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ON THE SPATIAL ANALYTICITY OF SOLUTIONS TO THE KELLER-SEGEL EQUATIONS

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Abstract. The regularizing rate of solutions to the Keller-Segel equations in the whole space is estimated just as for the heat equation. As an application of these rate estimates, it is proved that the solution is analytic in spatial variables. Spatial analyticity implies that the propagation speed is infinite, i.e., the support of the solution coincides with the whole space for any short time, even if the support of the initial datum is compact.

1. Introduction and main results. Let us consider the following system of coupled reaction-diffusion equations in $\mathbf{R}^n$ for $n \in \mathbf{N}$:

$$\text{(KS)} \qquad \begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) & \text{in } \mathbf{R}^n \times (0,T), \\ v_t = \Delta v - v + u & \text{in } \mathbf{R}^n \times (0,T), \\ u|_{t=0} = u_0, \qquad v|_{t=0} = v_0 & \text{in } \mathbf{R}^n. \end{cases}$$

In [10] E. F. Keller and L. A. Segel introduced (KS) as the mathematical model of the cell movement of mycetozoan by chemotaxis. Here $u := u(t) := u(x,t)$ denotes the density of cells, and $v := v(t) := v(x,t)$ stands for the concentration of chemoattractant at time $t \in (0,T)$ and location $x \in \mathbf{R}^n$; the initial data $u_0 := u_0(x)$ and $v_0 := v_0(x)$ are given non-negative functions. We use conventional notations: $u_t := \partial u/\partial t$, $\Delta := \sum_{i=1}^n \partial_i^2$, $\partial_i := \partial/\partial x_i$, $\nabla := (\partial_1, \dots, \partial_n)$, $\nabla \cdot F := \operatorname{div} F := \sum_{i=1}^n \partial_i F^i$ for a vector $F = (F^1, \dots, F^n)$.

There are a lot of works dealing with (KS). For example, local or global in time existence of smooth solutions in bounded domains with no-flux boundary conditions were obtained by [1]. Other related works can be found in references of [1].

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Our aim is to derive the regularizing rate of solutions (u,v) in short time, when $u_0 \in L^p(\mathbf{R}^n)$ and $\nabla v_0 \in (L^p(\mathbf{R}^n))^n$. Throughout this note we often discuss ∇v, instead of v itself. Here $L^p(\mathbf{R}^n)$ is the Lebesgue space for $p \in [1,\infty]$ with the norm denoted by $\|\cdot\|_p$. We sometimes suppress the domain $\mathbf{R}^n$ in the notation, i.e., $L^p = L^p(\mathbf{R}^n)$. Also, we often do not distinguish the function spaces of scalar valued and vector valued functions, if no confusion is likely.

We state the main theorem in this note on the existence, uniqueness and analyticity in x of solutions (u,v) to (KS) in L^p-framework:

THEOREM 1.1. *Let $n \in \mathbf{N}$ and $\max(1,n/2) < p_1 \le p_2 < \infty$ satisfying $1/p_1 + 1/p_2 \le 2/n$, and let $p_3 \in [1,\infty]$. Assume that $u_0 \in L^{p_1}(\mathbf{R}^n)$, $\nabla v_0 \in (L^{p_2}(\mathbf{R}^n))^n$, and that $v_0 \in L^{p_3}(\mathbf{R}^n)$. Then there exist $T_0 > 0$ and a unique solution (u,v) in the class*

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p_1}-\frac{1}{q})} u(t)] \in C([0,T_0]; L^q(\mathbf{R}^n)) \quad \text{for } q \in [p_1,\infty], \tag{1}$$

and

$$[t \mapsto t^{\frac{n}{2}(\frac{1}{p_2}-\frac{1}{q})} \nabla v(t)] \in C([0,T_0]; (L^q(\mathbf{R}^n))^n) \quad \text{for } q \in [p_2,\infty]. \tag{2}$$

Moreover, $u, v \in C^\infty(\mathbf{R}^n \times (0,T_0))$. Furthermore, u and v are analytic in x.

REMARK 1.2. (i) One can get mild solutions (the solution of the integral equations, see below) without the assumption $v_0 \in L^{p_3}$. We, however, need to show that (u,v) solves (KS) in the classical sense. It is not necessary to impose non-negativity of u_0 and v_0 in Theorem 1.1.

(ii) For u_0 or $\nabla v_0 \in L^\infty$ one can also get the similar results except the continuity in time at $t = 0$, since the heat semigroup $e^{t\Delta}$ is not a C_0-semigroup in L^∞. To obtain the continuity in time we restrict ourselves to u_0 or ∇v_0 in $BUC(\mathbf{R}^n)$ or $\dot{B}^0_{\infty,1}(\mathbf{R}^n)$, since $e^{t\Delta}$ is a bounded C_0-semigroup in these spaces, where BUC is the space of all bounded and uniformly continuous functions, and $\dot{B}^0_{\infty,1}$ is the homogeneous Besov space; see the details in [2, 5, 13].

(iii) Propagation speed is infinite. As the corollary of analyticity in x, it is shown that the propagation speed of solutions to (KS) is infinite. That is to say, the supports of $u(t)$ and $v(t)$ coincide with $\mathbf{R}^n$ for any small $t > 0$, even if the supports of u_0 and v_0 are compact.

(iv) Other equations. We may obtain the same results (in particular, analyticity in x) on the Cauchy problem of the following three equations of parabolic type.

(a) The generalized Keller-Segel equations:

$$\text{(GKS)} \qquad \begin{cases} u_t &= \Delta u - \nabla \cdot (u^q \nabla v), \\ \tau v_t &= \Delta v - \gamma v + u \end{cases}$$

for $q \in \mathbf{N}$, $\gamma \in \mathbf{R}$ and $\tau = 0, 1$ with initial data $u_0 \in L^p$ for $p \ge n$ if $\tau = 0$, and $u_0 \in L^{p_1}$, $\nabla v_0 \in (L^{p_2})^n$ if $\tau = 1$. For the case when $\tau = 0$ and $q = 1$, (GKS) is the so-called Nagai model, there exists a blow-up solution in finite time, see [11]. When $\gamma = 0$ and $q = 1$, one can obtain the global in time smooth solution for sufficiently small data, which was shown by H. Kozono and Y. Sugiyama, recently.

(b) The Fujita equation (semilinear heat equation) with algebraic nonlinearity:

$$u_t = \Delta u + |u|^{q-1}u \tag{F}$$

for $q \in \mathbf{N}$ with $u_0 \in BUC$; it is not necessary to impose any positivity.

(c) The Allen-Cahn equation:

$$u_t = \varepsilon\Delta u - u^3 + u \tag{AC}$$

for $\varepsilon > 0$ with suitable initial data, e.g. $u_0 \in BUC(\mathbf{R}^n)$ and $-1 \leq u_0 \leq 1$.

In the proof of Theorem 1.1 we use integral equations, which will be explained later. Existence is based on an iteration scheme, see e.g. [9], that is, we use successive approximations. Uniqueness follows from the Gronwall inequality. Involving the higher order derivatives, the iteration procedure works again to show that u and v are smooth. To get the analyticity of u and v in x we establish the rate estimate for the higher order derivatives.

Let us recall the heat equation:

$$w_t - \Delta w = 0, \quad w|_{t=0} = f \in L^r(\mathbf{R}^n). \tag{H}$$

Using the heat semigroup $e^{t\Delta} := G_t *$ and $G_t(x) := (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$, the solution w is given in the form $w = e^{t\Delta}f$ enjoying the following regularizing rate estimates:

$$\|\partial_x^\beta w(t)\|_q \leq C_0|\beta|^{\frac{|\beta|}{2}} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|f\|_r \quad \text{for} \quad t > 0 \quad \text{and} \quad 1 \leq r \leq q \leq \infty \tag{3}$$

with some constant C_0 depending only on n. Here $\partial_x^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$ and $|\beta| := \beta_1 + \cdots + \beta_n$ for multi-index $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{N}_0^n$, where $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. Obviously, (3) implies that $w(t)$ is analytic in x for all $t > 0$.

We are now in a position to introduce the notion of a mild solution. Historically, this notion was introduced by F. E. Browder to study equations of parabolic type, and by H. Fujita and T. Kato [3] for the nonstationary Navier-Stokes equations. By Duhamel's principle it is straightforward to solve (KS) in time to derive the integral equations:

$$\begin{cases} \quad u(t) &= e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\nabla \cdot (u(s)\nabla v(s))ds, \\ \nabla v(t) &= e^{t\Delta}\nabla v_0 - \int_0^t \nabla e^{(t-s)\Delta}(u(s) - v(s))ds. \end{cases} \tag{INT}$$

Obviously, (INT) is equivalent to (KS). The pair of solutions (u, v) to (INT) is often called *a mild solution*, we also use this terminology. In what follows we mainly deal with (INT) rather than (KS).

In this note we derive estimates of higher order derivatives of u and v similar to (3):

Proposition 1.3. *Let $n \in \mathbf{N}$, $p > 1$, $p \in [n, \infty)$, and let $u_0, \nabla v_0 \in L^p(\mathbf{R}^n)$. Assume that $(u(t), v(t))$ is a mild solution on $t \in (0, T)$ such that $u, \nabla v \in C^\infty(\mathbf{R}^n \times (0, T))$ and*

$$u,\ \nabla v \in C([0, T); L^p(\mathbf{R}^n)) \cap C((0, T); L^r(\mathbf{R}^n))$$

for some $T \in (0, 1]$ and some $r > \max(p, 2)$. Let M_j for $j = 1, \ldots, 4$ be constants satisfying

$$M_1 \geq \sup_{0 \leq t < T} \|u(t)\|_p, \qquad M_2 \geq \sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|u(t)\|_r,$$
$$M_3 \geq \sup_{0 \leq t < T} \|\nabla v(t)\|_p, \qquad M_4 \geq \sup_{0 < t < T} t^{\frac{n}{2}(\frac{1}{p}-\frac{1}{r})}\|\nabla v(t)\|_r.$$

Then there exist positive constants D_1 and D_2 depending only on n, p, r and M_j such that

$$\|\partial_x^\beta u(t)\|_q + \|\partial_x^\beta \nabla v(t)\|_q \le D_1(D_2|\beta|)^{|\beta|} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} \tag{4}$$

for all $q \in [p,\infty]$, $t \in (0,T)$ and $\beta \in \mathbf{N}_0^n$.

For simplicity we only deal with the solution when $p_1 = p_2$ in Theorem 1.1. From (4) it is easy to see that u and v are analytic in x. Indeed, one can estimate the radius of convergence of Taylor's expansion of u (=: $\rho(t)$) from below:

$$\rho(t) = \left[\limsup_{k\to\infty}\left(\frac{\|\partial_i^k u(t)\|_\infty}{k!}\right)^{1/k}\right]^{-1} \ge C\sqrt{t} \tag{5}$$

for some constant C for any $t \in (0,T)$ and $i = 1,\dots,n$. The estimate (5) follows from Cauchy-Hadamard's criterion and Stirling's formula, easily. It can be also shown that $v(t) \in C^\omega(\mathbf{R}^n)$.

To prove Proposition 1.3 we use the technique developed by Y. Giga and the author of this note [6]; see also [12]. We divide the time interval $(0,t)$ of integrals in (INT) into two parts, to distribute the singularity. To integrate them we press every differentiation against the heat kernel for $s \in (0,(1-\varepsilon)t)$, and against u and v for $s \in ((1-\varepsilon)t,t)$. Finally, we use the Gronwall type inequality (Lemma 2.3) to get (4). Also, smoothness of mild solutions can be shown by this technique.

This paper is organized as follows. In section 2 we recall several lemmata. Section 3 will be devoted to the proof of Theorem 1.1. In Proposition 3.1 we shall give a proof of existence and uniqueness of mild solutions. In Proposition 3.2 we shall verify that the mild solution is in the class C^ω in x.

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2. Preliminaries. In this section we give some lemmata. Firstly, we verify (4).

LEMMA 2.1. *Let $n \in \mathbf{N}$. Then there exists a constant $C_0 > 0$ depending only on n such that* (3) *holds for all $t > 0$, $1 \le r \le q \le \infty$, $\beta \in \mathbf{N}_0^n$ and $f \in L^r(\mathbf{R}^n)$.*

Proof. Although this lemma was proved by [6, Lemma 2.1], we give its proof for the readers' convenience. Since $\|\partial_i G_t\|_1 \le \pi^{-1/2}t^{-1/2}$, by Young's inequality we have

$$\|\partial_i e^{t\Delta} f\|_r \le \pi^{-\frac{1}{2}} t^{-\frac{1}{2}} \|f\|_r, \qquad \|e^{t\Delta} f\|_q \le (4\pi t)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\|f\|_r$$

for all $t > 0$, $1 \le r \le \infty$, $i = 1,\dots,n$ and $f \in L^r(\mathbf{R}^n)$. Notice that the heat semigroup and spatial differentiation commute. Using the semigroup property, for $1 < \theta, \theta' < \infty$

satisfying $\frac{1}{\theta}+\frac{1}{\theta'}=1$ we have

$$\begin{aligned}\|\partial_x^\beta e^{t\Delta}f\|_q &= \|e^{\frac{t}{\theta'}\Delta}\prod_{i=1}^n(\partial_i e^{\frac{t}{\theta|\beta|}\Delta})^{\beta_i}f\|_q\\ &\le \|e^{\frac{t}{\theta'}\Delta}\|_{\mathcal{L}(L^r\to L^q)}\Big\{\prod_{i=1}^n\|\partial_i e^{\frac{t}{\theta|\beta|}\Delta}\|^{\beta_i}_{\mathcal{L}(L^r\to L^r)}\Big\}\|f\|_r\\ &\le \left(4\pi\frac{t}{\theta'}\right)^{-\frac{n}{2}(\frac{1}{r}-\frac{1}{q})}\left\{\pi^{-\frac{1}{2}}\left(\frac{t}{\theta|\beta|}\right)^{-\frac{1}{2}}\right\}^{|\beta|}\|f\|_r\end{aligned}$$

for all $t>0$, $1\le r\le q\le\infty$ and $f\in L^r$. Finally, we take $\theta=\pi$ so that the constant $C_0:=(4\pi-4)^{-n/2}$ does not depend on β. ∎

Next, we recall an estimate for multiplication of multi-sequences with binomial coefficient, which has been proved by C. Kahane [8, Lemma 2.1]. That will be used to compute the nonlinear term.

Lemma 2.2. *Let $\delta>1/2$, and let $n\in\mathbf{N}$. Then there exists a positive constant λ depending only on δ and n such that*

$$\sum_{\gamma\le\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta}\le\lambda|\beta|^{|\beta|-\delta}$$

for all $\beta\in\mathbf{N}_0^n$. Here $\gamma\le\beta$ means $\gamma_i\le\beta_i$ for all $i=1,\dots,n$ for multi-indecies $\beta=(\beta_1,\dots,\beta_n)$ and $\gamma=(\gamma_1,\dots,\gamma_n)$, and $\binom{\beta}{\gamma}:=\prod_{i=1}^n\frac{\beta_i!}{\gamma_i!(\beta_i-\gamma_i)!}$.

The dependence of λ on δ is essentially $\lambda\sim\sum_{j=1}^\infty j^{-1/2-\delta}$, so we need $\delta>1/2$. The proof of Lemma 2.2 is based on Stirling's formula. We omit the proof for brevity.

At the end of this section, we refer to a Gronwall type inequality. Originally, the following lemma has been proved by M.-H. Giga and Y. Giga [4], and its modification (sequence version) is in [6].

Lemma 2.3. *Let $T>0$, $\alpha\in\mathbf{R}$, and let $\mu\in(0,1)$. Assume that ψ_0 is non-negative, measurable and locally integrable in $(0,T)$, and that $\{\psi_j\}_{j=1}^\infty$ be a sequence of non-negative measurable functions in $(0,T)$. Assume that $t^{-\alpha}\psi_0(t)$ is bounded in $(0,T)$. Let b_ε be non-increasing with respect to ε. Assume that there is a positive constant σ such that*

$$\psi_0(t)\le b_\varepsilon t^\alpha+\sigma\int_{(1-\varepsilon)t}^t(t-s)^{-\mu}s^{-1+\mu}\psi_0(s)ds$$

and

$$\psi_{j+1}(t)\le b_\varepsilon t^\alpha+\sigma\int_{(1-\varepsilon)t}^t(t-s)^{-\mu}s^{-1+\mu}\psi_j(s)ds$$

for all $j\in\mathbf{N}_0$, $t\in(0,T)$ and $\varepsilon\in(0,1)$. Let ε_0 be a unique positive number such that $I(2\varepsilon_0)=\min\{\frac{1}{2\sigma},I(1)\}$ with $I(\varepsilon):=\int_{1-\varepsilon}^1(1-\tau)^{-\mu}\tau^{\alpha-1+\mu}d\tau$. Then

$$\psi_j(t)\le 2b_{\varepsilon_0}t^\alpha$$

for all $j\in\mathbf{N}_0$ and $t\in(0,T)$.

We skip the proof, given in [6].

3. Proof of Theorem 1.1. We split the proof into three parts: (i) existence and uniqueness (by Proposition 3.1), (ii) smoothness, and (iii) analyticity (by Proposition 3.2).

Firstly, the time-local existence and uniqueness of mild solutions in L^n-framework. Although the proof of existence is based on the semigroup approach that is standard and explained in several books (e.g. [7]), for completeness we shall give the full proof. Throughout this section we only discuss the case $p_1 = p_2 = n$, since the proof with other exponents is essentially the same (or easier).

PROPOSITION 3.1. *Let $n \in \mathbf{N}$, $u_0 \in L^n(\mathbf{R}^n)$, and let $\nabla v_0 \in (L^n(\mathbf{R}^n))^n$. Then there exist $T_0 > 0$ and a unique solution (u, v) in the class* $(1) - (2)$ *for all $q \in [n, \infty]$.*

Proof. Firstly, we construct mild solutions by an iteration scheme. We define the successive approximations starting at

$$u_1(t) := e^{t\Delta} u_0, \qquad \nabla v_1(t) := e^{t\Delta} \nabla v_0,$$

and define $\{u_j\}_j$ and $\{\nabla v_j\}_j$ by

$$u_{j+1}(t) := u_1(t) - \int_0^t e^{(t-s)\Delta} \nabla \cdot (u_j(s) \nabla v_j(s)) ds \tag{6}$$

and

$$\nabla v_{j+1}(t) := \nabla v_1(t) - \int_0^t \nabla e^{(t-s)\Delta} (v_j(s) - u_j(s)) ds \tag{7}$$

for all $j \in \mathbf{N}$. We shall verify that the pair $\{(u_j, \nabla v_j)\}_j$ is a Cauchy sequence in the class (1)-(2), and has a unique limit $(u, \nabla v)$ which satisfies (INT).

Let $\delta \in (0,1)$. Put

$$K_j := K_j(T) := \sup_{0<t<T} t^{\frac{1-\delta}{2}} \|u_j(t)\|_{\frac{n}{\delta}}, \qquad L_j := L_j(T) := \sup_{0<t<T} t^{\frac{1-\delta}{2}} \|\nabla v_j(t)\|_{\frac{n}{\delta}}.$$

Remark that

$$K_1 \to 0, \quad L_1 \to 0 \quad \text{as} \quad T \to 0, \tag{8}$$

since C_0^∞ is a dense subset of L^p for $p \in [1, \infty)$.

We shall derive a priori estimate for u_j. Taking $\|\cdot\|_{\frac{n}{\delta}}$ into (6), and multiplying $t^{\frac{1-\delta}{2}}$, we have

$$\begin{aligned}
t^{\frac{1-\delta}{2}} \|u_{j+1}(t)\|_{\frac{n}{\delta}} &\le K_1 + t^{\frac{1-\delta}{2}} \int_0^t \|e^{(t-s)\Delta} \nabla \cdot (u_j(s) \nabla v_j(s))\|_{\frac{n}{\delta}} ds \\
&\le K_1 + t^{\frac{1-\delta}{2}} \int_0^t \|e^{(t-s)\Delta} \nabla \cdot \|_{\mathcal{L}(L^{\frac{n}{2\delta}} \to L^{\frac{n}{\delta}})} \|u_j(s) \nabla v_j(s)\|_{\frac{n}{2\delta}} ds \\
&\le K_1 + C t^{\frac{1-\delta}{2}} \int_0^t (t-s)^{-\frac{n}{2}(\frac{2\delta}{n} - \frac{\delta}{n}) - \frac{1}{2}} \|u_j(s)\|_{\frac{n}{\delta}} \|\nabla v_j(s)\|_{\frac{n}{\delta}} ds \\
&\le K_1 + C t^{\frac{1-\delta}{2}} K_j L_j \int_0^t (t-s)^{-\frac{n}{2}(\frac{2\delta}{n} - \frac{\delta}{n}) - \frac{1}{2}} s^{-1+\delta} ds \\
&\le K_1 + C_1 K_j L_j,
\end{aligned}$$

where the constant C_1 depends only on n and δ. Here we have used Hölder's inequality and Lemma 2.1. Taking $\sup_{0<t<T}$ on both sides, we thus have

$$K_{j+1} \le K_1 + C_1 K_j L_j. \tag{9}$$

Similarly, for (7) we derive

$$\begin{aligned} t^{\frac{1-\delta}{2}}\|\nabla v_{j+1}(t)\|_{\frac{n}{\delta}} &\le L_1 + t^{\frac{1-\delta}{2}}\int_0^t \|\nabla e^{(t-s)\Delta}(v_j(s)-u_j(s))\|_{\frac{n}{\delta}}ds \\ &\le L_1 + Ct^{\frac{1-\delta}{2}}\int_0^t (t-s)^{-\frac12}\|u_j(s)\|_{\frac{n}{\delta}}ds + Ct^{\frac{1-\delta}{2}}\int_0^t \|\nabla v_j(s)\|_{\frac{n}{\delta}}ds \\ &\le L_1 + Ct^{\frac{1-\delta}{2}}K_j\int_0^t (t-s)^{-\frac12}s^{-\frac{1-\delta}{2}}ds + Ct^{\frac{1-\delta}{2}}L_j\int_0^t s^{-\frac{1-\delta}{2}}ds \\ &\le L_1 + C_2\sqrt{t}K_j + C_3tL_j \end{aligned}$$

with some positive constants C_2 and C_3 which depend only on n and δ. So, we take $\sup_{0<t<T}$ to get

$$L_{j+1} \le L_1 + C_2\sqrt{T}K_j + C_3TL_j. \tag{10}$$

Notice that $\{u_j\}$ and $\{\nabla v_j\}$ are uniformly bounded with values in $L^{\frac{n}{\delta}}$ on $(0,T_0)$ for some small T_0 from (9) and (10). Indeed, by (8) for any $\lambda>0$ there exists T_λ such that $K_1\le\lambda$ and $L_1\le\lambda$ for all $T\le T_\lambda$. Set $\lambda=\lambda_* := \frac{1}{6C_1}$ and $T_0 := \min(T_{\lambda_*}, \frac{1}{4C_2^2}, \frac{1}{3C_3})$. We therefore obtain

$$\sup_{j\in\mathbf{N}} K_j(T) \le 2\lambda \quad \text{and} \quad \sup_{j\in\mathbf{N}} L_j(T) \le 3\lambda \quad \text{for} \quad T\le T_0. \tag{11}$$

Using uniform boundedness (11), one can show that u_j belongs to the class (1) as well as ∇v_j belongs to (2) for all $j\in\mathbf{N}$ and for all $q\in[n,\infty]$ at least when $T\le T_0$. One can also easily see that $\{t^{\frac{n}{2}(\frac1n-\frac1q)}u_j(t)\}$ and $\{t^{\frac{n}{2}(\frac1n-\frac1q)}\nabla v_j(t)\}$ are continuous in time with values in L^q for all $q\in[n,\infty]$. Moreover, they are Cauchy sequences in $C([0,T_0];L^q)$ if we again choose T_0 small enough. So, there exists limit $(u,\nabla v)$ satisfying (INT). Uniqueness follows from Gronwall's inequality, see e.g. [5]. This completes the proof of Proposition 3.1. ∎

Smoothness of a mild solution (u,v) is also obtained by a modification of the proof above. In order to get the ℓ-th derivative in x, we involve the quantities

$$K_j^{(\ell)} := \sup_{0<t<T} t^{\frac{1-\delta}{2}+\frac{\ell}{2}}\|\nabla^\ell u_j(t)\|_{\frac{n}{\delta}}, \quad L_j^{(\ell)} := \sup_{0<t<T} t^{\frac{1-\delta}{2}+\frac{\ell}{2}}\|\nabla^{\ell+1} v_j(t)\|_{\frac{n}{\delta}}$$

for $\ell\in\mathbf{N}$ in the iteration scheme. We use induction with respect to ℓ, and divide the time-interval of integrals into $(0,t/2)$ and $(t/2,t)$. For example, we derive the estimate of the second derivative of u_j by

$$\begin{aligned} &t^{\frac{3-\delta}{2}}\|\nabla^2 u_{j+1}(t)\|_{\frac{n}{\delta}} \\ &\le t^{\frac{3-\delta}{2}}\|\nabla^2 e^{t\Delta}u_0\|_{\frac{n}{\delta}} + t^{\frac{3-\delta}{2}}\left(\int_0^{t/2}+\int_{t/2}^t\right)\|\nabla^2 e^{(t-s)\Delta}\nabla\cdot(u_j(s)\nabla v_j(s))\|_{\frac{n}{\delta}}ds \\ &\le K_1^{(2)} + C(K_j^{(1)}L_j + K_jL_j^{(1)} + K_j^{(2)}L_j + K_j^{(1)}L_j^{(1)} + K_jL_j^{(2)}). \end{aligned}$$

It is natural that we have to choose T_ℓ small enough so that $K_j^{(\ell)}$ and $L_j^{(\ell)}$ are uniformly bounded in this way. Indeed, we may see $T_\ell \sim \ell^{-\ell}$ basically, if we use above arguments.

However, it is not necessary to choose T_ℓ depending on ℓ for deriving uniform bounds of $K_j^{(\ell)}$ and $L_j^{(\ell)}$ on j actually, if we use a modification of the proof of the next proposition; see the end of this section.

We may extend the time-interval when the mild solution exists up to T_0, since mild solution does not blow-up and is unique at least until T_0. So, it is shown that our mild solution $(u(t), v(t))$ belongs to $C^\infty(\mathbf{R}^n)$ for all $t \in (0, T_0)$. The analyticity of the heat semigroup implies that $u, v \in C^\infty(\mathbf{R}^n \times (0, T_0))$.

Next we establish the estimates for higher order derivatives of u and ∇v, which are formally equivalent to (4). Again, we only discuss the case $p = n$ in what follows.

PROPOSITION 3.2. *Suppose that all assumptions of Proposition 1.3 hold with $p = n$ and $n \geq 2$. Let $\delta \in (1/2, 1]$. Then there exist positive constants D_1 and D_2 depending only on n, r, δ and M_j such that*

$$\|\partial_x^\beta u(t)\|_q + \|\partial_x^\beta \nabla v(t)\|_q \leq D_1(D_2|\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})} \tag{12}$$

for all $q \in [n, \infty]$, $t \in (0, T)$ and $\beta \in \mathbf{N}_0^n$.

Proof. Let $n \geq 2$, $r > n$ and $\delta \in (1/2, 1]$. Assume that $u_0, \nabla v_0 \in L^n(\mathbf{R}^n)$. Let $T \leq \min(1, T_0)$, where $(0, T_0)$ is the time-interval when the unique solution exists ensured by Theorem 1.1. Using the above arguments, for $m_0 \in \mathbf{N}$ (fixed later) we choose D_1 large enough such that

$$\|\partial_x^\beta u(t)\|_q + \|\partial_x^\beta \nabla v(t)\|_q \leq D_1, \qquad t \in (0, T_0) \tag{13}$$

hold for all $\beta \in \mathbf{N}_0^n$ with $|\beta| \leq m_0$. Clearly, (13) implies (12) for $|\beta| \leq m_0$. We shall derive (12) by induction. Let $m \in \mathbf{N}$ and $m \geq m_0$. Assume that (12) holds for all $|\beta| \leq m - 1$. We now proceed to show (12) when $|\beta| = m$.

Let $\varepsilon \in (0, 1/2)$. By (INT) we estimate the L^q-norm of $\partial_x^\beta u$:

$$\begin{aligned}
\|\partial_x^\beta u(t)\|_q &\leq \|\partial_x^\beta e^{t\Delta} u_0\|_q + \left(\int_0^{(1-\varepsilon)t} + \int_{(1-\varepsilon)t}^t\right) \|\partial_x^\beta e^{(t-s)\Delta} \nabla \cdot \big(u(s)\nabla v(s)\big)\|_q ds \\
&=: U_1 + U_2 + U_3.
\end{aligned}$$

We derive the estimates for each term. For U_1 by Lemma 2.1 we have

$$U_1 \leq \tilde{C}_1 |\beta|^{\frac{|\beta|}{2}} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{n}-\frac{1}{q})}.$$

Here the constant $\tilde{C}_1 := C_0\|u_0\|_n \leq C_0 M_1$ does not depend on β and t. We estimate U_2 by

$$\begin{aligned}
U_2 &\leq \int_0^{(1-\varepsilon)t} \|\partial_x^\beta e^{(t-s)\Delta} \nabla \cdot\|_{\mathcal{L}(L^{\frac{n}{2}} \to L^q)} \|u(s)\nabla v(s)\|_{\frac{n}{2}} ds \\
&\leq C_0(|\beta|+1)^{\frac{|\beta|+1}{2}} \int_0^{(1-\varepsilon)t} (t-s)^{-\frac{|\beta|+1}{2}-\frac{n}{2}(\frac{2}{n}-\frac{1}{q})} \|u(s)\|_n \|\nabla v(s)\|_n ds \\
&\leq \tilde{C}_2(|\beta|+1)^{\frac{|\beta|+1}{2}} \varepsilon^{-\frac{|\beta|}{2}-\frac{3}{2}+\frac{n}{2q}} t^{-\frac{|\beta|}{2}-\frac{1}{2}+\frac{n}{2q}}.
\end{aligned}$$

Here $\tilde{C}_2 := 2C_0 M_1 M_3$. We divide U_3 into two parts:

$$\begin{aligned}
U_3 &= \int_{(1-\varepsilon)t}^{t} \|e^{(t-s)\Delta}\nabla\cdot\partial_x^\beta(u(s)\nabla v(s))\|_q ds \\
&\le \int_{(1-\varepsilon)t}^{t} \|e^{(t-s)\Delta}\nabla\cdot\|_{\mathcal{L}(L^{\frac{qr}{q+r}}\to L^q)} \Big\| \sum_{\gamma\le\beta}\binom{\beta}{\gamma}(\partial_x^\gamma u)(\partial_x^{\beta-\gamma}\nabla v)\Big\|_{\frac{qr}{q+r}} ds \\
&\le C_0\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac12-\frac1r}(\|\partial_x^\beta u\|_q\|\nabla v\|_r+\|u\|_r\|\partial_x^\beta\nabla v\|_q)ds \\
&\quad + C_0\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac12-\frac1r}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}\|\partial_x^\gamma u\|_q\|\partial_x^{\beta-\gamma}\nabla v\|_r ds \\
&=: U_{3a}+U_{3b}.
\end{aligned}$$

Here we used that $\gamma<\beta$ means $\gamma\le\beta$ and $\gamma_i<\beta_i$ for some i. By the definition of M_2 and M_4 we easily estimate U_{3a} by

$$U_{3a}\le\tilde{C}_3\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac12-\frac1r}s^{-\frac12+\frac1r}\left(\|\partial_x^\beta u(s)\|_q+\|\partial_x^\beta\nabla v(s)\|_q\right)ds$$

with $\tilde{C}_3:=C_0\max(M_2,M_4)$. For U_{3b} we use the assumptions of induction:

$$\begin{aligned}
U_{3b} &\le C_0\int_{(1-\varepsilon)t}^{t}(t-s)^{-\frac12-\frac1r}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}D_1(D_2|\gamma|)^{|\gamma|-\delta}s^{-\frac{|\gamma|}{2}-\frac n2(\frac1n-\frac1q)} \\
&\qquad\times D_1(D_2|\beta-\gamma|)^{|\beta-\gamma|-\delta}s^{-\frac{|\beta-\gamma|}{2}-\frac n2(\frac1n-\frac1r)}ds \\
&= C_0D_1^2D_2^{|\beta|-2\delta}J_{\varepsilon,|\beta|}t^{-\frac{|\beta|}{2}-\frac12+\frac{n}{2q}}\sum_{0<\gamma<\beta}\binom{\beta}{\gamma}|\gamma|^{|\gamma|-\delta}|\beta-\gamma|^{|\beta-\gamma|-\delta} \\
&\le C_0\lambda J_{\varepsilon,|\beta|}D_1^2D_2^{|\beta|-2\delta}|\beta|^{|\beta|-\delta}t^{-\frac{|\beta|}{2}-\frac12+\frac{n}{2q}}.
\end{aligned}$$

Here we have used Lemma 2.2, and $J_{\varepsilon,|\beta|}:=\int_{1-\varepsilon}^1(1-\tau)^{-\frac12-\frac1r}\tau^{-\frac{|\beta|}{2}-1+\frac n2(\frac1q+\frac1r)}d\tau$.

Analogously, we now estimate $\partial_x^\beta\nabla v$:

$$\begin{aligned}
\|\partial_x^\beta\nabla v(t)\|_q &\le \|\partial_x^\beta e^{t\Delta}\nabla v_0\|_q+\left(\int_0^{(1-\varepsilon)t}+\int_{(1-\varepsilon)t}^{t}\right)\|\partial_x^\beta\nabla e^{(t-s)\Delta}(v(s)-u(s))\|_q ds \\
&=: V_1+V_2+V_3.
\end{aligned}$$

Similarly to the case for U_1, we estimate V_1 by

$$V_1\le\tilde{C}_4|\beta|^{\frac{|\beta|}{2}}t^{-\frac{|\beta|}{2}-\frac n2(\frac1n-\frac1q)}$$

for $t\in(0,T)$ with the constant $\tilde{C}_4:=C_0\|\nabla v_0\|_n$. For V_2 we get

$$\begin{aligned}
V_2 &\le \int_0^{(1-\varepsilon)t}(\|\partial_x^\beta\nabla e^{(t-s)\Delta}\|_{\mathcal{L}(L^n\to L^q)}\|u(s)\|_n+\|\partial_x^\beta e^{(t-s)\Delta}\|_{\mathcal{L}(L^n\to L^q)}\|\nabla v(s)\|_n)ds \\
&\le C_0\int_0^{(1-\varepsilon)t}\left(M_1(|\beta|+1)^{\frac{|\beta|+1}{2}}(t-s)^{-\frac{|\beta|+1}{2}-\frac12+\frac{n}{2q}}+M_3|\beta|^{\frac{|\beta|}{2}}(t-s)^{-\frac{|\beta|}{2}-\frac12+\frac{n}{2q}}\right)ds \\
&\le \tilde{C}_5(|\beta|+1)^{\frac{|\beta|+1}{2}}\varepsilon^{-\frac{|\beta|}{2}-1+\frac{n}{2q}}t^{-\frac{|\beta|}{2}+\frac{n}{2q}}
\end{aligned}$$

for $t\in(0,T)$ with $\tilde{C}_5:=C_0(M_1+M_3)$, since $\varepsilon<1$ and $t\le T\le1$. We easily estimate

V_3 by

$$V_3 \le \int_{(1-\varepsilon)t}^{t} (C_0(t-s)^{-\frac{1}{2}} \|\partial_x^\beta u(s)\|_q + \|\partial_x^\beta \nabla v(s)\|_q) ds$$
$$\le \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}-\frac{1}{r}} s^{-\frac{1}{2}+\frac{1}{r}} (\|\partial_x^\beta u(s)\|_q + \|\partial_x^\beta \nabla v(s)\|_q) ds,$$

since $C_0 \le 1$.

Set $\psi(t) := \|\partial_x^\beta u(t)\|_q + \|\partial_x^\beta \nabla v(t)\|_q$. Summing up, we see that

$$\psi(t) \le (\tilde{C}_6 |\beta|^{\frac{|\beta|}{2}} + \tilde{C}_7 (|\beta|+1)^{\frac{|\beta|+1}{2}} \varepsilon^{-\frac{|\beta|}{2}-\frac{3}{2}+\frac{n}{2q}} + C_0 \lambda J_\varepsilon D_1^2 D_2^{|\beta|-2\delta} |\beta|^{|\beta|-\delta}) t^{-\frac{|\beta|}{2}-\frac{1}{2}+\frac{n}{2q}}$$
$$+ (\tilde{C}_3 + 1) \int_{(1-\varepsilon)t}^{t} (t-s)^{-\frac{1}{2}-\frac{1}{r}} s^{-\frac{1}{2}+\frac{1}{r}} \psi(s) ds$$

for $t \in (0,T)$ with some constants $\tilde{C}_6 := \tilde{C}_1 + \tilde{C}_4$ and $\tilde{C}_7 := \tilde{C}_2 + \tilde{C}_5$ independent of β and t.

Fix $m_0 \in \mathbf{N}$ with $m_0 \ge 2$ large enough so that $J_{m_0} \le 1/(2\tilde{C}_3 + 2)$, where $J_m := J_{1/m,m}$. Note that $\lim_{m\to\infty} J_m \to 0$, since $r > 2$. Recall that ε may depend on $|\beta|$, so we take $\varepsilon = 1/|\beta|$ for all $\beta \in \mathbf{N}_0^n$ with $m = |\beta| \ge m_0 \ge 2$. In this setting we can easily show that J_m is uniformly bounded in $m \ge 2$, indeed, $J_m \le \frac{4r\sqrt{e}}{r-2}$. Also, for $m \ge 2$ we see $m^{m/2} \le m^{m-\delta}$ and

$$(m+1)^{\frac{m+1}{2}} m^{\frac{m}{2}+\frac{3}{2}-\frac{n}{2q}} \le 2(8m)^{m-\delta}.$$

We now apply Lemma 2.3 to get

$$\psi(t) \le (\tilde{C}_8 8^{m-\delta} + \tilde{C}_9 D_1^2 D_2^{m-2\delta}) m^{m-\delta} t^{-\frac{m}{2}-\frac{1}{2}+\frac{n}{2q}}$$

with $\tilde{C}_8 := 2\tilde{C}_6 + 4\tilde{C}_7$ and $\tilde{C}_9 := 8C_0\lambda r\sqrt{e}/(r-2)$.

Finally, it should be shown

$$\tilde{C}_8 8^{m-\delta} + \tilde{C}_9 D_1^2 D_2^{m-2\delta} \le D_1 D_2^{m-\delta} \tag{14}$$

by suitable choice of the constants D_1 and D_2. In fact, let $D_1 \ge 2\tilde{C}_8$ satisfying (13) for all $\beta \in \mathbf{N}_0^n$ with $|\beta| \le m_0$. Also, let $D_2 \ge \max(8, (2\bar{C}_2 D_1)^{1/\delta})$, then (14) holds for all $m \ge m_0$. Now we obtain (12) for all $\beta \in \mathbf{N}_0^n$. This completes the proof of Proposition 3.2. ■

To end this note we show that

$$u(t), \nabla v(t) \in C^\infty(\mathbf{R}^n), \quad t \in (0, T_0) \tag{15}$$

by the modification of arguments above. We recall that the mild solution (u,v) is the limit function for successive approximations $(6)-(7)$ by iteration. Take $\beta \in \mathbf{N}_0^n$ arbitrary. We now define for $j \in \mathbf{N}$

$$\psi_j(t) := \|\partial_x^\beta u_j(t)\|_q + \|\partial_x^\beta \nabla v(t)\|_q,$$

and argue in the similar way to the proof of Proposition 3.2 to get $\partial_x^\beta u$ and $\partial_x^\beta \nabla v \in C(0,T_0;L^q)$ by applying Lemma 2.3 (sequence version). Indeed, there exist constants D_1' and D_2' such that

$$\psi_j(t) \le D_1' (D_2' |\beta|)^{|\beta|-\delta} t^{-\frac{|\beta|}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}$$

for all $t \in (0, T_0)$, $\beta \in \mathbf{N}_0^n$ and $j \in \mathbf{N}$. Since β is arbitrary, $u_j(t), \nabla v_j(t) \in C^\infty(\mathbf{R}^n)$ for $t \in (0, T_0)$ and $j \in \mathbf{N}$. Moreover, since u_j converges to u, and ∇v_j converges to ∇v in $(0, T_0)$, (15) holds true.

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