FUNCTION SPACES IX
BANACH CENTER PUBLICATIONS, VOLUME 92
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2011

LOCAL MEANS AND WAVELETS IN FUNCTION SPACES WITH LOCAL MUCKENHOUPT WEIGHTS

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Abstract. The paper deals with local means and wavelet bases in function spaces of Besov and Triebel-Lizorkin type with local Muckenhoupt weights.

1. Introduction. Both concepts, local means and wavelet characterization are widely studied in the context of Besov and Triebel-Lizorkin spaces. We refer to books by Hans Triebel for details and historical remarks, cf. [8], [9]. The close relation between the local means and wavelets is described in an unweighted case by H. Triebel in [10]. Our aim is to extend his results into function spaces with local Muckenhoupt weights $\mathcal{A}_\infty^{\text{loc}}$.

Local Muckenhoupt weights and corresponding weighted function spaces were introduced by V. Rychkov in 2001, cf. [6]. This class of weights is a generalization of the classical class of Muckenhoupt weights $\mathcal{A}_\infty$. Wavelet characterizations of function spaces with so called admissible weights were given by Haroske and Triebel in [2]. Later Haroske and Skrzypczak proved the characterization for spaces with Muckenhoupt weights, cf. [1]. Recently Izuki and Sawano have proved the result for function spaces with weights from the class $\mathcal{A}_\infty^{\text{loc}}$, cf. [5].

We follow the main idea of H. Triebel from [10], that Daubechies wavelets can serve both as atoms and kernels of local means. So, first we recall the atomic decomposition of function spaces with the local Muckenhoupt weights due to Izuki and Sawano, cf. [4], also [5]. Then we introduce local means and prove characterizations of function spaces. Our approach to wavelet decomposition is more direct than the one presented in [5] since we avoid some density arguments.

2010 *Mathematics Subject Classification*: Primary 46E35; Secondary 41A58, 42C40.
Key words and phrases: local means, Muckenhoupt weights, wavelets, Besov spaces, Triebel-Lizorkin spaces.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc92-0-28

2. Classes of weights. Let w be a nonnegative and locally integrable function on $\mathbb{R}^n$. These functions are called weights and for a measurable set E $w(E)$ denotes $\int_E w(x)\,dx$. Let $L_p^w(\mathbb{R}^n)$ be the space of p-integrable functions on $\mathbb{R}^n$ with the measure $w\,dx$.

2.1. Muckenhoupt weights. Let us recall the definition of Muckenhoupt weights.

A weight w belongs to $\mathcal{A}_p$, $w \in \mathcal{A}_p$, $1 < p < \infty$, if

$$A_p(w) := \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^p} \int_Q w(x)\,dx \Big(\int_Q w(x)^{1-p'}\,dx \Big)^{p-1} < \infty$$

and $w \in \mathcal{A}_1$ if

$$A_1(w) := \sup_{Q \subset \mathbb{R}^n} \frac{w(Q)}{|Q|} \left\| w^{-1} \right\|_{L_\infty(Q)} < \infty.$$

where supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

As an example we can take

$$w(x) = |x|^\alpha \in \mathcal{A}_p \quad \text{for } \begin{cases} -n < \alpha < n(p-1), & \text{if } 1 < p < \infty, \\ -n < \alpha \le 0, & \text{if } p = 1 \end{cases}$$

or weights with logarithmic part

$$v(x) = |x|^\alpha \log^{-\beta}(2 + |x|).$$

Then

$$v \in \mathcal{A}_1 \quad \text{if } \begin{cases} \beta \in \mathbb{R} & \text{and } -n < \alpha < 0, \\ \beta \ge 0 & \text{and } \alpha = 0, \end{cases}$$

and

$$v \in \mathcal{A}_p,\ 1 < p < \infty \quad \text{if } -n < \alpha < n(p-1),\ \beta \in \mathbb{R}.$$

2.2. Local Muckenhoupt weights

Definition 2.1 (Rychkov, 2001). We define a class of weights $\mathcal{A}_p^{\text{loc}}$ $(1 < p < \infty)$ to consist of all nonnegative locally integrable functions w defined on $\mathbb{R}^n$ for which

$$A_p^{\text{loc}}(w) := \sup_{|Q| \le 1} \frac{1}{|Q|^p} \int_Q w(x)\,dx \Big(\int_Q w(x)^{1-p'}\,dx \Big)^{p-1} < \infty$$

and $w \in \mathcal{A}_1^{\text{loc}}$

$$A_1^{\text{loc}}(w) := \sup_{|Q| \le 1} \frac{w(Q)}{|Q|} \left\| w^{-1} \right\|_{L_\infty(Q)} < \infty.$$

It follows directly from definitions that $\mathcal{A}_p \subset \mathcal{A}_p^{\text{loc}}$ and $A_p^{\text{loc}}(w) \le A_p(w)$ for any $w \in \mathcal{A}_p$, $1 \le p < \infty$.

Definition 2.2. We say that $w \in \mathcal{A}_\infty^{\text{loc}}$ if for any $\alpha \in (0,1)$

$$\sup_{|Q| \le 1} \left(\sup_{F \subset Q, |F| \ge \alpha |Q|} \frac{w(Q)}{w(F)} \right) < \infty,$$

where F is taken over all measurable sets in $\mathbb{R}^n$.

Remark. Any Muckenhoupt weight of the class $\mathcal{A}_p$ belongs to the class $\mathcal{A}_p^{\text{loc}}$. But local Muckenhoupt weights cover also so called admissible weights and locally regular weights, cf. [6], [2], [7].

As an example we can take

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \le 1, \\ \exp(|x|-1) & \text{for } |x| > 1, \end{cases}$$

for $-n < \alpha < n(p-1)$ if $1 < p < \infty$ and $-n < \alpha \le 0$ if $p = 1$. Then $w \in \mathcal{A}_p^{\mathrm{loc}}$.

2.3. Properties of classes $\mathcal{A}_p^{\mathrm{loc}}$. We would like to mention some important properties of classes $\mathcal{A}_p^{\mathrm{loc}}$.

LEMMA 2.3 (Rychkov, 2001). *Let $1 < p_1 < p_2 < \infty$. Then $\mathcal{A}_{p_1}^{\mathrm{loc}} \subset \mathcal{A}_{p_2}^{\mathrm{loc}} \subset \mathcal{A}_\infty^{\mathrm{loc}}$.*
Conversely, if $w \in \mathcal{A}_\infty^{\mathrm{loc}}$, then $w \in \mathcal{A}_p^{\mathrm{loc}}$ for some $p < \infty$.

The last lemma implies that $\mathcal{A}_\infty^{\mathrm{loc}} = \bigcup_{p \ge 1} \mathcal{A}_p^{\mathrm{loc}}$. In consequence we can define for $w \in \mathcal{A}_\infty^{\mathrm{loc}}$ a positive number

$$r_w = \inf\left\{1 \le p < \infty : w \in \mathcal{A}_p^{\mathrm{loc}}\right\}.$$

The next lemma shows us an important relation between $\mathcal{A}_p$ and $\mathcal{A}_p^{\mathrm{loc}}$ weights.

LEMMA 2.4 (Rychkov, 2001). *Let $1 \le p < \infty$, $w \in \mathcal{A}_p^{\mathrm{loc}}$ and I be a unit cube, i.e., $|I| = 1$. Then there exists a $\bar{w} \in \mathcal{A}_p$, such that $\bar{w} = w$ on I and*

$$A_p(\bar{w}) \le c A_p^{\mathrm{loc}}(w),$$

where constant c is independent of I.

An example of a weight, which is in $\mathcal{A}_p^{\mathrm{loc}} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$:

$$w(x) = \begin{cases} |x|^\alpha & \text{for } |x| \le 1, \\ |x|^\beta & \text{for } |x| > 1, \end{cases}$$

for $\alpha, \beta > -n$. For $\alpha < (p-1)n$ we have $w \in \mathcal{A}_p^{\mathrm{loc}}$ and $r_w = \frac{\max(0,\alpha)}{n} + 1$, for $\alpha, \beta < (p_1 - 1)n$ we have $w \in \mathcal{A}_{p_1}$ and $r_w = \frac{\max(0,\alpha,\beta)}{n} + 1$. Taking β big enough we get that w is in $\mathcal{A}_p^{\mathrm{loc}} \cap \mathcal{A}_\infty$, but not in $\mathcal{A}_p$.

DEFINITION 2.5. Let f be locally integrable. The operator

$$M^{\mathrm{loc}} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where supremum is taken over all cubes in $\mathbb{R}^n$ for which $|Q| \le 1$, is called *a local maximal function.*

The Fefferman-Stein maximal inequality holds for the operator M^{loc} and local Muckenhoupt weights.

LEMMA 2.6 (Rychkov, 2001). *Let $1 < p < \infty$, $1 < q \le \infty$ and $w \in \mathcal{A}_p^{\mathrm{loc}}$. Then for any sequence of measurable functions $\{f^j\}$ we have*

$$\big\|\{M^{\mathrm{loc}} f^j\}|L_p^w(l_q)\big\| \le c\big\|\{f^j\}|L_p^w(l_q)\big\|.$$

LEMMA 2.7 (Rychkov, 2001). *Let $w \in \mathcal{A}_p^{\mathrm{loc}}$ and $1 < p < \infty$. Then*

$$w(tQ) \le \exp(c_w t) w(Q) \quad t \ge 1,\ \|Q| = 1,$$

where $c_w > 0$ is a constant depending on n and $A_p^{\mathrm{loc}}(w)$.

It follows from the above lemma that classes $\mathcal{A}_p^{\mathrm{loc}}$ are independent of the upper bound for the cube size used in their definition, i.e. for any $C > 0$ we could have replaced $|Q| \le 1$ by $|Q| \le C$ in Definition 2.1.

3. Weighted function spaces. Following Rychkov we define Besov spaces with local Muckenhoupt weights, [6]. Because the class of tempered distributions $\mathcal{S}'$ is too narrow for this purpose, we introduce a class $\mathcal{S}'_e$ which is a topological dual to the following space:

$$\mathcal{S}_e := \{\psi \in C^\infty(\mathbb{R}^n) : q_N(\psi) < \infty \text{ for all } N \in \mathbb{N}\},$$

where the semi-norms q_N are given by

$$q_N(\psi) := \sup_{\alpha \in \mathbb{N}_0^n, |\alpha| \le N} \Big(\sup_{x \in \mathbb{R}^n} e^{N|x|} |D^\alpha \psi(x)| \Big).$$

We can identify the class $\mathcal{S}'_e$ with the set of those distributions $f \in \mathcal{D}'$ for which the estimate

$$|\langle f, \psi\rangle| \le A \sup\{|D^\alpha \psi(x)| \exp(N|x|) : x \in \mathbb{R}^n, |\alpha| \le N\} \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}^n),$$

is valid with some constants A, N depending on f. Such a distribution f can be extended to a continuous functional on $\mathcal{S}_e$.

We take a function $\varphi_0 \in \mathcal{D}$ such that $\int_{\mathbb{R}^n} \varphi_0(x)\,dx \ne 0$ and $\int_{\mathbb{R}^n} x^\beta \varphi_0(x)\,dx = 0$ for some $\beta \in \mathbb{N}_0^n$, $0 < |\beta| \le B$. We put $\varphi(x) = \varphi_0(x) - 2^{-n}\varphi_0(\frac{x}{2})$ and $\varphi_j(x) = 2^{(j-1)n}\varphi(2^{j-1}x)$ for $j = 1, 2, \ldots$. Then $\int_{\mathbb{R}^n} \varphi_j(x) x^\beta\,dx = 0$ if $|\beta| \le B$. We will write $B = -1$ if no vanishing moment conditions hold.

Definition 3.1 (Rychkov, 2001). Let $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{\mathrm{loc}}$. Let the function $\varphi_0 \in \mathcal{D}(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} \varphi_0(x)\,dx \ne 0$$

and

$$\int_{\mathbb{R}^n} x^\beta \varphi_0(x)\,dx = 0, \quad 0 < |\beta| < B,$$

where $B \ge [s]$. We define a *weighted Besov space* $B_{pq}^{s,w}(\mathbb{R}^n)$ to be a set of all $f \in \mathcal{S}'_e$ for which the quasi-norm

$$\big\|f|B_{pq}^{s,w}(\mathbb{R}^n)\big\|_{\varphi_0} = \Big(\sum_{j=0}^\infty 2^{jsq} \left\|\varphi_j * f|L_p^w\right\|^q\Big)^{1/q}$$

(with the usual modification if $q = \infty$) is finite, and a *weighted Triebel-Lizorkin space* $F_{pq}^{s,w}(\mathbb{R}^n)$ to be a set of all $f \in \mathcal{S}'_e$ for which the quasi-norm

$$\big\|f|F_{pq}^{s,w}(\mathbb{R}^n)\big\|_{\varphi_0} = \Big\|\Big(\sum_{j=0}^\infty 2^{jsq}|\varphi_j * f|^q\Big)^{1/q}|L_p^w(\mathbb{R}^n)\Big\|$$

(with the usual modification if $q = \infty$) is finite.

Remark. The definition of the above spaces is independent of a choice of the function φ_0, up to the equivalence of quasi-norms. The spaces are quasi-Banach and Banach spaces if $p \ge 1$ and $q \ge 1$.

REMARK. To simplify the notation we write $A^{s,w}_{pq}(\mathbb{R}^n)$ instead of $B^{s,w}_{pq}(\mathbb{R}^n)$ and $F^{s,w}_{pq}(\mathbb{R}^n)$, when both scales of spaces are meant simultaneously in some context.

4. Atomic decomposition. An important tool we will use to prove our main result is an atomic decomposition for weighted Besov and Triebel-Lizorkin spaces proved by M. Izuki and Y. Sawano in [4].

DEFINITION 4.1. Let $s \in \mathbb{R}$, $0 < p < \infty$, $K, L \in \mathbb{N}_0$ and $d \geq 1$. Then C^K-functions $a_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called (s,p)-*atoms* if

$$\operatorname{supp} a_{jm} \subset dQ_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

and there exist all (classical) derivatives $D^\alpha a_{jm} \in C(\mathbb{R}^n)$ with $|\alpha| \leq K$ such that

$$|D^\alpha a_{jm}(x)| \leq 2^{-j(s-n/p)+j|\alpha|}, \quad |\alpha| \leq K, \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n, \tag{1}$$

and

$$\int_{\mathbb{R}^n} x^\beta a_{jm}(x)\, dx = 0, \quad |\beta| < L, \ j \in \mathbb{N}, \ m \in \mathbb{Z}^n. \tag{2}$$

REMARK. Note that the last condition is omitted if $j = 0$.

DEFINITION 4.2. Let $0 < p < \infty$, $0 < q \leq \infty$ and $w \in \mathcal{A}^{\mathrm{loc}}_\infty$. Then b^w_{pq} is the collection of all sequences

$$\lambda = \{\lambda_{jm} \in \mathbb{C} : j \in \mathbb{N}_0, \ m \in \mathbb{Z}^n\} \tag{3}$$

such that

$$\|\lambda | b^w_{pq}\| = \Big\| \Big\{ \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi^{(p)}_{jm} \Big\}_{j \in \mathbb{N}_0} | l_q(L^w_p) \Big\| < \infty,$$

and let $0 < p < \infty$, $0 < q \leq \infty$ or $p = q = \infty$ then $f^w_{p,q}$ is the collection of all sequences λ according to (3) such that

$$\|\lambda | f^w_{p,q}\| = \Big\| \Big\{ \sum_{m \in \mathbb{Z}^n} \lambda_{jm} \chi^{(p)}_{jm} \Big\}_{j \in \mathbb{N}_0} | L^w_p(l_q) \Big\| < \infty,$$

where $\chi^{(p)}_{jm} = 2^{jn/p} \chi_{Q_{jm}}$. Once more we use the notation a^w_{pq}.

Izuki and Sawano proved in [4] that functions from $B^{s,w}_{pq}$ and $F^{s,w}_{pq}$ admit atomic decompositions, cf. also [5].

For $w \in \mathcal{A}^{\mathrm{loc}}_\infty$ let us define

$$\sigma_p(w) = n\Big(\frac{r_w}{\min(p, r_w)} - 1\Big) + (r_w - 1)n,$$

$$\sigma_q = \frac{n}{\min(1,q)} - n$$

and

$$\sigma_{pq}(w) = \max(\sigma_p(w), \sigma_q).$$

THEOREM 4.3 (Izuki, Sawano). *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}^{\mathrm{loc}}_\infty$. Let $K, L \in \mathbb{Z}$ satisfy*

$$K \geq (1 + [s])_+ \quad \textit{and} \quad L \geq \max(-1, [\sigma_p(w) - s])$$

when $A^{s,w}_{pq}$ *denotes* $B^{s,w}_{pq}$ *and*

$$K \ge (1+[s])_+ \quad \textit{and} \quad L \ge \max(-1, [\sigma_{pq}(w) - s])$$

when $A^{s,w}_{pq}$ *denotes* $F^{s,w}_{pq}$. *Let* $f \in A^{s,w}_{pq}(\mathbb{R}^n)$. *Then there exists a sequence of* (s,p)*-atoms* $\{a_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *and* $\lambda \in a^w_{pq}$ *such that*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm} \quad \textit{and} \quad \big\|\lambda | a^w_{pq}\big\| \le c \big\| f | A^{s,w}_{pq}(\mathbb{R}^n)\big\|$$

with convergence in $\mathcal{S}'_e$. *Conversely, let* $\{a_{jm}\}_{j\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *be a sequence of* (s,p)*-atoms and* $\lambda \in a^w_{pq}$. *Then the series*

$$f = \sum_{j=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{jm} a_{jm}$$

converges in $\mathcal{S}'_e$ *and belongs to* $A^{s,w}_{pq}(\mathbb{R}^n)$ *and*

$$\big\| f | A^{s,w}_{pq}(\mathbb{R}^n)\big\| \le c \big\|\lambda | a^w_{pq}\big\|.$$

5. Characterization by local means. Following Triebel in [10] we can define local means in Besov spaces with local Muckenhoupt weights.

Definition 5.1. Let $A, B \in \mathbb{N}_0$ and $C > 0$. Then C^A-functions $k_{jm} : \mathbb{R}^n \mapsto \mathbb{C}$ with $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, are called *kernels* if

$$\operatorname{supp} k_{jm} \subset CQ_{jm}, \quad j \in \mathbb{N}_0, \quad m \in \mathbb{Z}^n,$$

there exist all (classical) derivatives $D^\alpha k_{jm} \in C(\mathbb{R}^n)$ with $|\alpha| \le A$ such that

$$|D^\alpha k_{jm}(x)| \le 2^{jn+j|\alpha|}, \quad |\alpha| \le A,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{4}$$

and

$$\int_{\mathbb{R}^n} x^\beta k_{jm}(x)\,dx = 0, \quad |\beta| < B,\ j \in \mathbb{N},\ m \in \mathbb{Z}^n. \tag{5}$$

Since the kernels have finite smoothness we will work with distributions of finite order.

Let us consider a set $C^m_K(\mathbb{R}^n)$ of functions φ in $C^m(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset K$, where $K \subset \mathbb{R}^n$ is compact and a set $C^m_0(\mathbb{R}^n)$ consists of functions of order m with compact support.

Definition 5.2. A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is *of order* m if for every compact $K \subset \mathbb{R}^n$ there exists a constant c such that

$$|f(\varphi)| \le c \sum_{|\alpha|\le m} \sup_{x\in K} |D^\alpha \varphi(x)| \quad \text{for every } \varphi \in C^\infty_0(\mathbb{R}^n).$$

The set of all distributions of order m is denoted by $\mathcal{D}'_m(\mathbb{R}^n)$.

Theorem 5.3. *If* $f \in \mathcal{D}'_m(\mathbb{R}^n)$ *then* f *can be extended to a continuous linear functional on* $C^m_0(\mathbb{R}^n)$, *moreover* $(C^m_0(\mathbb{R}^n))' = \mathcal{D}'_m(\mathbb{R}^n)$.

The proof of the above theorem can be found in [3].

DEFINITION 5.4. Let $f \in \mathcal{D}'_A(\mathbb{R}^n) \cap \mathcal{S}'_e(\mathbb{R}^n)$. Let k_{jm} be kernels according to Definition 5.1 (with the same constant A). Then

$$k_{jm}(f) = (f, k_{jm}) = \int_{\mathbb{R}^n} k_{jm}(y) f(y)\, dy, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n, \tag{6}$$

are called *local means*. Furthermore, we put

$$k(f) = \{k_{jm}(f) : j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}. \tag{7}$$

DEFINITION 5.5. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $w \in \mathcal{A}_\infty^{\mathrm{loc}}$. Then $\bar{b}^{s,w}_{pq}$ is the collection of all sequences λ according to (3) such that

$$\|\lambda|\bar{b}^{s,w}_{pq}\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q}\Big(\int_{\mathbb{R}^n}\Big|\sum_{m\in\mathbb{Z}^n}\lambda_{jm}\chi^{(p)}_{jm}(x)\Big|^p w(x)\,dx\Big)^{q/p}\Big)^{1/q}$$

and $\bar{f}^{s,w}_{p,q}$ is the collection of all sequences λ according to (3) such that

$$\|\bar{f}^{s,w}_{p,q}\| = \Big\|\Big(\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jsq}\Big|\lambda_{jm}\chi^{(p)}_{jm}\Big|^q\Big)^{1/q}|L^w_p\Big\| < \infty.$$

LEMMA 5.6. *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $w \in \mathcal{A}_\infty^{\mathrm{loc}}$. Then $B^{s,w}_{pp}(\mathbb{R}^n) \subset \mathcal{D}'_l(\mathbb{R}^n)$ for any $l \ge \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$.*

Proof. Let $f \in B^{s,w}_{pp}(\mathbb{R}^n)$. From the atomic decomposition we have

$$f = \sum_{j,m} \lambda_{jm} a_{jm}$$

and $\lambda_{jm} \in b^w_{pp}$, with convergence in $\mathcal{D}'(\mathbb{R}^n)$. It means that we can approximate f by functions $f_k = \sum_{j\le k, |m|\le k} \lambda_{jm} a_{jm}$, i.e. $f = \lim_{k\to\infty} f_k$ in $\mathcal{D}'(\mathbb{R}^n)$, that is

$$f(\varphi) = \lim_{k\to\infty} f_k(\varphi)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$.

For $p > 1$ from the Hölder's inequality we have

$$|f_k(\varphi)| = \Big|\sum_{j,|m|\le k} \lambda_{jm} a_{jm}(\varphi)\Big| \le \sum_{j,|m|\le k} |\lambda_{jm}|\Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big|$$

$$\le \Big(\sum_{j,|m|\le k} 2^{jn}|\lambda_{jm}|^p w(Q_{jm})\Big)^{\frac{1}{p}} \Big(\sum_{j,|m|\le k} 2^{-jnp'/p} w(Q_{jm})^{-p'/p}\Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big|^{p'}\Big)^{\frac{1}{p'}}$$

Since $\lambda \in b^w_{pp}$, we have

$$\sum_{j=0}^{\infty}\sum_{m\in\mathbb{Z}^n} 2^{jn}|\lambda_{jm}|^p w(Q_{jm}) < \infty. \tag{8}$$

From $|a_{jm}(x)| \le 2^{-j(s-n/p)}$ we get

$$\Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big|^{p'} \le c|Q_{jm}|^{p'}\, 2^{-j(s-n/p)p'} \sup_x |\varphi(x)|^{p'}$$
$$\le c\, 2^{-j(s-n/p)p' - jnp'} \sup_x |\varphi(x)|^{p'}.$$

Let $\operatorname{supp}\varphi \subset K$, where K is a compact subset in $\mathbb{R}^n$. If $Q_{jm} \subset Q_{0,l}$ then

$$w(Q_{jm})^{-p'/p} \le c w(Q_{0,l})^{-p'/p}\, 2^{jnup'/p}, \tag{9}$$

since $w \in \mathcal{A}_u^{\mathrm{loc}}$ for some $r_w < u < \infty$. So

$$\sum_{m: Q_{jm}\cap K \ne \emptyset,\, |m|\le k} w(Q_{jm})^{-p'/p} \le 2^{jnup'/p+jn} \sum_{l: Q_{0,l}\cap K\ne\emptyset} w(Q_{0,l})^{-p'/p}. \tag{10}$$

Now we can keep on estimating

$$\begin{aligned}
&\sum_{j,|m|\le k} 2^{-jnp'/p} w(Q_{jm})^{-p'/p} \Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big|^{p'} \\
&\le c \sum_{j,|m|\le k} 2^{-j(s+n)p'} \sup_{x\in\mathbb{R}^n} |\varphi(x)|^{p'} w(Q_{jm})^{-p'/p} \\
&\le c \sum_{j\le k, l: Q_{0,l}\cap K\ne\emptyset} 2^{-j(s+n)p'} 2^{jnup'/p+jn} \sup_{x\in K} |\varphi(x)|^{p'} w(Q_{0,l})^{-p'/p} \\
&\le C_K \sup_{x\in K} |\varphi(x)|^{p'} \sum_{j\in\mathbb{N}_0} 2^{-j(s-nu/p+n/p)p'}.
\end{aligned}$$

For $s > \frac{nu}{p} - \frac{n}{p}$ we have

$$|f_k(\varphi)| \le C_K \sup_{x\in K} |\varphi(x)|,$$

where C_K depends only on K. Hence

$$|f(\varphi)| \le C_K \sup_{x\in K} |\varphi(x)|.$$

So f is a distribution of order 0 if $s > \frac{nu}{p} - \frac{n}{p}$. Now let $s \le \frac{nu}{p} - \frac{n}{p}$ and $l > -s + \frac{nu}{p} - \frac{n}{p}$. Using the Taylor expansion of φ and the moment conditions if $j > 0$ we get

$$\begin{aligned}
\Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big| &= c\Big|\int_{\mathbb{R}^n} a_{jm}(x) \sum_{|\alpha|=l} D^\alpha\varphi(x_0+\Theta(x-x_0))(x-x_0)^\alpha\,dx\Big| \\
&\le c\, 2^{-j(l+s-n/p+n)} \sum_{|\alpha|=l} \sup_{x\in\mathbb{R}^n} |D^\alpha\varphi(x)|.
\end{aligned} \tag{11}$$

Summing over $j, |m| \le k$, we get from (11) and (9)

$$\begin{aligned}
&\sum_{j,m} 2^{-jnp'/p} w(Q_{jm})^{-p'/p} \Big|\int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx\Big|^{p'} \\
&\qquad \le c_K \sum_{j\in\mathbb{N}_0} 2^{-j(l+s-nu/p+n/p)p'} \Big(\sum_{|\alpha|=l} \sup_{x\in K} |D^\alpha\varphi(x)|\Big)^{p'}.
\end{aligned}$$

Incorporating the term with $j = 0$ we get

$$|f_k(\varphi)| \le c_K \sum_{|\alpha|\le l} \sup_{x\in K} |D^\alpha\varphi(x)|.$$

For $0 < p \le 1$ we have an estimate

$$
\begin{aligned}
|f_k(\varphi)| &= \Big| \sum_{j,|m|\le k} \lambda_{jm} a_{jm}(\varphi) \Big| \\
&\le \Bigg(\sum_{\substack{j,|m|\le k \\ Q_{jm}\cap K \ne \emptyset}} |\lambda_{jm}|^p \Big| \int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx \Big|^p \Bigg)^{1/p} \\
&\le \Bigg(\sum_{\substack{j,|m|\le k \\ Q_{jm}\cap K \ne \emptyset}} 2^{jn} |\lambda_{jm}|^p w(Q_{jm}) \sup_{x\in K} |\varphi(x)|^p |Q_{jm}|^p w(Q_{jm})^{-1}\, 2^{-j(s-n/p)p} 2^{-jn} \Bigg)^{1/p} \\
&\le \sup_{x\in K} |\varphi(x)| \sup_{\substack{j,|m|\le k \\ Q_{jm}\cap K \ne \emptyset}} 2^{-j(s+n)} w(Q_{jm})^{-1/p} \Bigg(\sum_{\substack{j,|m|\le k \\ Q_{jm}\cap K \ne \emptyset}} 2^{jn} |\lambda_{jm}|^p w(Q_{jm}) \Bigg)^{1/p}.
\end{aligned}
$$

Using the fact that $\lambda \in b^w_{pp}$ we get

$$
|f_k(\varphi)| \le C \sup_{x\in K} |\varphi(x)| \sup_{j,|m|\le k, Q_{jm}\cap K \ne \emptyset} 2^{-j(s+n)} w(Q_{jm})^{-1/p}.
$$

In the same manner as in (9) we can see that

$$
\begin{aligned}
|f_k(\varphi)| &\le C \sup_{x\in K} |\varphi(x)| \sup_{j,l,Q_{0,l}\cap K \ne \emptyset} 2^{-j(s+n)} 2^{jnu/p} w(Q_{0,l})^{-1/p} \\
&\le C(K) \sup_{x\in K} |\varphi(x)| \sup_{j\in\mathbb{N}_0} 2^{-j(s-nu/p+n)}.
\end{aligned}
$$

For $s > \frac{nu}{p} - n$ we have

$$
|f_k(\varphi)| \le C \sup_{x\in K} |\varphi(x)|\,.
$$

For $s \le \frac{nu}{p} - n$ and $l > -s + \frac{nu}{p} - n$ using the above estimations and the same inequalities as in (11) we get

$$
\begin{aligned}
|f_k(\varphi)| &\le \Bigg(\sum_{j,|m|\le k} |\lambda_{jm}|^p \Big| \int_{\mathbb{R}^n} a_{jm}(x)\varphi(x)\,dx \Big|^p \Bigg)^{1/p} \\
&\le \sum_{|\alpha|=l} \sup_{x\in K} |D^\alpha \varphi(x)| \sup_{j,|m|\le k, K\cap Q_{jm}\ne\emptyset} 2^{-j(l+s+n)} w(Q_{jm})^{-1/p} \\
&\le c_K \sum_{|\alpha|=l} \sup_{x\in K} |D^\alpha \varphi(x)| \sup_{j\in\mathbb{N}_0} 2^{-j(l+s+n-nu/p)}.
\end{aligned}
$$

So f is a distribution of order l for any $l \ge \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$. ∎

Corollary 5.7. *Let a weight w belong to the class $\mathcal{A}^{\mathrm{loc}}_\infty$. The spaces $F^{s,w}_{pq}(\mathbb{R}^n)$ and $B^{s,w}_{pq}(\mathbb{R}^n)$ consist of distributions of finite order l for any $l \ge \max(0, [-s + \frac{nr_w}{p} - \frac{n}{p}] + 1)$.*

Proof. Let us choose $s' < s$ such that $l \ge \max(0, [-s' + \frac{nr_w}{p} - \frac{n}{p}] + 1)$. Then by the elementary embeddings and Lemma 5.6 we have

$$
F^{s,w}_{pq}(\mathbb{R}^n) \subset B^{s',w}_{pp}(\mathbb{R}^n) \subset \mathcal{D}'_l(\mathbb{R}^n).
$$

A similar argument works for Besov spaces. ∎

By the next theorem we have the characterization of Besov and Triebel-Lizorkin spaces with $\mathcal{A}_\infty^{\mathrm{loc}}$ weights by local means.

THEOREM 5.8. *Let* $0 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$. *Assume that* $w \in \mathcal{A}_\infty^{\mathrm{loc}}$. *Let* k_{jm} *be kernels according to Definition* 5.1, *where* $A, B \in \mathbb{N}_0$ *with*

$$A \ge \max(0, [-s + \sigma_p(w)], [\tfrac{nr_w}{p} - \tfrac{n}{p} - s] + 1), \quad B \ge \max(0, [s] + 1),$$

when $A^{s,w}_{pq}$ *denotes* $B^{s,w}_{pq}$ *and*

$$A \ge \max(0, [\sigma_{pq}(w) - s], [\tfrac{nr_w}{p} - \tfrac{n}{p} - s] + 1), \quad B \ge \max(0, [s] + 1),$$

when $A^{s,w}_{pq}$ *denotes* $F^{s,w}_{pq}$. *Let* $C > 0$ *be fixed. Let* $k(f)$ *be as in* (6) *and* (7). *Then for some* $c > 0$ *and all* $f \in A^{s,w}_{pq}(\mathbb{R}^n)$,

$$\big\|k(f)|\bar{a}^{s,w}_{pq}\big\| \le c\big\|f|A^{s,w}_{pq}(\mathbb{R}^n)\big\|.$$

Proof. We prove the theorem for Besov spaces. The proof in $F^{s,w}_{pq}$ case can be rewritten similarly to the unweighted case by using Lemma 2.6.

Let

$$f(x) = \sum_{r=0}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl}(x), \quad f \in B^{s,w}_{pq}(\mathbb{R}^n), \tag{12}$$

be an atomic decomposition according to Theorem 4.3 where

$$K = B \ge \max(0, [s] + 1) \quad \text{and} \quad L = A \ge \max(0, [-s + \sigma_p(w)], [\tfrac{nr_w}{p} - \tfrac{n}{p} - s] + 1).$$

For $j \in \mathbb{N}$ we split (12) into

$$f = f_j + f^j = \sum_{r=0}^{j} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl} + \sum_{r=j+1}^{\infty} \sum_{l \in \mathbb{Z}^n} \lambda_{rl} a_{rl}$$

and get

$$\int_{\mathbb{R}^n} k_{jm}(y) f(y)\, dy = \int_{\mathbb{R}^n} k_{jm}(y) f_j(y)\, dy + \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy.$$

Let $r \le j$ and $l \in l^j_r(m)$ where

$$l^j_r(m) = \{l : CQ_{jm} \cap DQ_{rl} \ne \emptyset\},$$

where $C, D \in \mathbb{R}$ are positive constants independent of j, r.

By the Taylor expansion of a_{rl} and properties of atoms (1) and local means (5) we have

$$\begin{aligned} &2^{j(s-n/p)}\Big|\int_{\mathbb{R}^n} k_{jm}(y) a_{rl}(y)\, dy\Big| \\ &\quad \le c\, 2^{j(s-n/p)} \sum_{|\gamma|=B} \sup_x |D^\gamma a_{rl}(x)| \int_{\mathbb{R}^n} |k_{jm}(y)|\big|y - 2^{-j}m\big|^B\, dy = c\, 2^{(j-r)(s-n/p-B)}. \end{aligned}$$

Thus for any $\varepsilon > 0$ we have

$$2^{j(s-n/p)p} |k_{jm}(f_j)|^p \le c \sum_{r=0}^{j} \sum_{l \in l^j_r(m)} |\lambda_{rl}|^p 2^{(j-r)(s-n/p-B+\varepsilon)p}.$$

Summing over $m \in \mathbb{Z}^n$ we get

$$\begin{aligned}
&2^{j(s-n/p)p} \sum_{m\in\mathbb{Z}^n} |k_{jm}(f_j)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r=0}^{j} 2^{(j-r)(s-n/p-B+\varepsilon)p} \sum_{m\in\mathbb{Z}^n} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&= c \sum_{r=0}^{j} 2^{(j-r)(s-n/p-B+\varepsilon)p} \sum_{l\in\mathbb{Z}^n} \sum_{m:l\in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r=0}^{j} 2^{(j-r)(s-B+\varepsilon)p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|}
\end{aligned} \tag{13}$$

where the last inequality is a consequence of the estimate $\operatorname{card} l_r^j(m) \sim 1$, which follows from the assumption $r \le j$.

Now let $r > j$. Using the Taylor expansion of k_{jm} and moment conditions of atoms (2) and (4) we have

$$\begin{aligned}
&2^{j(s-n/p)} \Big| \int_{\mathbb{R}^n} k_{jm}(y) a_{rl}(y)\, dy \Big| \\
&\le 2^{j(s-n/p)} \sum_{|\gamma|=A} \sup_x |D^\gamma k_{jm}(x)| \int_{\mathbb{R}^n} |a_{rl}(y)| \big|y - 2^{-r} l\big|^A \, dy = c\, 2^{(j-r)(s-n/p+n+A)}.
\end{aligned}$$

Thus for any $\varepsilon > 0$ we get

$$2^{j(s-n/p)p} \Big| \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy \Big|^p \le c \sum_{r>j} 2^{(j-r)(s-n/p+n+A-\varepsilon)p} \Big(\sum_{l\in l_r^j(m)} |\lambda_{rl}| \Big)^p.$$

From the Hölder's inequality and the estimates card $l_r^j(m) \sim 2^{n(r-j)}$

$$2^{j(s-n/p)p} \Big| \int_{\mathbb{R}^n} k_{jm}(y) f^j(y)\, dy \Big|^p \le c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p.$$

Summing over $m \in \mathbb{Z}^n$

$$\begin{aligned}
&2^{j(s-n/p)p} \sum_{m\in\mathbb{Z}^n} \big|k_{jm}(f^j)\big|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r>j} 2^{(j-r)(s+A-\varepsilon)p} \sum_{m\in\mathbb{Z}^n} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{jm})}{|Q_{jm}|} \\
&\le c \sum_{r>j} 2^{(j-r)(s+A+n/p-\varepsilon)p} \sum_{m\in\mathbb{Z}^n} \sum_{l\in l_r^j(m)} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \Big(\frac{|Q_{jm}|}{|Q_{rl}|} \Big)^u \\
&\le c \sum_{r>j} 2^{(j-r)(s+A+n/p-\varepsilon-nu/p)p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|},
\end{aligned} \tag{14}$$

where the second inequality follows from the fact that for $w \in \mathcal{A}_u^{\mathrm{loc}}$ and $Q_{rl} \subset Q_{jm}$ we have

$$w(Q_{jm}) \le c w(Q_{rl}) \Big(\frac{|Q_{jm}|}{|Q_{rl}|} \Big)^u.$$

Taking (13) and (14) together we get

$$2^{j(s-n/p)p} \sum_{m\in\mathbb{Z}^n} |k_{jm}(f)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \le c \sum_{r=0}^{j} 2^{(j-r)(s-B+\varepsilon)p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|}$$
$$+ c \sum_{r>j} 2^{(j-r)(s+A+n/p-\varepsilon-nr_w/p)p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \le c \sum_{r=0}^{\infty} 2^{-|j-r|\varkappa p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|},$$

where $\varkappa = \min(s + A + \frac{n}{p} - \frac{nr_w}{p} - \varepsilon, B - s - \varepsilon)$. Summing over j we have

$$\Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \Big(\sum_{m\in\mathbb{Z}^n} |k_{jm}(f)|^p \frac{w(Q_{jm})}{|Q_{jm}|} \Big)^{q/p} \Big)^{1/q}$$
$$\le c \Big(\sum_{j=0}^{\infty} \Big(\sum_{r=0}^{\infty} 2^{-|j-r|\varkappa p} \sum_{l\in\mathbb{Z}^n} |\lambda_{rl}|^p \frac{w(Q_{rl})}{|Q_{rl}|} \Big)^{q/p} \Big)^{1/q}.$$

Now using the Young inequalities for convolution of sequences if $\frac{q}{p} \ge 1$ or monotonicity of the l_p space if $\frac{q}{p} < 1$ we proved that

$$\big\| k(f) | \bar{b}_{pq}^{s,w} \big\| \le c \big\| \lambda | b_{pq}^{w} \big\| \le c \big\| f | B_{pq}^{s,w} \big\|,$$

where the constant c is independent of the given atomic decomposition. ∎

6. Characterization by wavelets. We are going to deal with Daubechies wavelets on $\mathbb{R}^n$. Let $\psi^F \in C^k(\mathbb{R})$ be Daubechies scaling function and $\psi^M \in C^k(\mathbb{R})$ Daubechies wavelet with $\int_{\mathbb{R}} \psi(x) x^v \, dx = 0$, $k \in \mathbb{N}$, $v \in \mathbb{N}_0$, $v < k$. We extend these wavelets from $\mathbb{R}$ to $\mathbb{R}^n$ by the usual tensor product procedure

$$\Psi_{jm}^G = 2^{jn/2} \prod_{r=1}^{n} \psi^{G_r}(2^j x_r - m_r), \tag{15}$$

where $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$, $G = (G_1, \ldots, G_n) \in G^j$ and $G^0 = \{F, M\}^n$ and for $j > 0$ $G^j = \{F, M\}^{n*}$, where $*$ indicates that at least one G_r must be an M.

$$\big\{ \Psi_{jm}^G : j \in \mathbb{N}_0, m \in \mathbb{Z}^n, G \in G^j \big\}$$

is an orthonormal basis in $L_2(\mathbb{R}^n)$, cf. [11].

Definition 6.1. Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $w \in \mathcal{A}_\infty^{\mathrm{loc}}$. Then $b_{pq}^{s,w}$ is the collection of all sequences λ according to (3) such that

$$\big\| \lambda | b_{pq}^{s,w} \big\| = \Big(\sum_{j=0}^{\infty} 2^{j(s-n/p)q} \sum_{G\in G^j} \Big(\int_{\mathbb{R}^n} \Big| \sum_{m\in\mathbb{Z}^n} \lambda_{jm}^G \chi_{jm}^{(p)}(x) \Big|^p w(x) \, dx \Big)^{q/p} \Big)^{1/q} < \infty$$

and $f_{pq}^{s,w}$ is the collection of all sequences λ according to (3) such that

$$\big\| \lambda | f_{pq}^{s,w} \big\| = \Big\| \Big(\sum_{j,m,G} 2^{jsq} |\lambda_{jm}^G \chi_{jm}^{(p)}|^q \Big)^{1/q} | L_p^w \Big\| < \infty.$$

THEOREM 6.2. *Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ and $w \in \mathcal{A}_\infty^{\rm loc}$. For wavelets defined in* (15) *we take*

$$k \geq \max\big(0, [s]+1, [\tfrac{nr_w}{p} - \tfrac{n}{p} - s]+1, [\sigma_p(w)-s]\big)$$

in $B^{s,w}_{pq}$ case and

$$k \geq \max\big(0, [s]+1, [\tfrac{nr_w}{p} - \tfrac{n}{p} - s]+1, [\sigma_{pq}(w)-s]\big)$$

in $F^{s,w}_{pq}$ case. Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$. Then $f \in A^{s,w}_{pq}(\mathbb{R}^n)$ if and only if it can be represented as

$$f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2} \Psi^G_{jm},$$

where $\lambda \in a^{s,w}_{pq}$ and the series converges in $S'_e(\mathbb{R}^n)$. This representation is unique with

$$\lambda^G_{jm} = 2^{jn/2}\left(f, \Psi^G_{jm}\right)$$

and

$$I : f \mapsto \left\{2^{jn/2}(f, \Psi^G_{jm})\right\}$$

is a linear isomorphism of $A^{s,w}_{pq}(\mathbb{R}^n)$ onto $a^{s,w}_{pq}$.

If $0 < p, q < \infty$ then the system $\left\{\Psi^G_{jm}\right\}_{j,m,G}$ is an unconditional basis in $A^{s,w}_{pq}(\mathbb{R}^n)$.

Proof.

Step 1. Let $f \in \mathcal{S}'_e(\mathbb{R}^n)$ and $f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2}\Psi^G_{jm}$. Then $a^G_{jm} = 2^{-j(s-\frac{n}{p})}2^{-j\frac{n}{2}}\Psi^G_{jm}$ is an (s,p)-atom. Indeed,

$$\operatorname{supp} a^G_{jm} \subset dQ_{jm} \qquad \text{and} \qquad \left|D^\alpha a^G_{jm}\right| \leq 2^{-j(s-n/p)+j|\alpha|}$$

for $|\alpha| \leq k$ and $k = K = L$ in the definition of atoms. So $f \in B^{s,w}_{pq}(\mathbb{R}^n)$ and

$$\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\| \leq c\left\|\left\{2^{j(s-n/p)}\lambda^G_{jm}\right\}_{j,m,G}|b^w_{pq}\right\| = c\left\|\lambda|b^{s,w}_{pq}\right\|.$$

Step 2. Now let $f \in B^{s,w}_{pq}(\mathbb{R}^n)$. We take $k^G_{jm} = 2^{jn/2}\Psi^G_{jm}$ as kernels of local means. Indeed

$$\operatorname{supp} k^G_{jm} \subset CQ_{jm} \quad \text{and} \quad \left|D^\alpha k^G_{jm}(x)\right| \leq 2^{jn+j|\alpha|},$$

where $|\alpha| \leq k$ and $A = B = k$. So from Theorem 5.8 we have

$$\left\|k(f)|b^{s,w}_{pq}\right\| \leq c\left\|f|B^{s,w}_{pq}(\mathbb{R}^n)\right\|. \tag{16}$$

From the atomic decomposition and (16) we have

$$g = \sum_{j,G,m} k^G_{jm}(f)\, 2^{-jn/2}\Psi^G_{jm} \in B^{s,w}_{pq}(\mathbb{R}^n).$$

It follows from Lemma 5.6 that $(g, \Psi^{G'}_{j'm'})$ make sense. By orthogonality of wavelet basis in $L_2(\mathbb{R}^n)$ we get

$$(g, \Psi^{G'}_{j'm'}) = \sum_{j,G,m} k^G_{jm}(f)\, 2^{-jn/2}(\Psi^G_{jm}, \Psi^{G'}_{j'm'}) = (f, \Psi^{G'}_{j'm'}).$$

This could be extended to finite linear combinations of $\Psi^{G'}_{j'm'}$. Both distributions f and g are locally contained in the space $B^\sigma_{pp}(\mathbb{R}^n)$ for any $\sigma < s - \frac{nr_w}{p} + \frac{n}{p}$. This follows easily from the corresponding result for the spaces with Muckenhoupt weights, cf. [1], since any local Muckenhoupt weight $w \in \mathcal{A}_p^{\rm loc}$ can be extended outside a fixed ball to a Muckenhoupt weight belonging to $\mathcal{A}_p$. Any $\varphi \in C_0^\infty(\mathbb{R}^n)$ has the unique $L_2(\mathbb{R}^n)$ wavelet representation. But we can choose σ such that $k > \max(-\sigma+\sigma_p, \sigma)$ so this representation converges in the dual space of $B^\sigma_{pp}(\mathbb{R}^n)$, cf. [10]. This implies that $(g,\varphi) = (f,\varphi)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ and $g = f$.

Step 3. By the above steps $f \in \mathcal{S}'_e(\mathbb{R}^n)$ belongs to $B^{s,w}_{pq}(\mathbb{R}^n)$ if and only if

$$f = \sum_{j,G,m} \lambda^G_{jm} 2^{-jn/2} \Psi^G_{jm} \quad \text{and} \quad \{\lambda^G_{jm}\} \in b^{s,w}_{pq}.$$

This representation is unique so $\lambda^G_{jm} = k^G_{jm}(f)$ and

$$\big\|f|B^{s,w}_{pq}(\mathbb{R}^n)\big\| \sim \big\|k(f)|b^{s,w}_{pq}\big\|.$$

It follows from the uniqueness of the coefficients that I is a monomorphism. We show that I is onto. Let $\{\lambda^G_{jm}\} \in b^w_{pq}$. Then by the atomic decomposition theorem

$$f = \sum_{j,G,m} \lambda^G_{jm} \Psi^G_{jm} \in B^{s,w}_{pq}(\mathbb{R}^n).$$

But the uniqueness of the coefficients implies that $\lambda^G_{jm} = (f, \Psi^G_{jm})$.

Let $\varepsilon_{j,G,m} = \pm 1$, then the sequence $\varepsilon_{j,G,m}\lambda^G_{jm}$ belongs to b^w_{pq} provided that $\{\lambda^G_{jm}\}$ belongs to b^w_{pq}. Thus the atomic decomposition theorem implies the convergence of the series

$$\sum_{j,G,m} \varepsilon_{j,G,m} \lambda^G_{jm} \Psi^G_{jm}$$

in $B^{s,w}_{pq}(\mathbb{R}^n)$. This implies the unconditional convergence.

In the case of $F^{s,w}_{pq}$ spaces the proof is similar. ∎

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