FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

DIEUDONNÉ OPERATORS ON THE SPACE OF BOCHNER INTEGRABLE FUNCTIONS

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Abstract. A bounded linear operator between Banach spaces is called a Dieudonné operator (=weakly completely continuous operator) if it maps weakly Cauchy sequences to weakly convergent sequences. Let (Ω, Σ, μ) be a finite measure space, and let X and Y be Banach spaces. We study Dieudonné operators $T : L^1(X) \to Y$. Let $i_\infty : L^\infty(X) \to L^1(X)$ stand for the canonical injection. We show that if X is almost reflexive and $T : L^1(X) \to Y$ is a Dieudonné operator, then $T \circ i_\infty : L^\infty(X) \to Y$ is a weakly compact operator. Moreover, we obtain that if $T : L^1(X) \to Y$ is a bounded linear operator and $T \circ i_\infty : L^\infty(X) \to Y$ is weakly compact, then T is a Dieudonné operator.

1. Introduction and preliminaries. Throughout the paper $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are real Banach spaces and X^*, Y^* and Z^* denote their Banach duals respectively. By $B(X)$, $B(Y)$ and $B(Z)$ we will denote the closed unit balls in X, Y and Z respectively. Let $\mathcal{L}(X,Y)$ stand for the space of all bounded linear operators from X to Y. We denote by $\sigma(L,K)$ the weak topology with respect to a dual pair $\langle L, K\rangle$. Recall that a subset A of L is said to be *conditionally* $\sigma(L,K)$*-compact* whenever each sequence in A contains a $\sigma(L,K)$-Cauchy subsequence. A Banach space X is said to be *almost reflexive* if every norm-bounded subset of X is conditionally weakly compact (see [C]). The fundamental ℓ^1-Rosenthal theorem [R] says that a Banach space X is almost reflexive if and only if it contains no isomorphic copy of ℓ^1. For terminology concerning vector lattices we refer to [AB]. By $\mathbb{N}$ and $\mathbb{R}$ we denote the sets of natural and real numbers.

A bounded linear operator $T : Z \to Y$ is called a *Dieudonné operator* (= *weakly completely continuous operator*) if it maps weakly Cauchy sequences in Z to weakly convergent sequences in Y (see [BC1], [BC2], [ABBL]).

2010 *Mathematics Subject Classification*: 47B38, 47B05, 46E40.
Key words and phrases: Dieudonné operators, weakly completely continuous operators, weakly compact operators, conditional compactness.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-19

In this paper we study Dieudonné operators T from the Banach space of Bochner integrable functions $L^1(X)$ (over a finite measure space) to Y. We prove that if X is an almost reflexive Banach space and $T : L^1(X) \to Y$ is a Dieudonné operator, then the restriction of T to $L^\infty(X)$ is a weakly compact operator (see Theorem 2.2 below). Moreover, we show that if the restriction to $L^\infty(X)$ of a bounded linear operator $T : L^1(X) \to Y$ is weakly compact, then T is a Dieudonné operator (see Theorem 2.4 below).

The following general characterization of Dieudonné operators between Banach spaces will be useful.

PROPOSITION 1.1. *For a bounded linear operator $T : Z \to Y$ the following statements are equivalent*:

(i) *T is a Dieudonné operator.*
(ii) *T maps conditionally weakly compact sets in Z into relatively weakly compact sets in Y.*

Proof. (i)⇒(ii) Assume that T is a Dieudonné operator, and let A be a conditionally weakly compact set in Z. We shall show that $T(A)$ is a relatively weakly sequentially compact set in Y. Indeed, let (y_n) be a sequence in $T(A)$, i.e., $y_n = T(z_n)$, where $z_n \in A$. Hence there exists a weakly Cauchy subsequence (z_{k_n}) of (z_n). It follows that $y_{k_n} = T(z_{k_n}) \to y \in Y$ for $\sigma(Y, Y^*)$. This means that $T(A)$ is relatively weakly sequentially compact in Y, and by the Eberlain-Šmulian theorem, $T(A)$ is relatively weakly compact in Y, as desired.

(ii)⇒(i) Assume that T maps conditionally weakly compact sets in Z to relatively weakly sequentially compact sets in Y. To show that T is a Dieudonné operator, assume that (z_n) is a weakly Cauchy sequence in Z. Since the set $\{z_n : n \in \mathbb{N}\}$ is conditionally weakly compact, the set $\{T(z_n) : n \in \mathbb{N}\}$ is relatively weakly compact in Y. Hence by the Eberlein-Šmulian theorem $\{T(z_n) : n \in \mathbb{N}\}$ is relatively weakly sequentially compact in Y. It follows that there exist a subsequence (z_{k_n}) of (z_n) and $y \in Y$ such that $T(z_{k_n}) \to y$ for $\sigma(Y, Y^*)$. On the other hand, since T is $(\sigma(Z, Z^*), \sigma(Y, Y^*))$-continuous (see [AB, Theorem 17.1]), we obtain that $(T(z_n))$ is a weakly Cauchy sequence in Y. It follows that $T(z_n) \to y$ for $\sigma(Y, Y^*)$, and this means that T is a Dieudonné operator. ∎

2. Dieudonné operators on $L^1(X)$. From now we assume that (Ω, Σ, μ) is a complete finite measure space. Let $\mathbb{1}_A$ denote the characteristic function of a set $A \in \Sigma$. By $L^0(X)$ we denote the set of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$. For $f \in L^0(X)$ let us set $\widetilde{f}(\omega) = \|f(\omega)\|_X$ for $\omega \in \Omega$. Let

$$L^1(X) = \Big\{ f \in L^0(X) : \|f\|_{L^1(X)} := \|\widetilde{f}\|_{L^1} = \int_\Omega \widetilde{f}(\omega)\, d\mu < \infty \Big\}$$

and

$$L^\infty(X) = \big\{ f \in L^0(X) : \|f\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{\omega \in \Omega} \widetilde{f}(\omega) < \infty \big\}.$$

If $X = \mathbb{R}$ we simply write L^1 and L^∞. For a subset H of $L^1(X)$ let

$$\widetilde{H} = \{\widetilde{f} : f \in H\}.$$

The following characterization of conditional weak compactness in $L^1(X)$ will be of importance (see [T, Corollary 9], [N, Theorem 2.7, Proposition 2.1]).

PROPOSITION 2.1. *Assume that X is an almost reflexive Banach space. Then for a subset H of $L^1(X)$ the following statements are equivalent:*

(i) *H is conditionally weakly compact in $L^1(X)$.*
(ii) *$\widetilde{H}$ is conditionally weakly compact in L^1.*
(iii) *$\widetilde{H}$ is a bounded and uniformly integrable subset of L^1.*
(iv) *$\widetilde{H}$ is a bounded subset of L^1 and the functional p_H on L^∞ defined for $v \in L^\infty$ by*

$$p_H(v) = \sup_{f\in H} \int_\Omega \widetilde{f}(\omega)|v(\omega)|\,d\mu$$

is an order continuous seminorm.

Now we are ready to establish a relationship between a Dieudonné operator $T : L^1(X) \to Y$ and the restriction of T to $L^\infty(X)$.

Let $i_\infty : L^\infty(X) \to L^1(X)$ denote the canonical injection.

THEOREM 2.2. *Let X be an almost reflexive Banach space and let Y be a Banach space. Let $T : L^1(X) \to Y$ be a Dieudonné operator. Then the operator $T \circ i_\infty : L^\infty(X) \to Y$ is weakly compact.*

Proof. In view of Proposition 1.1 we will prove that $B(L^\infty(X))$ is a conditionally weakly set in $L^1(X)$. Indeed, making use of Proposition 2.1 it is enough to show that the functional $p_{B(L^\infty(X))}$ on L^∞ defined for $v \in L^\infty$ by

$$p_{B(L^\infty(X))}(v) = \sup_{f\in B(L^\infty(X))} \int_\Omega \widetilde{f}(\omega)|v(\omega)|\,d\mu$$

is an order continuous seminorm. Note that $p_{B(L^\infty(X))}(v) = \|v\|_{L^1}$ for every $v \in L^\infty \subset L^1$. ∎

Before stating our next result we recall the following theorem (see [D, p. 227], [AB, Theorem 10.17]).

THEOREM 2.3 (A. Grothendieck). *A subset A of a Banach space Y is relatively weakly compact if and only if for each $\varepsilon > 0$ there exists a relatively weakly compact subset K_ε of Y with $A \subset \varepsilon B(Y) + K_\varepsilon$.*

THEOREM 2.4. *Let $T : L^1(X) \to Y$ be a bounded linear operator and assume that $T \circ i_\infty : L^\infty(X) \to Y$ is a weakly compact operator. Then $T : L^1(X) \to Y$ is a Dieudonné operator.*

Proof. Note that $T(B(L^\infty(X)))$ is relatively weakly compact in Y. Let H be a conditionally weakly compact subset of $L^1(X)$. Then $\widetilde{H}$ is a uniformly integrable subset of L^1 (see [BC, Theorem 2.2]). For $f \in L^1(X)$ and $\lambda > 0$ let

$$A_{f,\lambda} = \{\omega \in \Omega : \widetilde{f}(\omega) > \lambda\}.$$

Then

$$\lim_{\lambda\to\infty} \sup_{f\in H} \int_{A_{f,\lambda}} \widetilde{f}(\omega)\,d\mu = \lim_{\lambda\to\infty} \sup_{f\in H} \|\mathbb{1}_{A_{f,\lambda}} f\|_{L^\infty(X)} = 0.$$

Let $\varepsilon > 0$ be given. Then there exists $\lambda_\varepsilon > 0$ such that $\|\mathbb{1}_{A_{f,\lambda_\varepsilon}} f\|_{L^1(X)} \le \frac{\varepsilon}{\|T\|}$ for all $f \in H$. Hence for $f \in H$ we have $\|T(\mathbb{1}_{A_{f,\lambda_\varepsilon}} f)\|_Y \le \varepsilon$. Moreover, $\mathbb{1}_{\Omega\setminus A_{f,\lambda_\varepsilon}}(\omega)\widetilde{f}(\omega) \le \lambda_\varepsilon$ for $\omega \in \Omega$, so $\|\mathbb{1}_{\Omega\setminus A_{f,\lambda_\varepsilon}} f\|_{L^\infty(X)} \le \lambda_\varepsilon$, i.e., $\mathbb{1}_{\Omega\setminus A_{f,\lambda_\varepsilon}} f \in \lambda_\varepsilon B(L^\infty(X))$. Hence

$$T(f) = T(\mathbb{1}_{A_{f,\lambda_\varepsilon}} f) + T(\mathbb{1}_{\Omega\setminus A_{f,\lambda_\varepsilon}} f) \in \varepsilon B(Y) + \lambda_\varepsilon T(B(L^\infty(X))).$$

Hence, in view of Theorem 2.3, $T(H)$ is a relatively weakly compact subset of Y. By Proposition 1.1 T is a Dieudonné operator. ∎

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