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TRACTABLE EMBEDDINGS OF BESOV SPACES INTO ZYGMUND SPACES

HANS TRIEBEL

Mathematisches Institut, Friedrich-Schiller-Universität
D-07737 Jena, Germany
E-mail: hans.triebel@uni-jena.de

Abstract. The paper deals with dimension-controllable (tractable) embeddings of Besov spaces on n-dimensional cubes into Zygmund spaces. This can be expressed in terms of tractability envelopes.

1. Introduction. Let $W^1_p(\mathbb{R}^n)$ with $1 < p < \infty$ be the classical Sobolev spaces normed by

$$\|f\,|W^1_p(\mathbb{R}^n)\| = \Big(\int_{\mathbb{R}^n} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} + \Big(\int_{\mathbb{R}^n} |\nabla f(x)|^p\,\mathrm{d}x\Big)^{1/p}, \tag{1.1}$$

where $|\nabla f(x)|^2 = \sum_{j=1}^n \big|\frac{\partial f(x)}{\partial x_j}\big|^2$. Let

$$\mathbb{Q}^n = \big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ 0 < x_j < 1\big\} \tag{1.2}$$

be the unit cube in $\mathbb{R}^n$. Let $W^1_p(\mathbb{Q}^n)$ be the restriction of $W^1_p(\mathbb{R}^n)$ to $\mathbb{Q}^n$, normed by

$$\|f\,|W^1_p(\mathbb{Q}^n)\| = \Big(\int_{\mathbb{Q}^n} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} + \Big(\int_{\mathbb{Q}^n} |\nabla f(x)|^p\,\mathrm{d}x\Big)^{1/p} \tag{1.3}$$

and let $\mathring{W}^1_p(\mathbb{Q}^n)$ be the completion of $C_0^\infty(\mathbb{Q}^n) = D(\mathbb{Q}^n)$ in $W^1_p(\mathbb{Q}^n)$. Recall that $\mathring{W}^1_p(\mathbb{Q}^n)$ can be isometrically identified with the subspace

$$\widetilde{W}^1_p(\mathbb{Q}^n) = \big\{f \in W^1_p(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\mathbb{Q}^n}\big\} \tag{1.4}$$

of $W^1_p(\mathbb{R}^n)$, normed by (1.1). Let $f^*(t)$, $0 < t \le 1$, be the usual decreasing (= non-increasing) rearrangement of $f \in \widetilde{W}^1_p(\mathbb{Q}^n)$. Then

$$\Big(\int_0^1 (1 + |\log t|)^{p/2} f^*(t)^p\,\mathrm{d}t\Big)^{1/p} \le c\,\|f\,|W^1_p(\mathbb{R}^n)\| \tag{1.5}$$

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for all $f \in \mathring{W}^1_p(\mathbb{Q}^n)$, where the constant $c > 0$ is independent of the dimension $n \in \mathbb{N}$ (but may depend on p). This has been proved in [22, Section 7.1], based on [21]. It is our main aim to study corresponding inequalities for Besov spaces $\widetilde{B}^s_{pp}(\mathbb{Q}^n)$, in place of $\widetilde{W}^1_p(\mathbb{Q}^n) = \mathring{W}^1_p(\mathbb{Q}^n)$, where $\widetilde{B}^s_{pp}(\mathbb{Q}^n)$ is defined similarly as in (1.4), $1 < p < \infty$, $s > 0$. In particular if $1 < p < \infty$ and $0 < s < 1/p$, then one obtains as a special case

$$\begin{aligned} &\Big(\int_0^1 (1 + |\log t|)^{sp} f^*(t)^p \, \mathrm{d}t \Big)^{1/p} \\ &\le \lambda^{-1} \Big(\int_{\mathbb{Q}^n} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} + \Big(\int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x) - f(y)|^p}{(\lambda |x-y|)^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \\ &\le 2^{\varrho n} \Big(\int_{\mathbb{Q}^n} |f(x)|^p \, \mathrm{d}x + \int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \end{aligned} \tag{1.6}$$

for all $f \in B^s_{pp}(\mathbb{Q}^n) = \widetilde{B}^s_{pp}(\mathbb{Q}^n)$, with constants λ, $0 < \lambda \le 1$, and $\varrho > 0$ which are independent of the dimension $n \in \mathbb{N}$ (but may depend on s, p). Factors of type $2^{\varrho n}$ with ϱ independent of $n \in \mathbb{N}$ can be incorporated in the underlying Besov-norm by a suitable dimension-independent rescaling of the distance $|x| \to \kappa |x|$, $\kappa > 0$, and the Lebesgue measure. This may justify to call embeddings of type (1.6), and also (1.5), *tractable* in analogy to corresponding notation in common use in complexity theory, [23, 24].

The plan of the paper is the following. In Section 2 we comment first on so-called *logarithmic Sobolev inequalities* and how they are related to the above-described assertions. Otherwise we collect basic definitions and basic assertions for Zygmund spaces and Besov spaces. In Section 3 we prove (1.6) and a more general assertion for $\widetilde{B}^s_{pp}(\mathbb{Q}^n)$, $1 < p < \infty$, $s > 0$. The corresponding Theorems 3.3 and 3.7 may be considered as our main results. In Section 4 we introduce *tractability envelopes* in analogy to the nowadays well-established growth envelopes for function spaces, including $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. We have no final results, but some (maybe bold) conjectures which may serve (at the best) as a sound basis for future research.

2. Preliminaries

2.1. Logarithmic Sobolev inequalities. We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be Euclidean n-space, $\mathbb{R} = \mathbb{R}^1$. Let Ω be a domain in $\mathbb{R}^n$. Then $L_p(\Omega)$ with $1 \le p < \infty$ is the usual Banach space of all complex-valued Lebesgue-measurable functions such that

$$\|f \,|L_p(\Omega)\| = \Big(\int_\Omega |f(x)|^p \, \mathrm{d}x \Big)^{1/p}$$

is finite. We need $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{Q}^n$ where $\mathbb{Q}^n$ is the unit cube (1.2). Let $W^1_p(\mathbb{R}^n)$ be the above Sobolev space normed by (1.1). Then

$$\|f \,|L_r(\mathbb{R}^n)\| \le C \, \|f \,|W^1_p(\mathbb{R}^n)\| \quad \text{if} \quad 0 < \frac{1}{p} - \frac{1}{n} \le \frac{1}{r} \le \frac{1}{p},$$

are the classical Sobolev embeddings. If p is fixed and $n \to \infty$ then the admitted r tend to p. Asking for non-trivial target spaces which apply to $W^1_p(\mathbb{R}^n)$ or $W^1_p(\mathbb{Q}^n)$ for all dimensions $n \in \mathbb{N}$ one may think about Zygmund spaces of type $L_p(\log L)_a$, $a \ge 0$, as in (1.5), (1.6) or, more general Lorentz-Zygmund spaces $L_{pq}(\log L)_a$. As indicated in Introduction, such embeddings will be called *tractable* if the norm of the corresponding embedding operators can be estimated from above by $2^{\varrho n}$ where $\varrho \in \mathbb{R}$ is independent of $n \in \mathbb{N}$. The first decisive step in this direction was done in 1976 by L. Gross in his seminal paper [11]. Let ν be the Gauss measure in $\mathbb{R}^n$,

$$\nu(\mathrm{d}x) = (2\pi)^{-n/2}\, e^{-|x|^2/2}\, \mathrm{d}x, \qquad \text{hence} \quad \nu(\mathbb{R}^n) = 1.$$

Then for any $n \in \mathbb{N}$,

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)|\, \nu(\mathrm{d}x) \\ &\le \int_{\mathbb{R}^n} |\nabla f(x)|^2\, \nu(\mathrm{d}x) + \int_{\mathbb{R}^n} |f(x)|^2\, \nu(\mathrm{d}x) \ln\Big(\int_{\mathbb{R}^n} |f(x)|^2\, \nu(\mathrm{d}x)\Big)^{1/2} \end{aligned} \tag{2.1}$$

is one of the main results in [11]. Here $\ln$ is the natural logarithm to base e. Otherwise we take log always to base 2. Since $\nu(\mathbb{R}^n) = 1$ one can reformulate (2.1) as

$$\begin{aligned} \int_{\mathbb{R}^n} |f(x)|^2 \big(1 + \big|\ln |f(x)|\big|\big)\, \nu(\mathrm{d}x) \le \int_{\mathbb{R}^n} |\nabla f(x)|^2\, \nu(\mathrm{d}x) \\ + \left(1 + \ln\Big(\int_{\mathbb{R}^n} |f(x)|^2\, \nu(\mathrm{d}x)\Big)^{1/2}\right) \int_{\mathbb{R}^n} |f(x)|^2\, \nu(\mathrm{d}x) + e^{-1}. \end{aligned} \tag{2.2}$$

This is less elegant but avoids cancellations on the left-hand side and is nearer to assertions of type (1.5): If (R,μ) is a measure space with $\mu(R) = 1$ and $0 < p < \infty$, $a \in \mathbb{R}$, then

$$\int_R |f(x)|^p \Big(1 + \big|\log |f(x)|\big|\Big)^{ap} \mu(\mathrm{d}x) < \infty \tag{2.3}$$

if and only if

$$\Big(\int_0^1 (1 + |\log t|)^{ap}\, f^*(t)^p\, \mathrm{d}t \Big)^{1/p} < \infty, \tag{2.4}$$

[3, p. 252] (quoted also in [10, p. 66]). This makes clear that the left-hand sides of (1.5), (2.2) are related, but it remains to control the respective equivalence constants, what might be not so obvious. We do not comment on the history and the far-reaching applications of these so-called *logarithmic Sobolev inequalities*. One may consult [7, Chapter 2], the relevant parts of [1], the most recent survey [22], dealing with a large variety of different aspects of inequalities of the above type involving logarithmic terms, and [5]. But we formulate a few assertions which illustrate the situation, now in terms of the Lebesgue measure $\mathrm{d}x$ instead of the above Gauss measure. Recall that $\ln$ is the natural logarithm to base e. According to [12, p. 60] one has

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)|\, \mathrm{d}x \le \int_{\mathbb{R}^n} |\nabla f(x)|^2\, \mathrm{d}x - n\Big(\frac{1}{2} + \frac{1}{4} \ln(2\pi)\Big) \tag{2.5}$$

if $\|f\,|L_2(\mathbb{R}^n)\| = 1$, and, more general for $1 < p < \infty$,

$$\int_{\mathbb{R}^n} |f(x)|^p \ln |f(x)|\, \mathrm{d}x \le \int_{\mathbb{R}^n} |\nabla f(x)|^p\, \mathrm{d}x - \gamma_{n,p} \tag{2.6}$$

if $\|f\,|L_p(\mathbb{R}^n)\| = 1$, where $\gamma_{n,p}$ is explicitly known in terms of the Γ-function. If f has not L_p-norm 1, then one must replace $f(x)$ in (2.5), (2.6) by $f(x)\,\|f\,|L_p(\mathbb{R}^n)\|^{-1}$. The outcome is similar as in (2.1). But in contrast to the Gauss measure the Lebesgue measure of $\mathbb{R}^n$

is infinite. Then it is at least questionable whether a counterpart of (2.2) can be derived from (2.6) and the just indicated modification. Just on the contrary, this observation suggests to ask for corresponding logarithmic inequalities for spaces on bounded domains in $\mathbb{R}^n$. This is an additional motivation for the inequality (1.5) based on (1.4) with c independent of n. We refer again to [21, 22] for further assertions of this type in the context of isoperimetric inequalities and W^1_p-spaces. Similar results have been obtained in [19, 20]. It should be mentioned that my interest in problems of this type originated from a talk by M. Krbec at the conference *Function Spaces IX*, Kraków, July 2009, about this topic. Of course one can restrict (2.5), (2.6) to $f \in \widetilde{W}^1_p(\Omega)$ where $\widetilde{W}^1_p(\Omega)$ is the obvious counterpart of (1.4) with the bounded domain Ω in place of $\mathbb{Q}^n$. Then one can step from (2.6) to a counterpart of (2.2). But this does not result immediately in the sharp assertion (1.5) with (1.4). In [19, 20] one finds several inequalities of type (1.5) but with the Luxemburg norm of (2.3) on the left-hand side and some additional restrictions for the parameters involved.

2.2. Zygmund spaces. First we describe some basic assertions about Zygmund spaces defined in terms of rearrangements. The standard references are [3] and [8]. We rely mainly on [8], using some notation introduced there. Let (Γ, μ) be a set Γ furnished with a non-atomic measure μ such that $\mu(\Gamma) = 1$. Let $M_0(\Gamma, \mu)$ be the collection of all complex-valued μ-measurable μ-a.e. finite functions. The distribution function of $f \in M_0(\Gamma, \mu)$ is the map $\mu_f : [0, \infty) \to [0, \infty)$ given by

$$\mu_f(\lambda) = \mu\big(\{x \in \Gamma : |f(x)| > \lambda\}\big), \qquad \lambda \geq 0.$$

The (non-increasing) rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty)$ given by

$$f^*(t) = \inf\{\lambda \in [0, \infty) : \mu_f(\lambda) \leq t\}, \qquad t \geq 0.$$

For $f \in M_0(\Gamma, \mu)$ and $g \in M_0(\Gamma, \mu)$ and a suitable equimeasurable function $\widetilde{g}$ of g one has

$$\int_\Gamma \big|(fg)(\gamma)\big| \, \mu(\mathrm{d}\gamma) \leq \int_\Gamma \big|(f\widetilde{g})(\gamma)\big| \, \mu(\mathrm{d}\gamma) = \int_0^1 f^*(t) g^*(t) \, \mathrm{d}t, \tag{2.7}$$

[8, pp. 64, 77/78].

Definition 2.1. Let $0 < p < \infty$ and $a \in \mathbb{R}$. The *Zygmund space* $L_p(\log L)_a(\Gamma, \mu)$ is the collection of all $f \in M_0(\Gamma, \mu)$ such that

$$\|f \,|L_p(\log L)_a(\Gamma, \mu)\| = \Big(\int_0^1 \big(1 + |\log t|\big)^{ap} f^*(t)^p \, \mathrm{d}t\Big)^{1/p} < \infty. \tag{2.8}$$

Remark 2.2. Definitions of this type are well known since a long time. It is a special case of [8, pp. 95, 96]. If $1 \leq p < \infty$ then $L_p(\log L)_a(\Gamma, \mu)$ is a Banach space although (2.8) is only an equivalent quasi-norm. In particular it makes sense to ask for the dual space $L_p(\log L)_a(\Gamma, \mu)'$ of $L_p(\log L)_a(\Gamma, \mu)$ in the usual interpretation.

Proposition 2.3. *Let* (Γ, μ) *be as above,* $1 < p < \infty$, $a \in \mathbb{R}$. *Then*

$$L_p(\log L)_a(\Gamma, \mu)' = L_{p'}(\log L)_{-a}(\Gamma, \mu), \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

with equivalent (quasi-)norms, where the equivalence constants are independent of (Γ, μ), *but may depend on* p *and* a.

REMARK 2.4. This assertion follows from [8, Theorems 3.3.4, 3.4.41, Corollary 3.4.44, pp. 85, 114, 117], where the independence of the equivalent constants follows from Luxemburg's isometric reduction to a model case, where $\Gamma = I$, unit interval on $\mathbb{R}$, and the Lebesgue measure. As far as duality is concerned one may also consult [10, p. 68] with a reference to [2, Theorem 8.4, p. 30].

We need some extrapolation assertions. Let now $\Gamma = \Omega$ be a domain (= open set) in $\mathbb{R}^n$ furnished with the Lebesgue measure $\mu = \mu_L$ and $\mu_L(\Omega) = |\Omega| = 1$. In what follows we rely on [10, Section 2.6.2, pp. 69–75], based on [9], extending [29]. Let

$$\frac{1}{p^\sigma} = \frac{1}{p} + \frac{\sigma}{n} > 0, \qquad 0 < p < \infty, \quad \sigma \in \mathbb{R},$$

with the specifications

$$\frac{1}{p^{\sigma_j}} = \frac{1}{p} + \frac{2^{-j}}{n} \quad \text{and} \quad \frac{1}{p^{\lambda_j}} = \frac{1}{p} - \frac{2^{-j}}{n} > 0, \tag{2.9}$$

where $J \le j \in \mathbb{N}$, $J \in \mathbb{N}$, $2^J > p/n$. We simplify $L_p(\Omega, \mu_L)$ by $L_p(\Omega)$ and

$$L_p(\log L)_a(\Omega, \mu_L) \quad \text{by} \quad L_p(\log L)_a(\Omega).$$

PROPOSITION 2.5.
(i) *Let* $0 < p < \infty$ *and* $a < 0$. *Then* $L_p(\log L)_a(\Omega)$ *is the set of all complex-valued Lebesgue measurable functions* f *such that*

$$\sum_{j=1}^\infty 2^{jap} \| f \,|L_{p^{\sigma_j}}(\Omega)\|^p < \infty. \tag{2.10}$$

Furthermore,

$$\Big(\sum_{j=1}^\infty 2^{jap} \| f \,|L_{p^{\sigma_j}}(\Omega)\|^p \Big)^{1/p} \sim n^{-a} \Big(\int_0^1 \big(1 + |\log t|\big)^{ap} f^*(t)^p \, \mathrm{d}t \Big)^{1/p} \tag{2.11}$$

with equivalence constants which are independent of the dimension $n \in \mathbb{N}$ (*but may depend on* p, a).

(ii) *Let* $1 < p < \infty$, $a > 0$ *and* $2^J > p$ *where* $J \in \mathbb{N}$. *Then* $L_p(\log L)_a(\Omega)$ *is the set of all complex-valued Lebesgue measurable functions* g *which can be represented as*

$$g = \sum_{j=J}^\infty g_j, \qquad g_j \in L_{p^{\lambda_j}}(\Omega), \tag{2.12}$$

such that

$$\|\{g_j\}\|_{p,a} = \Big(\sum_{j=J}^\infty 2^{jap} \| g_j \,|L_{p^{\lambda_j}}(\Omega)\|^p \Big)^{1/p} < \infty. \tag{2.13}$$

Furthermore,

$$\inf \|\{g_j\}\|_{p,a} \sim n^{-a} \Big(\int_0^1 \big(1 + |\log t|\big)^{ap} g^*(t)^p \, \mathrm{d}t \Big)^{1/p} \tag{2.14}$$

where the infimum is taken over all representations (2.12), (2.13), *with equivalence constants which are independent of the dimension* $n \in \mathbb{N}$ (*but may depend on* p, a).

Proof. Step 1. The equivalence (2.11) follows from the explicit calculations in [10, p. 70/71, formulas (9), (12), Remark 1].

Step 2. For given p with $1 < p < \infty$ one can replace the summation over $j \in \mathbb{N}$ in (2.10) by $j \geq J$. Then one has $1 < p^{\lambda_j} < \infty$ for all $n \in \mathbb{N}$. Let

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{p^{\lambda_j}} + \frac{1}{q^{\sigma_j}} = 1, \qquad j \geq J.$$

Then part (ii) follows from the duality arguments in [10, pp. 72/73] and the above Proposition 2.3. ■

Remark 2.6. Later on we need part (ii) of the above proposition, proved by duality. But there are more direct proofs at least for the representation (2.12), (2.13) which apply to all $0 < p < \infty$. This can also be done in the context of Lorentz-Zygmund spaces $L_{pq}(\log L)_a(\Omega)$ where $0 < p, q \leq \infty$, $a > 0$ (and also $a < 0$ generalising (2.10), (2.11)). We refer to [18, 6] which are the decisive papers for problems of this type and might be also of some use for further developments in the context of this paper. But one has to care how the related equivalence constants depend on the dimensions $n \in \mathbb{N}$. Otherwise the standard references for Lorentz-Zygmund spaces are [3, 2, 8]. This may also pave the way to extend what follows to tractable embeddings of type

$$\mathbf{B}^s_{pq}(\mathbb{Q}^n) \hookrightarrow L_{pu}(\log L)_a(I), \qquad a > 0,$$

where $\mathbf{B}^s_{pq}(\mathbb{Q}^n)$ with $0 < p, q \leq \infty$, $s > 0$, are Besov spaces of measurable functions, treated as subspaces of $L_p(\mathbb{Q}^n)$. Details may be found in [31, Chapter 9] (based on numerous references given there), [17] and more recently [16, 25, 26, 33].

2.3. Besov spaces. We recall a few definitions and basic assertions for the classical Besov spaces B^s_{pq}. We will be especially interested in $B^s_{pp}(\mathbb{R}^n)$ and $B^s_{pp}(\mathbb{Q}^n)$ with $s > 0$, $1 < p < \infty$. Here $\mathbb{Q}^n$ is again the unit cube (1.2). Let

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \qquad (\Delta^{M+1}_h f)(x) = \Delta^1_h(\Delta^M_h f)(x),$$

where $x \in \mathbb{R}^n$, $h \in \mathbb{R}^n$, $M \in \mathbb{N}$, be the iterated differences in $\mathbb{R}^n$. Let $1 < p, q < \infty$, $0 < s < M \in \mathbb{N}$ and

$$\|f \,|B^s_{pq}(\mathbb{R}^n)_M\| = \|f\,|L_p(\mathbb{R}^n)\| + \left(\int_{|h|\leq 1} |h|^{-sq} \big\|\Delta^M_h f\,|L_p(\mathbb{R}^n)\big\|^q \frac{\mathrm{d}h}{|h|^n} \right)^{1/q}. \tag{2.15}$$

The classical Besov space $B^s_{pq}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ such that for fixed admitted M the norm (2.15) is finite. It is well known that these norms are equivalent to each other. Let $B^s_{pq}(\mathbb{Q}^n)$ be the restriction of $B^s_{pq}(\mathbb{R}^n)$ to $\mathbb{Q}^n$ and let

$$\widetilde{B}^s_{pq}(\mathbb{Q}^n) = \big\{ f \in B^s_{pq}(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{\mathbb{Q}^n} \big\}. \tag{2.16}$$

These spaces can be considered likewise as closed subspaces of $B^s_{pq}(\mathbb{R}^n)$ or as subspaces of $L_p(\mathbb{Q}^n)$ in the context of distributions on $\mathbb{Q}^n$. If $1 < p = q < \infty$, $0 < s < 1$, then

$B^s_{pp}(\mathbb{Q}^n)$ can be equivalently normed by

$$\|f\,|B^s_{pp}(\mathbb{Q}^n)\|_\lambda = \lambda^{-1}\Big(\int_{\mathbb{Q}^n} |f(x)|^p\,\mathrm{d}x\Big)^{1/p} + \bigg(\int_{\mathbb{Q}^n\times\mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{(\lambda|x-y|)^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/p} \tag{2.17}$$

where $0<\lambda<\infty$. Recall that

$$B^s_{pp}(\mathbb{Q}^n) = \widetilde{B}^s_{pp}(\mathbb{Q}^n) \quad \text{if} \quad 1<p<\infty, \quad 0<s<1/p. \tag{2.18}$$

If $\lambda = 1$ then we put $\|f\,|B^s_{pp}(\mathbb{Q}^n)\| = \|f\,|B^s_{pp}(\mathbb{Q}^n)\|_1$. We assume that the reader is familiar with these basic assertions about classical Besov spaces. We refer to [28, Theorem 2.5.1, p. 189; (8), Remark 2, pp. 323/324] as far as (2.15), (2.17) is concerned and in connection with (2.16), (2.18) to [31, Definition 1.95, p. 59, Section 1.11.6, p. 66].

3. Main results

3.1. Tractable embeddings, I. It is the main aim of this section to prove (1.6). We need some preparation. Let $\mathbb{Q}^n$ be the unit cube (1.2) and let $B^s_{pp}(\mathbb{Q}^n)$ with $1<p<\infty$, $0<s<1/p$, be the above Besov spaces, normed by (2.17). We rely on the Haar wavelet basis of $B^s_{pp}(\mathbb{Q}^n)$ according to [33, Section 2.3.3]. We give a brief description. Let

$$h_M(t) = \begin{cases} 1 & \text{if } 0\le t<1/2,\\ -1 & \text{if } 1/2\le t<1,\\ 0 & \text{if } t\notin[0,1), \end{cases} \tag{3.1}$$

$$h_F(t) = |h_M(t)|, \qquad t\in\mathbb{R}. \tag{3.2}$$

Let $n\in\mathbb{N}$,

$$G = (G_1,\ldots,G_n) \in G^0 = \{F,M\}^n$$

which means that G_r is either F or M. Let

$$G = (G_1,\ldots,G_n) \in G^j = \{F,M\}^{n*}, \qquad j\in\mathbb{N},$$

which means that G_r is either F or M where $*$ indicates that at least one of the components of G must be an M. Hence G^0 has 2^n elements, whereas G^j with $j\in\mathbb{N}$ has 2^n-1 elements. Let $\mathbb{Z}^n$ be the usual lattice in $\mathbb{R}^n$ consisting of all $m=(m_1,\ldots,m_n)\in\mathbb{R}^n$ with integer-valued components m_r. Let

$$\mathbb{P}^n_j = \big\{m\in\mathbb{Z}^n : 0\le m_r\le 2^j-1;\ r=1,\ldots,n\big\}, \qquad j\in\mathbb{N}_0.$$

Then

$$\big\{h^G_{jm} : j\in\mathbb{N}_0,\ G\in G^j,\ m\in\mathbb{P}^n_j\big\} \tag{3.3}$$

with

$$h^G_{jm}(x) = \prod_{r=1}^n h_{G_r}\big(2^j x_r - m_r\big) \tag{3.4}$$

is the well-known (L_∞-normalised) orthogonal Haar wavelet basis of $L_2(\mathbb{Q}^n)$. Let $\mathbb{C}$ be the complex plane. Let $b_{pp}(\mathbb{Q}^n)$ with $1<p<\infty$ be the spaces consisting of all sequences μ,

$$\mu = \big\{\mu^G_{jm}\in\mathbb{C} : j\in\mathbb{N}_0,\ G\in G^j,\ m\in\mathbb{P}^n_j\big\} \tag{3.5}$$

with

$$\|\mu\,|b_{pp}(\mathbb{Q}^n)\| = \Big(\sum_{j=0}^{\infty}\sum_{G\in G^j}\sum_{m\in\mathbb{P}^n_j}|\mu^G_{jm}|^p\Big)^{1/p} < \infty. \tag{3.6}$$

According to [33, Section 2.3.3, Theorem 2.26] the system $\{h^G_{jm}\}$ is a common unconditional basis in all spaces $B^s_{pp}(\mathbb{Q}^n)$, $1 < p < \infty$, $0 < s < 1/p$. In particular, any $f \in B^s_{pp}(\mathbb{Q}^n)$ can be uniquely represented as

$$f = \sum_{j=0}^{\infty}\sum_{G\in G^j}\sum_{m\in\mathbb{P}^n_j}\mu^G_{jm}\,2^{-j(s-n/p)}h^G_{jm}, \tag{3.7}$$

$$\mu^G_{jm} = \mu^G_{jm}(f) = 2^{j(s-n/p+n)}\int_{\mathbb{Q}^n} f(x)\,h^G_{jm}(x)\,\mathrm{d}x, \tag{3.8}$$

and

$$\|f\,|B^s_{pp}(\mathbb{Q}^n)\| \sim \|\mu\,|b_{pp}(\mathbb{Q}^n)\| \tag{3.9}$$

(equivalent norms). First we clarify how the sequence norms in (3.9) are related to the norms introduced in (2.17).

Proposition 3.1. *For $1 < p < \infty$, $0 < s < 1/p$, there is a constant $\lambda = \lambda(p,s)$, $0 < \lambda \le 1$, such that for all $n \in \mathbb{N}$ and all $f \in B^s_{pp}(\mathbb{Q}^n)$,*

$$\|\mu\,|b_{pp}(\mathbb{Q}^n)\| \le \|f\,|B^s_{pp}(\mathbb{Q}^n)\|_\lambda, \tag{3.10}$$

with $\mu = \{\mu^G_{jm}(f)\}$ as in (3.5), (3.8).

Proof. The starting term in (3.7) is the characteristic function of $\mathbb{Q}^n$. This gives the first term on the right-hand side of (2.17) (even with 1 in place of λ^{-1}). For all other terms we have

$$\operatorname{supp} h^G_{jm} = Q_{jm} = 2^{-j}m + 2^{-j}\mathbb{Q}^n, \qquad \int_{\mathbb{Q}^n} h^G_{jm}(x)\,\mathrm{d}x = 0.$$

It follows that

$$\begin{aligned}\Big|\int_{\mathbb{Q}^n} f(x)\,h^G_{jm}(x)\,\mathrm{d}x\Big| &= 2^{jn}\Big|\int_{Q_{jm}\times Q_{jm}}\big(f(x)-f(y)\big)\,h^G_{jm}(x)\,\mathrm{d}x\,\mathrm{d}y\Big| \\ &\le 2^{-jn+2jn/p}\Big(\int_{Q_{jm}\times Q_{jm}}\big|f(x)-f(y)\big|^p\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}.\end{aligned}$$

By (3.8) one has

$$|\mu^G_{jm}| \le 2^{js+jn/p}\Big(\int_{Q_{jm}\times Q_{jm}}\big|f(x)-f(y)\big|^p\,\mathrm{d}x\,\mathrm{d}y\Big)^{1/p}. \tag{3.11}$$

Let $\mathbb{D}^n = \big\{z = (x,x) : x \in \mathbb{R}^n\big\}$ be the n-dimensional diagonal hyper-plane in $\mathbb{R}^{2n}$. We use the Whitney decomposition of

$$\mathbb{R}^{2n}\setminus\mathbb{D}^n = \big\{(x,y) : x\in\mathbb{R}^n,\ y\in\mathbb{R}^n\big\}\setminus\big\{(x,x) : x\in\mathbb{R}^n\big\}$$

into dyadic $2n$-dimensional cubes as constructed in detail in [27, pp. 167–169]. By restriction to $\mathbb{Q}^{2n}$ one obtains a decomposition of $\mathbb{Q}^{2n}\setminus\big\{(x,x) : x\in\mathbb{Q}^n\big\}$ into tubular neighbourhoods T^l, $l \in \mathbb{N}$, of $\mathbb{Q}^{2n} \setminus \mathbb{D}^n$, consisting of $2n$-dimensional dyadic cubes $\widetilde{Q}_{l,\widetilde{m}}$, $\widetilde{Q}_{l,\widetilde{m}} \subset \mathbb{Q}^{2n}$, of side-length 2^{-l} and some $\widetilde{m} \in \mathbb{Z}^{2n}$. It follows from the construction that

$$c_1\, n^{-1/2} 2^{-l} \le |x-y| \le c_2\, n^{1/2}\, 2^{-l} \qquad \text{if} \quad (x,y) \in T^l,$$

where $0 < c_1 < c_2 < \infty$ are independent of $n \in \mathbb{N}$ and $l \in \mathbb{N}$. We transfer this decomposition by translation and dilation to $Q_{jm} \times Q_{jm}$. The corresponding $2n$-dimensional tubular neighbourhood T^l_{jm} of $(Q_{jm} \times Q_{jm}) \setminus \mathbb{D}^n$ consists again of cubes $\widetilde{Q}_{l,\widetilde{m}}$ of side-length 2^{-l} and some $\widetilde{m} \in \mathbb{Z}^{2n}$. Now $l = j+k$ with $k \in \mathbb{N}$. Then

$$\begin{aligned} &2^{j(s+n/p)} \Big(\int_{T^{j+k}_{jm}} |f(x) - f(y)|^p \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \\ &\qquad \le 2^{cn}\, 2^{-k(s+n/p)} \Big(\int_{T^{j+k}_{jm}} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \end{aligned} \tag{3.12}$$

for some $c > 0$ independent of $n \in \mathbb{N}$. We do not care about constants 2^{cn} with $c > 0$ in what follows. Then we may assume that a fixed cube $\widetilde{Q}_{l,\widetilde{m}} \subset T^l$, $l \in \mathbb{N}$, is involved in the above construction only for cubes $Q_{jm} \times Q_{jm}$ with $l > j \in \mathbb{N}_0$ and at most one $m \in \mathbb{P}^n_j$. Now it follows from (3.6), (3.11), (3.12) that

$$\|\mu\, |b_{pp}(\mathbb{Q}^n)\|^p \le \int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{(\lambda |x-y|)^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y$$

with some λ, $0 < \lambda \le 1$, independent of $n \in \mathbb{N}$ compensating factors 2^{cn}, say, $2^c \lambda^{1/p} = 1$. This proves (3.10). ∎

REMARK 3.2. If $\int_{\mathbb{Q}^n} f(x)\, \mathrm{d}x = 0$ then it follows from the above arguments that

$$\|\mu\, |b_{pp}(\mathbb{Q}^n)\| \le \Big(\int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{(\lambda|x-y|)^{n+sp}} \, \mathrm{d}x \, \mathrm{d}y \Big)^{1/p} \tag{3.13}$$

where $1 < p < \infty$, $0 < s < 1/p$.

THEOREM 3.3. *Let $1 < p < \infty$, $0 < s < 1/p$. There is a constant $\lambda = \lambda(p,s)$, $0 < \lambda \le 1$, such that for all $n \in \mathbb{N}$ and all $f \in B^s_{pp}(\mathbb{Q}^n)$ normed by* (2.17)

$$\Big(\int_0^1 \big(1 + |\log t|\big)^{sp} f^*(t)^p \, \mathrm{d}t \Big)^{1/p} \le \|f\, |B^s_{pp}(\mathbb{Q}^n)\|_\lambda. \tag{3.14}$$

Proof. Let f be given by (3.7), (3.8). The starting term in this expansion is

$$\chi_{\mathbb{Q}^n} \int_{\mathbb{Q}^n} f(x)\, \mathrm{d}x, \quad \text{where } \chi_{\mathbb{Q}^n} \text{ is the characteristic function of } \mathbb{Q}^n.$$

This contributes to $\lambda^{-1} \|f\, |L_p(\mathbb{Q}^n)\|$ in (3.14), (2.17) and also to the replacement of f by $f - \int_{\mathbb{Q}^n} f(x)\, \mathrm{d}x$. Hence we may assume that $\int_{\mathbb{Q}^n} f(x)\, \mathrm{d}x = 0$. Let

$$f = \sum_{j=J}^\infty f_{j-J}, \qquad f_l = \sum_{G \in G^l} \sum_{m \in \mathbb{P}^n_l} \mu^G_{lm}\, 2^{-l(s-n/p)} h^G_{lm}$$

with $2^J > p$ as in Proposition 2.5(ii). This index-shifting is unimportant. Again we do not care in estimates from above and from below about constants 2^{cn} where $c \in \mathbb{R}$ is independent of $n \in \mathbb{N}$ (but may depend on p, s). With p^{λ_j} as in (2.9) one has for $j \ge J$,

$$\big\| f_{j-J} \,|L_{p^{\lambda_j}}(\mathbb{Q}^n)\big\|^{p^{\lambda_j}} \le 2^{cn} \sum_{G \in G^j} \sum_{m \in \mathbb{P}^n_{j-J}} |\mu^G_{j-J,m}|^{p^{\lambda_j}}\, 2^{-j(s-n/p)p^{\lambda_j}}\, 2^{-jn}.$$

Using $s - \frac{n}{p} + \frac{n}{p^{\lambda_j}} = s - 2^{-j}$ and $p < p^{\lambda_j}$ one obtains that

$$2^{js} \big\| f_{j-J}\,|L_{p^{\lambda_j}}(\mathbb{Q}^n)\big\| \le 2^{cn} \Big(\sum_{G,m} |\mu_{j-J,m}|^{p^{\lambda_j}} \Big)^{1/p^{\lambda_j}} \le 2^{cn} \Big(\sum_{G,m} |\mu_{j-J,m}|^p \Big)^{1/p}.$$

By (2.13) and (3.6) one has

$$\big\| \{ f_{j-J} \} \big\|_{p,s} \le 2^{c'n} \big\| \mu | b_{pp}(\mathbb{Q}^n) \big\|.$$

Now (3.14) follows from (2.14) and (3.13). ∎

Remark 3.4. There might be a temptation to rescale the Euclidean distance $|x|$ in $\mathbb{R}^n$ by $\lambda |x|$ with some λ, $0 < \lambda \le 1$ (independent of $n \in \mathbb{N}$) or to rescale accordingly the Lebesgue measure $\mu_L(\Omega)$ by $\mu^\lambda_L(\Omega) = \lambda^n |\Omega|$ to get rid of λ in (2.17) and (3.14). But we have the impression that it is more natural to choose $\lambda = 1$ and to accept factors $2^{\varrho n}$ in the corresponding estimates. In other words, one can reformulate (3.14) by

$$\begin{aligned} &\Big(\int_0^1 \big(1 + |\log t|\big)^{sp}\, f^*(t)^p \,\mathrm{d}t \Big)^{1/p} \\ &\le 2^{\varrho n} \Big(\int_{\mathbb{Q}^n} |f(x)|^p \,\mathrm{d}x \Big)^{1/p} + 2^{\varrho n} \Big(\int_{\mathbb{Q}^n \times \mathbb{Q}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} \,\mathrm{d}x\,\mathrm{d}y \Big)^{1/p} \end{aligned} \tag{3.15}$$

for some $\varrho \in \mathbb{R}$ which is independent of $n \in \mathbb{N}$ (and of $f \in B^s_{pp}(\mathbb{Q}^n)$). This coincides with (1.6). Embeddings of this type will be called *tractable*. Both (3.10), (3.13) and (3.15) are counterparts of (1.5).

3.2. Tractable embeddings, II. We outline briefly how Theorem 3.3 can be extended from $1 < p < \infty$, $0 < s < 1/p$ to $1 < p < \infty$, $s > 0$. We deal with $\widetilde{B}^s_{pp}(\mathbb{Q}^n)$ according to (2.16) as a subspace of $B^s_{pp}(\mathbb{R}^n)$, normed by (2.15) for some fixed M with $0 < s < M \in \mathbb{N}$. If $s \ge 1/p$ then (2.18) is no longer valid. Instead of Haar wavelet bases we rely now on sufficiently smooth Daubechies wavelet bases. We give a brief description following [32, Sections 1.2.1, 1.2.2, pp. 13–17] where one finds further details and references to the literature. As usual, $C^u(\mathbb{R})$ with $u \in \mathbb{N}$ collects all continuous functions on $\mathbb{R}$ having continuous bounded derivatives up to order u (inclusively). Let

$$\psi_F,\ \psi_M \in C^u(\mathbb{R}) \quad \text{with} \quad \int_{\mathbb{R}} \psi_M(t)\, t^v \,\mathrm{d}t = 0, \quad v \in \mathbb{N}_0,\ v < u, \tag{3.16}$$

be real compactly supported Daubechies wavelets with L_2-norms 1. This is the smooth substitute of h_F, h_M in (3.1), (3.2). Then

$$\Big\{ \Psi^G_{jm} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n \Big\}$$

with

$$\Psi^G_{jm}(x) = \prod_{r=1}^n \psi_{G_r}\big(2^j x_r - m_r\big), \qquad G \in G^j,\ m \in \mathbb{Z}^n,\ j \in \mathbb{N}_0, \tag{3.17}$$

is the counterpart of (3.3), (3.4). It is a (L_∞-normalised) orthogonal basis in $L_2(\mathbb{R}^n)$. The $\mathbb{R}^n$-counterpart of the sequence space $b_{pp}(\mathbb{Q}^n)$ with (3.5), (3.6) is the space $b_{pp}(\mathbb{R}^n)$ consisting of all sequences

$$\mu = \left\{\mu^G_{jm} \in \mathbb{C} : j \in \mathbb{N}_0,\ G \in G^j,\ m \in \mathbb{Z}^n\right\}$$

with

$$\|\mu \,|b_{pp}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^\infty \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} |\mu^G_{jm}|^p\Big)^{1/p} < \infty.$$

According to [32, Theorem 1.20, p. 15] the system $\{\Psi^G_{jm}\}$ is a common unconditional basis in all spaces $B^s_{pp}(\mathbb{R}^n)$, $1 < p < \infty$, $0 < s < u$. In particular, any $f \in B^s_{pp}(\mathbb{R}^n)$ can be uniquely represented as

$$f = \sum_{j=0}^\infty \sum_{G\in G^j} \sum_{m\in\mathbb{Z}^n} \mu^G_{jm}\, 2^{-j(s-n/p)}\, \Psi^G_{jm}, \tag{3.18}$$

$$\mu^G_{jm} = \mu^G_{jm}(f) = 2^{j(s-n/p+n)} \int_{\mathbb{R}^n} f(x)\, \Psi^G_{jm}(x)\, \mathrm{d}x \tag{3.19}$$

and

$$\|f \,|B^s_{pp}(\mathbb{R}^n)\| \sim \|\mu \,|b_{pp}(\mathbb{R}^n)\|.$$

The counterpart of Proposition 3.1 relies on the following version of Whitney's approximation theorem.

Proposition 3.5. *Let $n \in \mathbb{N}$, $M \in \mathbb{N}$, $0 < \sigma \le 1$ and $0 < p < \infty$. Then there is a positive constant $\lambda = \lambda(M, \sigma, p)$ such that for any $f \in L^{\mathrm{loc}}_p(\mathbb{R}^n)$ and any cube Q in $\mathbb{R}^n$ with side-length $a > 0$ there is a polynomial $P(x)$ of degree less than M satisfying*

$$\int_Q |f(y) - P(y)|^p\, \mathrm{d}y \le 2^{\lambda n}\, a^{-n} \int_{|h|<\sigma a} \int_Q \big|\Delta^M_h f(y)\big|^p\, \mathrm{d}y\, \mathrm{d}h. \tag{3.20}$$

Remark 3.6. A detailed proof may be found in [17, Appendix, pp. 87–93] with a constant $C = C(M, \sigma, p, n)$ in place of $2^{\lambda n}$ with $\lambda = \lambda(M, \sigma, p)$. The authors did not care about the dependence on $n \in \mathbb{N}$. But the proof shows that one can choose $C = 2^{\lambda n}$. Assertions of type (3.20) have a long history which may be found in [17, pp. 87–90]. Special credit is given to the Russian school, in particular to some rare publications by Yu. Brudnyi. Fortunately enough there is a recent survey [4] by Yu. Brudnyi himself about this topic which may be consulted for further information.

Now we are in a similar position as in Section 3.1. Let $0 < s < M \in \mathbb{N}$. Then $\widetilde{B}^s_{pp}(\mathbb{Q}^n)_M$ means the space in (2.16) normed by (2.15).

Theorem 3.7. *Let $1 < p < \infty$, $0 < s < M \in \mathbb{N}$. Then there is a constant $\varrho = \varrho(s, p, M) > 0$ such that for all $n \in \mathbb{N}$ and all $f \in \widetilde{B}^s_{pp}(\mathbb{Q}^n)$,*

$$\Big(\int_0^1 (1 + |\log t|)^{sp} f^*(t)^p\, \mathrm{d}t\Big)^{1/p} \le 2^{\varrho n} \|f \,|B^s_{pp}(\mathbb{R}^n)_M\|. \tag{3.21}$$

Proof (*outline*). We are now in the same position as in the proof of Theorem 3.3. Let f be given by (3.18), (3.19) with $u = M$ in (3.16). In contrast to Haar bases there is now a moderate additional overlap. But this spoils the estimates only by additional factors of type 2^{cn}. In particular there are now 2^{cn} starting terms Ψ^j_{Gm} with $j=0$, $G=(F,\dots,F)$ having non-empty intersection with $\mathbb{Q}^n$. This results in the term $2^{cn}\|f\,|L_p(\mathbb{Q}^n)\|$. For the other wavelets one can apply the cancellation (3.16), (3.17) to the coefficients μ^G_{jm} in (3.19). Then it follows from Proposition 3.5 in the same way as in the proof of Proposition 3.1 that

$$\|\mu\,|b_{pp}(\mathbb{R}^n)\| \le 2^{cn}\|f\,|B^s_{pp}(\mathbb{R}^n)_M\|, \qquad f \in \widetilde{B}^s_{pp}(\mathbb{Q}^n).$$

Afterwards one can argue as in the proof of Theorem 3.3. ■

Remark 3.8. Recall that $\mathring{W}^1_2(\mathbb{Q}^n) = \widetilde{W}^1_2(\mathbb{Q}^n) = \widetilde{B}^1_{2,2}(\mathbb{Q}^n)$, where $\widetilde{W}^1_2(\mathbb{Q}^n)$ is normed by (1.3) and $\widetilde{B}^1_{2,2}(\mathbb{Q}^n)$ may be normed by (2.15), say, with $M=2$. It is remarkable that these different, but nevertheless distinguished, normings result in different tractable embeddings. With $p=2$ one has by (1.5),

$$\Big(\int_0^1 \big[(1+|\log t|)^w f^*(t)\big]^p\,\mathrm{d}t\Big)^{1/p} \le c\,\|f\,|W^1_p(\mathbb{R}^n)\|, \quad f \in \mathring{W}^1_p(\mathbb{Q}^n), \tag{3.22}$$

if $w \le 1/2$, and by (3.21),

$$\Big(\int_0^1 \big[(1+|\log t|)^a f^*(t)\big]^p\,\mathrm{d}t\Big)^{1/p} \le 2^{\varrho n}\big\|f\,|B^1_{pp}(\mathbb{R}^n)_2\big\|, \quad f \in \widetilde{B}^1_{pp}(\mathbb{Q}^n), \tag{3.23}$$

if $a \le 1$. We have c in (3.22) independent of $n \in \mathbb{N}$, but $2^{\varrho n}$ in (3.23) with $\varrho > 0$ independent of $n \in \mathbb{N}$. This comes from scaling. It cannot be the reason for the above remarkable difference. By the considerations in [21, 22], but also by respective counterparts in [20], one can expect that $w = 1/2$ in (3.22) is sharp for all $1<p<\infty$. The same applies to $a=1$ in (3.23) supported by the above considerations for all $1<p<\infty$.

It is well known that inequalities of type (3.21) can be reformulated in terms of *Hardy inequalities*. One may consult [30, Section 16] and the references given there. First we recall that the equivalence (2.3), (2.4) can be extended to $p=\infty$. For this purpose we complement Definition 2.1 as follows. Let $a<0$. Then $L_\infty(\log L)_a(\Gamma,\mu)$ is the collection of all $g \in M_0(\Gamma,\mu)$ such that

$$\|g\,|L_\infty(\log L)_a(\Gamma,\mu)\| = \sup_{0<t\le 1}\,(1+|\log t|)^a g^*(t) < \infty. \tag{3.24}$$

According to [10, p. 66] with a reference to [3, p. 252] a function g belongs to

$$L_\infty(\log L)_a(\Gamma,\mu) = L_{\exp,-a}(\Gamma,\mu)$$

if and only if there is a constant $\nu > 0$ such that

$$\int_\Gamma \exp\Big\{\big(\nu|g(x)|\big)^{-1/a}\Big\}\,\mu(\mathrm{d}x) < \infty.$$

We specify (Γ,μ) to $(\mathbb{Q}^n,\mu_L)$ omitting the Lebesgue measure μ_L in the notation.

Corollary 3.9. *Let p,s,M,ϱ be as in Theorem 3.7. Then*

$$\|fV\,|L_p(\mathbb{Q}^n)\| \le 2^{\varrho n}\,\|V\,|L_\infty(\log L)_{-s}(\mathbb{Q}^n)\| \cdot \|f\,|B^s_{pp}(\mathbb{R}^n)_M\| \tag{3.25}$$

for all $n \in \mathbb{N}$, all $f \in \widetilde{B}^s_{pp}(\mathbb{Q}^n)$ and all $V \in L_{\exp,s}(\mathbb{Q}^n)$.

Proof. By (2.7) and $(|f|^p)^* = f^{*p}$ one has

$$\int_{\mathbb{Q}^n} |f(x)V(x)|^p \,\mathrm{d}x \le \int_0^1 f^{*p}(t)\, V^{*p}(t)\,\mathrm{d}t.$$

Then (3.25) follows from (3.24) and (3.21). ∎

EXAMPLE 3.10. Let $1 < p < \infty$ and $0 < s < 1/p$. Then there is a constant $\lambda = \lambda(p,s)$, $0 < \lambda \le 1$, such that for all $n \in \mathbb{N}$ and all $f \in L_1(\mathbb{Q}^n)$,

$$\int_{\mathbb{Q}^n} \big|\log|x|\big|^{sp} \, \big|f(x) - f_{\mathbb{Q}^n}\big|^p \,\mathrm{d}x \le \int_{\mathbb{Q}^n\times\mathbb{Q}^n} \frac{|f(x)-f(y)|^p}{(\lambda|x-y|)^{n+sp}}\,\mathrm{d}x\,\mathrm{d}y \tag{3.26}$$

where $f_{\mathbb{Q}^n} = \int_{\mathbb{Q}^n} f(y)\,\mathrm{d}y$ is the mean value of f (admitting ∞ on the right-hand side or on both sides of (3.26)). This follows easily from Corollary 3.9, Theorem 3.3 and Remark 3.2.

4. Tractability envelopes

4.1. Envelopes. Growth and continuity envelopes measure the singularity behaviour of functions belonging to some function spaces in $\mathbb{R}^n$ or in domains Ω, $\Omega \subset \mathbb{R}^n$. This theory began some ten years ago, [13, 14, 30]. Nowadays there are numerous papers about this topic. We refer in particular to the recent book [15]. The above considerations suggest to complement this theory by *tractability envelopes*. First we recall very few notation and assertions about growth envelopes adapted to the above considerations.

We complement the Zygmund spaces as introduced in Definition 2.1 by Lorentz-Zygmund spaces where now $\Gamma = I = (0,1)$ is the unit interval and $\mu = \mu_L$ is the Lebesgue measure. Let

$$0 < r < \infty, \quad 0 < u \le \infty, \quad a \in \mathbb{R}.$$

Then $L_{ru}(\log L)_a(I)$ is the collection of all Lebesgue-measurable complex-valued functions f on I such that

$$\Big(\int_0^1 \big[t^{1/r}\big(1+|\log t|\big)^a f^*(t)\big]^u \frac{\mathrm{d}t}{t}\Big)^{1/u} < \infty \quad \text{if} \quad 0 < u < \infty$$

and

$$\sup_{t\in I} t^{1/r}\big(1+|\log t|\big)^a f^*(t) < \infty \quad \text{if} \quad u = \infty.$$

These are quasi-Banach spaces. The references are the same as in Remark 2.6. A short description and some discussions may be found in [30, Section 11.6, pp. 174–176]. The Zygmund spaces according to (2.8)

$$L_r(\log L)_a(I) = L_{rr}(\log L)_a(I) \quad \text{and the Lorentz spaces} \quad L_{ru}(I) = L_{ru}(\log L)_0(I)$$

are special cases. Let $0 < r < \infty$, $a \in \mathbb{R}$, $0 < u_0 < u_1 < \infty$. Then

$$\begin{aligned} \sup_{t\in I} t^{1/r}\big(1+|\log t|\big)^a f^*(t) &\le c_1 \Big(\int_0^1 \big[t^{1/r}\big(1+|\log t|\big)^a f^*(t)\big]^{u_1} \frac{\mathrm{d}t}{t}\Big)^{1/u_1} \\ &\le c_0 \Big(\int_0^1 \big[t^{1/r}\big(1+|\log t|\big)^a f^*(t)\big]^{u_0} \frac{\mathrm{d}t}{t}\Big)^{1/u_0} \end{aligned} \tag{4.1}$$

is a well-known monotonicity assertion of the Lorentz-Zygmund spaces, [30, p. 188] and the references given there.

Let $\mathbb{Q}^n$ be again the unit cube according to (1.2). Let $A^s_{pq}(\mathbb{Q}^n)$ with $A = B$ or $A = F$ be the well-known scales of spaces introduced by restriction of the corresponding spaces $A^s_{pq}(\mathbb{R}^n)$ to $\mathbb{Q}^n$ with

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > 0, \quad s - \frac{n}{p} = -\frac{n}{r} \quad \text{where} \quad 1 < r < \infty, \tag{4.2}$$

and let $\widetilde{A}^s_{pq}(\mathbb{Q}^n)$ be defined similarly as in (2.16). One may think about the above spaces $B^s_{pq}(\mathbb{Q}^n)$ and $\widetilde{B}^s_{pq}(\mathbb{Q}^n)$ as introduced in Section 2.3, or $W^1_p(\mathbb{Q}^n) = F^1_{p,2}(\mathbb{Q}^n)$ and $\widetilde{W}^1_p(\mathbb{Q}^n)$. Then

$$\mathcal{E}_G | A^s_{pq}(\mathbb{Q}^n)(t) = \sup\bigl\{ f^*(t) : \|f \,| A^s_{pq}(\mathbb{Q}^n)\| \le 1 \bigr\}, \qquad 0 < t \le 1, \tag{4.3}$$

is the *growth envelope function*. Similarly $\mathcal{E}_G | \widetilde{A}^s_{pq}(\mathbb{Q}^n)(t)$. Of interest is the qualitative behaviour for $t \downarrow 0$. This justifies to replace the growth envelope function by any other, say, decreasing continuous smooth function $\mathcal{E}_G A^s_{pq}(\mathbb{Q}^n)(t)$ which is equivalent near $t = 0$, hence

$$c_1 \mathcal{E}_G A^s_{pq}(\mathbb{Q}^n)(t) \le 1 + \mathcal{E}_G | A^s_{pq}(\mathbb{Q}^n)(t) \le c_2 \mathcal{E}_G A^s_{pq}(\mathbb{Q}^n)(t), \tag{4.4}$$

$0 < t \le 1$, for some $0 < c_1 < c_2 < \infty$. Recall that the monotonicity (4.1) remains valid for any real continuous monotonically increasing function $\psi(t)$ on the interval $[0,1]$ with $\psi(0) = 0$, $\psi(t) > 0$ if $0 < t \le 1$, in place of $t^{1/r}(1 + |\log t|)^a$, [30, Proposition 12.2, pp. 183–186]. One may choose in particular $\psi(t) = \mathcal{E}_G A^s_{pq}(\mathbb{Q}^n)(t)^{-1}$. Then the couple

$$\mathfrak{E}_G A^s_{pq}(\mathbb{Q}^n) = \bigl(\mathcal{E}_G A^s_{pq}(\mathbb{Q}^n)(\cdot), u \bigr)$$

is called the *growth envelope* for $A^s_{pq}(\mathbb{Q}^n)$, where u, $0 < u \le \infty$, is the infimum of all v, $0 < v \le \infty$, such that

$$\Bigl(\int_0^1 \bigl[\psi(t) f^*(t) \bigr]^v \frac{\mathrm{d}t}{t} \Bigr)^{1/v} \le c \, \|f \,| A^s_{pq}(\mathbb{Q}^n)\|$$

for some $c > 0$ (modification if $v = \infty$). This makes sense by the above comments. Detailed discussions may be found in [30] and in particular in [15]. Similarly for $\widetilde{A}^s_{pq}(\mathbb{Q}^n)$. With p, q, s, r as in (4.2) it comes out that

$$\mathfrak{E}_G B^s_{pq}(\mathbb{Q}^n) = \mathfrak{E}_G \widetilde{B}^s_{pq}(\mathbb{Q}^n) = \bigl(t^{-1/r}, q \bigr) \tag{4.5}$$

and

$$\mathfrak{E}_G F^s_{pq}(\mathbb{Q}^n) = \mathfrak{E}_G \widetilde{F}^s_{pq}(\mathbb{Q}^n) = \bigl(t^{-1/r}, p \bigr). \tag{4.6}$$

Let now

$$\bigl\{ A^s_{pq}(\mathbb{Q}^n) : n \in \mathbb{N} \bigr\}, \quad \bigl\{ \widetilde{A}^s_{pq}(\mathbb{Q}^n) : n \in \mathbb{N} \bigr\}, \qquad 1 < p, q < \infty, \quad s > 0,$$

be sequences (by dimension) of spaces of the above type with fixed norms. One may think about the spaces $\widetilde{B}^s_{pq}(\mathbb{Q}^n)$ according to (2.16) normed by (2.15) where $M \in \mathbb{N}$ with $s < M$ is fixed. Similarly one can fix the norms of $\widetilde{F}^s_{p,2}(\mathbb{Q}^n) = \widetilde{W}^s_p(\mathbb{Q}^n)$ by

$$\|f \,| W^s_p(\mathbb{Q}^n)\| = \sum_{|\alpha| \le s} \bigl\| D^\alpha f \,| L_p(\mathbb{Q}^n) \bigr\|, \qquad s \in \mathbb{N}, \quad 1 < p < \infty. \tag{4.7}$$

This coincides essentially with (1.3) if $s = 1$. Let $\varrho \in \mathbb{R}$. Then

$$\mathcal{E}_T | \bigl\{ A^s_{pq}(\mathbb{Q}^n) \bigr\}(t)_\varrho = \sup\bigl\{ f^*(t) : 2^{\varrho n} \bigl\| f \,| A^s_{pq}(\mathbb{Q}^n) \bigr\| \le 1, \ n \in \mathbb{N} \bigr\} \le \infty, \tag{4.8}$$

$0 < t \le 1$, is the counterpart of (4.3) (admitting ∞). Let

$$\varrho_0\big(\{A^s_{pq}(\mathbb{Q}^n)\}\big) = \inf\big\{\varrho : \mathcal{E}_T|\{A^s_{pq}(\mathbb{Q}^n)\}(t)_\varrho < \infty,\ 0 < t \le 1\big\} \le \infty$$

(admitting ∞). Similarly for $\{\widetilde{A}^s_{pq}(\mathbb{Q}^n)\}$. If $\varrho_0 < \infty$ and $\varrho_0 < \varrho < \infty$ then (4.8) is called a *tractability envelope function*. Similarly as in (4.4) it is reasonable to admit decreasing continuous smooth function $\mathcal{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}(t)$ which are equivalent to the tractability envelope function near $t = 0$, hence

$$c_1\,\mathcal{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}(t)_\varrho \le 1 + \mathcal{E}_T|\{A^s_{pq}(\mathbb{Q}^n)\}(t)_\varrho \le c_2\,\mathcal{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}(t)_\varrho$$

for some $0 < c_1 < c_2$ and $0 < t \le 1$. Let again $\psi(t) = \mathcal{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}(t)^{-1}_\varrho$.

Definition 4.1. Let $1 < p, q < \infty$, $s > 0$. Let $\varrho_0\big(\{A^s_{pq}(\mathbb{Q}^n)\}\big) < \infty$ and $\varrho_0 < \varrho < \infty$. The couple

$$\mathfrak{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}_\varrho = \big(\mathcal{E}_T\{A^s_{pq}(\mathbb{Q}^n)\}(\cdot)_\varrho, u\big) \tag{4.9}$$

is called the *tractability envelope* for $\{A^s_{pq}(\mathbb{Q}^n)\}$, where u, $0 < u \le \infty$, is the infimum of all v, $0 < v \le \infty$, such that

$$\Big(\int_0^1 \big[\psi(t)\, f^*(t)\big]^v \,\frac{\mathrm{d}t}{t}\Big)^{1/v} \le 2^{\varrho n}\|f\,|A^s_{pq}(\mathbb{Q}^n)\| \tag{4.10}$$

for all $f \in A^s_{pq}(\mathbb{Q}^n)$ and $n \in \mathbb{N}$ (modification if $v = \infty$). Similarly for $\{\widetilde{A}^s_{pq}(\mathbb{Q}^n) : n \in \mathbb{N}\}$.

Remark 4.2. If one wishes to have a closer look at ϱ_0 and ϱ then it might be reasonable to replace $2^{\varrho n}$ in (4.8), (4.10) by $c\,2^{\varrho n}$ with some $c > 0$ independent of $n \in \mathbb{N}$ (to avoid that small $n \in \mathbb{N}$ spoil the values of ϱ_0 and ϱ). The assumption $\varrho > \varrho_0$ seems to be reasonable to exclude some limiting situations in those cases where (4.8) with $\varrho = \varrho_0$ is also finite for any t, $0 < t \le 1$.

4.2. Conjectures. Recall that

$$\mathfrak{E}_G L_p(\mathbb{Q}^n) = \big(t^{-1/p}, p\big), \qquad 1 < p < \infty. \tag{4.11}$$

This can be checked easily, but it can also be found in [15, Section 4.2, pp. 66/67] together with corresponding assertions for Lorentz spaces and Lorentz-Zygmund spaces. Let $s > 0$, $1 < p < \infty$ be fixed. If $n \to \infty$ then one has $r \downarrow p$ in (4.2), (4.5), (4.6). On the other hand, if one applies (4.1) with $r = u_1 = p$ and $a = s$ to (3.21) then one obtains that

$$\sup_{t \in I} t^{1/p}\big(1 + |\log t|\big)^s f^*(t) \le 2^{\varrho n}\,\|f\,|B^s_{pp}(\mathbb{R}^n)_M\|$$

for all $f \in \widetilde{B}^s_{pp}(\mathbb{Q}^n)$, $n \in \mathbb{N}$. Similarly (1.5) gives

$$\sup_{t \in I} t^{1/p}\big(1 + |\log t|\big)^{1/2} f^*(t) \le c\,\|f\,|W^1_p(\mathbb{R}^n)\|$$

for all $f \in \widetilde{W}^1_p(\mathbb{Q}^n)$, $n \in \mathbb{N}$. There are good reasons to believe that not only these estimates, but also (3.21), (1.5) are sharp. This can be formulated as follows.

Conjecture 4.3.
(i) *Let $1 < p < \infty$ and $0 < s < M \in \mathbb{N}$. Let $\widetilde{B}^s_{pp}(\mathbb{Q}^n)$ according to (2.16) normed by (2.15). Let $\varrho > 0$ be sufficiently large. Then*

$$\mathfrak{E}_T\{\widetilde{B}^s_{pp}(\mathbb{Q}^n)\}_\varrho = \big(t^{-1/p}(1 + |\log t|)^{-s}, p\big).$$

(ii) *Let* $1 < p < \infty$. *Let* $\widetilde{W}^1_p(\mathbb{Q}^n)$ *according to* (1.4) *normed by* (1.3). *Let* $\varrho \ge 0$. *Then*

$$\mathfrak{E}_T\big\{\widetilde{W}^1_p(\mathbb{Q}^n)\big\}_\varrho = \big(t^{-1/p}(1+|\log t|)^{-1/2}, p\big).$$

REMARK 4.4. Compared with (4.11) there is a non-trivial gain by powers of log which is in good agreement with (2.1), (2.2) going back to [11]. One may ask for tractability envelopes for other scales of spaces. First candidates are $\widetilde{W}^s_p(\mathbb{Q}^n)$, $s \in \mathbb{N}$, $1 < p < \infty$, normed by (4.7), and $\widetilde{B}^s_{pq}(\mathbb{Q}^n)$, $1 < p, q < \infty$, $s > 0$, normed by (2.15) with $0 < s < M \in \mathbb{N}$. We do not know what is the outcome. But

$$\mathfrak{E}_T\big\{\widetilde{W}^s_p(\mathbb{Q}^n)\big\}_\varrho = \big(t^{-1/p}(1+|\log t|)^{-s/2}, p\big), \qquad s \in \mathbb{N}, \quad 1 < p < \infty,$$

and

$$\mathfrak{E}_T\big\{\widetilde{B}^s_{pq}(\mathbb{Q}^n)\big\}_\varrho = \big(t^{-1/p}(1+|\log t|)^{-s}, u\big), \qquad s > 0, \quad 1 < p, q < \infty,$$

with $u = p$ or $u = q$ look at least reasonable, worth to be considered.

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References

[1] W. Beckner, *Geometric inequalities in Fourier analysis*, in: Essays on Fourier Analysis in Honor of Elias M. Stein, Princeton Math. Ser. 42, Princeton Univ. Press, Princeton, 1995, 36–68.

[2] C. Bennett, K. Rudnick, *On Lorentz-Zygmund spaces*, Dissertationes Math. (Rozprawy Mat.) 175 (1980).

[3] C. Bennett, R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, Boston, 1988.

[4] Yu. Brudnyi, *Sobolev spaces and their relatives: local polynomial approximation*, in: Sobolev Spaces in Mathematics II, Int. Math. Ser. (N.Y.) 9, Springer, New York, 2009, 31–68.

[5] A. Cianchi, L. Pick. *Optimal Gaussian Sobolev embeddings*, J. Funct. Anal. 256 (2009), 3588–3642.

[6] F. Cobos, L. M. Fernández-Cabrera, A. Manzano, A. Martínez, *Logarithmic interpolation spaces between quasi-Banach spaces*, Z. Anal. Anwend. 26 (2007), 65–86.

[7] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge Univ. Press, Cambridge, 1990.

[8] D. E. Edmunds, W. D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer Monogr. Math., Springer, Berlin, 2004.

[9] D. E. Edmunds, H. Triebel, *Logarithmic Sobolev spaces and their applications to spectral theory*, Proc. London Math. Soc. (3) 71 (1995), 333–371.

[10] D. E. Edmunds, H. Triebel, *Function Spaces, Entropy Numbers, Differential Operators*, Cambridge Tracts in Math. 120, Cambridge Univ. Press, Cambridge, 1996.

[11] L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. 97 (1975), 1061–1083.

[12] J. Gunson, *Inequalities in Mathematical Physics*, in: Inequalities (Birmingham, 1987), Lect. Notes in Pure and Appl. Math. 129, Dekker, New York, 1991, 53–79.

[13] D. D. Haroske, *Envelopes in function spaces—a first approach*, Jenaer Schriften Math. Informatik, Math/Inf/16/01, Jena, 2001.

[14] D. D. Haroske, *Limiting embeddings, entropy numbers and envelopes in function spaces*, Habilitationsschrift, Friedrich-Schiller-Universität Jena, Germany, 2002.

[15] D. D. Haroske, *Envelopes and Sharp Embeddings of Function Spaces*, Chapman Hall/CRC Res. Notes Math. 437, Chapman & Hall/CRC, Boca Raton, FL, 2007.

[16] D. D. Haroske, C. Schneider, *Besov spaces with positive smoothness on* $\mathbb{R}^n$*, embeddings and growth envelopes*, J. Approx. Theory 161 (2009), 723–747.

[17] L. I. Hedberg, Yu. Netrusov, *An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation*, Mem. Amer. Math. Soc. 188 (2007), no. 882.

[18] G. E. Karadzhov, M. Milman, *Extrapolation theory: new results and applications*, J. Approx. Theory 133 (2005), 38–99.

[19] M. Krbec, H.-J. Schmeisser, *A limiting case of the uncertainty principle*, in: Proceedings of Equadiff 11, Proceedings of minisymposia and contributed talks (Bratislava, 2005), Bratislava, 2007, 181–187.

[20] M. Krbec, H.-J. Schmeisser, *Dimension-invariant Sobolev imbeddings*, this volume, 205–217.

[21] J. Martín, M. Milman, *Isoperimetry and symmetrization for logarithmic Sobolev inequalities*, J. Funct. Anal. 256 (2009), 149–178.

[22] J. Martín, M. Milman, *Pointwise symmetrization inequalities for Sobolev functions and applications*, Adv. Math. 225 (2010), 121–199.

[23] E. Novak, H. Woźniakowski, *Tractability of Multivariate Problems, I: Linear Information*, EMS Tracts Math. 6, European Math. Soc., Zürich, 2009.

[24] E. Novak, H. Woźniakowski, *Tractability of Multivariate Problems, II: Standard Information for Functionals*, EMS Tracts Math. 12, European Math. Soc., Zürich, 2010.

[25] C. Schneider, *On dilation operators in Besov spaces*, Rev. Mat. Complut. 22 (2009), 111–128.

[26] C. Schneider, *Spaces of Sobolev type with positive smoothness in* $\mathbb{R}^n$*, embeddings and growth envelopes*, J. Function Spaces Appl. 7 (2009), 251–288.

[27] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton Univ. Press, Princeton, 1970.

[28] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland Math. Library 18, North-Holland, Amsterdam, 1978.

[29] H. Triebel, *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces*, Proc. London Math. Soc. (3) 66 (1993), 589–618.

[30] H. Triebel, *The Structure of Functions*, Monogr. Math. 97, Birkhäuser, Basel, 2001.

[31] H. Triebel, *Theory of Function Spaces III*, Monogr. Math. 100, Birkhäuser, Basel, 2006.

[32] H. Triebel, *Function Spaces and Wavelets on Domains*, EMS Tracts Math. 7, European Math. Soc., Zürich, 2008.

[33] H. Triebel, *Bases in Function Spaces, Sampling, Discrepancy, Numerical Integration*, EMS Tracts Math. 11, European Math. Soc., Zürich, 2010.