ORLICZ CENTENARY VOLUME II
BANACH CENTER PUBLICATIONS, VOLUME 68
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2005

ON SOME GLOBAL AND LOCAL GEOMETRIC PROPERTIES OF CALDERÓN-LOZANOVSKIĬ SPACES

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Abstract. Criteria for full k-rotundity ($k \in \mathbb{N}$, $k \geqslant 2$) and uniform rotundity in every direction of Calderón-Lozanovskiĭ spaces are formulated. A characterization of H_μ-points in these spaces is also given.

Introduction. First we introduce the notations and define the notions used in this paper. Let $(X, \|\ \|)$ be a real Banach space and $S(X)$, $B(X)$ denote the unit sphere and the (closed) unit ball of the space X, respectively.

A Banach space X is called *fully k-rotund* (kR-*space* for short), where $k \in \mathbb{N}$, $k \geqslant 2$, if any sequence (x_n) in $B(X)$ such that

$$\|x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k)}\| \to k$$

for arbitrary subsequences $(x_n^{(1)}), (x_n^{(2)}), \dots, (x_n^{(k)})$ as $n \to \infty$, is a Cauchy sequence (see [FG]). It is known that any kR-space is a $(k+1)R$-space ($k \geqslant 2$).

A Banach space X is said to be *compactly fully k-rotund* (CkR-space for short) if every sequence (x_n) in $B(X)$ satisfying

$$\|x_n^{(1)} + x_n^{(2)} + \dots + x_n^{(k)}\| \to k$$

for any subsequences $(x_n^{(1)}), (x_n^{(2)}), \dots, (x_n^{(k)})$ as $n \to \infty$, is a relatively compact sequence. Compact full k-rotundity of a Banach space X implies reflexivity (see [CHK]) and approximative compactness of the space X (see [HW]). A Banach space X is fully k-rotund iff it is compactly fully k-rotund and rotund (see [CHK]).

We say that a Banach space X is *uniformly convex in every direction* ($URED$-space for short) if for any $\varepsilon \in (0,1)$ and $z \in S(X)$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that $\|(x+y)/2\| \leqslant 1 - \delta(\varepsilon, z)$ for any $x, y \in B(X)$ with $x - y = \varepsilon z$ or equivalently, if for any

2000 *Mathematics Subject Classification*: 46E30, 46B20, 46B30, 46C05.
Key words and phrases: Köthe space, Calderón-Lozanovskiĭ space, Orlicz function, delta two type condition, fully k-rotundity, uniform rotundity in every direction, H_μ-points.

The paper is in final form and no version of it will be published elsewhere.

$\varepsilon \in (0,1)$ and $z \in S(X)$ there exists $\delta(\varepsilon, z) \in (0,1)$ such that inequality $\|y + \varepsilon z/2\| \leqslant 1 - \delta(\varepsilon, z)$ holds whenever $y \in B(X)$ and $\|y + \varepsilon z\| \leqslant 1$.

Recall that if a Banach space X is $URED$, then it has normal structure and so it has the weak fixed point property (see [CCHS]).

Let (T, Σ, μ) be a complete and σ-finite measure space and $L^0 = L^0(T, \Sigma, \mu)$ be the space of all (equivalence classes of) Σ-measurable real functions defined on T.

A Banach space $(E, \| \|_E)$ is said to be a *Köthe space* (see [KA]) if $E \subset L^0$ and:

(i) for every $x \in L^0$ and $y \in E$ with $|x(t)| \leqslant |y(t)|$ for μ-a.e. $t \in T$, we have $x \in E$ and $\|x\|_E \leqslant \|y\|_E$,

(ii) there is a function $x \in E$ such that $x(t) > 0$ for any $t \in T$.

By E^+ we denote the positive cone of E, that is, $E^+ = \{x \in E : x \geqslant 0\}$.

A Köthe space E is said to be *uniformly monotone* if for any $\varepsilon \in (0,1)$ there is $\delta(\varepsilon) \in (0,1)$ such that $\|x - y\|_E \leqslant 1 - \delta(\varepsilon)$ whenever $0 \leqslant y \leqslant x$, $\|x\|_E = 1$ and $\|y\|_E \geqslant \varepsilon$. For the conditions that are equivalent to this definition we refer to [HKM2].

We say that a Köthe space E has the *Fatou property* ($E \in (FP)$ for short) if for any $x \in L^0$ and (x_n) in E^+ such that $x_n \uparrow |x|$ μ-a.e. and $\sup_n \|x_n\|_E < \infty$, we have $x \in E$ and $\|x_n\|_E \to \|x\|_E$ (see [Bi] and [KA]).

A point $x \in E$ is said to have *order continuous norm* if for any sequence (y_n) in E such that $0 \leqslant y_n \leqslant |x|$ $(n \in \mathbb{N})$ and $y_n \to 0$ μ-a.e., we have $\|y_n\|_E \to 0$. If every point of E has order continuous norm, then we say that the space E is *order continuous*.

A point $x \in E$ is said an H_μ*-point* if for any sequence $(x_n) \subset E$ such that $x_n \to x$ locally in measure and $\|x_n\|_E \to \|x\|_E$, we have $\|x_n - x\|_E \to 0$. If every point $x \in E$ is H_μ-point, then we say that the space E has H_μ*-property* (see [HM]).

A function $\varphi : [0, \infty) \to [0, \infty]$ is said to be an *Orlicz function* if φ is convex, vanishing and continuous at zero, left continuous on $(0, \infty)$ and not identically equal to zero (see [Ch], [KR], [Lu], [Ma], [Mu] and [RR]). If the Orlicz function φ vanishes only at zero, then we will write $\varphi > 0$ and if φ takes only values from $[0, \infty)$, then we will write $\varphi < \infty$.

Given a real Köthe space E and an Orlicz function φ, we define on L^0 the convex modular

$$\varrho_\varphi(x) = \begin{cases} \|\varphi \circ |x|\,\|_E & \text{if } \varphi \circ |x| \in E, \\ \infty & \text{otherwise.} \end{cases}$$

The *Calderón-Lozanovskiĭ space* E_φ generated by the couple (E, φ) is defined as the set of those $x \in L^0$ such that $\varrho_\varphi(\lambda x) < +\infty$ for some $\lambda > 0$. The norm in E_φ is defined by

$$\|x\|_\varphi = \inf\{\lambda > 0 : \varrho_\varphi(x/\lambda) \leqslant 1\}$$

(see [CHM] and [Ma]; cf. [Ca] and [Lo]). If E has the Fatou property, then also E_φ has this property, whence it follows that E_φ is a Banach space. This class of Köthe spaces is a subclass of the more general class of Köthe spaces $\Psi(E, F)$ that are interpolation spaces between two Köthe spaces E and F over the same measure space generated by concave and homogeneous functions $\Psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$. Köthe spaces constructed in such a way by Lozanovskiĭ (see [Lo]) are generalizations of the interpolation spaces constructed by Calderón (see [Ca]).

In the remaining part of the paper we will assume that E is a Köthe space with the Fatou property.

We say an Orlicz function φ satisfies *condition* $\Delta_2(0)$ ($\varphi \in \Delta_2(0)$ for short) if there exist $K > 0$ and $u_0 > 0$ such that $0 < \varphi(u_0)$ and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \in [0, u_0]$.

We say a function φ satisfies *condition* $\Delta_2(\infty)$ ($\varphi \in \Delta_2(\infty)$ for short) if there exist $K > 0$ and $u_0 > 0$ such that $\varphi(u_0) < \infty$ and the inequality $\varphi(2u) \leqslant K\varphi(u)$ holds for all $u \geqslant u_0$.

If there exists $K > 0$ such that $\varphi(2u) \leqslant K\varphi(u)$ for all $u \geqslant 0$, then we say that φ satisfies *condition* $\Delta_2(\mathbb{R}_+)$ ($\varphi \in \Delta_2(\mathbb{R}_+)$ for short).

For a Köthe space E and an Orlicz function φ we say that φ satisfies *condition* Δ_2^E ($\varphi \in \Delta_2^E$ for short) if:

1) $\varphi \in \Delta_2(0)$ whenever $E \hookrightarrow L^\infty$,
2) $\varphi \in \Delta_2(\infty)$ whenever $L^\infty \hookrightarrow E$,
3) $\varphi \in \Delta_2(\mathbb{R}_+)$ whenever neither $L^\infty \hookrightarrow E$ nor $E \hookrightarrow L^\infty$

(see [HKM1]).

LEMMA 1. *If E_φ is a Calderón-Lozanovskiĭ space and $x \in E_\varphi$, then:*

(*i*) *if $\|x\|_\varphi \leqslant 1$, then $\varrho_\varphi(x) \leqslant \|x\|_\varphi$,*
(*ii*) *if $\|x\|_\varphi > 1$, then $\varrho_\varphi(x) \geqslant \|x\|_\varphi$.*

LEMMA 2 (see [CHM], [FH1] and [FH2]). *If φ is an Orlicz function such that $\varphi < \infty$, $\varphi \in \Delta_2^E$ and E is a Köthe space, then for any $x \in E_\varphi$ and any sequence (x_n) in E_φ, we have:*

(*i*) *$\varrho_\varphi(x) = 1$ whenever $\|x\|_\varphi = 1$,*
(*ii*) *$\varrho_\varphi(x_n) \to 1$ whenever $\|x_n\|_\varphi \to 1$,*
(*iii*) *$\varrho_\varphi(\lambda x) < \infty$ for any $\lambda \geqslant 0$.*

LEMMA 3 (see [CHM], [FH1] and [FH2]). *Let φ be an Orlicz function such that $\varphi > 0$ and $\varphi \in \Delta_2^E$. Then for any sequence (x_n) in the Calderón-Lozanovskiĭ space E_φ, we have $\|x_n\|_\varphi \to 0$ whenever $\varrho_\varphi(x_n) \to 0$.*

REMARK 1. For any real numbers a, b we have:

(*i*) if $ab \geqslant 0$, then $|a+b| = |a| + |b|$ and $|a-b| = \big||a| - |b|\big|$,
(*ii*) if $ab < 0$, then $|a+b| = \big||a| - |b|\big|$ and $|a-b| = |a| + |b|$.

Results

PROPOSITION 1. *Let E be a uniformly monotone Köthe space and φ be an Orlicz function with $\varphi > 0$, $\varphi < \infty$ and $\varphi \in \Delta_2^E$. If E is fully k-rotund, then E_φ is fully k-rotund ($k \geqslant 2$).*

Proof. Let (x_n) be a sequence in $B(E_\varphi)$ such that

$$(1) \qquad \|x_n^{(1)} + x_n^{(2)} + \cdots + x_n^{(k)}\|_\varphi \to k \quad \text{as } n \to \infty$$

for any subsequences $(x_n^{(1)}), (x_n^{(2)}), \ldots, (x_n^{(k)})$ of (x_n). By the assumptions that $\varphi \in \Delta_2^E$ and $\varphi < \infty$, we have $\varphi \circ |x_n| \in B(E)$ for any $n \in \mathbb{N}$ and

$$\left\|\varphi \circ \left|\frac{x_n^{(1)} + x_n^{(2)} + \cdots + x_n^{(k)}}{k}\right|\right\|_E \to 1 \quad \text{as } n \to \infty.$$

(see Lemmas 1 and 2) and therefore,

$$\frac{1}{k}\|\varphi \circ |x_n^{(1)}| + \varphi \circ |x_n^{(2)}| + \cdots + \varphi \circ |x_n^{(k)}|\,\|_E \to 1 \quad \text{as } n \to \infty. \tag{2}$$

The space E is fully k-rotund, so (2) implies that $(\varphi \circ |x_n|)$ is a Cauchy sequence in E that is

$$\|\varphi \circ |x_m| - \varphi \circ |x_l|\,\|_E \to 0 \quad \text{as } m, l \to \infty.$$

Using superadditivity of the function φ we have

$$\varphi \circ |\,|x_m| - |x_l|\,| \leqslant |\varphi \circ |x_m| - \varphi \circ |x_l|\,|$$

so the previous condition yields

$$\varrho_\varphi(|x_m| - |x_l|) = \|\varphi \circ |\,|x_m| - |x_l|\,|\,\|_E \to 0 \quad \text{as } m, l \to \infty$$

and, by $\varphi > 0$ and $\varphi \in \Delta_2^E$, we get

$$\|\,|x_m| - |x_l|\,\|_\varphi \to 0 \quad \text{as } m, l \to \infty \tag{3}$$

(see Lemma 3). Observe that condition (1) yields

$$\|x_m + x_l\|_\varphi \to 2 \quad \text{as } m, l \to \infty. \tag{4}$$

Let us define for any $i, j \in \mathbb{N}$

$$A_{ij} = \{t \in T : x_i(t) \cdot x_j(t) < 0\}.$$

We will show that

$$\|(|x_m| + |x_l| - |x_m - x_l|)\chi_{A_{ml}}\|_\varphi \to 0 \quad \text{as } m, l \to \infty. \tag{5}$$

If we suppose, on the contrary, that condition (5) is not true, then there exist increasing sequences (m_n), (l_n) of natural numbers such that

$$\|(|x_{m_n}| + |x_{l_n}| - |x_{m_n} - x_{l_n}|)\chi_{A_{m_n l_n}}\|_\varphi \geqslant \delta$$

for some $\delta > 0$ and any $n \in \mathbb{N}$. The uniform monotonicity of E and the assumptions concerning φ imply uniform monotonicity of E_φ (see [CHM]). So, there exists $\eta > 0$ such that $\|z_n + y_n\|_\varphi \geqslant 1 + \eta$ for $n \in \mathbb{N}$ large enough, whenever $(z_n), (y_n) \subset E_\varphi^+$, $\|z_n\| \to 1$ and $\|y_n\| \geqslant \frac{\delta}{2}$ $(n \in \mathbb{N})$. Then, by (4) and Remark 1, we have

$$1 \geqslant \left\|\frac{|x_{m_n}| + |x_{l_n}|}{2}\right\|_\varphi = \left\|\left(\frac{|x_{m_n}| + |x_{l_n}|}{2}\right)\chi_{T\setminus A_{m_n l_n}} + \left(\frac{|x_{m_n}| + |x_{l_n}|}{2}\right)\chi_{A_{m_n l_n}}\right\|_\varphi$$

$$= \left\|\frac{|x_{m_n} + x_{l_n}|}{2}\chi_{T\setminus A_{m_n l_n}} + \left(\frac{|x_{m_n}| + |x_{l_n}|}{2} - \frac{|x_{m_n} + x_{l_n}|}{2}\right)\chi_{T\setminus A_{m_n l_n}} + \frac{|x_{m_n} + x_{l_n}|}{2}\chi_{A_{m_n l_n}}\right\|_\varphi$$

$$= \left\|\frac{|x_{m_n} + x_{l_n}|}{2} + \left(\frac{|x_{m_n}| + |x_{l_n}|}{2} - \frac{|x_{m_n} + x_{l_n}|}{2}\right)\chi_{T\setminus A_{m_n l_n}}\right\|_\varphi \geqslant 1 + \eta$$

for $n \in \mathbb{N}$ large enough, a contradiction. This means that condition (5) holds.

Using again Remark 1, we get the inequalities

$$\begin{aligned}
&\| |x_m| - |x_l| \|_\varphi + \|(|x_m| + |x_l|)\chi_{A_{ml}} - |x_m + x_l|\chi_{A_{ml}}\|_\varphi \\
&\quad \geqslant \| | |x_m| - |x_l| |\chi_{T\setminus A_{ml}} + | |x_m| - |x_l| |\chi_{A_{ml}} + (|x_m| + |x_l|)\chi_{A_{ml}} - |x_m + x_l|\chi_{A_{ml}}\|_\varphi \\
&\quad = \| | |x_m| - |x_l| |\chi_{T\setminus A_{ml}} + (|x_m| + |x_l|)\chi_{A_{ml}}\|_\varphi \\
&\quad \geqslant \| |x_m - x_l|\chi_{T\setminus A_{ml}} + |x_m - x_l|\chi_{A_{ml}}\|_\varphi = \| |x_m - x_l| \|_\varphi = \|x_m - x_l\|_\varphi,
\end{aligned}$$

which, by (3) and (5), yield

$$\|x_m - x_l\|_\varphi \to 0 \quad \text{as } m, l \to \infty. \ \blacksquare$$

Analogously we can prove

PROPOSITION 2. *Let E be a uniformly monotone Köthe space and φ be an Orlicz function with $\varphi > 0$, $\varphi < \infty$ and $\varphi \in \Delta_2^E$. If E is compactly fully k-rotund, then E_φ is compactly fully k-rotund ($k \geqslant 2$).*

REMARK 2. In the proof of Proposition 1 it is shown that for any Köthe space E if the positive cone E^+ is (compactly) fully k-rotund and E is uniformly monotone, then E is (compactly) fully k-rotund.

PROPOSITION 3. *If E is a uniformly monotone Köthe space and φ is a strictly convex Orlicz function satisfying the Δ_2^E-condition, then E_φ is a URED-space.*

Proof. Let us fix $\varepsilon \in (0, 1)$ and $z \in \varepsilon S(E_\varphi)$. Let $y \in B(E_\varphi)$ be such that $\|y + z\|_\varphi \leqslant 1$.

Since the space E is uniformly monotone, $\varphi \in \Delta_2^E$ and φ is strictly convex, so E_φ is uniformly monotone (see [CHM]) and in consequence, E_φ is order continuous (see [Bi]). Therefore, we can find a measurable set A with positive finite measure and a number $k > 0$ such that

$$1/k \leqslant |z(t)| \leqslant k \quad \text{for any} \quad t \in A \quad \text{and} \quad \|z\chi_A\|_\varphi \geqslant 4\varepsilon/5.$$

Now we see that $\chi_A \in E$ and, since $\varphi > 0$, we have $\varrho_\varphi(z\chi_A) > 0$. Note that $\varrho_\varphi(y) \leqslant \|y\|_\varphi \leqslant 1$ (see Lemma 1). In the following we will consider two cases separately.

1° Assume first that A is not an atom. Let U be an arbitrary subset of A such that $0 < \mu(U) < \mu(A)$. Since E is a strictly monotone space (because it is uniformly monotone), we have

$$\|\chi_A\|_E - \|\chi_U\|_E =: \delta_1 > 0.$$

Let us choose $l > 0$ such that

$$\varphi(l)\|\chi_U\|_E > 1$$

and define $B = \{t \in A : |y(t)| \leqslant l\}$. If we suppose that $\|\chi_{A\setminus B}\|_E > \|\chi_U\|_E$, then we have

$$\varrho_\varphi(y) \geqslant \varrho_\varphi(y\chi_{A\setminus B}) = \|\varphi \circ |y|\chi_{A\setminus B}\|_E \geqslant \varphi(l)\|\chi_{A\setminus B}\|_E > \varphi(l)\|\chi_U\|_E > 1,$$

a contradiction. Therefore, $\|\chi_{A\setminus B}\|_E \leqslant \|\chi_U\|_E$, and, in consequence,

$$\|\chi_B\|_E = \|\chi_A - \chi_{A\setminus B}\|_E \geqslant \|\chi_A\|_E - \|\chi_{A\setminus B}\|_E \geqslant \|\chi_A\|_E - \|\chi_U\|_E = \delta_1$$

and

$$\varrho_\varphi(z\chi_B) = \|\varphi \circ |z|\chi_B\|_E \geqslant \varphi(1/k)\|\chi_B\|_E \geqslant \varphi(1/k)\delta_1 =: \delta_2 > 0.$$

2° Now we consider the case when A is an atom. Let $l > 0$ be such that

$$\varphi(l)\|\chi_A\|_E > 1.$$

Denote again $B = \{t \in A : |y(t)| \leqslant l\}$. If $\mu(A \setminus B) = \mu(A)$, then $\chi_A = \chi_{A\setminus B}$ and

$$\varrho_\varphi(y) \geqslant \varrho_\varphi(y\chi_{A\setminus B}) = \|\varphi \circ |y|\chi_{A\setminus B}\|_E = \|\varphi \circ |y|\chi_A\|_E \geqslant \varphi(l)\|\chi_A\|_E > 1.$$

But we have $\varrho_\varphi(y) \leqslant \|y\|_\varphi \leqslant 1$. Therefore, $\mu(A) = \mu(B)$ and $\varrho_\varphi(z\chi_B) = \varrho_\varphi(z\chi_A) > 0$.

We have shown that there exist numbers l, $\delta > 0$ (independent of y) such that, for the set $C = \{t \in A : |y(t)| \leqslant l\}$, we have

$$\varrho_\varphi(z\chi_C) \geqslant \delta. \tag{6}$$

Observe that

$$\max\{|y(t) + z(t)|, |y(t)|\} \leqslant k + l$$

and

$$|(y(t) - z(t)) - y(t)| = |z(t)| \geqslant 1/k$$

for μ-a.e $t \in C$. So, by strict convexity of φ there exists $p \in (0, 1)$, depending on k, l (i.e. depending on z and ε) only, such that

$$\varphi\left(\left|y(t) + \frac{1}{2}z(t)\right|\right) \leqslant \frac{1-p}{2}[\varphi(|y(t) + z(t)|) + \varphi(|y(t)|)]$$

for μ-a.a. $t \in C$. Therefore, we have

$$\begin{aligned}\varphi \circ \left|y + \frac{1}{2}z\right| &= \varphi \circ \left|\frac{(y+z)+y}{2}\right| \\ &\leqslant \frac{1}{2}\varphi \circ |y+z|\chi_{T\setminus C} + \frac{1}{2}\varphi \circ |y|\chi_{T\setminus C} + \frac{1-p}{2}(\varphi \circ |y+z|\chi_C + \varphi \circ |y|\chi_C) \\ &\leqslant \frac{1}{2}\varphi \circ |y+z| + \frac{1}{2}\varphi \circ |y| - \frac{p}{2}\varphi \circ |y+z|\chi_C - \frac{p}{2}\varphi \circ |y|\chi_C.\end{aligned} \tag{7}$$

If we define $D = \{t \in C : |z(t)| \geqslant \frac{\delta}{4}\max\{|y(t) + z(t)|, |y(t)|\}\}$, then the inequality

$$\|\varphi \circ |z|\chi_{C\setminus D}\|_E \leqslant \frac{\delta}{4}\|\varphi \circ |y+z|\chi_{C\setminus D} + \varphi \circ |y|\chi_{C\setminus D}\|_E \leqslant \frac{\delta}{2} \tag{8}$$

holds and, in viev of (6), it gives

$$\varrho_\varphi(z\chi_D) \geqslant \frac{\delta}{2}.$$

Assume now that $L_\infty \hookrightarrow E$. Since $\varphi \in \Delta_2^E$ and $\varphi > 0$, there exist v, $K > 0$ such that $\|\varphi(v)\chi_T\|_E \leqslant \delta/4$ and $\varphi(2u) \leqslant K\varphi(u) + \varphi(v)$ for any $u \in [0, \infty)$. Then we have

$$\begin{aligned}\frac{\delta}{2} \leqslant \varrho_\varphi(z\chi_D) = \|\varphi \circ |z + y - y|\chi_D\|_E &\leqslant \left\|\varphi \circ \left(\frac{1}{2}|2(y+z)| + \frac{1}{2}|2y|\right)\chi_D\right\|_E \\ &\leqslant \frac{K}{2}\|\varphi \circ |y+z|\chi_D + \varphi \circ |y|\chi_D\|_E + \|\varphi(v)\chi_T\|_E \\ &\leqslant \frac{K}{2}\|\varphi \circ |y+z|\chi_D + \varphi \circ |y|\chi_D\|_E + \frac{\delta}{4}\end{aligned}$$

and, in consequence,

$$\left\| \frac{p}{2}\varphi\circ|y+z|\chi_D + \frac{p}{2}\varphi\circ|y|\chi_D \right\|_E \geqslant \frac{\delta p}{4K}. \tag{9}$$

The uniform monotonicity of E and conditions (7) and (9) imply that there exists $\eta > 0$ (depending on p, δ and K only) such that

$$\varrho_\varphi\left(y + \frac{1}{2}z\right) \leqslant 1 - \eta.$$

Now, by the Δ_2^E-condition for φ there exists $\beta > 0$, depending only on η, such that $\|x\|_\varphi \leqslant 1-\beta$ whenever $\varrho_\varphi(x) \leqslant 1-\eta$ for any $x \in E_\varphi$. Finally, we have

$$\left\| y + \frac{1}{2}z \right\|_\varphi \leqslant 1 - \beta.$$

If $E \hookrightarrow L_\infty$, then $\|x\|_\infty \leqslant M$ for every $x \in B(E_\varphi)$ and some $M > 0$. Since $\varphi \in \Delta_2^E$ and φ takes on only finite values, there exists $K_1 > 0$ such that $\varphi(2u) \leqslant K_1\varphi(u)$ for $u \in [0, M]$. Hence we have

$$\frac{\delta}{2} \leqslant \varrho_\varphi(z\chi_D) = \|\varphi\circ|z+y-y|\chi_D\|_E \leqslant \frac{K_1}{2}\|\varphi\circ|y+z|\chi_D + \varphi\circ|y|\chi_D\|_E$$

and

$$\left\| \frac{p}{2}\varphi\circ|y+z|\chi_D + \frac{p}{2}\varphi\circ|y|\chi_D \right\|_E \geqslant \frac{\delta p}{2K_1}.$$

Now we deduce, as above, that there exists $\beta_1 > 0$ such that

$$\|y + z/2\|_\varphi < 1 - \beta_1.$$

The remaining case when neither $L_\infty \hookrightarrow E$ nor $E \hookrightarrow L_\infty$ is analogous and even easier to handle because the Δ_2^E-condition means in this case the Δ_2-condition on the whole $\mathbb{R}_+$. ∎

We say that $x \in E^+$ is an *H_μ^+-point* if for any sequence (x_n) in E^+ such that $x_n \xrightarrow{\mu\,(\text{loc})} x$ (locally in measure) and $\|x_n\|_E \to \|x\|_E$, we have $\|x_n - x\|_E \to 0$. If all points $x \in E^+$ are H_μ^+-points, then we say that E has *H_μ^+-property*.

In Proposition 1 in [HM] it was proved that any order continuous Köthe space has the H_μ-property if and only if it has the H_μ^+-property. The next lemma is a local version of that proposition.

LEMMA 4. *For any order continuous Köthe space E, a point $x \in E$ is an H_μ-point if and only if $|x|$ is an H_μ^+-point.*

Proof. Sufficiency. We may assume that $x \in S(E)$. Let (x_n) be an arbitrary sequence in E such that

$$x_n \xrightarrow{\mu\,(\text{loc})} x \quad \text{and} \quad \|x_n\|_E \to 1 = \|x\|_E. \tag{10}$$

We will show that $\|x_n - x\|_E \to 0$ (by the assumption that $|x|$ is an H_μ^+-point). Observe that condition $x_n \xrightarrow{\mu\,(\text{loc})} x$ yields

$$|x_n| \xrightarrow{\mu\,(\text{loc})} |x|.$$

The point $|x|$ is an H_μ^+-point, so we have

$$\|\,|x_n| - |x|\,\|_E \to 0$$

Therefore, there exist $y \in E^+$ and an increasing sequence (n_k) of natural numbers such that

$$(11) \qquad |\,|x_{n_k}| - |x|\,| \leqslant y$$

for any $k \in \mathbb{N}$ (see Lemma 2 in [KA], p. 141). We may assume additionally that

$$(12) \qquad x_{n_k} \to x \ \mu\text{-a.e. on } T.$$

Applying (11) we have the inequality

$$(13) \qquad |x_{n_k} - x| \leqslant y + 2|x|$$

for any $k \in \mathbb{N}$. Conditions (12) and (13) together with the order continuity of E give

$$\|x_{n_k} - x\|_E \to 0.$$

Now it remains to apply the double extract subsequence theorem to obtain

$$\|x_n - x\|_E \to 0$$

and to end the proof of sufficiency.

Necessity. Let x be an H_μ-point and (x_n) be an arbitrary sequence in E^+ such that $x_n \xrightarrow{\mu\,(\mathrm{loc})} |x|$ and $\|x_n\|_E \to \|x\|_E$. Define $y_n := fx_n$ $(n \in \mathbb{N})$, where $f(t) = 1$ if $x(t) \geqslant 0$ and $f(t) = -1$ if $x(t) < 0$ $(t \in T)$. Then, we have

$$|y_n - x| = |fx_n - f|x|\,| = |x_n - |x|\,|$$

for any $n \in \mathbb{N}$. Therefore, $y_n \xrightarrow{\mu\,(\mathrm{loc})} x$. Moreover, $\|y_n\|_E = \|x_n\|_E \to \|x\|_E$. So, $\|y_n - x\|_E \to 0$ and in consequence, $\|x_n - |x|\,\|_E \to 0$. This means that $|x|$ is an H_μ^+-point. ■

PROPOSITION 4. *Let E be an order continuous Köthe space and φ be an Orlicz function with $\varphi > 0$, $\varphi < \infty$ and $\varphi \in \Delta_2^E$. An element $x \in E_\varphi$ is an H_μ-point if and only if $\varphi \circ |x|$ is an H_μ^+-point in E.*

Proof. Sufficiency. Without loss of generality, we may assume that $x \in S(E_\varphi)$. The order continuity of E and conditions $\varphi > 0$ and $\varphi \in \Delta_2^E$ imply that E_φ is order continuous (see [FH1]). Therefore, by Lemma 4, it suffices to show that $|x|$ is H_μ^+-point. Let (x_n) be an arbitrary sequence in E_φ^+ such that

$$(14) \qquad x_n \xrightarrow{\mu\,(\mathrm{loc})} |x| \quad \text{and} \quad \|x_n\|_\varphi \to 1.$$

So, in view of $\varphi \in \Delta_2^E$ and $\varphi < \infty$, we have

$$\varrho_\varphi(x_n) = \|\varphi \circ x_n\|_E \to 1 = \|\varphi \circ |x|\,\|_E$$

(see Lemma 2). Condition (14) also yields

$$(15) \qquad \varphi \circ x_n \xrightarrow{\mu\,(\mathrm{loc})} \varphi \circ |x|.$$

Indeed, if $x_n \xrightarrow{\mu\,(\mathrm{loc})} |x|$, then $x_{n_k} \to |x|$ μ-a.e. on T for some increasing sequence (n_k) of natural numbers. Hence, by continuity of the function φ, we get $\varphi \circ x_{n_k} \to \varphi \circ |x|$ μ-a.e. on

T which implies $\varphi \circ x_{n_k} \xrightarrow{\mu\,(\mathrm{loc})} \varphi \circ |x|$. Applying the double extract subsequence theorem we obtain condition (15).

The element $\varphi \circ |x|$ is an H_μ^+-point in E, so we obtain

$$\|\varphi \circ x_n - \varphi \circ |x|\,\|_E \to 0$$

and in consequence,

$$\varrho_\varphi(x_n - |x|) = \|\varphi \circ |x_n - |x|\,|\,\|_E \leqslant \|\varphi \circ x_n - \varphi \circ |x|\,\|_E \to 0,$$

by superadditivity of φ on $\mathbb{R}_+$. But $\varphi \in \Delta_2^E$ and $\varphi > 0$, so

$$\|x_n - x\|_\varphi \to 0$$

(see Lemma 3), which means that $|x|$ is an H_μ^+-point.

Necessity. We may assume that $x \in S(E_\varphi)$. Then, by $\varphi \in \Delta_2^E$ and $\varphi < \infty$, we have $\|\varphi \circ |x|\,\|_E = 1$. Let us choose an arbitrary sequence (y_n) in E^+ such that $y_n \xrightarrow{\mu\,(\mathrm{loc})} \varphi \circ |x|$ and $\|y_n\|_E \to 1$. The function φ is an injection, so we can define $x_n := \varphi^{-1} \circ y_n$ for all $n \in \mathbb{N}$. We have $x_n \in E_\varphi^+$ and $\|x_n\|_\varphi \to 1$ because $\varrho_\varphi(x_n) = \|y_n\|_E \to 1$ (see Lemma 1). Moreover, condition $y_n \xrightarrow{\mu\,(\mathrm{loc})} \varphi \circ |x|$, continuity of φ^{-1} and the double extract subsequence theorem give

$$\varphi^{-1} \circ y_n = x_n \xrightarrow{\mu\,(\mathrm{loc})} |x| = \varphi^{-1} \circ \varphi \circ |x|.$$

From the assumption that x is an H_μ-point in E_φ we have that $|x|$ is an H_μ^+-point in E_φ (see Lemma 4), so

$$\|x_n - |x|\,\|_\varphi \to 0.$$

By Lemma 2 in [KA] (page 141), there exist $z \in E_\varphi^+$ and an increasing sequence (n_k) of natural numbers such that

$$|x_{n_k} - |x|\,| \leqslant z$$

for all $k \in \mathbb{N}$. Then, we have

$$x_{n_k} + |x| \leqslant z + 2|x| \ (k \in \mathbb{N}). \tag{16}$$

The conditions $\varphi \in \Delta_2^E$, $\varphi < \infty$ and Lemma 2 yield $\|\varphi \circ (z + 2|x|)\|_E < \infty$, which means, by $E \in (FP)$, that $\varphi \circ (z + 2|x|) \in E$. Let (n_m) be a subsequence of (n_k) such that

$$y_{n_m} \to \varphi \circ |x| \ \ \mu\text{-a.e. on } T. \tag{17}$$

Now, by condition (16) and superadditivity of the function φ, we get

$$y_{n_m} = \varphi \circ x_{n_m} = \varphi \circ |(x_{n_m} + |x|) - |x|\,| \leqslant |\varphi \circ |x_{n_m} + |x|\,| - \varphi \circ |x|\,| \leqslant$$
$$\varphi \circ |x_{n_m} + |x|\,| + \varphi \circ |x| \leqslant \varphi \circ (z + 2|x|) + \varphi \circ |x|.$$

Therefore, the order continuity of E and condition (17) imply that $\|y_{n_m} - \varphi \circ |x|\,\|_E \to 0$. Finally, applying the double extract subsequence theorem, we obtain $\|y_n - |x|\,\|_E \to 0$, which means that $\varphi \circ |x|$ is H_μ^+-point in E. ■

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