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ON ω-CONVEX FUNCTIONS

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Abstract. In Orlicz spaces theory some strengthened version of the Jensen inequality is often used to obtain nice geometrical properties of the Orlicz space generated by the Orlicz function satisfying this inequality. Continuous functions satisfying the classical Jensen inequality are just convex which means that such functions may be described geometrically in the following way: a segment joining every pair of points of the graph lies above the graph of such a function. In the current paper we try to obtain a similar geometrical description of the aforementioned inequality.

1. Introduction. We deal here with some modified version of convexity. The usual convexity of a real function may be described in the following way: a function f is convex if and only if every segment which connects two points of the graph of f lies above the graph of f. We would like to obtain a similar description of some inequality used in Orlicz spaces theory. E. F. Beckenbach in [1] introduced a more general definition of convexity where segments are replaced by pieces of graphs of functions from a given family. Recently some kind of generalized convexity has been considered by M. Bessenyei and Z. Páles in [3] where the authors use just two functions ω_1, ω_2 to construct such families.

On the other hand, in papers devoted to Orlicz spaces some strengthened version of the Jensen inequality is considered. Namely the right-hand side of this inequality is multiplied by some constant smaller than one, i.e. the inequality

$$f\Big(\frac{x+y}{2}\Big) = \gamma(a)[f(x)+f(y)] \tag{1}$$

(with $\gamma(a) \in \big(0, \frac{1}{2}\big)$) postulated for all $x, y \in \mathbb{R}$ such that $|x| \le a|y|$ where $a \in (0,1)$. It is used in order to obtain some properties of Orlicz spaces generated by Orlicz functions satisfying this inequality (see for example [4], [5] and [6]).

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In the current paper we provide the first step to obtain a geometrical description of inequality (1) similar to the mentioned description of the Jensen inequality. To this end we introduce here a convexity with respect to a given function. Then we study some basic properties of this new notion.

And since the classical examples of functions satisfying inequality (1) are functions $f(x) = |x|^p$ with $p > 1$, we use the function $\omega(x) = |x|^p$. It turns out that under some assumptions ω-convexity with respect to this function implies the inequality (1).

2. Results

DEFINITION 2.1. We say that a function $\omega : \mathbb{R} \supset D \to \mathbb{R}$ has the "joining points" property (is a JP-function) if it is continuous and satisfies the following condition

$$\bigwedge_{(a,b)\in\mathbb{R}^2, a>0} \ \overset{1}{\bigvee_{x\in\mathbb{R}}} [x, x+a] \subset D \text{ and } f(x+a) - f(x) = b.$$

For the sake of simplicity in the whole paper we shall assume that the domain of ω is an infinite and open interval. First we shall have a closer look at JP-functions.

THEOREM 2.2. *Let $I \subset \mathbb{R}$ be an infinite open interval and $\omega : I \to \mathbb{R}$ be a JP-function. Then ω is either convex or concave.*

Proof. Take an $a \in (0, \infty)$ and consider the function $\omega_a : I_a \to \mathbb{R}$ given by the formula $\omega_a(x) := \omega(x+a) - \omega(x)$ where $I_a := I \cap (I - a)$. The function ω_a is a continuous injection and therefore it must be monotone. Assuming that this function is increasing, we shall show that in this case ω must be convex.

To this end take $u, v \in I$, $u < v$ and $\lambda \in (0, 1)$. Put $x_0 = \lambda u + (1 - \lambda)v$, we are going to show that

$$\omega(x_0) \le \lambda\omega(u) + (1 - \lambda)\omega(v).$$

The function $\omega_{a/2}$ must be increasing or decreasing. First we shall show that $\omega_{a/2}$ is also an increasing function. For the indirect proof suppose that $\omega_{a/2}$ is decreasing. Taking $x \in I$, such that $[x, x + 3/2\,a] \subset I$ we obtain

$$\omega(x + a/2) - \omega(x) > \omega(x + a) - \omega(x + a/2)$$

and

$$\omega(x + a) - \omega(x + a/2) > \omega(x + 3/2\,a) - \omega(x + a).$$

From the above equalities we get

$$\omega(x + a) - \omega(x) > \omega(x + 3/2\,a) - \omega(x + a/2)$$

which contradicts the fact that ω_a is increasing. Thus we have shown that if ω_a is an increasing function then $\omega_{a/2}$ is also increasing. One can easily show that if k is a given positive integer then $\omega_{a/2^k}$ is increasing.

Now take $n = 2^k$, where k is a given positive integer, and put

$$a_0 := u,\ a_1 := u + \frac{a}{n},\ \ldots,\ a_l := u + l\frac{a}{n},\ a_{l+1} := u + (l+1)\frac{a}{n},$$

where $l \in \mathbb{N}$ is such that $a_l < v$ and $v - a_{l+1} < \frac{a}{n}$. Let further $p \in \{1, \ldots, l-1\}$ be such that $|x_0 - a_p| \le \frac{a}{n}$.

Since the function $g_{a/n}$ is increasing, we obtain

$$\begin{aligned}\omega(a_l)-\omega(a_p) &= [\omega(a_l)-\omega(a_{l-1})]+[\omega(a_{l-1})-\omega(a_{l-2})]+\ldots+[\omega(a_{p+1})-\omega(a_p)]\\ &\geq (l-p)[\omega(a_{p+1})-\omega(a_p)] \geq (l-p)[\omega(a_p)-\omega(a_{p-1})] = \frac{l-p}{p}p[\omega(a_p)-\omega(a_{p-1})]\\ &\geq \frac{l-p}{p}\Big([\omega(a_p)-\omega(a_{p-1})]+[\omega(a_{p-1})-\omega(a_{p-2})]+\ldots+[\omega(a_1)-\omega(a_0)]\Big)\\ &= \frac{l-p}{p}[\omega(a_p)-\omega(a_0)].\end{aligned}$$

Multiplying the obtained inequality by $\frac{p}{l}$ we obtain

$$\frac{l-p}{l}[\omega(a_p)-\omega(a_0)] \leq \frac{p}{l}[\omega(a_l)-\omega(a_p)],$$

i.e.

$$\omega(a_p) \leq \frac{p}{l}\omega(a_l)+\frac{l-p}{l}\omega(a_0) = \frac{p}{l}\omega(a_l)+\frac{l-p}{l}\omega(u).$$

Obviously if $k\to\infty$ then the left-hand side of this inequality tends to $\omega(x_0)$ and the limit of the right-hand side is equal to

$$\lambda\omega(u)+(1-\lambda)\omega(v),$$

hence we obtain the convexity of the function ω.

One can similarly show that if for some positive constant a the function g_a is decreasing, then f is concave. ∎

Corollary 2.3. *Let $(u,v)=I\subset\mathbb{R}$ be an infinite and open interval. A function $\omega : I\to\mathbb{R}$ is a JP-function iff it satisfies the following conditions*

(i) *ω is strictly convex (or concave)*
(ii) $\lim_{x\to v}\omega'_+(x)=\infty$ *(resp.* $-\infty$*)*
(iii) $\lim_{x\to u}\omega'_-(x)=-\infty$ *(resp.* ∞*).*

Proof. First we shall show that every JP-convex function ω has property (i). Let $\omega : I\to\mathbb{R}$ be a JP-function. From Theorem 2.2 we know that ω must be convex or concave. Therefore it is enough to observe that if

$$\frac{\omega(x)+\omega(y)}{2}=\omega\Big(\frac{x+y}{2}\Big),$$

for some $x,y\in I$, $x<y$, then $\omega(y)-\omega(\frac{x+y}{2})=\omega(\frac{x+y}{2})-\omega(x)$, which contradicts the assumption that ω is a JP-function since $y-\frac{x+y}{2}=\frac{x+y}{2}-x$.

Now we are going to show that ω satisfies (ii) and (iii). Suppose that condition (ii) is not satisfied. This means that ω'_+ is bounded above. In such a case the function $\varphi(x):=\omega(x+1)-\omega(x)$ is bounded by the same constant and, in consequence, it does not take all real values. It is a contradiction with the fact that ω is a JP-function. If, on the other hand, condition (iii) were not satisfied we would in the same way obtain that φ is bounded below and also is not a surjection, which contradicts the fact that ω is a JP-function.

It remains to show that a function which satisfies the conditions (i)–(iii) is a JP-function. To this end fix a function ω which satisfies these conditions and a pair of real numbers (a,b), $a>0$. Define the function $\psi(x):=\omega(x+a)-\omega(x)$. Since ω is strictly convex, ψ is strictly increasing. Moreover, convexity of ω implies its continuity and consequently also continuity of ψ. Now from the continuity of ψ and from conditions (ii) and (iii) we infer that ψ is a surjection. Summarizing we have shown that ψ is a bijection and so there exists exactly one $x_0 \in I \cap (I - a)$ such that $\psi(x_0) = b$, which means that ω is a JP-function. ■

PROPOSITION 2.4. *Let $\omega : I \to \mathbb{R}$ be a JP-function. Then for every pair of real numbers (a, b) and for every $x, y \in I$ there exists exactly one pair $(\alpha, \beta) \in \mathbb{R}^2$ such that*

$$\omega(x + \alpha) + \beta = a \quad \textit{and} \quad \omega(y + \alpha) + \beta = b.$$

In view of the above proposition we may now introduce the following

DEFINITION 2.5. Let $I \subset \mathbb{R}$ be an infinite open interval, let $\omega : I \to \mathbb{R}$ be a JP-function, and let $\lambda \in (0, 1)$ be a given number. We say that $f : \mathbb{R} \to \mathbb{R}$ is a (λ, ω)-*convex function* if for all $x, y \in \mathbb{R}$ we have

$$f(\lambda x + (1 - \lambda)y) \le \omega(\lambda x + (1 - \lambda)y + \alpha) + \beta, \tag{2}$$

where $\alpha, \beta \in \mathbb{R}$ are such that $\omega(x + \alpha) + \beta = f(x)$ and $\omega(y + \alpha) + \beta = f(y)$.

If a function f is (λ, ω)-convex for all $\lambda \in (0, 1)$, then we say that f is ω-convex.

Let us now have a closer look at the shape of the JP-function. We observe that an important class of JP-functions are functions of the type $\omega(x) = |x|^c$, $x \in \mathbb{R}$, where $c > 1$. Such functions are important in the Orlicz spaces theory and therefore we shall later study functions which are ω-convex with ω of this type. However in the case $c = 2$ we are able to say much more about such functions.

THEOREM 2.6. *A function $f : \mathbb{R} \to \mathbb{R}$ is x^2-convex if and only if the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) := f(x) - x^2$ is convex.*

Proof. Take $x, y \in \mathbb{R}$, define $\omega(t) := t^2$ and consider the function $f : \mathbb{R} \to \mathbb{R}$ of the form $f(t) = t^2 + g(t)$. Now we define a new function

$$\omega_1(t) := t^2 + (t - x)\frac{g(y) - g(x)}{y - x} + g(x).$$

Note that $\omega_1(x) = f(x)$, $\omega_1(y) = f(y)$ and

$$\omega_1(t) = \omega(t + \alpha) + \beta$$

for some real numbers α and β. This means that f is ω-convex iff we have $f(z) \le \omega_1(z)$ for all $z \in (x, y)$. Using here the form of f we obtain

$$z^2 + g(z) \le z^2 + \frac{z - x}{y - x}[g(y) - g(x)] + g(x), \quad z \in (x, y). \tag{3}$$

From inequality (3) we have

$$\frac{g(z) - g(x)}{z - x} \le \frac{g(y) - g(x)}{y - x}, \quad z \in (x, y),$$

which is equivalent to the convexity of g. ■

EXAMPLE 2.7. There exist JP-functions ω and functions of the form $f = \omega + g$ where g is convex but f is not ω-convex.

Take $\omega(x) := |x|^3$, and define f by $f(x) = |x|^3 + 4x + 2x^2$. Now take $x = 0$, $y = 1$; the function $\omega_1(t) := |t+1|^3 - 1$ satisfies the equalities

$$\omega_1(x) = f(x), \quad \omega_1(y) = f(y).$$

We shall show that

$$f(z) > \omega_1(z), \quad \text{for } z \in (x, y).$$

To this end fix a $z \in (0,1)$ and write

$$f(z) = z^3 + 4z + 2z^2 > z^3 + 3z + 3z^2 = z^3 + 3z^2 + 3z + 1 - 1 = (z+1)^3 - 1 = \omega_1(z).$$

In the classical case it is easy to show that if a function f satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

with $\lambda = \frac{1}{2}$, then it satisfies this inequality with every rational number λ. The next theorem yields a similar result for ω-convexity (for dyadic numbers).

THEOREM 2.8. *A function f which is Jensen ω-convex is (λ, ω)-convex with every dyadic number $\lambda \in (0,1)$.*

Proof. Take an infinite and open interval $I \subset \mathbb{R}$, $\omega : I \to \mathbb{R}$ satisfying the assumptions of the theorem and $x, y \in I$, $x < y$. Let further $a_1 \in \mathbb{R}$ be such that

$$\omega(a_1 + y - x) - \omega(a_1) = f(y) - f(x).$$

Now we define the function $\omega_1 : [x, y] \to \mathbb{R}$ by the formula

$$\omega_1(t) := \omega(t + a_1 - x) - \omega(a_1) + f(x), \ \ t \in [x, y].$$

We need to show that for every λ of the form $\frac{m}{2^k}$ the inequality

$$f(\lambda x + (1-\lambda)y) < \omega_1(\lambda x + (1-\lambda)y) \tag{4}$$

is true. The proof will be done by induction over k. For $k = 1$ our assertion follows directly from the assumption. Consequently let us assume that for some k this condition is satisfied. Fix $m \in \{1, \ldots, 2^{k+1}\}$ and put $\lambda := \frac{m}{2^{k+1}}$. Without loss of generality we may assume that m is an odd number. This means that both numbers $m+1$ and $m-1$ are even. Now define $x_0 := \lambda x + (1-\lambda)y$, $x_1 := \frac{m-1}{2^k}x + \frac{2^k-(m-1)}{2^k}y$ and $x_2 := \frac{m+1}{2^k}x + \frac{2^k-(m+1)}{2^k}y$. We can find $a_2 \in \mathbb{R}$ such that

$$\omega(a_2 + x_2 - x_1) - \omega(a_2) = f(x_2) - f(x_1)$$

and similarly as before we define ω_2 by the formula

$$\omega_2(t) := \omega(t + a_2 - x_1) - \omega(a_2) + f(x_1), \quad t \in [x_1, x_2].$$

We have $x_0 = \frac{x_1 + x_2}{2}$, i.e.

$$f(x_0) \leq \omega_2(x_0). \tag{5}$$

Now it suffices to show that $\omega_2(x_0) \leq \omega_1(x_0)$. Suppose for the indirect proof that $\omega_2(x_0) > \omega_1(x_0)$, then

$$(\omega_2 - \omega_1)(x_0) > 0.$$

Write $(\omega_2 - \omega_1)(x_1) = f(x_1) - \omega_1(x_1) \leq f(x_1) - f(x_1) = 0$. A similar inequality is satisfied in the point x_2, so using the continuity of $\omega_2 - \omega_1$ we obtain the existence of points: $x_3 \in [x_1, x_0)$ and $x_4 \in (x_0, x_2]$, where functions ω_1 and ω_2 take the same values. This means that

$$\omega(x_3 + a_1 - x) - \omega(a_1) + f(x) = \omega(x_3 + a_2 - x_1) - \omega(a_2) + f(x_1)$$

and

$$\omega(x_4 + a_1 - x) - \omega(a_1) + f(x) = \omega(x_4 + a_2 - x_1) - \omega(a_2) + f(x_1).$$

Subtracting these inequalities we obtain

$$\omega(x_3 + a_1 - x) - \omega(x_4 + a_1 - x) = \omega(x_3 + a_2 - x_1) - \omega(x_4 + a_2 - x_1)$$

and

$$\omega(x_3 + a_1 - x) - \omega(x_3 + a_2 - x_1) = \omega(x_4 + a_1 - x) - \omega(x_4 + a_2 - x_1).$$

Now as we can see the differences of the arguments on both sides are equal which in view of the fact that $x_3 \neq x_4$ is not possible since ω is a JP-function. ∎

As a direct consequence of this theorem we obtain the following corollary which can already be found in [2] in a more general situation.

Corollary 2.9. *Every continuous Jensen ω-convex function is ω-convex.*

Now from Proposition 2.4 we obtain the following

Remark 2.10. Let $\omega : I \to \mathbb{R}$ be a JP-function. Then for every pair of real numbers (a, b) and for every $x, y \in D$ there exists exactly one pair $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\omega(x + \alpha) - \omega(x) + \beta = a \quad \text{and} \quad \omega(y + \alpha) - \omega(y) + \beta = b.$$

Proposition 2.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a given function. The function f is ω-convex if and only if the function $g(x) := f(x) - \omega(x)$ satisfies the condition*

$$g(z) \leq \omega(z + \alpha) - \omega(z) + \beta \tag{6}$$

for all x, y such that $x < y$ and for every $z \in (x, y)$, where α, β are such that $\omega(x + \alpha) - \omega(x) + \beta = g(x)$ and $\omega(y + \alpha) - \omega(y) + \beta = g(y)$.

Remark 2.12. Let $\omega : \mathbb{R} \to \mathbb{R}$ be a convex function and let $x, y \in \mathbb{R}$, $x < y$. If $\alpha, \beta \in \mathbb{R}$ are such that

$$\omega(x + \alpha) - \omega(x) + \beta = a \quad \text{and} \quad \omega(y + \alpha) - \omega(y) + \beta = b,$$

where $a < b$ are some real numbers, then $\alpha > 0$.

Proof. We have $\alpha \neq 0$. So, for the indirect proof assume that $\alpha < 0$. Subtracting the assumed equalities and using the fact that $a < b$ we obtain $\omega(x + \alpha) - \omega(x) < \omega(y + \alpha) - \omega(y)$. Dividing this inequality by α we get

$$\frac{\omega(x + \alpha) - \omega(x)}{\alpha} > \frac{\omega(y + \alpha) - \omega(y)}{\alpha}$$

and since $x < y$, this inequality contradicts the convexity of ω. ∎

Now we shall turn our attention to Orlicz functions. By an Orlicz function we mean a convex and even function $f : \mathbb{R} \to \mathbb{R}$ which vanishes at zero and is not identically equal to zero (the definitions may slightly differ in different papers devoted to this topic). Using an Orlicz function one can define a certain Banach space which is called an Orlicz space.

In the theory of Orlicz spaces it is important to find connections between the properties of a given Orlicz function and the geometrical properties of the Orlicz space which is generated by this function. One of the conditions which is often used to this purpose (see [5]) is the following:

$$\bigvee_{a\in(0,1)}\ \bigvee_{\gamma\in(0,1)}\ \bigwedge_{x,y\in\mathbb{R}^+;y\le ax} f\Big(\frac{x+y}{2}\Big)\le\gamma[f(x)+f(y)]. \tag{7}$$

Let us also mention that if we take here the function $f(x)=|x|^c$, then the above inequality is satisfied. Moreover, if we take $y=ax$, then we have the equality

$$f(x)=\Big(\frac{x+ax}{2}\Big)^c=\frac{(1+a)^c}{2^c(1+a^c)}[x^c+(ax)^c]=\frac{(1+a)^c}{2^c(1+a^c)}[f(x)+f(ax)]$$

and the function $a\mapsto\frac{(1+a)^c}{2^c(1+a^c)}$ occurring in this equation will later be denoted by γ_c.

So we shall study functions which are ω-convex for $\omega(x)=|x|^c$ and we would like to obtain some condition similar to the inequality (7). In other words our aim is to find a geometrical description of the properties of this kind.

First we need the following

COROLLARY 2.13. *If a function $f:\mathbb{R}^+\to\mathbb{R}$ is $|x|^c$-convex, where $c>2$ is a given number, the function g is given by the formula $g(x):=f(x)-x^c$, and $x,y\in\mathbb{R}^+$, $x<y$, are such that $g(x)<g(y)$, then for every $\lambda\in(0,1)$ we have the inequality*

$$g(\lambda x+(1-\lambda)y)\le\lambda g(x)+(1-\lambda)g(y).$$

Proof. Take a function f satisfying the above assumptions, $x,y\in\mathbb{R}^+$ and $z=\lambda x+(1-\lambda)y$ for $\lambda\in(0,1)$. Then from Proposition 2.11 for the function g we have

$$g(z)\le(z+\alpha)^c-z^c+\beta,$$

where α and β are such that $(x+\alpha)^c-x^c+\beta=g(x)$ and $(y+\alpha)^c-y^c+\beta=g(y)$. Moreover, note that $x,y,z>0$, from Remark 2.12 we know that also $\alpha>0$ and therefore we may omit the absolute value signs. This means that it is enough to show that the function $h(x):=(x+\alpha)^c-x^c$ is convex. To this end let us determine the derivative h' of h:

$$h'(x)=c(x+\alpha)^{c-1}-cx^{c-1}=c[(x+\alpha)^{c-1}-x^{c-1}]$$

and the second derivative h'' of h:

$$h''(x)=c(c-1)[(x+\alpha)^{c-2}-x^{c-2}].$$

The function h'' is clearly positive which means that h is a convex function. ∎

Now we will prove the main result of the paper which means that in order to obtain inequality (7) it is enough (in some circumstances) to assume ω-convexity with $\omega(x)=|x|^c$.

THEOREM 2.14. *Let $\varphi:\mathbb{R}\to\mathbb{R}^+$ be an Orlicz function. Let d be a positive number, let $c>2$ and the set A_d be defined as follows*

$$A_d:=\{x\in\mathbb{R}:\varphi(x)\le dx^c\}.$$

If $\varphi_{|[0,\infty)}$ is $|x|^c$-convex, then there exists a function $\gamma : (0,1) \to (0,1/2)$ such that the inequality

$$\varphi\Big(\frac{x+y}{2}\Big) \le \gamma(a)[\varphi(x)+\varphi(y)]$$

is satisfied for every $a \in (0,1)$ and all $x, y \in A_d$ such that $x \le ay$.

Proof. First we shall show that if the function φ satisfies the assumptions of the theorem, then $\psi(x) := \varphi(x) - x^c$ has nonnegative values. For the indirect proof, suppose that for some x_0 we have $\psi(x_0) < 0$. In such a case, since $\psi(0) = 0$, we have

$$\psi(x) \le |x+\alpha|^c - |x|^c - |\alpha|^c$$

for $x \in (0, x_0)$, where α is some negative real number. This means that for such values of x we have

$$\varphi(x) = |x|^c + \psi(x) \le |x+\alpha|^c - |\alpha|^c$$

and in consequence $\varphi(x) < 0$ for some $x > 0$, which contradicts the assumption that φ is an Orlicz function.

Now one can easily see that ψ is a strictly increasing function. Indeed, take x_1, x_2 satisfying $x_1 < x_2$. Then $x_1 = \lambda x_2$ for some $\lambda \in (0,1)$, $\psi(x_2) > \psi(0) = 0$ and, in view of Corollary 2.13, we get

$$\psi(x_1) \le (1-\lambda)\psi(0) + \lambda\psi(x_2) < \psi(x_2).$$

Now, using Corollary 2.13 once more, we obtain the convexity of ψ.

Now we shall show that if $x, ax \in \mathbb{R}^+$ are such that $x, ax \in A_d$, then

$$\varphi\Big(\frac{x+ax}{2}\Big) \le \gamma(a)[\varphi(x)+\varphi(ax)] \tag{8}$$

for some function $\gamma : (0,1) \to \big(0, \frac{1}{2}\big)$.

Indeed, we have

$$\varphi\Big(\frac{x+ax}{2}\Big) = \Big(\frac{x+ax}{2}\Big)^c + \psi\Big(\frac{x+ax}{2}\Big) = \gamma_c(a)[x^c + (ax)^c] + \psi\Big(\frac{x+ax}{2}\Big).$$

Thus in view of the convexity of ψ we obtain

$$\varphi\Big(\frac{x+ax}{2}\Big) \le \gamma_c(a)[x^c + (ax)^c] + \frac{1}{2}[\psi(x)+\psi(ax)]. \tag{9}$$

Defining $\gamma(a) := \frac{d/2+\gamma_c(a)}{1+d}$, we see that γ has values in the interval $\big(0, \frac{1}{2}\big)$ and

$$\frac{\gamma(a)-\gamma_c(a)}{1/2-\gamma(a)} = d \ge \frac{\psi(x)+\psi(ax)}{x^c+(ax)^c},$$

which means that

$$\gamma(a)[x^c+(ax)^c] - \gamma_c(a)[x^c+(ax)^c] \ge \frac{1}{2}[\psi(x)+\psi(ax)] - \gamma(a)[\psi(x)+\psi(ax)].$$

This inequality together with (3) means that

$$\varphi\Big(\frac{x+ax}{2}\Big) \le \gamma(a)[\varphi(x)+\varphi(ax)].$$

Now take $x, y \in A_d$ such that $x \le ay$ and observe that the function γ_c is increasing and consequently also γ defined as above must be increasing. This means that

$$\varphi\Big(\frac{x+y}{2}\Big) \le \gamma(x/y)[\varphi(x)+\varphi(y)] \le \gamma(a)[\varphi(x)+\varphi(y)]. \ \blacksquare$$

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