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NARROW OPERATORS (A SURVEY)

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Abstract. Narrow operators are those operators defined on function spaces which are "small" at signs, i.e., at $\{-1, 0, 1\}$-valued functions. We summarize here some results and problems on them. One of the most interesting things is that if E has an unconditional basis then each operator on E is a sum of two narrow operators, while the sum of two narrow operators on L_1 is narrow. Recently this notion was generalized to vector lattices. This generalization explained the phenomena of sums: the set of all regular narrow operators is a band in the vector lattice of all regular operators (in particular, a subspace). In L_1 all operators are regular, and in spaces with unconditional bases narrow operators with non-narrow sum are non-regular. Nevertheless, a new lattice approach has led to new interesting problems.

1. Introduction. Most classes of operators which are not isomorphic embeddings are characterized by some kind of a "smallness" conditions. Narrow operators are those operators defined on function spaces which are "small" at signs, i.e. at $\{-1, 0, 1\}$-valued functions. The idea to consider such operators has led to interesting problems which can be applied to Geometric Functional Analysis.

We present here the most significant results and open problems on narrow operators, as to our point of view. The simplest ones are accompanied with proofs, and some more complicated results are explained by sketches.

1.1. History of the notion. Formally the notion of narrow operators was introduced by Plichko and the author in 1990 (see [PlPo] and [Pop3]) for operators acting from a rearrangement invariant (r.i.) function F-space with an absolutely continuous norm to an F-space. But in fact, these operators were studied by several authors before 1990,

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including Bourgain and Rosenthal [BoRo], Johnson, Maurey, Schechtman and Tzafriri [JMST]. Ghoussoub and Rosenthal [GhRo] (1983) considered the operators from L_1 that are exactly non-narrow, and called them “norm-sign-preserving” operators. Also several papers of Rosenthal [Ros3], [Ros4], [Ros5] contain results on narrow operators. The first systematic study of narrow operators was done by Plichko and Popov in the above mentioned paper [PlPo]. The notion of narrow operators was extended to $C(K)$-spaces by V. Kadets and the author [KaPo2] (1996). Next, a new notion of narrow operators defined on Banach spaces with the Daugavet property was introduced by V. Kadets, Shvidkoy and Werner in [KSW] (2001). Recently, the notion of narrow operators was extended to vector lattices by O. Maslyuchenko, Mykhaylyuk and Popov in [MMyP2] (2009).

In this survey we do not deal with narrow operators in $C(K)$-spaces introduced by V. Kadets and the author, and narrow operators in Banach spaces with the Daugavet property introduced by V. Kadets, Shvidkoy and Werner.

1.2. Our notation. Throughout the paper E is assumed to be a *Köthe F-space* on a finite atomless[1] measure space (Ω, Σ, μ). This means that E is a metric linear space with an invariant[2] metric ρ of equivalence classes of Σ-measurable functions $x : \Omega \to \mathbb{K}$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, such that if $x \in E$, $y \in L_0(\mu)$ and $|y| \le |x|$ then $y \in E$ and $\|y\| \le \|x\|$ (here $L_0(\mu)$ denotes the set of all equivalence classes of Σ-measurable functions; $\|x\| = \rho(x, 0)$ and the inequality $u \le v$ for $u, v \in L_0(\mu)$ means that $u(\omega) \le v(\omega)$ holds a.e. on Ω). It is also supposed that $\mathbf{1} = \mathbf{1}_\Omega \in E$ (by $\mathbf{1}_A$ we denote the characteristic function of a set $A \in \Sigma$). Σ^+ means $\{A \in \Sigma : \mu(A) > 0\}$; for a given $A \in \Sigma$ we set $\Sigma(A) = \{B \in \Sigma : B \subseteq A\}$, $\Sigma(A)^+ = \Sigma(A) \cap \Sigma^+$ and $E(A) = \{x \in E : \operatorname{supp} x \subseteq A\}$. For the Borel σ-algebra on $[0, 1]$ we fix the notation $\mathcal{B}$, and by λ we denote the Lebesgue measure on $\mathcal{B}$.

The letters E, F are reserved for function spaces or vector lattices, and X, Y for general Banach or F-spaces, for which by $\mathcal{L}(X, Y)$ we denote the set of all continuous linear operators; $\mathcal{L}(X) = \mathcal{L}(X, X)$. B_X stands for the closed unit ball of a Banach space X.

According to Semenov [Sem] (1964), a Köthe Banach space E is called a *symmetric space* if the following condition holds

$$\text{for each } x \in L_0(\mu) \text{ and } y \in E \text{ if } d_{|x|} = d_{|y|} \text{ then } x \in E \text{ and } \|x\| = \|y\|. \qquad (*)$$

Here $d_z : \mathbb{R} \to [0, \mu(\Omega)]$ is the distribution function of a real function $z \in L_0(\mu)$ defined as

$$d_z(t) = \mu\{\omega \in \Omega : z(\omega) < t\}.$$

We remark that if E is a Banach space then $E \subseteq L_1(\mu)$ by the definition of Köthe function space. Nevertheless, we consider a more general case.

Definition 1.1. A Köthe function F-space E is called a *symmetric F-space* if $(*)$ holds.

All the spaces $L_p(\mu)$ with $0 \le p \le \infty$ are symmetric F-spaces. For more information on symmetric F-spaces see [PlPo].

[1] = non-atomic; to our point of view, the word “atomless” reflexes its sense better

[2] i.e. $\rho(x, y) = \rho(x + z, y + z)$

A close notion is a *rearrangement invariant function space*. The difference is that in the definition of a r.i. function space given in [LiTz, p. 118] one has two additional conditions which are unessential for our purposes. Moreover, we consider symmetric spaces on arbitrary atomless finite measure spaces. The restriction to finite measure spaces is just for convenience of notation in definitions and formulation of the results (in the definition of narrow operator on a σ-finite or even arbitrary infinite measure space one should consider the sets of finite measure only).

We remark that a symmetric space E on $[0,1]$ with an absolutely continuous norm and the normalized condition $\|\mathbf{1}_{[0,1]}\|_E = 1$ is a r.i. function space.

1.3. Definitions and useful observations. We say that a Köthe F-space E has *an absolutely continuous norm* if for each $x \in E$ and each decreasing sequence $A_n \in \Sigma$ with empty intersection one has $\lim\limits_{n\to\infty} \|x \cdot \mathbf{1}_{A_n}\| = 0$. The space $L_p(\mu)$ has an absolutely continuous norm if $0 < p < \infty$ and does not have if $p = \infty$.

An element $x \in L_0(\mu)$ is called *a sign* if x takes values in the set $\{-1, 0, 1\}$, and *a sign on* $A \in \Sigma$ if it is a sign with $\operatorname{supp} x = A$. A sign x is said to be *of mean zero*, provided $\displaystyle\int_\Omega x\, d\mu = 0$.

DEFINITION 1.2. Let E be a Köthe F-space and X be an F-space. An operator $T \in \mathcal{L}(E,X)$ is called *narrow* if for each $A \in \Sigma^+$ and each $\varepsilon > 0$ there exists a mean zero sign x on A such that $\|Tx\| < \varepsilon$. If for each $A \in \Sigma^+$ there exists a mean zero sign x on A such that $Tx = 0$ then T is called *strictly narrow*[3].

The same definition can be applied to a larger class of mappings. In particular, to prove that every order-to-norm continuous AM-compact operator (see Subsection 7.1 for the definitions) from a vector lattice without the assumption about absolute continuity of its norm to a Banach space is narrow, the authors in [MMyP2] (2009) used narrow non-linear maps.

The following lemma gives a useful sufficient condition for an operator to be narrow.

LEMMA 1.3. *Let E have an absolutely continuous norm. Then an operator $T \in \mathcal{L}(E,X)$ is narrow if and only if for each $A \in \Sigma$ and each $\varepsilon > 0$ there are $B \in \Sigma(A)$ and a sign x on B such that $\mu(B) \geq \mu(A)/2$ and $\|Tx\| < \varepsilon$. In particular, the condition on a sign to be of mean zero in the definition is not necessary.*

The "only if" part is evident. One can prove "if" by a recursive construction of measurable subsets $B_1 \subseteq A$, $B_2 \subseteq A \setminus B_1$, … [PlPo, p. 54]. On the other hand, we do not know whether Lemma 1.3 is true without the absolute continuity of the norm of E. In particular, the following problem is unsolved.

PROBLEM 1.4. *Does Definition 1.2 remain the same for $E = L_\infty$ if one omits the condition on a sign to be of mean zero?*

Problem 1.4 is open for both narrow and strictly narrow operators. The answer is affirmative for order-to-norm continuous operators [KMMMP] (2009) (see Subsection 7.1 for the definitions).

[3] actually, the strict narrowness is a property of the operator kernel.

The following lemma is a useful reformulation of the definition of narrow operator.

LEMMA 1.5. *Let $T \in \mathcal{L}(E, X)$ be a narrow operator. Then for each $A \in \Sigma$, $\varepsilon > 0$ and an integer $n \geq 1$ there is a partition $A = A' \sqcup A''$ to disjoint subsets of measures $\mu(A') = (1 - 2^{-n})\, \mu(A)$ and $\mu(A'') = 2^{-n}\mu(A)$ such that $\|Th\| < \varepsilon$, where $h = \mathbf{1}_{A'} - (2^n - 1)\, \mathbf{1}_{A''}$.*

For the proof one should use the definition n times.

2. Each "small" operator is narrow. From now on, up to Section 7 unless otherwise stated, we always assume that the norm of E is absolutely continuous.

Recall that a linear operator T from a vector lattice (in particular, from a Köthe F-space) E to an F-space X is called *AM-compact* if it sends order bounded sets from E to relatively compact sets in X. If E is a Banach lattice then an AM-compact operator is automatically continuous. Obviously, each compact operator is AM-compact, but not conversely (for example, the conditional expectation operator in $L_p(\mu)$ for $1 \leq p < \infty$ with respect to a purely atomic sub-σ-algebra is AM-compact but not compact).

PROPOSITION 2.1. *Let E be a Köthe F-space with an absolutely continuous norm and let X be an F-space. Then each AM-compact operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Given any $A \in \Sigma^+$ and $\varepsilon > 0$, we consider a Rademacher system (r_n) in $E(A)$ (see [AbAl, p. 497]). Then the set $\{Tr_n : n \in \mathbb{N}\}$ is relatively compact and hence, there are numbers $n \neq m$ such that $\|Th\| < \varepsilon$ where $h = (r_n - r_m)/2$. Since h is a sign on some $B \in \Sigma(A)$ with $\mu(B) = \mu(A)/2$, by Lemma 1.3, T is narrow. ∎

Recall that an operator $T \in \mathcal{L}(X, Y)$ between Banach spaces is called a Dunford-Pettis operator if T sends weakly null sequences from X to norm null sequences in Y.

PROPOSITION 2.2. *Let E be a symmetric Banach space with an absolutely continuous norm and X be a Banach space. Then every Dunford-Pettis operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Given any $A \in \Sigma^+$, consider a Rademacher system (r_n) in $E(A)$ (see [AbAl, p. 497]). Since $r_n \xrightarrow{\mathrm{w}} 0$ [LiTz, p. 160], we have that $\|Tr_n\| \to 0$. ∎

Recall that an operator $T \in \mathcal{L}(L_1(\mu), X)$ is said to be *representable* if there is $y \in L_\infty(X)$ such that

$$Tx = \int_\Omega xy\, d\mu$$

for all $x \in L_1(\mu)$. For more information on representable operators we refer the reader to [DiUh].

PROPOSITION 2.3. *Let X be a Banach space. Then each representable and hence, each weakly compact operator $T \in \mathcal{L}\big(L_1(\mu), X\big)$ is narrow.*

Proof. Each representable operator is Dunford-Pettis [DiUh, p. 74] and each weakly compact operator defined on $L_1(\mu)$ is representable [DiUh, p. 75]. ∎

PROPOSITION 2.4. *Let E be a Köthe Banach space and X be a Banach space. Suppose that for each $A \in \Sigma^+$ there exists a Rademacher type system on A which is equivalent* *to the unit vector basis of ℓ_2. Then each absolutely summing operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Fix $A \in \Sigma^+$. Let (r_k) be a Rademacher type system on A equivalent to the unit vector basis of ℓ_2. The unconditional convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n} r_n$ implies that $\sum_{n=1}^{\infty} \frac{1}{n} \|Tr_n\| < \infty$ and hence $\liminf_n \|Tr_n\| = 0$. ■

By using the definition, it is easy to prove the following property of narrow operators.

PROPOSITION 2.5 ([PlPo]). *If E is a symmetric Banach space, X is a Banach space and an operator $T \in \mathcal{L}(E, X)$ is narrow then for each $\varepsilon > 0$ there exists a subspace E_0 of E isometrically isomorphic to the closed linear span of the Haar system $(h_n)_{n=2}^{\infty}$ without the first term such that the restriction $T|_{E_0}$ of T to E_0 is compact and $\|T|_{E_0}\| < \varepsilon$.*

Let ω_α be any cardinal. We consider the measure space $(\{-1,1\}^{\omega_\alpha}, \sigma_\alpha, \mu_\alpha)$ on the ω_α-th power of the two-point set $\{-1,1\}$ which is a compact Abelian group with the Haar measure μ_α at the cylindrical σ-algebra σ_α (for more details see [PlPo]). For any F-space X by $\dim X$ we mean the least cardinality of subsets of X with dense linear span.

THEOREM 2.6. *Let E be a symmetric F-space on $(\{-1,1\}^{\omega_\alpha}, \sigma_\alpha, \mu_\alpha)$ and X be an F-space with $\dim X < \aleph_\alpha$. Then every operator $T \in \mathcal{L}(E, X)$ is narrow.*

Proof. Fix any $A \in \sigma_\alpha^+$ and $\varepsilon > 0$. One of the consequences of Maharam's theorem ([Mah] (1942), [PlPo]) says that there exists a "Rademacher" system $(r_\beta)_{\beta<\omega_\alpha}$ in $L_0(A)$ of cardinality $\aleph_\alpha$. Since $\dim X < \aleph_\alpha$, there are indices $\beta \neq \gamma < \omega_\alpha$ such that $\|Th\| < \varepsilon$ where $h = (r_\beta - r_\gamma)/2$. By Lemma 1.3, T is narrow. ■

3. A "very" non-compact narrow operator.

If E is a Köthe Banach space and $\mathcal{F}$ is a purely atomic sub-σ-algebra of Σ with the atoms $(A_i)_{i\in I}$ then the conditional expectation operator

$$M^{\mathcal{F}} x = \sum_{i\in I} \Big(\frac{1}{\mu(A_i)} \int_{A_i} x \, d\mu \Big) \cdot \mathbf{1}_{A_i}$$

from E to E is an AM-compact operator which is narrow and non-compact if I is infinite. Now we give a more interesting example.

EXAMPLE 3.1. *Consider E on the square $[0,1]^2$, and its elements as functions of two variables and define for any $x \in E$*

$$(Px)(s,t) = \int_{[0,1]} x(s,t')\, dt'. \tag{3.1}$$

For a large class of Köthe Banach spaces E, the operator $P : E \to E$ is correctly defined and bounded. For instance, it is correctly defined and bounded on every r.i. space [LiTz, p. 122]. The operator P can also be considered as the conditional expectation operator $M^{\mathcal{F}_0}$ with respect to the sub-σ-algebra $\mathcal{F}_0 = \{A \times [0,1] : A \in \mathcal{B}\}$ of the Borel σ-algebra on $[0,1]^2$. It is a good exercise to show that if P is correctly defined in E then P is narrow (even strictly narrow!). This example can be extended to any symmetric Banach space on any finite atomless measure space [PlPo] (1990). In [DoPo] (2008) the authors give necessary and sufficient condition on an atomless sub- σ-algebra $\mathcal{F}$ of the Borel σ-algebra $\mathcal{B}$ on $[0,1]$ such that $M^{\mathcal{F}}$ is narrow (= strictly narrow).

Example 3.1 is the basis of the following nice result.

THEOREM 3.2. *Let E be a symmetric Banach space. Then there exists a subspace E_0 of E isometrically isomorphic to E and two projections of E onto E_0, one of which is strictly narrow and another is not narrow.*

In particular, it follows that the property of an operator to be narrow is not a property of its image. In [PlPo] (1990) the reader can find a proof that Example 3.1 gives a narrow projection of E onto the subspace E_0 of E consisting of all functions $x(s,t) \in E$ depending only of the first variable s. An example of a non-narrow projection of E onto E_0 is constructed in [PoRa] (2002). Then in [Kra] (2009) it is shown that P in Example 3.1 is strictly narrow in L_p for any p, $1 \le p \le \infty$, and in [DoPo] (2008) it is proved that P is strictly narrow in any Köthe space on $[0,1]$ on which it is bounded.

4. Some deep results on narrow operators

4.1. Sufficient conditions for narrow operators in L_p for $1 \le p < 2$. One of the deepest results on narrow operators was announced by Bourgain [Bou, p. 54] (1981) without a proof. He gave here only a citation to the Johnson-Maurey-Schechtman-Tzafriri book [JMST] for the idea how one can prove it. In a private communication G. Schechtman kindly confirmed that one can deduce the proof from ideas of [JMST], and even outlined me a sketch of the proof.

THEOREM 4.1. *Let $1 \le p < 2$ and $T \in \mathcal{L}(L_p)$. If for each complemented subspace X of L_p isomorphic to L_p the restriction $T|_X$ is not an into isomorphism then T is narrow.*

We remark that this sufficient condition is not necessary (cf. Theorem 3.2). For $p = 1$ Theorem 4.1 was obtained by Rosenthal [Ros5, Theorem 1.5] (1984) and also can be deduced from the results of Enflo-Starbird's paper [EnSt] (1979). For $p = 2$ this result holds evidently, and for $p > 2$ is false (consider, for example, the composition operator $J_r \circ I_p$ from the remark after Theorem 6.10).

The peculiarity of L_1 permits one to obtain weaker sufficient conditions for an operator to be narrow. The following two results of Rosenthal give necessary and sufficient ones.

THEOREM 4.2 ([Ros5]). *An operator $T \in \mathcal{L}(L_1)$ is narrow if and only if for any $A \in \mathcal{B}^+$ the restriction $T|_{L_1(A)}$ is not an into isomorphism.*

THEOREM 4.3 ([GhRo], [Ros4]). *Let X be any Banach space. An operator $T \in \mathcal{L}(L_1, X)$ is narrow if and only if for each $A \in \mathcal{B}^+$ and each $\varepsilon > 0$ there exist a set $B \in \mathcal{B}(A)$ and a sign x on B such that $\|Tx\| < \varepsilon \|x\|$.*

One should compare the last condition to that of Lemma 1.3. Another deep result is due to Ghoussoub and Rosenthal. Recall that an injective operator $T \in \mathcal{L}(X,Y)$ between Banach spaces is called a *G_δ-embedding* if TK is a G_δ-set for each closed bounded $K \subset X$.

THEOREM 4.4 ([GhRo]). *No G_δ-embedding $T \in \mathcal{L}(L_1, X)$ is narrow.*

4.2. For what spaces X is every operator $T \in \mathcal{L}(L_1, X)$ narrow? This question can be reformulated in a more convenient notion. According to Rosenthal [Ros3] (1981), [Ros4] (1983), an injective operator $T \in \mathcal{L}(L_1, X)$ is called a *sign-embedding* if

$$\|Tx\| \geq \delta \|x\| \tag{$**$}$$

for some $\delta > 0$ and every sign $x \in L_1$. It is said that L_1 *sign-embeds* in a Banach space X provided there exists a sign-embedding $T \in \mathcal{L}(L_1, X)$.

It is not hard to show that the injectivity assumption for T is essential in this definition. Moreover, in [MyPo] (2006) the authors constructed an example of a projection in L_1 which satisfies $(**)$ and with kernel isomorphic to L_1. One more question naturally arising from the definition is the following: if condition $(**)$ holds for mean zero signs only, is it sufficient for the operator to be a sign-embedding? In [MyPo] it is shown that the answer is negative (compare to Lemma 1.3). On the other hand, if an operator $T \in \mathcal{L}(L_1, X)$ (not necessary injective) possesses $(**)$ for each mean zero sign then there exists $A \in \mathcal{B}^+$ for which the restriction $T|_{L_1(A)}$ is a sign-embedding, i.e. in this case L_1 sign-embeds in X [MyPo].

What is the connection between sign-embeddings and narrow operators? Obviously, the notions of sign-embeddings and narrow operators are mutually exclusive, but $Tx = x\,\mathbf{1}_{[0,1/2]}$ is an example of an operator in $\mathcal{L}(L_1)$ which is neither a sign-embedding nor narrow. On the other hand, the following statement holds.

Proposition 4.5. *For a Banach space X the following assertions are equivalent:*

(i) *L_1 does not sign-embed in X,*
(ii) *every operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Indeed, (ii) trivially implies (i); the converse constitutes Lemma 3 of Rosenthal's paper [Ros4] (1983). It is clear that if L_1 embeds in X then L_1 sign-embeds in L_1. Rosenthal asked ([Ros3] (1981), [Ros4]) whether the converse is true. Talagrand [Tal] (1990), in solving another problem, constructed a counterexample. Actually, he constructed a subspace Y of L_1 such that both Y and L_1/Y have no isomorphic copy of L_1. We claim that the quotient map $T : L_1 \to X$ is not narrow. Indeed, if it were narrow, by Proposition 2.5, one would construct a sequence $(h_n)_{n=2}^{\infty}$ in L_1 with $[h_n]$ isomorphic to L_1 with $\|Th_n\| < 2^{-n}$. This implies that Y has a subspace isomorphic to L_1.

Now we list several results saying that, under some conditions on X, every operator $T \in \mathcal{L}(L_1, X)$ is narrow. Let X, Y, Z be Banach spaces. Recall that an operator $T \in \mathcal{L}(X, Y)$ *fixes a copy of* Z if there exists a subspace X_0 of X isomorphic to Z such that the restriction $T|_{X_0}$ is an into-isomorphism. Otherwise it is said that T *fixes no copy of* Z, or is *Z-singular*.

Theorem 4.6 ([BoRo]). *Every ℓ_1-singular operator $T \in \mathcal{L}(L_1, X)$ is narrow. Hence, if X contains no subspace isomorphic to ℓ_1 then every operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

Ghoussoub and Rosenthal in [GhRo] (1983) defined a class $\mathcal{G}$ as the minimal class of separable Banach spaces closed under G_δ-embeddings and containing L_1. In particular, $\mathcal{G}$ contains all separable duals.

THEOREM 4.7 ([GhRo]). *If $X \in \mathcal{G}$ then every L_1-singular operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

PROBLEM 4.8. *Let $1 < p < \infty$, $p \neq 2$. Is every strictly singular operator $T \in \mathcal{L}(L_p, X)$ narrow for every Banach space X?*

The case $p = 1$ is excluded, because in view of Theorem 4.6, the answer is affirmative.

PROBLEM 4.9. *Let $1 \le p < \infty$, $p \neq 2$. Is every ℓ_2-singular operator $T \in \mathcal{L}(L_p, X)$ narrow for every Banach space X?*

A partial answer to these problems for regular operators was given by Flores and Ruiz in [FlRu]. In particular, they proved that, if F is a Banach lattice containing no isomorph of c_0, then every ℓ_2-singular operator $T \in \mathcal{L}(L_1, X)$ is narrow.

5. Ideal properties of narrow operators. Is the sum of two narrow operators narrow? It is a shocking and the most interesting phenomenon on narrow operators that if a symmetric Banach space on $[0,1]$ has an unconditional basis then the answer is negative, while for $E = L_1$ it is affirmative. Not less interesting is that the first fact is quite simple, and the second is quite involved.

PROPOSITION 5.1. *Let E be a symmetric Banach space with an unconditional basis. Then the identity* Id *on E is a sum of two narrow projections.*

Proof. Decompose the integers into two infinite parts $\mathbb{N} = N_1 \sqcup N_2$ and set

$$E_j = \left[h_i : 2^{m-1} \le i \le 2^m - 1,\ m \in N_j\right],\ j = 1, 2,$$

where (h_n) is the Haar system which is unconditional in E by [LiTz, p. 156]. Then $E = E_1 \oplus E_2$. Now it is a technical exercise to show that both corresponding projections are narrow. ∎

Observe that if $T \in \mathcal{L}(E, X)$ is narrow then for any Banach space Y and any $S \in \mathcal{L}(X, Y)$ the composition $S \circ T$ is narrow. Hence, we obtain the following consequence.

COROLLARY 5.2. *Let E be a symmetric Banach space with an unconditional basis. Then every operator $T \in \mathcal{L}(E)$ is a sum of two narrow projections.*

THEOREM 5.3. *The sum of two narrow operators in L_1 is narrow.*

This theorem appeared in [PlPo] (1990), but the proof contained a gap. Later it was proved in different ways by different authors [Shv2] (2001), [KaPo3] (2003). In [MMyP1] (2006) much more was proved, namely that the set of all narrow operators in L_1 is a band in $\mathcal{L}(L_1)$ (see Section 7 for more details).

In general, one can easily show that the sum of a narrow and a Dunford-Pettis operator is narrow.

In contrast to composition from the left, the set of all narrow operators does not have a right-hand-side ideal property.

PROPOSITION 5.4. *Let E be a symmetric Banach space. There are operators $T, S \in \mathcal{L}(E)$ with T narrow such that $T \circ S$ is not narrow.*

Proof. Let T be a narrow projection of E with the range $X = T(E)$ isomorphic to E (see Theorem 3.2) and $S : E \to X$ be an isomorphism. Then $T \circ S = S$ is not narrow. ■

We remark that in this case the operator $S^{-1} \circ T : E \to E$ is narrow and onto. Hence, the conjugate operator $(S^{-1} \circ T)^* \in \mathcal{L}(E^*)$ is an into isomorphism and is not narrow. Thus, we obtain the following assertion.

PROPOSITION 5.5. *Let both E and E^* be symmetric Banach spaces. Then there is a narrow operator $T \in \mathcal{L}(E)$ with non-narrow conjugate operator $T^* \in \mathcal{L}(E^*)$.*

PROBLEM 5.6. *Is the sum of two narrow operators in $\mathcal{L}(L_1, X)$ narrow, for every Banach space X?*

6. Some applications of narrow operators. Here we describe some applications of narrow operators to different branches of Geometric Functional Analysis.

6.1. The only narrow operator on $L_p(\mu)$ is zero when $0 < p < 1$. Let $0 < p < 1$ and X be an F-space. Recall that the F-norm on $L_p(\mu)$ is given by

$$\|x\| = \int_\Omega |x|^p \, d\mu.$$

THEOREM 6.1. *If $T \in \mathcal{L}\big(L_p(\mu), X\big)$ is narrow then $T = 0$.*

Proof. It is enough to prove that $T\mathbf{1}_A = 0$ for each $A \in \Sigma^+$. Given any A, choose by Lemma 1.5 a sequence (h_n) in E such that $\lim_n \|Th_n\| = 0$, $h_n = \mathbf{1}_{A'_n} - (2^n - 1)\,\mathbf{1}_{A''_n}$, $A'_n \sqcup A''_n = A$ and $\mu(A'') = 2^{-n}\mu(A)$. Then

$$\lim_{n\to\infty} \|\mathbf{1}_A - h_n\| = \lim_{n\to\infty} \|2^n \mathbf{1}_{A''_n}\| = \lim_{n\to\infty} 2^{n(p-1)}\mu(A) = 0.$$

Thus, we obtain that $\mathbf{1}_A = \lim_{n\to\infty} h_n$ and $\|T\mathbf{1}_A\| = \lim_{n\to\infty} \|Th_n\| = 0$. ■

The following immediate consequence of Theorems 6.1 and 2.6 implies that the space $L_p\{-1,1\}^{\omega_\alpha}$ with $\alpha > 0$ has no separable quotient space (to the best of our knowledge, it is still unknown, whether there exists an infinite-dimensional Banach space with no separable infinite-dimensional quotient space).

COROLLARY 6.2. *Let X be an F-space with $\dim X < \aleph_\alpha$. Then*

$$\mathcal{L}\big(L_p\{-1,1\}^{\omega_\alpha}, X\big) = \{0\}.$$

The following consequence of Theorem 6.1 and Proposition 2.1 was known before for compact operators due to Kalton [Kal1] (1976), [Kal2] (1977), Pallaschke [Pal] (1973) and Turpin [Tur] (1973).

COROLLARY 6.3. *If $T \in \mathcal{L}(L_p(\mu), X)$ is AM-compact then $T = 0$.*

6.2. Isomorphic and near isometric classification of $L_p(\mu)$-spaces. An isomorphic classification of $L_p(\mu)$-spaces on finite atomless measure spaces for $1 \le p < \infty$ is given by Lindenstrauss [Lac, p. 130], and for $p = \infty$ by Rosenthal [Ros1] (1970). For isomorphic and near isometric classifications, it is clearly enough to consider probability measure spaces only, i.e. with $\mu(\Omega) = 1$. For a measure space (Ω, Σ, μ) by $\hat{\Sigma}$ we denote the Boolean σ-algebra of equivalence classes of sets from Σ, equal up to measure null sets with the natural operations $\vee$ and $\wedge$. Recall that two probability spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, are called *isomorphic* if there exists a measure preserving Boolean isomorphism of $\hat{\Sigma}_1$ onto $\hat{\Sigma}_2$. For example, every two separable (i.e. having separable $L_1(\mu_i)$) atomless probability spaces are isomorphic (Caratheodory's theorem). A complete characterization of arbitrary atomless probability spaces is given by Maharam's theorem [Mah] (1942), [Lac, p. 128], [PlPo].

THEOREM 6.4 ([Pop1], [PlPo]). *Let $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, be atomless probability spaces and $0 < p < 1$. Then the following assertions are equivalent:*

(i) *$L_p(\mu_1)$ and $L_p(\mu_2)$ are isomorphic;*
(ii) *$L_p(\mu_1)$ and $L_p(\mu_2)$ are isometrically isomorphic;*
(iii) *$(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are isomorphic, up to some positive multipliers.*

Of course, implications (iii)$\Rightarrow$(ii)$\Rightarrow$(i) are obvious.

An analogous result is the following near isometric classification of $L_p(\mu)$-spaces for $1 \le p < \infty$, $p \ne 2$.

THEOREM 6.5 ([PlPo]). *For each $p \in [1, 2) \cup (2, +\infty)$ there exists a constant $k_p > 1$, $k_1 = 2$ such that for any atomless probability spaces $(\Omega_i, \Sigma_i, \mu_i)$, $i = 1, 2$, if the Banach-Mazur distance satisfies $d(L_p(\mu_1), L_p(\mu_2)) < k_p$ then the spaces $L_p(\mu_1)$ and $L_p(\mu_2)$ are isometrically isomorphic.*

For separable measure spaces admitting atoms, the same result was earlier obtained by Benyamini [Ben] (1975), and for $p = 1$ and any atomless probability spaces by Cambern [Cam] (1980).

The main point for the isomorphic classification when $0 < p < 1$ was the absence of non-zero narrow operators, and the main point for the $1 \le p < \infty$, $p \ne 2$, case is the following theorem asserting that any non-identity projection of $L_p(\mu)$ having "small" kernel must have "large" norm.

THEOREM 6.6 ([Pop2]). *For each $p \in [1, 2) \cup (2, +\infty)$ there exists a constant $k_p > 1$, $k_1 = 2$, such that if $P \ne \mathrm{Id}$ is a projection of $L_p(\mu)$ with narrow complement projection $\mathrm{Id} - P$ then $\|P\| \ge k_p$.*

Franchetti [Fra] (1992) showed that the best value of k_p is the norm of the following codimension-one projection P_0 of L_p

$$P_0 x = x - \Big(\int_{[0,1]} x \, d\lambda \Big) \cdot \mathbf{1} \quad \text{where} \quad \mathbf{1}(t) \equiv 1.$$

Theorem 6.6 was generalized from the spaces L_p to Lorentz spaces $L_{w,p}$ on [0,1] with $p > 2$ in [PoRa] (2002).

6.3. The Daugavet property. Lozanovskii [Loz] (1966) proved that the space L_1 has the so-called Daugavet property for compact operators [4] which first was discovered by Daugavet [Dau] (1963) for $C[a, b]$, i.e. that

$$\|\mathrm{Id} + K\| = 1 + \|K\| \tag{6.1}$$

[4] now it is known that, equivalently, one can replace compact with weakly compact, or even with rank-one operators [KSSW] (2000)

for each compact operator $K \in \mathcal{L}(L_1)$. A more general result on L_1 is that (6.1) is satisfied for all narrow operators. But this is not the most general result; for a complete characterization of operators $K \in \mathcal{L}(L_1)$ satisfying (6.1) see [Shv1] (2001).

Using narrow operators, the author in [Pop4] (2008) strengthened the Daugavet property by replacing the identity with any "small" into isomorphism of L_1.

THEOREM 6.7 ([Pop4]). *Let $T \in \mathcal{L}(L_1)$ be a narrow operator and $J \in \mathcal{L}(L_1)$ be an into isomorphism with $d = \|J\|\|J^{-1}\| < 2$. Then*

$$\|J + T\| \geq \|T\| + \|J\|\Big(\frac{2}{d} - 1\Big). \tag{6.2}$$

Moreover, the inequality in (6.2) *is sharp.*

There is another inequality that holds for operators in L_p with $1 < p < \infty$, $p \neq 2$.

THEOREM 6.8 ([PlPo]). *Let $1 < p < \infty$, $p \neq 2$. For each $\varepsilon > 0$ there exists $\delta_p(\varepsilon) > 0$ such that if $T \in \mathcal{L}(L_p)$ is narrow then*

$$\|\mathrm{Id} + T\| \geq 1 + \delta_p(\|T\|). \tag{6.3}$$

This result was earlier obtained by Benyamini and P. K. Lin in [BeLi] (1985) for compact operators. We remark also that Boyko and V. Kadets proved that $\delta_p(t) \to t$ as $p \to 1$ [BoKa] (2004), and so, the Daugavet equation in L_1 is a limiting case of the Benyamini-Lin L_p theorem, as the title of [BoKa] says.

6.4. Ranges of vector measures. By the well known Lyapunov theorem, the range $\nu(\Sigma)$ of a vector measure $\nu : \Sigma \to X$ (here X is a Banach space) is convex if $\dim X < \infty$. It is not hard to see that if $\dim X = \infty$ then there exists a countably additive X-valued measure of bounded variation with non-convex range. But if we consider the closure $\overline{\nu(\Sigma)}$ then the question of whether it is convex becomes non-trivial.

THEOREM 6.9 ([KaPo1]). *For any Banach space X the following assertions are equivalent*

(i) $\overline{\nu(\Sigma)}$ *is convex for each countably additive X-valued measure of bounded variation*;
(ii) *every operator $T \in \mathcal{L}(L_1, X)$ is narrow.*

6.5. An analogue of the Pitt compactness theorem. By using the notion of infratype for Banach spaces, the following result was obtained in [KaPo1] (1992).

THEOREM 6.10 ([KaPo1]). *If $1 \leq p < 2$ and $p < r < \infty$ then every operator $T \in \mathcal{L}(L_p, L_r)$ is narrow.*

We remark that Theorem 6.10 is false for any other values of p and r. If $p \geq 2$ then the composition $J_r \circ I_p$ of the identity embedding $I_p : L_p \to L_2$ and the isomorphic embedding $J_r : L_2 \to L_r$ is evidently not narrow. And if $1 \leq p < 2$ and $1 \leq r \leq p$ then the identity embedding of L_p into L_r is not narrow.

Using the usual Pitt theorem, saying that every operator from ℓ_p to ℓ_r is compact if $1 \leq r < p < \infty$, one can obtain the following proposition.

PROPOSITION 6.11. *Let E be a symmetric Banach space such that the closed linear span of the Rademacher system in E is isomorphic to ℓ_2, and let $1 \leq r < 2$. Then every operator $T \in \mathcal{L}(E, \ell_r)$ is narrow.*

Proof. Given any $A \in \mathcal{B}^+$, consider a Rademacher system (r_n) in $E(A)$. Since $[r_n]$ is isomorphic to ℓ_2, by the Pitt theorem, the restriction $T|_{[r_n]}$ is compact. Since $r_n \xrightarrow{\mathrm{w}} 0$, we have that $\|Tr_n\| \to 0$. ∎

THEOREM 6.12 ([KaPo1]). *If $1 \le p < \infty$ then every operator $T \in \mathcal{L}(L_p, c_0)$ is narrow.*

A similar result was recently obtained for narrow operators defined on the space L_∞, the norm of which is not absolutely continuous, and thus where the usual technique does not work (see Subsection 7.1 for the definitions).

THEOREM 6.13 ([KMMMP]). *Every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, c_0)$ is narrow.*

We remark that there is an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, c_0)$ which cannot be extended to L_p for some $p < \infty$ [KMMMP] (2009), so, this theorem cannot be obtained from Theorem 6.12. Likewise, there exists an order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, c_0)$ which is not AM-compact, and so, Theorem 6.13 does not follow from Theorem 7.19 below.

THEOREM 6.14 ([KMMMP]). *Let $1 \le p < 2$. Then every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ is narrow.*

PROBLEM 6.15. *Let $2 \le p < \infty$. Is every order-to-norm continuous operator $T \in \mathcal{L}(L_\infty, \ell_p)$ narrow?*

6.6. Rich subspaces and subspaces of L_p isomorphic to L_p

DEFINITION 6.16. Let E be a Köthe F-space on (Ω, Σ, μ). A subspace $X \subseteq E$ is called *rich* if the quotient map $T : E \to E/X$ is narrow.

In other words, $X \subseteq E$ is rich if for each $A \in \Sigma$ and $\varepsilon > 0$ there are an element $x \in X$ and a mean zero sign y on A such that $\|x - y\| < \varepsilon$.

Theorem 2.6 gives the following consequence.

COROLLARY 6.17. *Let E be a Köthe F-space on (Ω, Σ, μ) with an absolutely continuous norm. Then every subspace $X \subseteq E$ with*

$$\operatorname{codim} X < \min\{\dim E(A) : A \in \Sigma^+\}$$

is rich.

Here and in the sequel by $\operatorname{codim} X$ we denote the codimension of a subspace $X \subseteq E$ in E, i.e. $\operatorname{codim} X = \dim E/X$.

The following statement is a consequence of Theorem 6.1.

COROLLARY 6.18. *Let $0 < p < 1$ and (Ω, Σ, μ) be an atomless measure space. Then the only rich subspace of $L_p(\mu)$ is $L_p(\mu)$ itself.*

Using Proposition 2.5 one can prove that if E is a symmetric Banach space then any rich subspace of E contains some isomorph of the corresponding space E_0 on $[0,1]$ (for the definition of the corresponding symmetric space, which is too complicated, we refer the reader to [PlPo]; here we just say that, for example, the corresponding space to $L_p(\mu)$ is L_p).

PROPOSITION 6.19. *Let X be a rich subspace of a symmetric Banach space E on (Ω, Σ, μ). Then there exists a subspace $Y \subseteq X$ which is isomorphic to the corresponding space E_0 on $[0,1]$ and complemented in E.*

A symmetric space E is said to be s-concave with $1 \le s < \infty$ if there is a constant $M > 0$ such that for any $n \in \mathbb{N}$ and any vectors $(x_i)_1^n$ in E one has

$$\Big(\sum_{i=1}^{n} \|x_i\|^s\Big)^{1/s} \le M \Big\| \Big(\sum_{i=1}^{n} |x_i|^s\Big)^{1/s} \Big\|.$$

Proposition 6.19 together with Corollary 9.2 from [JMST, p. 240] implies the following result.

COROLLARY 6.20. *Let E be a symmetric Banach space on $[0,1]$. Suppose that:*

(i) *E is s-concave for some $s < \infty$;*
(ii) *E does not contain, for all integers n, almost isometric copies of ℓ_1^n spanned by disjoint elements having the same distribution;*
(iii) *the Haar system in E is not equivalent to a sequence of disjoint elements of E.*

Then every complemented rich subspace of E is isomorphic to E.

We remark that the condition on the lower Boyd index in [JMST, Corollary 9.2] has been replaced with an equivalent condition (ii), in accordance with [LiTz, p. 141].

Theorem 5.1 implies the following.

COROLLARY 6.21. *Let E be a separable symmetric Banach space on $[0,1]$ with an absolutely continuous norm and an unconditional basis. Then E is a sum of two rich subspaces $E = E_1 \oplus E_2$.*

Theorem 3.2 has the following reformulation in terms of rich subspaces.

COROLLARY 6.22. *Let E be a symmetric Banach space with an absolutely continuous norm on (Ω, Σ, μ). Then there exists a complemented subspace E_0 of E isometric to E and two decompositions $E = E_0 \oplus Y = E_0 \oplus Z$ where Y is rich and Z is not rich.*

The property of a subspace to be rich is not preserved when passing to dual spaces. More precisely, let X be a subspace of a Banach space E. By $X^\perp$ we denote the annihilator of X, i.e. the subspace of E^* consisting of all functionals vanishing on X.

Observe that each decomposition $E = X \oplus Y$ of a Banach space into subspaces produces the dual decomposition $E^* = Y^\perp \oplus X^\perp$. Consequently, it is natural to ask whether a subspace Y of E is rich in E if and only if $X^\perp$ is rich in E^* (of course, if in both spaces E and E^* rich subspaces are well defined). The answer is no.

PROPOSITION 6.23. *Let E be a symmetric Banach space on (Ω, Σ, μ) with an absolutely continuous norm such that the dual space E^* is also a symmetric Banach space on (Ω, Σ, μ) with an absolutely continuous norm. Then there exists a decomposition $E = X \oplus Y$ such that both subspaces X, Y are not rich and the annihilator $X^\perp$ is rich in E^*.*

Formally, it cannot be deduced from Corollary 6.22. But if we recall the construction of these decompositions (see Theorem 3.2), we can show that E has a decomposition $E = E_0 \oplus E_1$ with E_0 and E_1 not rich and $E_0^\perp$ rich in E^*.

One famous theorem of Enflo asserts that, if the space L_p, $1 \le p < \infty$, is decomposed into closed subspaces $L_p = X \oplus Y$ then, at least, one of X, Y is isomorphic to L_p (see [EnSt] for the case $p = 1$ and [Mau] (1974) for the other values of p). Alspach asked [Als], whether there exists a constant $M_p > 1$ such that if P is a projection of L_p onto an infinite-dimensional subspace X of L_p with $\|P\| < M_p$ then X is isomorphic to L_p. Applying the technique developed in Subsection 6.2 to rich subspaces, one can obtain a related result.

Theorem 6.24 ([PlPo]). *For each $p \in [1,2) \cup (2,+\infty)$ there is a constant $k_p > 1$, $k_1 = 2$, such that if $P \neq 0$ is a projection of L_p onto X with $\|\mathrm{Id} - P\| < k_p$ then X is isomorphic to L_p.*

6.7. Best estimation for the numerical index of L_p. We start this subsection with some preliminaries. Let X be a Banach space. The numerical radius of an operator $T \in \mathcal{L}(X)$ is a semi-norm defined as

$$v(T) = \sup\bigl\{|x^*(Tx)| : x \in S_X,\ x^* \in S_{X^*},\ x^*(x) = 1\bigr\},$$

and the numerical index of X is the following constant

$$n(X) = \inf\bigl\{v(T) : T \in \mathcal{L}(X),\ \|T\| = 1\bigr\}.$$

These notions were first studied in the paper [DGPW] of Duncan, McGregor, Pryce and White (1970), with a remark that they are due to Lumer (1968) (see also monographs of Bonsall and Duncan [BoDu1] (1971), [BoDu2] (1973) and survey of V. Kadets, Martín and Payá [KMP] (2006)). Obviously, $0 \le n(X) \le 1$ and $n(X) > 0$ means that $v(T)$ is a norm on $\mathcal{L}(X)$ equivalent to the operator norm. It is also not hard to see that $n(X^*) \ge n(X)$. There are lots of spaces with the numerical index one (among classical ones, for instance, $L_1(\mu)$ and $C(K)$), and some attractive open problems on them [KMP]. It is interesting to remark that properties like that are different for the real and complex cases. So, for every complex Banach space one has that $n(X) \ge 1/e$ (and the inequality is sharp), nevertheless, $n(X) = 0$ for some real Banach spaces, for example, $n(\ell_2) = 0$. Recently it was proved by Martín, Merí and Popov that $n(L_p) > 0$ in the real case for every $1 < p \neq 2$ [MMeP] (2010). More precisely, in [MMeP] the authors introduced the notions of absolute numerical radius of an operator $T \in \mathcal{L}(L_p)$ and absolute numerical index of $L_p(\mu)$ on a measure space (Ω, Σ, μ) as follows. For any $T \in \mathcal{L}(L_p(\mu))$ the *absolute numerical radius* is the following number:

$$|v|(T) = \sup\Bigl\{\int_\Omega |x^* T x| : x \in S_{L_p(\mu)},\ x^* \in S_{L_q(\mu)},\ \int_\Omega x^* x = 1\Bigr\},$$

and the absolute numerical index of $L_p(\mu)$ is the following constant

$$|n|(L_p) = \inf\bigl\{|v|(T) : T \in \mathcal{L}(L_p(\mu)),\ \|T\| = 1\bigr\}.$$

The main results of [MMeP] assert that, given an operator T on the *real* space $L_p(\mu)$, we have

$$v(T) \geq \frac{M_p}{6}\,|v|(T) \qquad \text{and} \qquad |v|(T) \geq \frac{n\big(L_p^{\mathbb{C}}(\mu)\big)}{2}\|T\|,$$

where $n\big(L_p^{\mathbb{C}}(\mu)\big)$ is the numerical index of the *complex* space $L_p(\mu)$ and

$$M_p = \max_{\tau\in[0,1]} \frac{|\tau^{p-1}-\tau|}{1+\tau^p} = \max_{\tau\geq 0} \frac{|\tau^{p-1}-\tau|}{1+\tau^p} > 0.$$

Since $n\big(L_p^{\mathbb{C}}(\mu)\big) \geq 1/e$ (as for any complex space, see [BoDu1, Theorem 4.1]), the above two inequalities together give, in particular, the following inequality:

$$n\big(L_p(\mu)\big) \geq \frac{M_p}{12e}\,. \tag{6.4}$$

In a more recent unpublished paper, Martín, Merí and the author have found an exact value for the absolute numerical index of L_p.

THEOREM 6.25. *For every $1 < p < \infty$ one has the equality*

$$|n|(L_p) = \frac{1}{p^{1/p}q^{1/q}}\,.$$

The authors have established a stronger lower estimate for the numerical radius than the general one gives. More precisely, define

$$n_{\mathrm{nar}}(L_p) = \inf\big\{v(T) : T \in \mathcal{L}(E),\ \|T\| = 1,\ T \text{ is narrow}\big\}.$$

Then the following result holds.

THEOREM 6.26. *For every $1 < p < \infty$ one has*

$$n_{\mathrm{nar}}(L_p^{\mathbb{R}}) \geq \max_{\tau>0} \frac{\kappa_p \tau^{p-1} - \tau}{1+\tau^p} \qquad \textit{and} \qquad n_{\mathrm{nar}}(L_p^{\mathbb{C}}) \geq \kappa_p^2$$

where by $L_p^{\mathbb{R}}$ and $L_p^{\mathbb{C}}$ we denote the real and the complex spaces respectively.

It is a natural conjecture that the numbers $n(L_p)$ and $n_{\mathrm{nar}}(L_p)$ are equal. Nevertheless, the problem of whether this is true seems to be quite involved.

PROBLEM 6.27. *Let $1 < p < \infty$, $p \neq 2$. Is $n(L_p) = n_{\mathrm{nar}}(L_p)$?*

7. Narrow operators on vector lattices. A lattice approach to narrow operators which was used in [MMyP2] (2009) allows us to give an answer to the question why in "good" spaces the sum of narrow operators need not be narrow, while in L_1 the sum is narrow. The answer is: because in L_1 there are "few" operators (all of them are regular, i.e. the difference of two positive operators), and in "good" spaces there are a lot of operators (including non-regular). Nevertheless, in all spaces the sum of two regular narrow operators is narrow. This fact is quite deep and involved, and our aim is to present it after some preliminaries on vector lattices. Note that *all vector spaces are considered over the reals* in this section. All the results and new notions presented in this section (unless other authors are cited) are due to O. Maslyuchenko, Mykhaylyuk and the author [MMyP2].

7.1. Preliminaries on vector lattices. We use standard notation and terminology as in the Aliprantis-Burkinshaw book [AlBu]. A vector lattice is said to be *order complete*[5] if each order bounded from above non-empty set has the exact upper bound. We assume that all vector lattices are Archimedean, which is the case for order complete vector lattices [Sch, p. 64]. A subset F of a vector lattice E is said to be *order closed*, if for any subset $G \subseteq F$ the existence of $y = \sup G \in E$ (or $y = \inf G \in E$) implies that $y \in F$. The set of all positive elements in E is denoted by E^+. Two elements $x, y \in E$ are called disjoint (or orthogonal) if $|x| \wedge |y| = 0$ and this fact is written as $x \perp y$. Two subsets $A, B \subseteq E$ are disjoint if $x \perp y$ for each $x \in A$ and $y \in B$. For any subset $A \subseteq E$ by A^d we denote the set $A^d = \{x \in E : A \text{ and } \{x\} \text{ are disjoint}\}$. The equality $x = \bigsqcup\limits_{k=1}^{n} x_k$ in a vector lattice means that $x = \sum\limits_{k=1}^{n} x_k$ and $x_i \perp x_j$ if $i \neq j$.

A subset A of a vector lattice E is said to be *solid* if for any $x \in A$ and $y \in E$ the condition $|y| \leq |x|$ implies that $y \in A$. A solid vector subspace is called an *ideal*. An order closed ideal is said to be a *band*. A band I of a vector lattice E is called a *projection band* if $E = I \oplus I^d$. For an arbitrary vector lattice E and any subset $A \subseteq E$ by $\mathrm{Band}(A)$ we denote the least[6] band in E which contains A.

LEMMA 7.1 ([Sch, p. 62]). *Let E be an order complete vector lattice, $A \subseteq E$ be any subset. Then A^d is a band and $E = \mathrm{Band}(A) \oplus A^d$. In particular, each band is a projection band.*

If E is a Banach lattice (i.e. a vector lattice and a Banach space such that $|x| \leq |y|$ implies that $\|x\| \leq \|y\|$ for each $x, y \in E$) and I is a projection band in E then the corresponding projections of E onto I and I^d are of norm one; this immediately follows from the definition of Banach lattice.

According to [AbAl, p. 86], an element $u > 0$ of a vector lattice E is called an *atom*, whenever $0 \leq x \leq u$, $0 \leq y \leq u$ and $x \wedge y = 0$ imply that either $x = 0$ or $y = 0$. Since we deal with order complete lattices only, we equivalently reformulate this notion as follows. A non-zero element x of an order complete vector lattice E is an *atom* if for each $y \in E$ the equality $|x| = |y|$ is possible if and only if $y = x$ or $y = -x$. A vector lattice E is *atomless* if there is no atom $x \in E$. An element y of a vector lattice E is called a *component* of an element $x \in E$, provided $y \perp (x - y)$. The notation $y \sqsubseteq x$ means that y is a component of x. Evidently, a non-zero element $x \in E$ is an atom if and only if the only components of x are 0 and x itself. Hence, a vector lattice E is atomless if each non-zero element $x \in E$ has a proper component $y \sqsubseteq x$, that is, y is a component of x such that $0 \neq y \neq x$. Clearly, a r.i. F-space on an atomless measure space is an atomless vector lattice.

We consider *order convergence* in a vector lattice E. A decreasing net (x_α) *converges to zero* in E (notation $x_\alpha \downarrow 0$) provided that $\inf\limits_\alpha x_\alpha = 0$. More generally, a net (x_α) in E *order converges to an element* $x \in E$ (notation $x_\alpha \xrightarrow{\mathrm{o}} x$) if there exists a decreasing net $u_\alpha \downarrow 0$ in E with $|x_\alpha - x| \leq u_\alpha$ for all α.

[5]= Dedekind complete

[6]obviously, the intersection of bands is a band, so this object is well defined

Let E, F be vector lattices. The set of all linear operators $T : E \to F$ is denoted by $L(E,F)$. An operator $T \in L(E,F)$ is said to be *positive* if $T(E^+) \subseteq F^+$ (in this case we write $T \geq 0$), and *regular* if $T = T_1 - T_2$ for some positive operators $T_1, T_2 \in L(E,F)$. We define the order $T \leq S$ by $S - T \geq 0$, and $L_r(E,F)$ denotes the set of all regular operators in $L(E,F)$. Then $L_r(E,F)$ becomes an ordered vector space, and a vector lattice when F is order complete. If E, F are Banach lattices then $L_r(E,F) \subseteq \mathcal{L}(E,F)$. For $E = F$ the part (E,F) of any above notation is replaced with (E). An operator $T \in L(E,F)$ is said to be

— *disjointness preserving* if T sends disjoint elements from E to disjoint elements from F;
— *order continuous* provided T maps order convergent nets to order convergent nets.

Let X be a Banach space. An operator $T \in L(E,X)$ is said to be *order-to-norm continuous* if T sends order converging nets in E to norm converging nets in X.

7.2. Two new definitions of narrow operator. There are two definitions of narrow operator for vector lattices, depending on whether the range space is a Banach space or a Banach lattice.

DEFINITION 7.2. Let E be an atomless order complete vector lattice, and let X be a Banach space. A linear operator[7] $T : E \to X$ is called *narrow* if for every $x \in E^+$ and every $\varepsilon > 0$ there exists some $y \in E$ such that $|y| = x$ and $\|Ty\| < \varepsilon$.

Note that there is no need to restrict to the atomless case in this definition, but evidently, a narrow map must send "atoms" to zero. Of course, the new definition must be equivalent to the old one, at least, for most natural cases.

PROPOSITION 7.3. *Let E be a Köthe Banach space with an absolutely continuous norm, and let X be a Banach space. For an operator $T \in \mathcal{L}(E,X)$ Definitions 1.2 and 7.2 of narrow operator are equivalent.*

Sketch of proof. If T is narrow in the sense of Definition 1.2 and $x \in E^+$ is any element of the form $x = \sum\limits_{k=1}^{n} a_k \mathbf{1}_{A_k}$ then the desired $y \in E$ for Definition 7.2 can be found using Definition 1.2. Then we pass to the general case using absolute continuity of the norm. The converse implication is trivial in view of Lemma 1.3 which also uses the absolute continuity of the norm. ∎

The method of the proof allows us to state one "half" of Proposition 7.3 if E satisfies a weaker assumption.

PROPOSITION 7.4. *Let E be a Köthe Banach space such that finite valued functions from E are dense in E. If an operator $T \in \mathcal{L}(E,X)$ is narrow in the sense of Definition 1.2 then it is also narrow in the sense of Definition 7.2.*

For example, the space $E = L_\infty$, the norm of which is not absolutely continuous, satisfies the assumption of Proposition 7.4, so every operator $T \in \mathcal{L}(L_\infty, X)$ which is narrow in the sense of Definition 1.2 is also narrow in the sense of Definition 7.2 (cf. Problem 1.4).

[7] this can be applied for non-linear maps also

PROBLEM 7.5. *Are Definitions 1.2 and 7.2 equivalent for $E = L_\infty$?*

The answer is affirmative for order-to-norm continuous operators [KMMMP] (2009).

Now we give a definition of narrow operator for the case when the range space is a vector lattice. For most interesting cases it is equivalent to Definition 7.2, but in general, the definitions do not have the same meaning, as Proposition 7.8 below shows.

DEFINITION 7.6. Let E, F be vector lattices with E atomless. A linear operator $T : E \to F$ is called *order narrow* if for every $x \in E^+$ there exists a net (x_α) in E such that $|x_\alpha| = x$ for each α and $Tx_\alpha \xrightarrow{\mathrm{o}} 0$.

PROPOSITION 7.7. *Let E be an atomless vector lattice and F be a Banach lattice. Then each narrow linear operator $T : E \to F$ is order narrow.*

Proof. If $|x_n| = x$ and $\|Tx_n\| \le 2^{-n}$ then one can show that $Tx_n \xrightarrow{\mathrm{o}} 0$. Indeed, for $z_n = \sum\limits_{k=n}^{\infty} |Tx_k|$ we have that $|Tx_n| \le z_n \downarrow 0$. ■

However, the converse is not true.

PROPOSITION 7.8. *There exists an order narrow positive operator $T \in \mathcal{L}(L_\infty)$ that is not narrow.*

We remark that the operator given in [MMyP2] (2009) is not narrow in the sense of both definitions 1.2 and 7.2.

Nevertheless, for operators with values in an order continuous Banach lattice the two notions of narrow operator coincide. Recall that a Banach lattice E is said to be *order continuous* if for each net (x_α) in E the condition $x_\alpha \downarrow 0$ implies $\|x_\alpha\| \to 0$. Note that in this case the condition $x_\alpha \xrightarrow{\mathrm{o}} 0$ also implies $\|x_\alpha\| \to 0$.

PROPOSITION 7.9. *Let E be an atomless vector lattice and F be an order continuous Banach lattice. Then a linear operator $T : E \to F$ is order narrow if and only if it is narrow.*

Proof. Let T be order narrow. Given $x \in E^+$, choose a net (x_α) in E such that $|x_\alpha| = x$ for each α and $Tx_\alpha \xrightarrow{\mathrm{o}} 0$. By the definition of order continuous Banach lattice, $\|Tx_\alpha\| \to 0$, and thus, T is narrow. In view of Proposition 7.7, the proof is completed. ■

7.3. The main problem

PROBLEM 7.10. *Let E, F be order complete vector lattices with E atomless. Is the set*

(i) *$N_r(E, F)$ of all narrow regular operators*
(ii) *$ON_r(E, F)$ of all order narrow regular operators*

a band in the vector lattice $L_r(E, F)$ of all regular operators from E to F?

In general, Problem 7.10(i) has a negative answer.

THEOREM 7.11. *The set $N_r(L_\infty)$ is not a band in $L_r(L_\infty)$.*

To prove this theorem, the authors constructed a sequence $T_n \in N_r(L_\infty)$, the supremum of which does not belong to $N_r(L_\infty)$. However, the following problem remains unsolved.

PROBLEM 7.12. *Is the sum of two regular narrow operators in* L_∞ *narrow?*

This problem is open for both definitions 1.2 and 7.2. But if one omits the word "regular" in Problem 7.12 then the answer is known to be negative [Kra] (2009).

PROPOSITION 7.13. *The sum of two narrow operators*[8] *in* L_∞ *need not be narrow.*

Sketch of proof. Using Proposition 5.1, one can show that the identity embedding $J : L_\infty \to L_2$ is a sum of two narrow operators $T, S : L_\infty \to L_2$. Let $U : L_2 \to L_\infty$ be any isomorphic embedding. Then the operators $U \circ T$ and $U \circ S$ are narrow members of $\mathcal{L}(L_\infty)$. It is not hard to show that their sum $V = U \circ T + U \circ S = U \circ J_2$ is not narrow. ∎

For vector lattices E, F by $A_r(E,F)$ we denote the set of all disjointness preserving operators in $L_r(E,F)$ (the letter A is reserved because Rosenthal in [Ros5] called these operators *atoms*, and we mainly follow ideas from [Ros5] (1984)). The set $A_r(E,F)$ is not a linear subspace, but nevertheless, it is solid in $L_r(E,F)$ [MMyP2] (2009). And for an $M \subseteq L_r(E,F)$ by $B(M)$ we mean the minimal band in $L_r(E,F)$ which contains M.

The following two deep results give particular affirmative answers to both parts of Problem 7.10. The first of them is restricted to order continuous Banach lattices.

THEOREM 7.14. *Let* E, F *be order continuous Banach lattices with* E *atomless. Then* $N_r(E,F)$ *and* $B\big(A_r(E,F)\big)$ *are orthogonal bands in* $L_r(E,F)$.

The second theorem concerns a more general case of vector lattices but is restricted to order continuous operators only. Let $L_r^{\mathrm{oc}}(E,F)$, $ON_r^{\mathrm{oc}}(E,F)$ and $A_r^{\mathrm{oc}}(E,F)$ denote the intersections of the set of all order continuous operators from E to F with the sets $L_r(E,F)$, $ON_r(E,F)$ and $A_r(E,F)$ respectively.

THEOREM 7.15. *Let* E, F *be order complete vector lattices such that* E *is atomless and* F *is an ideal of some order continuous Banach lattice. Then* $ON_r^{\mathrm{oc}}(E,F)$ *and* $B\big(A_r^{\mathrm{oc}}(E,F)\big)$ *are orthogonal bands in* $L_r^{\mathrm{oc}}(E,F)$.

7.4. Is every AM-compact operator from a vector lattice to a Banach space narrow? The following striking example gives a negative answer even in the sense of Definition 1.2 concerning Köthe spaces without the assumption of absolute continuity of the norm (cf. Proposition 2.1).

EXAMPLE 7.16. *There exists a bounded linear functional*[9] $f : L_\infty \to \mathbb{R}$ *which is not order-to-norm continuous and not narrow.*

Proof. Denote by $\hat{\mathcal{B}}$ the Boolean algebra of the Borel subsets of $[0,1]$ which are equal, up to measure null sets. Let $\mathcal{U}$ be any ultrafilter on $\hat{\mathcal{B}}$. Then the linear functional $f_{\mathcal{U}} : E \to \mathbb{R}$ defined by

$$f_{\mathcal{U}}(x) = \lim_{A \in \mathcal{U}} \frac{1}{\mu(A)} \int_A x \, d\mu$$

is obviously bounded and AM-compact. However, it is not narrow. Indeed, for each $x \in L_\infty$ of the form $x = \mathbf{1}_A - \mathbf{1}_B$ where $[0,1] = A \sqcup B$ one has $f_{\mathcal{U}}(x) = \pm 1$ depending of whether $A \in \mathcal{U}$ or $B \in \mathcal{U}$.

[8] in sense of Definition 1.2, that is stronger

[9] hence, AM-compact

Now we prove that $f_{\mathcal{U}}$ is not order-to-norm continuous. Indeed, consider a nested sequence (A_n) of members of $\mathcal{U}$ with $\mu(A_n) \to 0$. Then $\mathbf{1}_{A_n} \downarrow 0$, however, $f(\mathbf{1}_{A_n}) = 1$ for each $n \in \mathbb{N}$. ∎

Another example of a non-narrow continuous linear operator on L_∞ was discovered by V. Kadets (private communication). Let $f \in L_\infty^*$ be any non-zero multiplicative functional (i.e. $f(x \cdot y) = f(x) \cdot f(y)$ for each $x, y \in L_\infty$). Then it is non-narrow. Indeed, for each $A \in \Sigma$ one has $f(\mathbf{1}_A) = f(\mathbf{1}_A \cdot \mathbf{1}_A) = (f(\mathbf{1}_A))^2$, thus, either $f(\mathbf{1}_A) = 1$ or $f(\mathbf{1}_A) = 0$. Since $f \neq 0$, there exists $A \in \Sigma$ such that $f(\mathbf{1}_A) = 1$. If f were narrow, then it would exist a sign $x \in L_\infty$ with $x^2 = \mathbf{1}_A$ and $|f(x)| < 1$. Then $1 = f(\mathbf{1}_A) = f(x^2) = (f(x))^2 < 1$, a contradiction.

The following questions concerning functional on L_∞ are open in any sense of narrowness.

Problem 7.17. *Does there exist a narrow but not strictly narrow continuous linear functional on L_∞?*

Problem 7.18. *Is a sum of two narrow continuous linear functionals on L_∞ narrow?*

The following positive result shows that, in most natural cases, the answer to the question asked in the title of the subsection, is affirmative.

Theorem 7.19. *Let E be an atomless order complete vector lattice and X be a Banach space. Then every AM-compact order-to-norm continuous linear operator $T : E \to X$ is narrow.*

Now the obvious technique of the Rademacher system (see Proposition 2.1) cannot be applied. This led to a quite involved proof in [MMyP2] (2009). In particular, it follows that every AM-compact order-to-norm continuous linear operator $T : L_\infty \to X$ is narrow. We remark that the last fact can be proved directly in a shorter way [KMMMP] (2009).

7.5. Sketch of the proofs of two main results. To prove the main results, the authors introduced several new notions of narrow operator which themselves may be of interest. We are going to present the idea of proofs of Theorems 7.14 and 7.15 which consists of several steps. First, we need some more definitions. For an arbitrary set J, a series $\sum\limits_{j \in J} x_j$ of elements $x_j \in E$ is said to be *order convergent* and the family $(x_j)_{j \in J}$ is said to be *order summable* if the net $(y_s)_{s \in J^{<\omega}}$, $y_s = \sum\limits_{j \in s} x_j$, order converges to some $y_0 \in E$ where $J^{<\omega}$ is the net of all finite subsets $s \subseteq J$ ordered by inclusion. In this case y_0 is called the *order sum of the series* $\sum\limits_{j \in J} x_j$ and we write $y_0 = \sum\limits_{j \in J} x_j$. A series $\sum\limits_{j \in J} x_j$ is said to be *absolutely order convergent* and the family $(x_j)_{j \in J}$ is said to be *absolutely order summable* if the series $\sum\limits_{j \in J} |x_j|$ order converges.

Let A be a solid subset of a vector lattice E. Denote by $\mathrm{Abs}(A)$ the set of all sums of absolutely order convergent series $\sum\limits_{j \in J} x_j$ of elements $x_j \in A$.

Definition 7.20. Let E, F be vector lattices with F order complete. An operator $T \in L_r(E, F)$ is said to be

— *pseudo-embedding* if $T \in \mathrm{Abs}\big(A_r(E,F)\big)$;
— *pseudo-narrow* if there is no non-zero disjointness preserving operator $S \in L_r^+(E,F)$ with $S \le |T|$.

Our terminology "pseudo-embedding" and "pseudo-narrow operator" is explained by two theorems of Rosenthal concerning operators in L_1. One of them asserts that a non-zero operator in L_1 is a pseudo-embedding if and only if it is an almost isometrically isomorphic embedding when restricted to a suitable $L_1(A)$ (see Theorem 7.23 below). Another theorem implies, in particular, that an operator in L_1 is narrow if and only if it is pseudo-narrow.

The set of all pseudo-embeddings from E to F will be denoted by $L_{pe}(E,F)$. Thus, $L_{pe}(E,F) = \mathrm{Abs}\big(A_r(E,F)\big)$ by the definitions. The set of all pseudo-narrow operators $T \in L_r(E,F)$ will be denoted by $L_{pn}(E,F)$.

Step 1. Generalization of the Kalton-Rosenthal representation theorem for operators on L_1 to vector lattices. The following theorem generalizes Rosenthal's representation theorem for operators on L_1 (Rosenthal considered his theorem as a version of Kalton's representation theorem [Kal3] (1978), so he did not provide a proof) to vector lattices [Ros5] (1984).

THEOREM 7.21. *Let E, F be vector lattices with F order complete. Then*

(i) $B\big(A_r(E,F)\big) = L_{pe}(E,F)$;
(ii) $A_r(E,F)^d = L_{pn}(E,F)$;
(iii) $L_{pe}(E,F)$ *and* $L_{pn}(E,F)$ *are orthogonal bands.*

Since $L_{pe}(E,F) = \mathrm{Abs}\big(A_r(E,F)\big)$ is a band, it equals the set of all sums of (not necessary absolutely) order convergent series $\sum\limits_{j\in J} x_j$ of elements $x_j \in A_r(E,F)$.

So, to prove Theorem 7.14, we need to prove that an operator $T \in L_r(E,F)$ is narrow if and only if it is pseudo-narrow (this is the most difficult part of the proof). It is not easy in both directions. Let us explain how it can be done for the simplest case $E = F = L_1$. For an operator $T \in \mathcal{L}(L_1)$, by $\lambda_T(t)$ we denote the *Enflo-Starbird maximal function*

$$\lambda_T(t) = \lim_{n\to\infty} \max_{1\le i\le 2^n} \big|T\mathbf{1}_{I_n^i}\big|(t), \quad t \in [0,1], \text{ where } I_n^i = \Big[\frac{i-1}{2^n}, \frac{i}{2^n}\Big).$$

THEOREM 7.22 (Kalton and Rosenthal, [Kal3], [Ros5]). *For an operator $T \in \mathcal{L}(L_1)$ the following conditions are equivalent*

(i) *T is narrow*;
(ii) *T is pseudo-narrow*;
(iii) $\lambda_T(t) = 0$ *a.e. on* $[0,1]$;
(iv) *for each $A \in \mathcal{B}$ the restriction $T\big|_{L_1(A)}$ is not an into isomorphism.*

The equivalence (i)⇔(iv) is stated in Theorem 4.2 above. Condition (iii) plays an essential role in Rosenthal's proof of the equivalence (i) ⇔(ii). Conditions (iii) and (iv) may also be used to give new definitions of narrow operator in different contexts.

The equivalence (ii)⇔(iii)⇔(iv) can be deduced from Kalton's paper [Kal3] (1978). The implication (iv)⇒(iii) is due to Enflo and Starbird [EnSt] (1979). Finally, the equivalence of (i) with all other conditions was established by Rosenthal in [Ros5] (1984). Observe that one of the main steps in Rosenthal's proof was the following property of pseudo-embeddings which is evident for disjointness preserving operators and not evident for sums of disjointness preserving operators.

THEOREM 7.23 (Rosenthal, [Ros5]). *Let $T \in \mathcal{L}(L_1)$ be a non-zero pseudo-embedding. Then for each $\varepsilon > 0$ there exists an $A \in \mathcal{B}^+$ such that the restriction $S = T\big|_{L_1(A)}$ is an into isomorphism with $\|S\| \geq \|T\| - \varepsilon$ and $\|S\|\|S^{-1}\| \leq 1 + \varepsilon$.*

Step 2. Reducing to positive operators. Here we reduce the general case to the case of positive narrow operators. For the sake of generality, we deal with order narrow operators. On the other hand, the method of proof uses order continuity of operators.

THEOREM 7.24. *Let E, F be order complete vector lattices such that E is atomless and F is an ideal of some order continuous Banach lattice. Then, every order continuous regular operator $T : E \to F$ is order narrow if and only if $|T|$ is.*

Note that the domination property of narrow operators [10] in the sense of Definition 1.2 was established earlier by Flores and Ruiz in [FlRu] (2003). In view of Proposition 7.9, one obtains the following consequence.

COROLLARY 7.25. *Let E be an atomless order complete vector lattice and F be an order continuous Banach lattice. Then, every order continuous regular operator $T : E \to F$ is narrow if and only if $|T|$ is.*

Step 3. λ-narrow positive operators and conditions under which they coincide with order narrow operators. We are going to give a new definition of narrow operator acting from a vector lattice E to a vector lattice F. In general, the class of operators for which the new definition is given, is incomparable with that of Definition 7.2. But on their intersection (i.e. when the range space is a Banach lattice) it is proved that in some cases both definitions are equivalent.

For any $x \in E^+$, by Π_x we denote the system of all finite sets $\pi \subseteq E^+$ such that $x = \bigsqcup\limits_{u \in \pi} u$. For $\pi', \pi'' \in \Pi_x$ we write $\pi' \leq \pi''$ provided for each $u \in \pi'$ there is a subset $\pi''_u \subseteq \pi''$ such that $u = \bigsqcup\limits_{v \in \pi''_u} v$. Clearly, Π_x is a directed set.

Let E, F be vector lattices with F order complete. For a linear operator $T : E \to F$ we define a function $\lambda_T : E^+ \to F^+$ by setting

$$\lambda_T(x) = \inf_{\pi \in \Pi_x} \sup_{u \in \pi} |Tu|. \tag{7.1}$$

Since F is order complete, λ_T is well defined. Because of the similarity of the properties of the function λ_T and the function introduced by Enflo and Starbird in [EnSt] (1979) and studied also by Kalton in [Kal3] (1978) and Rosenthal in [Ros5] (1984), we call it the *Enflo-Starbird function* of an operator T.

DEFINITION 7.26. Let E, F be order complete vector lattices with E atomless. A linear operator $T : E \to F$ is called *λ-narrow* if $\lambda_T = 0$.

[10] i.e. if $0 \leq S \leq T$ and T is narrow then S is

Obviously, if T is regular then $\lambda_T(x) \leq \lambda_{|T|}(x)$ for each $x \in E^+$, hence if $|T|$ is λ-narrow then so is T.

THEOREM 7.27. *Let E, F be order complete vector lattices such that E is atomless and F is an ideal of some order continuous Banach lattice. Then a positive operator $T : E \to F$ is λ-narrow if and only if it is order narrow.*

Step 4. Positive pseudo-narrow operators are exactly positive λ-narrow operators. The most difficult step in all the proof is to show that every positive pseudo-narrow operator $T : E \to F$ is λ-narrow. We assume that $\lambda_T(x) > 0$ for some $x \in E^+$, and our purpose is to construct a disjointness preserving operator S with $0 < S \leq T$.

THEOREM 7.28. *Let E, F be order complete vector lattices with E atomless. Then a positive order continuous operator $T : E \to F$ is λ-narrow if and only if it is pseudo-narrow.*

We remark that one implication here is almost obvious, as the following statement shows.

PROPOSITION 7.29. *Let E, F be order complete vector lattices with E atomless. If a positive operator $T : E \to F$ is λ-narrow then it is pseudo-narrow.*

Proof. Suppose that T is not pseudo-narrow. Let $0 < S \leq T$ be a disjointness preserving operator and $x \in E^+$ be such that $Sx > 0$. Then for each representation $x = \bigsqcup_{k=1}^{n} x_k$ we have

$$Sx = \bigsqcup_{k=1}^{n} Sx_k = \sup_{1 \leq k \leq n} Sx_k \leq \sup_{1 \leq k \leq n} Tx_k,$$

and hence, one obtains that $\lambda_T(x) \geq Sx > 0$ and T is not λ-narrow. ∎

Thus, combining the results in all steps, we obtain Theorems 7.14 and 7.15.

7.6. Some open problems on narrow operators on vector lattices. In [MMyP2] the authors gave an affirmative answer to Problem 7.10 for order continuous Banach lattices, and showed that for $E = F = L_\infty$ the problem has a negative answer. So, Problem 7.10 remains unsolved for other cases.

PROBLEM 7.30. *Is the set of all order narrow regular operators $T : L_\infty \to L_\infty$ a band in the vector lattice $L_r(L_\infty)$ of all regular linear operators in L_∞?*

Note that by Theorem 7.15, the set of all order narrow regular order continuous operators in L_∞ is a band in the vector lattice $L_r^{\mathrm{oc}}(L_\infty)$ of all regular order continuous linear operators on L_∞.

PROBLEM 7.31. *Can one remove the condition of order continuity on T in Theorem 7.15?*

PROBLEM 7.32. *Let E be an order continuous Banach lattice and $T \in L_r(E)$. Is T narrow if and only if for each band $F \subseteq E$ the restriction $T|_F$ is not an isomorphic embedding?*

Rosenthal in [Ros5] (1984) proved that this is the case for $E = L_1$ (see Theorem 7.22). Note that for $E = L_p$, $2 < p \leq \infty$, there exists a non-regular operator $T \in \mathcal{L}(L_p)$ which is not narrow while for each band $F \subseteq E$ the restriction $T|_F$ is not an isomorphic embedding. Indeed, the composition $T = S \circ I$ of the inclusion operator $I : L_p \to L_2$ and an isomorphic embedding $S : L_2 \to L_p$ has the desired properties.

One of the most interesting problems in the isomorphic theory of Banach spaces that remains still unsolved is whether every infinite-dimensional complemented subspace of L_1 is isomorphic to either L_1, or ℓ_1? On the other hand, if $1 < p < \infty$, $p \neq 2$ then there are a lot of isomorphic types of complemented subspaces of L_p. Let us say that a subspace E of L_p is *regularly complemented* if there is a regular projection of L_p onto E. The following problem extends the mentioned above problem to the setting of L_p-spaces.

PROBLEM 7.33. *Let $1 \leq p < \infty$, $p \neq 2$. Is every infinite-dimensional regularly complemented subspace of L_p isomorphic to either L_p, or ℓ_p?*

8. Hereditarily narrow operators. Different facts concerning the Daugavet property for L_1 give some further developments of the famous Pełczyński theorem on the impossibility of embedding the space L_1 into a Banach space with an unconditional basis. V. Kadets and R. Shvidkoy (1999) proved that for any Banach space X any into isomorphism $J : L_1 \to X$ cannot be represented as a pointwise unconditionally convergent series of compact operators [KaSh] (1999). Moreover, if a Banach space has the Daugavet property relative to a linear subspace $\mathcal{M} \subseteq \mathcal{L}(X)$ then the identity of X cannot be represented as a pointwise unconditionally convergent series of operators in $\mathcal{M}$ [KaPo3] (2003). In [KaPo3] one can find the following statement.

THEOREM 8.1. *A pointwise unconditionally convergent series of narrow operators in $\mathcal{L}(L_1)$ is narrow.*

The following refinement of the notion of narrow operator introduced by V. Kadets, Kalton and Werner [KKW] (2005) allows one to get much more in this direction (originally it was applied for operators on L_p).

DEFINITION 8.2. Let E be a Köthe Banach space on a finite atomless measure space. An operator $T \in \mathcal{L}(E, X)$ is called *hereditarily narrow* if for each atomless σ-algebra $\Sigma_1 \subseteq \Sigma$ of subsets of any set $A \in \Sigma^+$ the restriction of T to $E(\Sigma_1)$ is narrow.

Of course, each hereditarily narrow operator is narrow, however the converse is not true: the conditional expectation operator with respect to an atomless sub- σ-algebra is a counterexample. If every operator $T \in \mathcal{L}(E, X)$ is narrow then clearly, every operator $T \in \mathcal{L}(E, X)$ is hereditarily narrow. Proposition 2.1 (Proposition 2.2) yields that if the norm of E is absolutely continuous then each AM-compact (each Dunford-Pettis) operator $T \in \mathcal{L}(E, X)$ is hereditarily narrow. If there exists a disjoint weakly null normalized sequence (x_n) in E then the conditional expectation operator with respect to the purely atomic sub-σ-algebra generated by $(\operatorname{supp} x_n)$ is a hereditarily narrow operator which is not Dunford-Pettis. In the space L_1 (which contains no such a sequence) an analogous example gives the so-called biased coin convolution operator constructed by Rosenthal in [Ros2] (1975). Indeed, this operator $S \in \mathcal{L}(L_1)$ is L_1-singular (and by Theorem 4.2 is hereditarily narrow) and satisfies $Sx = x$ for any element $x \in R$ of the subspace R of L_1 spanned by the Rademacher system $R = [r_n]$ (and hence is not Dunford-Pettis).

The notion of hereditarily narrow operator is a refinement of the notion of narrow operator in the sense of the following two results.

PROPOSITION 8.3 ([KKW]). *The sum of two hereditarily narrow operators in a Köthe Banach space is hereditarily narrow.*

Note that the proof of this proposition in [KKW], which concerns the L_p-spaces, makes sense for any Köthe Banach space.

We remark that Theorem 8.1 trivially holds for hereditarily narrow operators. Moreover, the following strong version of it holds.

THEOREM 8.4 ([KKW]). *Let X be a Banach space. Then the sum of a pointwise unconditionally convergent series of hereditarily narrow operators in $\mathcal{L}(L_1, X)$ is hereditarily narrow.*

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