k'(t)A_k,$$

for every $t \in T$, then

$$\operatorname{dist}(F_n(t), F(t)) \to 0, \quad \operatorname{dist}\big((\partial F_n)(t), (\partial F)(t)\big) \to 0 \text{ for every } t \in T,$$

and $(\partial F) \in X_{Y,\mathcal{C}(T)}$.

If $F, G \in X_{Y,\mathcal{C}(T)}$ (see (7)), $F(t) = \sum_{k=1}^{\infty} f_k(t)A_k$, $G(t) = \sum_{k=1}^{\infty} g_k(t)B_k$ for every $t \in T$, $a \in \mathbb{R}$, then we define $F + G$ and aF as follows

$$(F+G)(t) = \sum_{k=1}^{\infty}(f_k(t)A_k + g_k(t)B_k), \qquad (aF)(t) = \sum_{k=1}^{\infty}(af_k(t))A_k$$

for every $t \in T$.

It is easy to see that $X_{Y,\mathcal{C}^1(T)}$ is X-linear. If $F \in X_{Y,\mathcal{C}^1(T)}$ then we say that ∂F is a *multiderivative of* F. If $f_k'(t) \ge 0$ for every $t \in T$ and $k \in \mathbb{N}$, then it is an X-linear operator.

DEFINITION 3.1. Let $F, F_n \in X_{Y,\mathcal{C}(T)}$ for every $n \in \mathbb{N}$. Let

$$F(t) = \sum_{k=1}^{\infty} f_k(t) A_k, \quad F_n(t) = \sum_{k=1}^{\infty} f_k^n(t) A_k^n$$

for all $t \in T$, $n \in \mathbb{N}$.

We write $F_n \longrightarrow F$ iff $f_k^n \to f_k$ in $\mathcal{C}(T)$ for every $k \in \mathbb{N}$, $\operatorname{dist}(A_k^n, A_k) \to 0$ as $n \to \infty$ for every $k \in \mathbb{N}$, $\operatorname{dist}(F_n(t), F(t)) \to 0$ as $n \to \infty$, for every $t \in T$.

DEFINITION 3.2. Let $F_n \in X_{Y,\mathcal{C}(T)}$ for every $n \in \mathbb{N}$. Let

$$F_n(t) = \sum_{k=1}^{\infty} f_k^n(t) A_k^n \text{ for all } t \in T,\ n \in \mathbb{N}.$$

We say that $\{F_n\}$ is a *Cauchy sequence in* $X_{Y,\mathcal{C}(T)}$ iff

- $\{f_k^n\}$ are Cauchy sequences in $\mathcal{C}(T)$, for every $k \in \mathbb{N}$,
- there is an $M > 0$ such that $|f_k^n(t)| \le M$ for all $k, n \in \mathbb{N}$, $t \in T$,
- $\{g_n(t)\} = \{\{f_k^n(t)\}\}$ is the Cauchy sequence in l^1 for every $t \in \mathbb{R}$,
- there are $A_k \in kC(Y)$ such that

$$\sum_{k=1}^{\infty} \operatorname{dist}(A_k^n, A_k) \to 0 \text{ as } n \to \infty.$$

THEOREM 3.3. *If $\{F_n\}$ is a Cauchy sequence in $X_{Y,\mathcal{C}(T)}$, then there is an $F \in X_{Y,\mathcal{C}(T)}$ such that $F_n \longrightarrow F$.*

Proof. By the assumptions there are $f_k \in \mathcal{C}(T)$, $k \in \mathbb{N}$, such that $f_k^n \to f_k$ in $\mathcal{C}(T)$ for every $k \in \mathbb{N}$ and

$$\sum_{k=1}^{\infty} |f_k(t)| < \infty \text{ for every } t \in T.$$

By the assumptions we have $|A_k| \le 1$ for every $k \in \mathbb{N}$.

Let

$$F(t) = \sum_{k=1}^{\infty} f_k(t) A_k \text{ for every } t \in T.$$

We have for every $t \in T$

$$\begin{aligned}
\operatorname{dist}(F_n(t), F(t)) &\le \sum_{k=1}^{\infty} \operatorname{dist}(f_k^n(t) A_k^n, f_k(t) A_k) \\
&\le \sum_{k=1}^{\infty} \operatorname{dist}(f_k^n(t) A_k^n, f_k^n(t) A_k) + \sum_{k=1}^{\infty} \operatorname{dist}(f_k^n(t) A_k, f_k(t) A_k) \\
&\le \sum_{k=1}^{\infty} |f_k^n(t) - f_k(t)||A_k| + \sum_{k=1}^{\infty} |f_k^n(t)| \operatorname{dist}(A_k^n, A_k) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

so $F \in X_{Y,\mathcal{C}(T)}$. ■

4. Generalized Orlicz-Sobolev spaces of multifunctions. Let $W^k_\varphi(T)$ denote the generalized Orlicz–Sobolev space (see [Mu2, pp. 66–68]), let $\|\cdot\|^k_\varphi$ denote the norm in $W^k_\varphi(T)$, $\|\cdot\|_\varphi$ denote the Luxemburg norm in $L^\varphi(T)$ and $Y=\mathbb{R}$. Let $\mathcal{D}^a x$ denote the generalized derivative of order $a\le k$ of $x\in W^k_\varphi(T)$. Let

$$X_m=\{F\in X : F \text{ is measurable}\},$$
$$X_{m,\varphi}=\{f\in X_m : \underline{f}(F),\overline{f}(F)\in L^\varphi(T)\}.$$

If $F\in X_m$, then we define $\operatorname{conv} F$ by $(\operatorname{conv} F)(t)=\operatorname{conv}(F(t))$ for every $t\in T$.

Let

$$X_{1,\varphi,k}=\{F\in X_{kc} : \underline{f}(F),\overline{f}(F)\in W^k_\varphi(T)\},$$
$$\tilde{X}_{\varphi,k}=\{F\in X_m : \operatorname{conv} F\in X_{1,\varphi,k}\}.$$

It is easy to see that $X_{1,\varphi,k}$, $\tilde{X}_{\varphi,k}$ are linear subsets of X and we will call them the *generalized Orlicz–Sobolev spaces of multifunctions*.

If $F\in X_{1,\varphi,k}$, then we define the generalized derivative of order $a\le k$ of F by

$$D^aF(t)=\mathcal{D}^a\underline{f}(F)(t)+\mathcal{D}^a(\overline{f}(F)(t)-\underline{f}(F)(t))[0,1] \quad \text{for every } t\in T.$$

If $F\in\tilde{X}_{\varphi,k}$, then we define the generalized derivative of order $a\le k$ of F by $D^aF=D^a(\operatorname{conv} F)$.

Let $F_1,F_2\in X_{1,\varphi,k}$ and

$$F_1(t)=f_1(t)+g_1(t)[0,1],\qquad F_2(t)=f_2(t)+g_2(t)[0,1],$$

for every $t\in T$. We define

$$\rho(F_1,F_2)=\|f_1-f_2\|^k_\varphi+\|g_1-g_2\|^k_\varphi.$$

It is easy to see that ρ is the metric in $X_{1,\varphi,k}$ and $(X_{1,\varphi,k},\rho)$ is a complete linear metric space.

Let $F_1,F_2\in\tilde{X}_{\varphi,k}$ and let

$$\varrho(F_1,F_2)=\rho(\operatorname{conv} F_1,\operatorname{conv} F_2)+\big\|\operatorname{dist}\big(F_1(\cdot),F_2(\cdot)\big)\big\|_\varphi.$$

It is easy to see that ϱ is a metric in $\tilde{X}_{\varphi,k}$.

THEOREM 4.1. *$(\tilde{X}_{\varphi,k},\varrho)$ is a complete metric space.*

Proof. By [K1, Theorem 1] we deduce that $(X_{m,\varphi},\|\operatorname{dist}(F(\cdot),G(\cdot))\|_\varphi)$ is a complete metric space.

Let $\{F_n\}$ be a Cauchy sequence in $\tilde{X}_{\varphi,k}$, then $\{F_n\}$ is a Cauchy sequence in $X_{m,\varphi}$. So there is an $F\in X_{m,\varphi}$ such that

$$\|\operatorname{dist}(F_n(\cdot),F(\cdot))\|_\varphi\longrightarrow 0 \text{ as } n\to\infty.$$

Also it is easy to see that $\underline{f}(F),\overline{f}(F)\in W^k_\varphi(T)$ so $F\in\tilde{X}_{\varphi,k}$. ■

It is well known that

REMARK 4.2. A function $u\in L^1([0,b),Y)$ possesses derivatives of every order in the distributional sense.

Let now $Y = \mathbb{R}^n$. Define

$$X_{L^1,Y} = \{F \in X_m : |F| \in L^1([0,b),Y)\}.$$

It is easy to see that $X_{L^1,Y}$ is a linear space. Let $F, G \in X_{L^1,Y}$, define

$$\rho(F,G) = \|\mathrm{dist}(F(\cdot),G(\cdot))\|.$$

THEOREM 4.3. *$(X_{L^1,Y},\rho)$ is a complete metric space.*

Proof. Let $\{F_n\}$ be a Cauchy sequence in $X_{L^1,Y}$, then there is an $F \in X$ such that

$$\mathrm{dist}(F_n(t),F(t)) \to 0 \text{ as } n \to \infty$$

in measure (because $(B(Y),\mathrm{dist})$ is a complete metric space). So there is a subsequence $\{F_{n_k}\}$ of the sequence $\{F_n\}$ such that

$$\mathrm{dist}(F_{n_k}(t),F(t)) \to 0 \text{ a.e.}$$

Applying the Fatou lemma, we obtain

$$\|\mathrm{dist}(F_n(\cdot),F(\cdot))\| \to 0 \text{ as } n \to \infty,$$

also we have

$$\int_0^b \mathrm{dist}(F(t),\{\theta\})\,dt \le \int_0^b \mathrm{dist}(F(t),F_n(t))\,dt + \int_0^b \mathrm{dist}(F_n(t),\{\theta\})\,dt < +\infty,$$

so $|F| \in L^1([0,b),Y)$, hence $F \in X_{L^1,Y}$. ■

Let $K(\theta,1)$ denote the closed unit ball in Y. Define

(9) $$X_{L^1,\mathrm{ball}} = \{F \in X_{L^1,Y} : F(t) = g(t) + |f(t)|K(\theta,1) \text{ for every } t \in [0,b)\},$$

where $f \in L^1([0,b),\mathbb{R})$, $g \in L^1([0,b),Y)$.

We easily obtain the following:

THEOREM 4.4. *$(X_{L^1,\mathrm{ball}},\rho)$ is a complete metric space.*

For $F \in X_{L^1,\mathrm{ball}}$ we define a generalized derivative of order k by

$$D^kF(t) = D^kg(t) + |D^kf(t)|K(\theta,1) \text{ for every } t \in [0,b).$$

Let now $Y = \mathbb{R}$. Let

(10) $$X_{1,L^1} = \{F \in X : F(t) = f(t) + g(t)[0,1] \text{ for every } t \in [0,b)\},$$

where $f, g \in L^1([0,b),\mathbb{R})$.

We easily obtain the following:

THEOREM 4.5. *(X_{1,L^1},ρ) is a complete linear metric space.*

For $F \in X_{1,L^1}$ we define a generalized derivative of order k by

$$D^kF(t) = D^kf(t) + D^kg(t)[0,1] \text{ for every } t \in [0,b).$$

It is easy to notice that the space X_{1,L^1} given by (10) is more comfortable than the space $X_{\infty,\mathcal{C}^1(T)}$ given by (6).

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