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ŁOJASIEWICZ-SICIAK CONDITION FOR THE PLURICOMPLEX GREEN FUNCTION

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Abstract. A compact set $K \subset \mathbb{C}^N$ satisfies Łojasiewicz-Siciak condition if it is polynomially convex and there exist constants $B, \beta > 0$ such that

$$V_K(z) \geq B(\mathrm{dist}(z,K))^\beta \quad \text{if} \quad \mathrm{dist}(z,K) \leq 1. \tag{ŁS}$$

Here V_K denotes the pluricomplex Green function of the set K. We cite theorems where this condition is necessary in the assumptions and list known facts about sets satisfying inequality (ŁS).

1. Introduction. Let $\mathcal{L}(\mathbb{C}^N)$ denote the family of all plurisubharmonic functions in $\mathbb{C}^N$ of minimal growth in the infinity, i.e. such plurisubharmonic functions u that

$$(u(z) - \log|z|) \leq O(1) \quad \text{as} \quad |z| \to \infty.$$

Let K be a compact set in $\mathbb{C}^N$. We are interested in some properties of its pluricomplex Green function V_K with pole at infinity, given by the formula

$$V_K(z) := \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^N) \text{ and } u|_K \leq 0\}, \quad z \in \mathbb{C}^N$$

(for the background see [Kl]). In the one-dimensional case if K is non-polar and Ω is the unbounded component of $\widehat{\mathbb{C}} \setminus K$, the function V_K coincides with $g_\Omega(\cdot,\infty)$, where $g_\Omega(\cdot,\cdot)$ is the generalized Green function of the domain Ω.

We say that a compact set K has *Hölder Continuity Property* if there exist constants $A, \alpha > 0$ such that

$$V_K(z) \leq A(\mathrm{dist}(z,K))^\alpha \quad \text{if} \quad \mathrm{dist}(z,K) \leq 1. \tag{HCP}$$

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If the inequality (HCP) is satisfied, we write $K \in \mathrm{HCP}(\alpha)$ or $K \in \mathrm{HCP}$ for short. This property is equivalent to the Hölder continuity of V_K in the whole space (see [Si]). There are quite large families of sets known to satisfy the inequality (HCP) (for examples and a rich list of references see [Pl]).

In [Ge] (see also [BG]) Gendre defined a *Łojasiewicz-Siciak condition*. A compact set K satisfies this condition if it is polynomially convex and there exist constants $B, \beta > 0$ such that

$$V_K(z) \geq B(\operatorname{dist}(z,K))^{\beta} \quad \text{if} \quad \operatorname{dist}(z,K) \leq 1. \tag{ŁS}$$

If the inequality (ŁS) is satisfied, we write $K \in \mathrm{ŁS}(\beta)$ or $K \in \mathrm{ŁS}$. This condition has not been investigated as thoroughly as (HCP) and became an object of interest only recently.

Gendre needed both inequalities (HCP) and (ŁS) to prove a theorem on approximation of functions in regular holomorphic Carleman classes on some compact sets. He posed an open problem to find more examples of sets satisfying Łojasiewicz-Siciak condition, since he did not know too many of them.

Throughout the paper $|\cdot|$ denotes the Euclidean norm in $\mathbb{K}^N$ ($\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $N \in \{1, 2, \dots\}$). By $\mathcal{P}_n(\mathbb{K}^N)$ we mean the space of polynomials of N variables with coefficients from $\mathbb{K}$ and of degree not greater than n. If E is a set and $f : E \to \mathbb{C}$ is a bounded function, then $\|f\|_E := \sup\{|f(z)| : z \in E\}$.

2. Approximation in Carleman classes. We say that a compact set $K \subset \mathbb{C}^N$ is *Whitney 1-regular* if it is pathwise-connected and there exists a constant $C > 0$ such that any two points x, y in K can be joined by a rectifiable arc of length not greater than $C|x-y|$ (i.e. if $\varrho(x,y)$ denotes the geodesic distance between the points x and y, then $\varrho(x,y) \leq C|x-y|$).

For a fixed $t_0 \gg 0$ choose a function $m \in \mathcal{C}^\infty(t_0, \infty)$ such that

- the functions m, m' and m'' are strictly positive;
- $\lim\limits_{t\to\infty} m'(t) = +\infty$;
- $\exists \delta > 0\ \forall t > t_0 \quad m''(t) \leq \delta$.

Put $M(t) := e^{m(t)}$. If $K \subset \mathbb{C}^N$ is compact and has a non-empty interior define

$$\mathcal{H}_M(K) := \big\{ f \in \mathcal{O}(\operatorname{Int} K) \cap \mathcal{C}^\infty(K) : \exists C, \varrho > 0\ \forall \alpha \in \mathbb{N}^N \\ \big(|\alpha| > t_0 \Rightarrow \|D^\alpha f\|_K \leq C\varrho^{|\alpha|} M(|\alpha|)\big)\big\}.$$

If $m(t) := t\log t$, then $\mathcal{H}_M(K) = \mathcal{O}(K)$. To have $\mathcal{O}(K) \subset \mathcal{H}_M(K)$ we take $m(t) := t\log t + t\mu(t)$, where $\mu \in \mathcal{C}^\infty(t_0, \infty)$ is such that

$$\mu' > 0, \quad \lim_{t\to\infty} \mu(t) = +\infty \quad \text{and} \quad \exists a : \mu(t) \leq at.$$

Put $\overline{\mu}(t) := \log t + \mu(t)$ and define $\overline{\omega}$ by the formula

$$\overline{\omega}(s) := \log(\inf_{t > t_0} s^t e^{t\overline{\mu}(t)}), \quad s \gg 0.$$

We are ready to cite the result mentioned in the introduction (see [Ge, Théorème 7] or [BG]).

THEOREM 2.1. *Let K be a Whitney 1-regular compact set in $\mathbb{C}^N$ with a non-empty interior. Assume that $K \in$ HCP and $K \in$ ŁS. Then*

(i) *For every $f \in \mathcal{H}_M(K)$*

$$\exists C, D > 0\ \exists (P_k)_{k\in\mathbb{N}} \subset \mathcal{P}_n(\mathbb{C}^N) : \|f - P_k\|_K \le Ce^{-D\overline{\omega}(k)}. \tag{1}$$

(ii) *If $f \in \mathcal{C}(K)$ satisfies (1), then $f \in \mathcal{H}_M(K)$.*

In the proof of this theorem a special consequence of inequalities (HCP) and (ŁS) was used. We will cite it here since it gives some information about the geometrical properties of sets satisfying these conditions.

Fix $s \ge 1$. We say that a compact set $K \subset \mathbb{C}^N$ is *s-H convex* if there exists a constant $A > 0$ such that for every $\varepsilon > 0$ there exists a pseudoconvex open set $U_\varepsilon \subset \mathbb{C}^N$ with

$$\{z \in \mathbb{C}^N : \operatorname{dist}(z, K) < A\varepsilon^s\} \subset U_\varepsilon \subset \{z \in \mathbb{C}^N : \operatorname{dist}(z, K) < \varepsilon\}.$$

All convex sets are 1-H convex. Note that an s-H convex compact set admits a suitable basis of pseudoconvex neighbourhoods. We have (see [BG] or [Ge, Lemme 16])

PROPOSITION 2.2. *Let K be a compact set in $\mathbb{C}^N$. If $K \in$ HCP(α), $K \in$ ŁS(β) and $s := \beta/\alpha$, then K is s-H convex.*

3. Two types of Markov properties. A compact set $K \subset \mathbb{K}^N$ is said to have the *global Markov property* if there exist constants $M, m > 0$ such that

$$\forall P \in \mathcal{P}_n(\mathbb{K}^N) \quad \|\operatorname{grad} P\|_K \le Mn^m \|P\|_K.$$

If this condition is satisfied, we write $K \in$ GMP(m) or $K \in$ GMP for short.

Put $B(x_0, r) := \{x \in \mathbb{K}^N : |x - x_0| \le r\}$ for $x_0 \in \mathbb{K}^N$ and $r > 0$.

We say that a compact set $K \subset \mathbb{R}^N$ admits the *local Markov inequality* if there exist constants $M > 0$ and $m, s \ge 1$ such that

$$\forall x_0 \in K\ \forall r \in (0,1]\ \forall n \in \mathbb{N}\ \forall P \in \mathcal{P}_n(\mathbb{R}^N)\ \forall \alpha \in \{0,1,2,\dots\}^N$$
$$|D^\alpha P(x_0)| \le \Big(\frac{Mn^s}{r^m}\Big)^{|\alpha|} \|P\|_{K\cap B(x_0,r)}.$$

If this condition is satisfied, we write $K \in$ LMI(m) or $K \in$ LMI for short.

In the definition above D^α denotes the real partial derivative. Below we deal with the holomorphic polynomials of one complex variable. We say that a compact set $K \subset \mathbb{C}$ has the *local Markov property* if there exist constants $M > 0$ and $m, s \ge 1$ such that

$$\forall z_0 \in K\ \forall r \in (0,1]\ \forall n \in \mathbb{N}\ \forall P \in \mathcal{P}_n(\mathbb{C}) \quad \|P'\|_{K\cap B(z_0,r)} \le \frac{Mn^s}{r^m} \|P\|_{K\cap B(z_0,r)}.$$

If this condition is satisfied, we write $K \in$ LMP(m) or $K \in$ LMP for short.

Bos and Milman proved in [BM] that $K \in$ GMP $\iff$ $K \in$ LMI for any compact set $K \subset \mathbb{R}^N$. A natural question whether $K \in$ GMP $\iff$ $K \in$ LMP for any compact set $K \subset \mathbb{C}$ is the object of investigations of Białas-Cież and Eggink. The answer is negative unless we assume that $K \in$ ŁS, as was shown by an example given in [BCE]. This fact underlines the importance of the Łojasiewicz-Siciak condition.

Inequality (ŁS) is also a necessary condition for Jackson property in the complex plane, which in its turn is an important tool in the investigations of Białas-Cież and

Eggink. If K is a compact set in $\mathbb{C}$, by $\mathcal{H}^\infty(K)$ we denote the space of all functions of the class $\mathcal{C}^\infty(\mathbb{C})$ which are holomorphic in an open neighbourhood of the set K. For any $\delta > 0$ put $K_\delta := \{z \in \mathbb{C} : \operatorname{dist}(z, K) \le \delta\}$. We say that a compact set $K \subset \mathbb{C}$ admits the *Jackson property* if there exist constants $c, C > 0$ and $s, v \ge 1$ such that

$$\forall l \ge 1\ \forall \delta \in (0,1]\ \forall n \in \mathbb{N}\ \forall f \in \mathcal{H}^\infty(K) \quad n^l \operatorname{dist}(f, \mathcal{P}_n(\mathbb{C})) \le \Big(\frac{Cl^v}{\delta^s}\Big)^l \delta^{-c} \|f\|_{K_\delta}.$$

If this condition is satisfied, we write $K \in \mathrm{JP}(s,v)$ or $K \in \mathrm{JP}$ for short.

The following two results can be found in [BCE] (see also [Eg, Theorem 8.24 and Proposition 8.26]).

THEOREM 3.1 (Jackson's theorem in the complex plane). *Let $K \subset \mathbb{C}$ be compact and $s \ge 1$. If $K \in \text{ŁS}(s)$ and $K \in \mathrm{HCP}$, then $K \in \mathrm{JP}(s,1)$.*

PROPOSITION 3.2. *Let $K \in \mathbb{C}$ be compact, $s, v \ge 1$ and $\beta > s$. Then*

- $K \in \mathrm{JP}(s,v) \Longrightarrow K \in \text{ŁS}(\beta)$,
- $K \in \mathrm{JP}(s,1) \Longrightarrow K \in \text{ŁS}(s)$.

4. Sets in $\mathbb{C}^N$ with (ŁS). Note first that the inequality (ŁS) is satisfied for all pluripolar compact sets because their pluricomplex Green functions are equal to plus infinity outside the sets.

It is not so easy to give the straightforward formula of the pluricomplex Green function. We will list now some sets for which the formulas are known.

Let $|||\cdot|||$ be a complex norm in $\mathbb{C}^N$. Put $K := K(a,r) := \{z \in \mathbb{C}^N : |||z-a||| \le r\}$ for $a \in \mathbb{C}^N$ and $r > 0$. Then K is compact, polynomially convex and (see [Kl, Example 5.1])

$$V_K(x) = \log^+ \frac{|||z-a|||}{r}, \quad z \in \mathbb{C}^N.$$

Therefore $K \in$ ŁS.

Let $f = (f_1, \ldots, f_N) : \mathbb{C}^N \to \mathbb{C}^N$ be a polynomial mapping. By $\hat{f}_j$ denote the homogeneous part of f_j of the highest degree (i.e. of degree $\deg(f_j)$). We say that f is *regular* if $(\hat{f}_1, \ldots, \hat{f}_N)^{-1}(0) = \{0\}$. The set

$$E(f) := \{z \in \mathbb{C}^N : |f_j(z)| \le 1,\ j \in \{1, \ldots, N\}\}$$

is a polynomial polyhedron. We have (see [Kl, Corollary 5.3.2])

PROPOSITION 4.1. *If $f : \mathbb{C}^N \to \mathbb{C}^N$ is a regular polynomial mapping, then*

$$V_{E(f)} = \max_{j \in \{1,\ldots,N\}} \frac{1}{\deg(f_j)} \log^+ |f_j|.$$

Therefore $E(f) \in$ ŁS too.

If K is a compact subset of $\mathbb{R}^N = \mathbb{R}^N + 0i \subset \mathbb{C}^N = \mathbb{R}^N + i\mathbb{R}^N$, then K is polynomially convex in $\mathbb{C}^N$ (see [Kl, Lemma 5.4.1]). Define

$$\mathbf{h} : \mathbb{C} \setminus [-1,1] \ni \zeta \mapsto \zeta + (\zeta^2 - 1)^{1/2} \in \{\xi \in \mathbb{C} : |\xi| > 1\},$$

where the branch of the square root is chosen so that $h(t) > 1$ for $t > 1$. Prolong the function putting $\mathbf{h}|_{[-1,1]} \equiv 1$. We have (see [Kl, Corollary 5.4.5])

$$V_{[-1,1]^N}(z_1, \ldots, z_N) = \max_{j \in \{1,\ldots,N\}} \log |\mathbf{h}(z_j)|, \quad z = (z_1, \ldots, z_N) \in \mathbb{C}^N.$$

Hence $[-1,1]^N \in$ ŁS. Furthermore, if $\mathbb{B} := \{x \in \mathbb{R}^N : |x| \le 1\}$, then

$$V_{\mathbb{B}}(z) = \frac{1}{2}\log \mathbf{h}(|z|^2 + |z_1^2 + \ldots + z_N^2 - 1|), \qquad z = (z_1, \ldots, z_N) \in \mathbb{C}^N,$$

(see [Kl, Theorem 5.4.6]). Thus $\mathbb{B} \in$ ŁS too.

Let us finally note the following observations which can lead to constructions of some other examples.

PROPOSITION 4.2. *Let $a, b \in \mathbb{N}$, the sets $E \subset \mathbb{C}^a$ and $F \subset \mathbb{C}^b$ be compact. If $E \in$ ŁS(β_1) and $F \in$ ŁS(β_2), then $E \times F \in$ ŁS$(\max(\beta_1, \beta_2))$.*

Proof. Take $(z, w) \in \mathbb{C}^a \times \mathbb{C}^b$ with $\operatorname{dist}((z,w), E \times F) \le 1$. We have

$$V_E(z) \ge B_1(\operatorname{dist}(z,E))^{\beta_1} \quad \text{and} \quad V_F(w) \ge B_2(\operatorname{dist}(w,F))^{\beta_2}$$

where B_1, B_2 do not depend on the choice of (z, w).

Put $\beta := \max(\beta_1, \beta_2)$ and $B := \sqrt{2}^{\beta} \min(B_1, B_2)$.

There exist $z_0 \in E$ and $w_0 \in F$ with $\operatorname{dist}(z, E) = |z - z_0|$ and $\operatorname{dist}(w, F) = |w - w_0|$. Assume e.g. that $|z - z_0| \ge |w - w_0|$. From [Kl, Theorem 5.1.8] it follows that $V_{E\times F}(z, w) = \max(V_E(z), V_F(w))$. Hence

$$\begin{aligned} V_{E\times F}(z,w) \ge V_E(z) \ge B_1|z - z_0|^{\beta} &= B_1(\max(|z - z_0|, |w - w_0|))^{\beta} \\ &\ge B|(z,w) - (z_0, w_0)|^{\beta} \ge B(\operatorname{dist}((z,w), E \times F))^{\beta}. \ \blacksquare \end{aligned}$$

PROPOSITION 4.3. *Let $E_1, \ldots, E_k$ be compact, convex and symmetric subsets of $\mathbb{R}^N$ with non-empty interiors and define $E := E_1 \cap \ldots \cap E_k$. If $E_1, \ldots, E_k \in$ ŁS, then $E \in$ ŁS.*

Proof. In view of [Ba, Proposition 3.3] we have

$$V_E = \max_{j \in \{1,\ldots,k\}} V_{E_j}.$$

Hence if $E_j \in$ ŁS(β_j) for every $j \in \{1, \ldots, k\}$ and $\beta := \max\{\beta_j : j \in \{1, \ldots, k\}\}$, then $E \in$ ŁS(β). ∎

We can apply Proposition 4.3 to construct another example. Consider

$$K := \{(x, y) \in \mathbb{R}^2 : |x| + |y| \le 1\}.$$

Since $K = \phi([-1,1]^2)$, where

$$\phi : \mathbb{C}^2 \ni (z, w) \mapsto \Big(\frac{z+w}{\sqrt{2}}, \frac{z-w}{\sqrt{2}}\Big) \in \mathbb{C}^2,$$

it is easy to check that $K \in$ ŁS (because $[-1,1]^2 \in$ ŁS as mentioned above). Therefore by Proposition 4.3 the symmetric octagon $K \cap [-1,1]^2$ belongs to ŁS too.

5. One-dimensional case. We start with another concrete positive example.

$$V_{[-1,1]\cup[-i,i]} = \frac{1}{4}\log \mathbf{h}(|z^4| + |z^4 - 1|), \qquad z \in \mathbb{C},$$

by [Kl, Example 5.4.8], therefore $[-1,1] \cup [-i,i] \in$ ŁS.

The examples from the previous section and the one mentioned here are almost all quite regular, the sets satisfy Hölder Continuous Property. However an example given by

Siciak shows that there exist regular sets without Łojasiewicz-Siciak property. Namely

$$K := \{\zeta \in \mathbb{C} : |\zeta - 2| \le 2\} \cup \{\zeta \in \mathbb{C} : |\zeta + 2| \le 2\}$$

does not satisfy (ŁS) (see [BCK, Example 1.1]). However $K \in \mathrm{HCP}$.

This example is not so surprising if one knows the following characterization. Let us recall that a simply connected domain D in $\widehat{\mathbb{C}}$ (such that its complement has at least two points) is a *Hölder domain*, if the conformal map $f : \{\zeta \in \mathbb{C} : |\zeta| < 1\} \longrightarrow D$ is Hölder continuous up to the closed disk $\{\zeta \in \mathbb{C} : |\zeta| \le 1\}$ (see [Po, §4.6]). We are ready now to state the characterization given in [BCE].

THEOREM 5.1. *Let $K \subset \mathbb{C}$ be compact and simply connected and let it have at least two points. Then*

$$K \in \text{ŁS} \iff \widehat{\mathbb{C}} \setminus K \textit{ is a Hölder domain.}$$

For a contrast we recall now a construction of a big family of totally disconnected compact sets satisfying Łojasiewicz-Siciak condition. Consider a family of contracting similarities of $\mathbb{C}$, i.e. the system $(f_1, \dots, f_m)$ of functions $f_j : \mathbb{C} \to \mathbb{C}$ such that there exist constants $a_1, \dots, a_m \in (0, 1)$ with

$$|f_j(z) - f_j(w)| = a_j|z - w|, \qquad z, w \in \mathbb{C},\ j \in \{1, \dots, m\}.$$

If

$$f(K) := \bigcup_{j=1}^{m} f_j(K), \quad \mathbb{C} \supset K \text{ compact},$$

then there exists a unique non-empty compact set $E \subset \mathbb{C}$ satisfying $f(E) = E$. This set is called the *attractor* of the iterated function system $(f_1, \dots, f_m)$ (for the background see [Hu]). In [BR] the condition COSC was defined: we say that the iterated function system $(f_1, \dots, f_m)$ satisfies the *closed open set condition*, COSC for short, if there exists an open set U such that

- $f_j(U) \subset U$, $j \in \{1, \dots, m\}$;
- $f_j(\overline{U}) \cap f_l(\overline{U}) = \emptyset$, $j \ne l$, $j, l \in \{1, \dots, m\}$.

In this situation the attractor is a totally disconnected set ([BCK]) and it is uniformly perfect ([BBRR]). It satisfies the Łojasiewicz-Siciak condition ([BCK]).

The most known example of the sets obtained in the way described above is the classical ternary Cantor set. However this set is contained in $\mathbb{R}$, therefore the fact that it satisfies the Łojasiewicz-Siciak condition follows also from another result obtained independently (see [BCE] or [Eg, Proposition 8.27]).

THEOREM 5.2. *If $K \subset \mathbb{R}$ is compact, then $K \in \text{ŁS}(1)$.*

Let now Ω be a non-empty proper open subset of $\mathbb{C}$ and let $\partial\Omega$ denote its boundary. For any $z_0 \in \Omega$ we define $D(z_0) := \{z \in \mathbb{C} : |z - z_0| < \operatorname{dist}(z_0, \partial\Omega)\}$. Let $a > 0$. We say that Ω is *a-admissible* if for every $z_0 \in \Omega$ there exists a $w_0 \in \Omega$ with $\operatorname{dist}(w_0, \partial\Omega) = a$ and $D(z_0) \subset D(w_0)$. We have (see [Ku] or [Ga])

THEOREM 5.3. *Let Ω be a non-empty bounded open set in $\mathbb{C}$, $a > 0$ and u be a positive superharmonic function on Ω. If Ω is a-admissible, then $\exists B > 0 : u \ge B \operatorname{dist}(\cdot, \partial\Omega)$.*

We are ready to prove the last result.

PROPOSITION 5.4. *Let K be a regular polynomially convex compact set in $\mathbb{C}$ and $a > 0$. If $\mathbb{C} \setminus K$ is a-admissible, then $K \in$ ŁS(1).*

Proof. Take an $r > 0$ such that $K \subset B(0,r) = \{z \in \mathbb{C} : |z| \le r\}$. Put

$$\Omega := (\mathbb{C} \setminus K) \cap \{z \in \mathbb{C} : |z| < r + 3\}.$$

Then Ω is a-admissible and $V_K|_\Omega$ is positive and harmonic, hence we can apply Theorem 5.3. ■

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