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APPROXIMATION OF SOLUTIONS OF NONLINEAR INITIAL-VALUE PROBLEMS BY B-SPLINE FUNCTIONS

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Abstract. This note is motivated by [GGG], where an algorithm finding functions close to solutions of a given initial value-problem has been proposed (this algorithm has been recalled in Theorem 2.2). In this paper we present a commonly used definition and basic facts concerning B-spline functions and use them to improve the mentioned algorithm. This leads us to a better estimate of the Cauchy problem solution under some additional assumption on f appearing in the Cauchy problem. We also estimate the accuracy of the method (Theorem 2.6).

1. Introduction. In this note we will approximate solutions of the initial-value problem by an algorithm proposed in [GGG], improved by applying B-spline functions instead of usual Schauder basis in $\mathcal{C}([\alpha,\alpha+\beta])$.

The initial value problem is formulated in the following way:

Given $\alpha\in\mathbb{R}$, $\beta>0$, $l\geq 1$, $x_0\in\mathbb{R}^l$, $f\in\mathcal{C}([\alpha,\alpha+\beta]\times\mathbb{R}^l,\mathbb{R}^l)$, *find* $x\in\mathcal{C}^1([\alpha,\alpha+\beta],\mathbb{R}^l)$ *such that*

$$\begin{cases} x'(t)=f(t,x(t)), & t\in[\alpha,\alpha+\beta],\\ x(\alpha)=x_0. \end{cases}\tag{1.1}$$

We additionally assume that there exists $M\in\mathbb{R}$ such that

$$\forall t\in[\alpha,\alpha+\beta]\ \forall x,y\in\mathbb{R}^l\quad \|f(t,x)-f(t,y)\|_\infty\leq M\|x-y\|_\infty$$

The equivalent reformulation of this problem in an integral way is to find the unique fixed

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point of the operator $T : \mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l) \to \mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l)$ given by the equation

$$(Tx)(t) := x_0 + \int_\alpha^t f(s, x(s))\, ds, \quad t \in [\alpha, \alpha+\beta], \quad x \in \mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l),$$

where the norm in $\mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l)$ is

$$\|x\|_\infty := \sup_{t \in [\alpha, \alpha+\beta]} \|x(t)\|_\infty$$

Before we proceed, let us recall after [KC] the definition of B-spline functions of order k.

Let us take an infinite sequence of knots in $\mathbb{R}$ such that

$$\ldots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \ldots,$$

where $\lim_{i \to -\infty} t_i = -\infty$ and $\lim_{i \to \infty} t_i = \infty$.

B-spline functions of order 0 are the functions B_i^0, $i \in \mathbb{Z}$, defined by

$$B_i^0(x) := \begin{cases} 1, & t_i \le x < t_{i+1} \\ 0, & x < t_i \text{ or } x \ge t_{i+1}, \end{cases}$$

B-splines function of order $k > 0$ are defined in the recursive way

$$B_i^k(x) := \frac{x - t_i}{t_{i+k} - t_i} B_i^{k-1}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1}^{k-1}(x), \quad k \ge 1.$$

Let us recall after [KC] one basic fact concerning B-spline functions.

THEOREM 1.1. *If a real-valued function $x(\cdot)$ is defined on $[\alpha, \alpha+\beta] = [t_0, t_n]$, $k \ge 2$, and*

$$g := \sum_{i=-\infty}^{\infty} x(t_{i+2}) B_i^k$$

(*we assume that $x(t_i) = x(a)$ if $i < 0$, and $x(t_i) = x(b)$ if $i > n$*), *then*

$$\max_{t \in [\alpha, \alpha+\beta]} |x(t) - g(t)| \le k\omega(x; \delta),$$

where

$$\delta := \max_{-k \le i \le n+1} (t_i - t_{i-1}),$$

and

$$\omega(x; \delta) := \max_{|s-t| \le \delta} |x(s) - x(t)|.$$

The above ω is called *the module of continuity of x*.

2. Using B-spline functions in an algorithm finding solutions of nonlinear initial-value problems. Let us consider a continuous function $x(\cdot) : [\alpha, \alpha+\beta] \to \mathbb{R}$. Fix $k, n \in \mathbb{N}_2$ and choose $\{t_i\}_{i=0}^n$ such that

$$\alpha = t_0 < t_1 < \ldots < t_n = \alpha + \beta.$$

Let

$$\delta := \max_{i=1,\ldots,n} (t_i - t_{i-1}). \tag{2.1}$$

Additionally let us take

$$t_{-i} := t_0 - i \cdot \delta, \qquad i = 1, 2, \dots$$

and

$$t_{n+i} := t_n + i \cdot \delta, \qquad i = 1, 2, \dots .$$

We define these points in order to be able to define B-spline functions B_i^k for each $i \in \mathbb{Z}$, especially for $i = -k, \dots, n-1$.

Having n and $\{t_i\}_{i=-\infty}^{\infty}$, let us define an operator

$$Q_n : \mathcal{C}([\alpha, \alpha+\beta]) \ni x \mapsto Q_n(x) \in \mathcal{C}([\alpha, \alpha+\beta])$$

by the formula

$$Q_n(x) := \sum_{i=-k}^{n-1} x(t_{i+2}) B_i^k. \tag{2.2}$$

where $\mathcal{C}([\alpha, \alpha+\beta])$ is endowed with the sup-norm.

Remark 2.1. Using the above notation, we have

$$\max_{t \in [\alpha, \alpha+\beta]} |x(t) - Q_n(x)(t)| \leq k\omega(x; \delta).$$

Proof. Because of the properties of the B-spline functions of order k we have

$$B_i^k(t) = 0, \qquad t \notin (t_i, t_{i+k+1})$$

It gives us that if $t \in [\alpha, \alpha+\beta]$, then

$$B_i^k(t) = 0 \text{ for } i \in \{\dots, -k-2, -k-1\} \cup \{n, n+1, n+2, \dots\},$$

and so

$$g(t) = \sum_{i=-\infty}^{\infty} x(t_{i+2}) \cdot B_i^k(t) = \sum_{i=-k}^{n-1} x(t_{i+2}) \cdot B_i^k(t) = Q_n(x)(t) \text{ for } t \in [\alpha, \alpha+\beta].$$

On the other hand, for our chosen points t_i, we have that

$$\max_{i=-k,\dots,n+1} (t_i - t_{i-1}) = \max_{i=1,\dots,n} (t_i - t_{i-1}).$$

Applying Theorem 1.1 we get the desired conclusion. ∎

At this point we would like to recall after [GGG] the result which inspired our research. In this theorem it is assumed that $\hat{Q}_n : \mathcal{C}([\alpha, \alpha+\beta]) \to \mathcal{C}([\alpha, \alpha+\beta])$ is a projection defined by

$$\tilde{Q}_n(x) = \sum_{i=1}^{n} \lambda_i \cdot \Gamma_i, \tag{2.3}$$

where $\{\Gamma_i\}_{i \geq 1}$ is the Schauder basis associated with a dense sequence of distinct points $\{s_i\}_{i \geq 1}$ in $[\alpha, \alpha+\beta]$ such that $s_1 = \alpha$, $s_2 = \alpha+\beta$ and $\{\lambda_i\}_{i \geq 1}$ is the unique sequence of scalars such that $x = \sum_{i=0}^{\infty} \lambda_i \cdot \Gamma_i$.

Theorem 2.2. *Let $l \geq 1$, $f = (f_q)_{q=1,\dots,l} \in \mathcal{C}([\alpha, \alpha+\beta] \times \mathbb{R}^l, \mathbb{R}^l)$, $x_0 \in \mathbb{R}^l$. Let $T : \mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l) \to \mathcal{C}([\alpha, \alpha+\beta], \mathbb{R}^l)$ be the integral operator*

$$Tx(t) = x_0 + \int_{\alpha}^{t} f(s, x(s))\, ds.$$

Let $\overline{x} : [\alpha, \alpha+\beta] \to \mathbb{R}^l$ be a continuous function. Let $m \geq 1$ and $n_1, n_2, \ldots, n_m \geq 1$.

Let us consider the functions

$$y_0(t) := \overline{x}(t), \qquad t \in [\alpha, \alpha+\beta],$$
$$L_{r-1}(t) := f(t, y_{r-1}(t)), \qquad t \in [\alpha, \alpha+\beta], \quad r = 1, \ldots, m,$$

and

$$y_r(t) := x_0 + \Big(\int_\alpha^t \hat{Q}_{n_r}(L_{r-1}(s))_q \, ds \Big)_{q=1,\ldots,l}.$$

If u is the solution of the Cauchy problem, then

$$\|u - y_m\| \leq \frac{(M\beta)^m}{m!} e^{M\beta} \|T\overline{x} - \overline{x}\| + \sum_{r=1}^m \|Ty_{r-1} - y_r\| \frac{(M\beta)^{m-r}}{(m-r)!}$$

REMARK 2.3. The claim of the above theorem is true for Q_n (defined by (2.2)) used instead of $\hat{Q}_n$.

Observe that at no point of the proof of Theorem 2.2 given in [GGG] the authors have used the fact that $\hat{Q}_n$ is a projection. The key fact was that $\|\hat{Q}_n\| = 1$, which for Q_n defined by (2.2) will be shown in the proof of Lemma 2.4. However, application of Q_n instead of $\hat{Q}_n$ will significantly improve the estimate of $\|Ty_{r-1} - y_r\|$, which will be shown in Theorem 2.6.

From now on (particularly in the lemma below) we will be considering the algorithm defined in Theorem 2.2 with Q_n given by formula (2.2) instead of $\hat{Q}_n$ given by the formula (2.3).

LEMMA 2.4. *If f is a $\mathcal{C}^1$ function such that $\frac{\partial}{\partial t} f(t,x)$ and $\frac{\partial}{\partial x} f(t,x)$ satisfy global Lipschitz condition with respect to the second variable, then the sequence of the derivatives $\{L_r'\}_{r\geq 1}$ is uniformly bounded.*

Proof. Notice that for $x \in \mathcal{C}([\alpha, \alpha+\beta])$

$$\|Q_j(x)\| = \Big\| \sum_{i=-k}^{n-1} x(t_{i+2}) B_i^k \Big\| \leq \|x\| \sum_{i=-\infty}^{\infty} B_i^k = \|x\|,$$

since $\sum_{i=-\infty}^{\infty} B_i^k = 1$, and so $\|Q_j\| \leq 1$. On the other hand, taking $x \equiv 1$ we have $\|x\| = 1$ and $\|Q_j(x)\| = 1$, and what follows, $\|Q_j\| = 1$. The rest of the proof follows the proof of Lemma 3 in [GGG]. ∎

At this point we will use

THEOREM 2.5 (see [KC, p. 360]). *Let p be a natural number such that $p < k < n$, $x \in \mathcal{C}^p([\alpha, \alpha+\beta])$. Then for δ given by (2.1) we have*

$$\operatorname{dist}(x, S_n^k) \leq k^p \delta^p \|x^{(p)}\|,$$

where S_n^k is the subspace of $(\mathcal{C}([\alpha, \alpha+\beta]), \|\cdot\|_\infty)$ generated by $\{B_i^k | [t_0, t_n] : -k \leq i \leq n-1\}$.

Now we will prove the main result of this note.

THEOREM 2.6. *If the assumptions of Theorem 2.2 are satisfied and, moreover, $f \in \mathcal{C}^p([\alpha, \alpha+\beta] \times \mathbb{R}^l, \mathbb{R}^l)$ for some $p \in \mathbb{N}$ such that $p < k < n$, and the partial derivatives* *of f up to the order p satisfy the global Lipschitz condition with the constant $M > 0$ with respect to the second variable, then*

$$\sum_{r=1}^{m} \|Ty_{r-1} - y_r\| \frac{(M\beta)^{m-r}}{(m-r)!} \leq \beta k^p \delta^p \max_{r=1,\dots,m} \|L_{r-1}^{(p)}\| e^{M\beta} \tag{2.4}$$

and the sequence of the derivatives $\{L_r^{(p)}\}_{r=1}^{\infty}$ is uniformly bounded.

Proof. Let us notice first that $L_{r-1} \in \mathcal{C}^p([\alpha, \alpha+\beta])$ for $r \geq 2$. Indeed, $L_{r-1}^{(p)}$ consists of sums and products of partial derivatives of f up to the order p and derivatives of y_d $(d = 1, \dots, r-1)$ up to the order p. By our assumption, all the derivatives of f up to the order p are continuous. Notice that $y'_{r-1}(t) = Q_{n_{r-1}}(f(t, y_{r-2}(t)))$ and—because of the derivative properties of B-spline functions— $Q_{n_{r-1}}(f(t, y_{r-2}(t)))$ has continuous derivatives up to the order $k-1$ and so it has continuous derivatives up to the order p, since $p \leq k-1$.

Now, let us notice that

$$\|Ty_{r-1} - y_r\| \leq \beta \|L_{r-1} - (Q_{n_r}(L_{r-1})_q)_{q=1,\dots,l}\|.$$

Due to Theorem 2.5

$$\|L_{r-1} - (Q_{n_r}(L_{r-1})_q)_{q=1,\dots,l}\| \leq k^p \delta^p \|L_{r-1}^{(p)}\|.$$

Hence

$$\begin{aligned}
\sum_{r=1}^{m} \|Ty_{r-1} - y_r\| \frac{(M\beta)^{m-r}}{(m-r)!} &\leq \sum_{r=1}^{m} \beta k^p \delta^p \|L_{r-1}^{(p)}\| \frac{(M\beta)^{m-r}}{(m-r)!} \\
&\leq \beta k^p \delta^p \max_{r=1,\dots,m} \|L_{r-1}^{(p)}\| \sum_{r=1}^{m} \frac{(M\beta)^{m-r}}{(m-r)!} \\
&\leq \beta k^p \delta^p \max_{r=1,\dots,m} \|L_{r-1}^{(p)}\| e^{M\beta}.
\end{aligned}$$

Now we will prove the uniform boundedness of the sequence $\{L_r^{(p)}\}_{r=1}^{\infty}$. For $p = 1$ it follows from Lemma 2.4. Notice that $L_r^{(p)}$, despite of its complicated formula, consists only on finite number of sums and products of

$$y_{r-1}^{(a)} \quad \text{and} \quad \frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, y_{r-1}(t)) \quad \text{for} \quad 1 \leq a \leq p,\ 1 \leq i, j \leq p,\ i+j \leq p.$$

Since $f \in \mathcal{C}^p([\alpha, \alpha+\beta] \times \mathbb{R}^l, \mathbb{R}^l)$, we have $\|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, 0)\| \leq K$ for some $K > 0$. Notice that by the global Lipschitz condition

$$\begin{aligned}
\Big\|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, y_{r-1}(t))\Big\| \leq \Big\|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, y_{r-1}(t)) - \frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, 0)\Big\| + \Big\|\frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, 0)\Big\| \\
\leq M\|y_{r-1}(t)\| + K.
\end{aligned}$$

We get the same estimate for f, taking f instead of $\frac{\partial^{i+j} f}{\partial t^i \partial x^j}(t, y_{r-1}(t))$ in the above calculations.

On the other hand, since

$$y_r(t) = x_0 + \Big(\int_\alpha^t Q_{n_r}(L_{r-1}(s))_q \, ds\Big)_{q=1,\dots,l},$$

we have

$$y_r'(t) = \big(Q_{n_r}(L_{r-1}(t))_q\big)_{q=1,\dots,l} = \big(Q_{n_r}(f(t, y_{r-1}(t)))_q\big)_{q=1,\dots,l}$$
$$= \sum_{i=-k}^{n-1} f(t_{i+2}, y_{r-1}(t_{i+2})) \cdot B_i^k(t)$$

and for $p > 1$

$$y_r^{(p)}(t) = \sum_{i=-k}^{n-1} f(t_{i+2}, y_{r-1}(t_{i+2})) \cdot (B_i^k)^{(p-1)}(t).$$

It is a well known fact for B-spline functions (see [KC, p. 347]) that $B_i^k \in \mathcal{C}^{k-1}(\mathbb{R})$. Since $p < k$, $(B_i^k)^{(p-1)}$ are continuous on $\mathbb{R}$ for all $-k \le i \le n-1$ and so they are bounded on $[\alpha, \alpha+\beta]$.

Keeping in mind that

$$\|f(t, y_{r-1}(t))\| \le M\|y_{r-1}(t)\| + K$$

we get

$$\|y_r^{(p)}\| \le \sum_{i=-k}^{n-1} (M\|y_{r-1}\| + K) \cdot (B_i^k)^{(p-1)}(t) = (M\|y_{r-1}\| + K) \cdot D$$

where D is an upper bound of $\sum_{i=-k}^{n-1} (B_i^k)^{(p-1)}(t)$ on $[\alpha, \alpha+\beta]$.

In order to complete the proof, we only need to show that $\{y_r\}_r$ is a uniformly bounded sequence.

We will show that for $r \ge 2$

$$\|y_r(t)\| \le \|x_0\| + \Big(\|y_0\| + \frac{K}{M}\Big)\frac{(M(t-\alpha))^r}{r!} + \sum_{j=1}^{r-1}\Big(\|x_0\| + \frac{K}{M}\Big)\frac{(M(t-\alpha))^j}{j!} \tag{2.5}$$

leading us to the conclusion that

$$\|y_r\| \le \|x_0\| + \Big(\|y_0\| + \frac{K}{M}\Big)\frac{(M\beta)^r}{r!} + \sum_{j=1}^{r-1}\Big(\|x_0\| + \frac{K}{M}\Big)\frac{(M\beta)^j}{j!}\,,$$

which proves the uniform boundedness of $\{y_r\}_r$.

Applying again the fact that $\|f(t, y_{r-1}(t))\| \le M\|y_{r-1}(t)\| + K$, let us notice that

$$\|y_r(t)\| \le \|x_0\| + \int_\alpha^t (M\|y_{r-1}(s)\| + K)\,ds \tag{2.6}$$

which gives us the estimate

$$\|y_1(t)\| \le \|x_0\| + (M\|y_0\| + K)(t-\alpha)$$
$$\le \|x_0\| + (M\|y_0\| + K)\beta.$$

It is easy to check that

$$\|y_2(t)\| \le \|x_0\| + (\|y_0\| + \frac{K}{M})\frac{(M(t-\alpha))^2}{2!} + (\|x_0\| + \frac{K}{M})\frac{(M(t-\alpha))}{1!}\,.$$

Assuming that the inequality (2.5) is true for r, we will show that it remains true for $r+1$.

By the inequality (2.6), we get

$$\begin{aligned}
\|y_{r+1}(t)\| &\le \|x_0\| + \int_\alpha^t (M\|y_r(s)\| + K)\,ds \\
&\le \|x_0\| + \int_\alpha^t \biggl(M\Bigl[\|x_0\| + \Bigl(\|y_0\| + \frac{K}{M}\Bigr)\frac{(M(s-\alpha))^r}{r!} \\
&\qquad + \sum_{j=1}^{r-1}\Bigl(\|x_0\| + \frac{K}{M}\Bigr)\frac{(M(s-\alpha))^j}{j!}\Bigr] + K\biggr)\,ds \\
&= \|x_0\| + M\Bigl(\|y_0\| + \frac{K}{M}\Bigr)\frac{M^r(t-\alpha)^{r+1}}{r!(r+1)} + M(\|x_0\| + K)(t-\alpha) \\
&\qquad + M\sum_{j=1}^{r-1}\Bigl(\|x_0\| + \frac{K}{M}\Bigr)\frac{M^j(s-\alpha)^{j+1}}{j!(j+1)} \\
&= \|x_0\| + \Bigl(\|y_0\| + \frac{K}{M}\Bigr)\frac{(M(t-\alpha))^{r+1}}{(r+1)!} + \sum_{j=1}^{r}\Bigl(\|x_0\| + \frac{K}{M}\Bigr)\frac{(M(t-\alpha))^j}{j!}.
\end{aligned}$$

This completes the proof of the theorem. ■

COROLLARY 2.7. *With the assumptions of Theorem* 2.6, *if* u *is the solution of the initial-value problem defined by* (1.1), *then*

$$\|u - y_m\| \le \frac{(M\beta)^m}{m!}e^{M\beta}\|T\overline{x} - \overline{x}\| + \beta k^p\delta^p \max_{r=1,\ldots,m}\|L_{r-1}^{(p)}\|e^{M\beta}. \tag{2.7}$$

It is worth mentioning that using the operator Q_n defined by (2.2) in the algorithm proposed in Theorem 2.2 instead of the operator $\tilde{Q}_n$ defined by (2.3), we can obtain a better estimate of the left-hand side of inequality (2.7) (compare with the inequality

$$\|u - y_m\| \le \frac{(M\beta)^m}{m!}e^{M\beta}\|T\overline{x} - \overline{x}\| + 2\beta\delta \max_{r=1,\ldots,m}\|L'_{r-1}\|e^{M\beta}$$

achieved in [GGG]), which enables us to better estimate the solutions of Cauchy problem (1.1), if f is a function of higher regularity, whose derivatives satisfy global Lipschitz condition with respect to the second variable.

I have focused on B-spline functions to improve estimate of the solutions of the Cauchy problem, because of their simple recursive formula and nice calculation properties, very useful in numerical analysis.

Observe that the proof of Theorem 2.6 is valid not only for operators Q_n defined by formula (2.2). Instead of Q_n we can use any sequence of operators $P_n : \mathcal{C}([\alpha, \alpha + \beta]) \to \mathcal{C}([\alpha, \alpha+\beta])$ such that the norm of $\{P_n\}$ is uniformly bounded and the image of $\{P_n\}$ is regular enough. For example $\{P_n\}$ can be taken as the Bernstein operators, the Fejér operators and their various modifications. Also projections associated with other Schauder bases can be considered in Theorem 2.6. More information about Schauder bases in Banach spaces can be found in [BFGG, M, S].

Theorem 2.2, [GGG], as well as Theorem 2.6 provide us with a new numerical method, which does not need to solve systems of algebraic equations—collocation methods—or to use quadrature formulas, especially useful for $l > 1$.

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