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ON SOME PROPERTIES FOR DUAL SPACES OF MUSIELAK-ORLICZ FUNCTION SPACES

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Abstract. We will present relationships between the modular ρ^* and the norm in the dual spaces $(L_\Phi)^*$ in the case when a Musielak-Orlicz space L_Φ is equipped with the Orlicz norm. Moreover, criteria for extreme points of the unit sphere of the dual space $(L_\Phi^o)^*$ will be presented.

1. Introduction. The triple (T, Σ, μ) stands for a positive, nonatomic, σ-finite and complete measure space. By $L^0 = L^0(\mu)$ we denote the space of all (equivalence classes of) Σ-measurable real functions x defined on T. A mapping $\Phi : T \times \mathbb{R} \longrightarrow \mathbb{R}_+$ is said to be a *Musielak-Orlicz function* if it satisfies the *Carathéodory conditions*, i.e. for any $u \in \mathbb{R}$, the function $\Phi(\cdot, u)$ is Σ-measurable and there is a set $T_0 \in \Sigma$ with $\mu(T_0) = 0$ such that for any $t \in T \setminus T_0$ the function $\Phi(t, \cdot)$ is an Orlicz function, i.e. it is convex, even, vanishing at zero and satisfying $\Phi(t, u)/u \to +\infty$ as $u \to +\infty$.

For every Musielak-Orlicz function Φ we define its *complementary function in the sense of Young* $\Psi : T \times \mathbb{R} \to [0, \infty)$ by the formula

$$\Psi(t, v) = \sup_{u>0}\{u\,|v| - \Phi(t, u)\}$$

for every $v \in \mathbb{R}$ and $t \in T$. Given any Musielak-Orlicz function Φ define on L^0 a convex modular I_Φ by

$$I_\Phi(x) = \int_T \Phi(t, x(t))\, d\mu$$

for every $x \in L^0$. Then the *Musielak-Orlicz function space* L_Φ and its subspace E_Φ are

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defined as follows:

$$L_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < +\infty \quad \text{for some } \lambda > 0\},$$
$$E_\Phi = \{x \in L^0 : I_\Phi(\lambda x) < +\infty \quad \text{for any } \lambda > 0\}.$$

It is easy to see that E_Φ is the subspace of order continuous elements in L_Φ.

The spaces L_Φ and E_Φ coincide if and only if Φ satisfies the so-called Δ_2-*condition*. Recall that Φ satisfies the Δ_2-*condition* ($\Phi \in \Delta_2$ for short), if there are a set T_0 of measure zero, a constant $K > 0$ and a Σ-measurable nonnegative function h defined on T such that $\int_T h(t)\,d\mu < +\infty$ and $\Phi(t, 2u) \le K\Phi(t, u) + h(t)$ for every $t \in T \setminus T_0$ and $u \in \mathbb{R}$.

For any $t \in T \setminus T_0$, by $p(t, u)$ and $q(t, u)$ we denote the right derivatives of $\Phi(t, \cdot)$ and $\Psi(t, \cdot)$ at any fixed point $u \in \mathbb{R}$, respectively. For every $u, v \in \mathbb{R}$ and all $t \in T \setminus T_0$, we have the following *Young inequality*

$$uv \le \Phi(t, u) + \Psi(t, v),$$

and for a given $t \in T \setminus T_0$ the equality $uv = \Phi(t, u) + \Psi(t, v)$ holds whenever $u \in \mathbb{R}$ and $v \in [q(t, u), p(t, u)]$.

For any $x \in L_\Phi$ the *Luxemburg norm* is defined by

$$\|x\| = \inf\left\{k > 0 : I_\Phi(x/k) \le 1\right\},$$

and the *Orlicz norm* is defined by

$$\|x\|^o = \sup\Big\{\int_T x(t)y(t)\,d\mu : I_\Psi(y) \le 1\Big\}.$$

Let us note that the Orlicz norm on L_Φ can be also defined by the very useful *Amemiya formula* [F]:

$$\|x\|^o = \inf_{k>0} \frac{1}{k}(1 + I_\Phi(kx)).$$

LEMMA 1.1 (see [K]). *Let Φ be a Musielak-Orlicz function. Then there exists an ascending sequence $(T_n)_{n=1}^\infty$ of measurable sets with $0 < \mu(T_n) < +\infty$ such that $\sup_{t \in T_n} \Phi(t, \lambda) < +\infty$ for every $\lambda > 0$, for any $n \in \mathbb{N}$ and $\mu\left(T \setminus \bigcup_{n=1}^\infty T_n\right) = 0$.*

This yields that χ_{T_n} (the characteristic function of T_n) belongs to E_Φ for any $n \in N$.

We denote by L_Φ^* the dual space of L_Φ and $\varphi \in L_\Phi^*$ is called a singular functional ($\varphi \in F$ for short), if $\varphi(E_\Phi) = \{0\}$, that is, $\varphi(x) = 0$ for any $x \in E_\Phi$.

Any functional $f \in L_\Phi^*$ has the unique decomposition

$$f = v + \varphi \qquad (v \in L_\Psi,\ \varphi \in F),$$

where v means in fact the regular functional defined by the function v from L_Ψ by the formula $\langle v, x\rangle = \int_T v(t)x(t)\,d\mu$ for any $x \in L_\Phi$.

Let us define for each $f \in L_\Phi^*$:

$$\|f\|_\Psi^o = \|f\|^o = \sup\left\{f(u) : \|u\| = 1\right\}, \qquad \|f\|_\Psi = \|f\| = \sup\{f(u) : \|u\|^o = 1\}.$$

The following results are due to H. Hudzik and Z. Zbąszyniak [HZ]:

LEMMA 1.2. *Let* $f \in L_\Phi^*$. *Then*

$$\|f\|^o = \|v\|^o + \|\varphi\|^o,$$
$$\|f\| = \inf\{\lambda > 0 : I_\Psi(v/\lambda) + \|\varphi\|/\lambda \le 1\}.$$

LEMMA 1.3. *For any* $\varphi \in F$,

$$\|\varphi\| = \|\varphi\|^o = \sup\{\varphi(u) : I_\Phi(u) < +\infty\}.$$

2. Result

PROPOSITION 2.1. *Let* $f \in L_\Phi^*$. *If* $\|f\|^o \le 1$ *then*

(i) $I_\Phi(q(|v|)) \le 1$,
(ii) $\rho^*(f) \le \|f\|^o$.

Proof. By virtue of Lemma 1.1 we can repeat the proof of Proposition 2.2 in [CHL] with the sequence of sets $(T_n)_{n=1}^\infty$ from Lemma 1.1. ■

PROPOSITION 2.2. *Let* Φ *be a Musielak-Orlicz function satisfying condition* $\Phi(t,u)/u \to 0$ *as* $u \to 0$ *for* μ-*a.e.* $t \in T$. *Then the convergence to zero in the Orlicz norm* $\|\cdot\|^o$ *and in the modular in* L_Φ^* *are equivalent if and only if* $\Psi \in \Delta_2$.

Proof. We can repeat here the proof of the necessity of Proposition 2.3 from [CHL].

Sufficiency. Let $f_n \in L_\Phi^*$, where $f_n = v_n + \varphi_n$ for any $n \in \mathbb{N}$ and $\rho^*(f_n) = I_\Psi(v_n) + \|\varphi_n\| \to 0$. Then $I_\Psi(v_n) \to 0$ and $\|\varphi_n\| \to 0$. Since $\Psi \in \Delta_2$ and Ψ vanishes only at zero, we can deduce from Theorem 3.3 in [KH] that $\|v_n\|^o \to 0$, and consequently $\|f_n\|^o = \|v_n\|^o + \|\varphi_n\|^o \to 0$. ■

The proofs of the next three propositions and of Proposition 2.6 (2)–(5) can proceed analogously as the respective proofs in [CHL].

PROPOSITION 2.3. *Let* $f \in L_\Phi^*$. *If there exists* $k > 0$ *such that*

$$\int_T \Phi\big(t, q(t, k\,|v(t)|)\big)\, d\mu = 1,$$

then $\|f\|^o = \int_T |v(t)|\, q(t, k\,|v(t)|)\, d\mu + \|\varphi\| = \frac{1}{k}(1 + \rho^*(kf))$.

PROPOSITION 2.4. *If* $f \in L_\Phi^*$, *then*

$$\|f\|^o = \inf_{k>0} \frac{1}{k}(1 + \rho^*(kf)).$$

PROPOSITION 2.5. *If* $f \in L_\Phi^*$, *then*

(1) $\|f\| \le 1 \implies \rho^*(f) \le \|f\|$,
(2) $\|f\| > 1 \implies \rho^*(f) > \|f\|$,
(3) $\|f\| \le \|f\|^o \le 2\,\|f\|$.

PROPOSITION 2.6. *Suppose* $\Psi \in \Delta_2$, $\Phi(t,u)/u \to 0$ *as* $u \to 0$ *for* μ-*a.e.* $t \in T$ *and* $f_n, f \in L_\Phi^*$, *where* $f_n = v_n + \varphi_n$, $f = v + \varphi$, $(v_n, v \in L_\Psi,\ \varphi_n, \varphi \in F)$. *Then*

(1) $\rho^*(f_n) \to \infty \implies \|f_n\| \to \infty$,
(2) $\|f\| = 1 \implies \rho^*(f) = 1$,

(3) $\forall \varepsilon > 0\ \exists \delta > 0 \quad (\|f\| \ge \varepsilon \implies \rho^*(f) \ge \delta)$,
(4) $\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1) \quad (\rho^*(f) \le 1 - \varepsilon \implies \|f\| \le 1 - \delta)$,
(5) $\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1) \quad (\rho^*(f) \ge 1 + \varepsilon \implies \|f\| \ge 1 + \delta)$.

Proof. Let us prove implication (1). Suppose that $\Psi \in \Delta_2$, where the function h is defined on T and $\Psi(t,2u) \le K\Psi(t,u) + h(t)$ for every $t \in T \setminus T_0$ with $\mu(T_0) = 0$ and $u \in \mathbb{R}$. We have for a constant $L > 1$

$$\Psi(t, Lv) \le \widetilde{K}\Psi(t,v) + \widetilde{h}(t)$$

for all $t \in T \setminus T_0$ and $v \in \mathbb{R}$ with a positive constant $\widetilde{K}$ and a nonnegative integrable function $\widetilde{h}$.

Then $\|f\| \le L$ implies that $\|f/L\| \le 1$, whence

$$\rho^*(f/L) = I_\Psi(v/L) + \|\varphi/L\| \le 1$$

and so

$$I_\Psi(v/L) \le 1 \quad \text{and} \quad \|\varphi/L\| \le 1.$$

In consequence, setting $M = \int_T \widetilde{h}(t)\, d\mu$, we obtain

$$\begin{aligned}
\rho^*(f) = I_\Psi(v) + \|\varphi\| &= \int_T \Psi\Big(t, \frac{1}{L}Lv(t)\Big)\, d\mu + \|\varphi\| \\
&\le \int_T \Big[\widetilde{K}\Psi\Big(t, \frac{1}{L}v(t)\Big) + \widetilde{h}(t)\Big]\, d\mu + \|\varphi\| \\
&= \widetilde{K} I_\Psi(v/L) + \int_T \widetilde{h}(t)\, d\mu + \|\varphi\| \le \widetilde{K} + M + L.
\end{aligned}$$

By the transposition law this finishes the proof of (1). ∎

PROPOSITION 2.7. *If the Musielak-Orlicz function Φ satisfies the condition $\Phi(t,u)/u \to 0$ as $u \to 0$ for μ-a.e. $t \in T$ and $\Psi \in \Delta_2$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $f, g \in L^*_\Phi$ with $\rho^*(f) \le L$ and $\rho^*(g) \le \delta$ the inequality $|\rho^*(f+g) + \rho^*(f)| < \varepsilon$ is satisfied.*

Proof. Since $\Psi(t,u)$ vanishes only at zero for μ-a.e. $t \in T$, by virtue of Lemma 1.6 in [H] there exists a set $A \in \Sigma$ with $\mu(A) = 0$ such that for any $\varepsilon > 0$ there exist a function $h_\varepsilon(\cdot) \ge 0$, $\int_T h_\varepsilon(t)\, d\mu \le \varepsilon$ and a constant $M_\varepsilon \ge 2$, such that

$$\Psi(t,2u) \le M_\varepsilon \Psi(t,u) + h_\varepsilon(t)$$

for any $u \in \mathbb{R}$ and $t \in T \setminus A$. Therefore

$$\begin{aligned}
\rho^*(2f) = I_\Psi(2v) + 2\,\|\varphi\| &= \int_T \Psi(t, 2v(t))\, d\mu + 2\,\|\varphi\| \\
&\le M_\varepsilon \int_T \Psi(t, v(t))\, d\mu + \int_T h_\varepsilon(t)\, d\mu + 2\,\|\varphi\| \\
&\le M_\varepsilon I_\Psi(v) + \varepsilon + M_\varepsilon\,\|\varphi\| \\
&= M_\varepsilon(I_\Psi(v) + \|\varphi\|) + \varepsilon = M_\varepsilon \rho^*(f) + \varepsilon,
\end{aligned}$$

which means that the modular ρ^* satisfies in L^*_Φ the condition Δ^s_2 defined in [CH]. By Lemma 2.1 in [CH] the proof is complete. ∎

THEOREM 2.8. *Let Φ be a Musielak-Orlicz function such that $\Phi(t,u)/u \to 0$ as $u \to 0$ for μ-a.e. $t \in T$. Then a functional $\varphi \in S(F)$ is an extreme point of $B(L_\Phi^*)$ if and only if $\|\varphi|_E\| = \|\varphi|_{T\setminus E}\| = 0$ for every $E \in \Sigma$.*

Proof. The proof proceeds in the same way as the proof of Theorem 3.1 in [CHL]. ■

THEOREM 2.9. *A functional $f = v + \varphi \in S(L_\Phi^*)$ is an extreme point of $B(L_\Phi^*)$ if and only if the following conditions are satisfied:*

(1) $\rho^*(f) = 1$,
(2) *$v(t)$ is a point of strict convexity of Ψ for μ-a.e. $t \in T$,*
(3) *$\varphi/\|\varphi\|$ is an extreme point of $B(L_\Phi^*)$.*

Proof. The sufficiency follows in the same way as the sufficiency of Theorem 3.2 in [CHL].

Necessity. Let $f = v + \varphi \in \operatorname{Ext} B(L_\Phi^*)$ and let us assume that condition (1) is not satisfied. If $\varepsilon = 1 - \rho^*(f) > 0$, then we can choose $E \in \Sigma$ such that

$$0 < \int_E \Psi(t, 2v(t))\, d\mu \le \varepsilon.$$

Define

$$v_1(t) = \begin{cases} v(t) & \text{for } t \in T\setminus E \\ 0 & \text{for } t \in E \end{cases} \qquad \text{and} \qquad v_2(t) = \begin{cases} v(t) & \text{for } t \in T\setminus E \\ 2v(t) & \text{for } t \in E. \end{cases}$$

Then $v_1 \ne v_2$ and $v_1 + v_2 = 2v$. Defining $f_1 = v_1 + \varphi$ and $f_2 = v_2 + \varphi$, we have

$$\begin{aligned} \rho^*(f_1) &= I_\Psi(v_1) + \|\varphi\| < I_\Psi(v_2) + \|\varphi\| = \rho^*(f_2) \\ &< I_\Psi(v) + \varepsilon + \|\varphi\| = \rho^*(f) + \varepsilon = 1, \end{aligned}$$

whence $\|f_1\| \le 1$ and analogously $\|f_2\| \le 1$. We have $1 = \|f\| = \|(f_1+f_2)/2\| \le \frac{1}{2}(\|f_1\| + \|f_2\|) \le 1$ and consequently $\|f_1\| = \|f_2\| = 1$, which contradicts the assumption that $f \in \operatorname{Ext} B(L_\Phi^*)$.

Assume now that condition (2) is not satisfied. Then there exist a set $A \in \Sigma$ with $\mu(A) > 0$ and two numbers a, b with $0 < a < b < \infty$ such that $a < v(t) < b$ and $\Phi(t,\cdot)$ is affine on $[a,b]$ for all $t \in A$. Further there exist $\varepsilon > 0$ and $K \in \Sigma$ with $K \subset A$ and $\mu(K) > 0$ such that $a + \varepsilon < v(t) < b - \varepsilon$ for all $t \in K$. Let us write $\Psi(t,u)$ for $(t,u) \in K \times [a,b]$ in the form $\Psi(t,u) = \alpha(t)u + \beta(t)$ with $\alpha(t) > 0$, $\beta(t) > 0$ for μ-a.e. $t \in K$.

Next, let us define on $\Sigma \cap K$ the measure

$$\mu_\alpha(B) = \int_B \alpha(t)\, d\mu \quad (\forall B \in \Sigma \cap K).$$

This is an atomless measure, so there exist two sets $K_1, K_2 \in \Sigma \cap K$ such that $K_1 \cap K_2 = \varnothing$, $K_1 \cup K_2 = K$ and $\mu_\alpha(K_1) = \mu_\alpha(K_2)$. This means that

$$\int_{K_1} \alpha(t)\, d\mu = \int_{K_2} \alpha(t)\, d\mu.$$

Let us define two functions v_1 and v_2 by

$$v_1(t) = \begin{cases} v(t) & \text{for } t \in T \setminus (K_1 \cup K_2) \\ v(t) - \varepsilon & \text{for } t \in K_1 \\ v(t) + \varepsilon & \text{for } t \in K_2, \end{cases} \qquad v_2(t) = \begin{cases} v(t) & \text{for } t \in T \setminus (K_1 \cup K_2) \\ v(t) + \varepsilon & \text{for } t \in K_1 \\ v(t) - \varepsilon & \text{for } t \in K_2. \end{cases}$$

Then $v_1 \neq v_2$ and $v_1 + v_2 = 2v$. Let us define $f_1 = v_1 + \varphi$, $f_2 = v_2 + \varphi$. Then

$$\begin{aligned} I_\Psi(v_1) &= \int_{T\setminus K} \Psi(t, v(t))\, d\mu + \int_{K_1} [\alpha(t)(v(t) - \varepsilon) + \beta(t)]\, d\mu \\ &\quad + \int_{K_2} [\alpha(t)(v(t) + \varepsilon) + \beta(t)]\, d\mu \\ &= \int_{T\setminus K} \Psi(t, v(t))\, d\mu + \int_{K} [\alpha(t)v(t) + \beta(t)]\, d\mu = \int_T \Psi(t, v(t))\, d\mu = I_\Psi(v), \end{aligned}$$

which gives

$$\rho^*(f_1) = I_\Psi(v_1) + \|\varphi\| = I_\Psi(v) + \|\varphi\| = \rho^*(f) = 1.$$

Similarly we deduce that $\rho^*(f_2) = 1$, whence it follows that $\|f_1\| = \|f_2\| = 1$. Since $f_1 \neq f_2$, this contradicts the assumption that f is an extreme point of $B(L_\Phi^*)$.

If (3) does not hold, then there exist $\varphi_1, \varphi_2 \in S(F)$, $\varphi_1 \neq \varphi_2$ such that $2\frac{\varphi}{\|\varphi\|} = \varphi_1 + \varphi_2$. Let $\varphi_1' = \|\varphi\|\, \varphi_1$, $\varphi_2' = \|\varphi\|\, \varphi_2$. Then $\|\varphi_1'\| = \|\varphi_2'\| = \|\varphi\|$. Defining $f_1 = v + \varphi_1'$, $f_2 = v + \varphi_2'$, we have $\rho^*(f_1) = I_\Psi(v) + \|\varphi_1'\| = I_\Psi(v) + \|\varphi\| = \rho^*(f) = 1$ and similarly $\rho^*(f_2) = 1$, which contradicts the assumption that $f \in \operatorname{Ext} B(L_\Phi^*)$. ∎

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