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TESTING FLATNESS AND COMPUTING RANK OF A MODULE USING SYZYGIES

BY

OSWALDO LEZAMA (Bogotá)

Abstract. Using syzygies computed via Gröbner bases techniques, we present algorithms for testing some homological properties for submodules of the free module A^m, where $A = R[x_1, \dots, x_n]$ and R is a Noetherian commutative ring. We will test if a given submodule M of A^m is flat. We will also check if M is locally free of constant dimension. Moreover, we present an algorithm that computes the rank of a flat submodule M of A^m and also an algorithm that computes the projective dimension of an arbitrary submodule of A^m. All algorithms are illustrated with examples.

1. Introduction. Gröbner bases introduced by Bruno Buchberger ([4]) have been studied intensively in the last years, and there are a lot of interesting applications in many branches of mathematics, including homological algebra, commutative algebra, algebraic geometry, differential algebra, graph theory, etc. (see [1], [2], [5], [6], [7], [11], [12] and [17]).

In this paper we present some applications of Gröbner bases of modules in homological algebra using some results of [14] and [15]. If R is a Noetherian commutative ring, $A = R[x_1, \dots, x_n]$, and A^m is the free module of vector columns of length $m \geq 1$ with entries in $R[x_1, \dots, x_n]$, we will test if a given submodule M of A^m is flat. We will also check if M is locally free of constant dimension. Moreover, we will present an algorithm that computes the rank of a flat submodule M of A^m and also an algorithm that computes the projective dimension of an arbitrary submodule of A^m. All algorithms will be illustrated with examples.

In [15] we computed presentations of $\mathrm{Ext}_A^r(M,N)$ and $\mathrm{Tor}_r^A(M,N)$, where M is a submodule of A^m and N is a submodule of A^l, with $m, l \geq 1$ and $r \geq 0$. The technique used in [15] is very simple: presentations of submodules of A^m were computed using syzygies and Gröbner bases as in [14], and then free resolutions and their modules of homology were calculated. All computations in [14] and [15] were done manually or using the computer algebra system

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`CoCoa`. In the present paper we will apply the results and computations of [15] and also the Fitting ideals of a module in order to make computations announced in the previous paragraph.

The paper is divided into five sections. In the second section we will present a test for checking if a given submodule M of A^m is flat using the Ext_A^r modules. As we will see, this test could also be used for checking if M is projective, or equivalently, locally free. In the third section we will check local freeness of constant dimension for finitely presented modules over arbitrary commutative rings. The idea is to compute a finite presentation, and then to compute the Fitting ideals of the matrix presentation. The result will be applied in the fourth section to Noetherian rings. The fourth section is dedicated to computing the rank of flat submodules of A^m using again matrix presentations and their Fitting ideals. Finally, in the last section, we will present an algorithm that computes the projective dimension of a given submodule M of A^m.

2. Test for flatness. In this section we will present a test for checking if a given submodule M of A^m is flat. As we will see below, this test could also be used for checking if M is projective, or equivalently, locally free. By the results in [15] we know how to compute the $\mathrm{Ext}_A^r(M,N)$ modules, so the central idea now is to compute $\mathrm{Ext}_A^r(M,\mathrm{Syz}(M))$, where $\mathrm{Syz}(M)=\mathrm{Syz}[\boldsymbol{f}_1\cdots\boldsymbol{f}_s]$ and $M=\langle\boldsymbol{f}_1,\ldots,\boldsymbol{f}_s\rangle$.

We will use the following notation. If S is a commutative ring, $\mathrm{Spec}(S)$ denotes the set of prime ideals of S, and $\mathrm{Max}(S)$ the set of maximal ideals of S. Moreover, if $P\in\mathrm{Spec}(S)$, then S_P is the localization of S with respect to P and M_P is the localization of M with respect to P. If M is a free module over S we will also say that M is *S-free*, and the *dimension* of M over S, denoted by $\dim_S(M)$, is the number of elements of any basis of M.

Some definitions and well known preliminary results are needed to formulate the test for flatness.

Definition 1. Let S be a commutative ring and M a finitely generated S-module.

(i) M is *locally free* if M_P is S_P-free for each $P\in\mathrm{Spec}(S)$.
(ii) M is *locally free of constant dimension* $r\geq 0$ if M_P is S_P-free of dimension r for each $P\in\mathrm{Spec}(S)$.

It is clear that any locally free module of constant dimension is locally free. The following trivial example shows that the converse is not always true. Another nontrivial example will be presented later.

EXAMPLE 2. In $\mathbb{Z}_{10}$, the principal ideal $\langle 2\rangle$ is locally free but is not locally free of constant dimension. In fact, $\operatorname{Spec}(\mathbb{Z}_{10}) = \{\langle 2\rangle, \langle 5\rangle\}$, $\langle 2\rangle_{\langle 2\rangle} = 0$ and $\langle 2\rangle_{\langle 5\rangle} = (\mathbb{Z}_{10})_{\langle 5\rangle}$, thus $\langle 2\rangle_{\langle 2\rangle}$ and $\langle 2\rangle_{\langle 5\rangle}$ are free, but $\dim_{(\mathbb{Z}_{10})_{\langle 2\rangle}}(\langle 2\rangle_{\langle 2\rangle}) = 0$ and $\dim_{(\mathbb{Z}_{10})_{\langle 5\rangle}}(\langle 2\rangle_{\langle 5\rangle}) = 1$.

The following well known result will be needed.

PROPOSITION 3 ([10]). *Let S be a commutative local ring. Then the following conditions are equivalent*:

(i) *M is a finitely generated flat module.*
(ii) *M is free of finite dimension.*
(iii) *M is a finitely generated projective module.*

The test for flatness is supported by the following theorem (cf. [10, Proposition 7.3.15]).

THEOREM 4. *Let S be a Noetherian commutative ring and M a finitely generated S-module such that the sequence*

$$0 \to K \to S^n \xrightarrow{F_0} M \to 0$$

is exact. Then the following conditions are equivalent:

(i) *M is flat.*
(ii) *M is projective.*
(iii) *M is locally free.*
(iv) $\operatorname{Ext}^1_S(M, K) = 0$.

Proof. (i)$\Leftrightarrow$(ii). This equivalence is evident since any projective module is flat, and moreover, every finitely generated flat module over a Noetherian ring is projective (see [18, Corollary 4.3]).

(i)$\Leftrightarrow$(iii). This follows from Proposition 3 and the fact that flatness is a local-global property (see [10, Proposition 7.4.1]).

(ii)$\Rightarrow$(iv). This implication is well known: see [19, Corollary 10.2.9].

(iv)$\Rightarrow$(ii). From the given sequence we get the exact sequence

$$0 \to \operatorname{Hom}_S(M, K) \to \operatorname{Hom}_S(M, S^n) \xrightarrow{F_0^*} \operatorname{Hom}_S(M, M) \to \operatorname{Ext}^1_S(M, K) = 0$$

(see [18, Theorem 7.3]). Thus, F_0^* is surjective and there is $f \in \operatorname{Hom}_S(M, S^n)$ such that $F_0^*(f) = i_M$, i.e., $F_0 f = i_M$. This means that $S^n \cong K \oplus M$, i.e., M is projective. ∎

Now we are able to present an algorithm that tests flatness.

Test for Flatness

`Input`: $M=\langle \boldsymbol{f}_1,\dots,\boldsymbol{f}_s\rangle\subseteq A^m$ with $\boldsymbol{f}_i\neq 0\,(1\le i\le s)$
`Output`: TRUE if M is a flat module, FALSE otherwise
`Initialization`: Compute $\mathrm{Syz}\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_s\}$
    $K:=\mathrm{Syz}\{\boldsymbol{f}_1,\dots,\boldsymbol{f}_s\}$
    Compute $\mathrm{Ext}^1_A(M,K)$
`if` $\mathrm{Ext}^1_A(M,K)=0$ `then`
        `return` TRUE
`else`
        `return` FALSE

The following examples illustrate the algorithm.

Example 5. Let $M=\langle \boldsymbol{f}_1,\boldsymbol{f}_2\rangle\subseteq(\mathbb{Z}_{10}[x,y])^2$, where

$$\boldsymbol{f}_1=(3x^2y+3x,xy-2y),\quad \boldsymbol{f}_2=(7xy^2+y,y^2-4x).$$

We will test flatness for M. In [15] we calculated $\mathrm{Syz}(M)=\langle(5y,5x)\rangle$. According to the algorithm we must compute $\mathrm{Ext}^1_A(M,K)$ with $K=\langle(5y,5x)\rangle$ and $A=\mathbb{Z}_{10}[x,y]$. We will follow the procedure described in [15], i.e., we compute presentations of M and K, next we compute a free resolution of M by a sequence of matrices, and finally we compute $\mathrm{Ext}^1_A(M,K)$. Since $\mathrm{Syz}(K)=\langle 2\rangle$, the presentations of M and K are

$$M\cong A^2/\mathrm{Syz}(M)=A^2/\langle(5y,5x)\rangle,\qquad K\cong A/\mathrm{Syz}(K)=A/\langle 2\rangle.$$

A free resolution $\{F_i\}_{i\ge 0}$ of M was calculated in [15]:

$$\cdots\to A\xrightarrow{[5]}A\xrightarrow{[2]}A\xrightarrow{\begin{bmatrix}5y\\5x\end{bmatrix}}A^2\xrightarrow{\begin{bmatrix}3x^2y+3x & 7xy^2+y\\ xy-2y & y^2-4x\end{bmatrix}}M\to 0;$$

in particular, $F_1=\left[\begin{smallmatrix}5y\\5x\end{smallmatrix}\right]$ and $F_2=[2]$; according to [15] we must compute $\mathrm{Syz}[\mathrm{Syz}[I_t\otimes F_2^T]|I_t\otimes F_1^T]$ where p_1 is the number of generators of the module $\mathrm{Syz}[I_t\otimes F_2^T]$ and t is the number of generators of K. Thus, in this case, $t=1$, $\mathrm{Syz}[I_1\otimes F_2^T]=\mathrm{Syz}[1\otimes[2]]=\mathrm{Syz}[2]=[5]$ and $p_1=1$. Since

$$\mathrm{Syz}[\mathrm{Syz}[I_1\otimes F_2^T]|I_1\otimes F_1^T]=\mathrm{Syz}[5\ \ 5y\ 5x]=\begin{bmatrix}2 & -y & -x & 0 & 0 & 0\\ 0 & 1 & 0 & 2 & 0 & x\\ 0 & 0 & 1 & 0 & 2 & -y\end{bmatrix},$$

we have $\mathrm{Ext}^1_A(M,K)\cong A/\langle 2,-y,-x\rangle\neq 0$ (we take the entries of the first row of the above matrix, see [15]). By the algorithm, M is not flat, or equivalently, M is not projective, or equivalently, M is not locally free.

EXAMPLE 6. Let $M = \langle \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3 \rangle \subseteq (\mathbb{Z}[x,y])^4$ with $\boldsymbol{f}_1 = (1, x, -1, 0)$, $\boldsymbol{f}_2 = (xy + x + y, -y^2, 0, -x - y)$ and $\boldsymbol{f}_3 = (-x^2y - x^2 - xy, xy^2, 0, x^2 + xy)$. Then M has a finite free resolution given by

$$0 \to A \xrightarrow{F_1} A^3 \xrightarrow{F_0} M \to 0,$$

where $A = \mathbb{Z}[x,y]$,

$$F_0 = \begin{bmatrix} 1 & xy + x + y & -x^2y - x^2 - xy \\ x & -y^2 & xy^2 \\ -1 & 0 & 0 \\ 0 & -x - y & x^2 + xy \end{bmatrix} \quad \text{and} \quad F_1 = \begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix}.$$

Hence the free resolution has $F_i = 0$ for $i \geq 2$. We now compute $\mathrm{Ext}^1_A(M, K)$, where $K = \langle (0, x, 1) \rangle$. Using the notation of the previous example we see that in this case $t = 1$ and $p_1 = 1$ is the number of generators of the module $\mathrm{Syz}[I_1 \otimes F_2^T] = \mathrm{Syz}[0] = [1]$. Thus

$$\mathrm{Syz}[\mathrm{Syz}[I_1 \otimes F_2^T] | I_1 \otimes F_1^T] = \mathrm{Syz}[1\ 0\ x\ 1] = \begin{bmatrix} 0 & 1 & x \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

We take the entries of the first row of the above matrix to conclude that

$$\mathrm{Ext}^1_A(M, K) = A / \langle 0, 1, x \rangle = A/A = 0.$$

This means that M is flat, or equivalently, M is projective, or equivalently, M is locally free.

3. Locally free modules of constant dimension. In this section we will check local freeness of constant dimension for modules over arbitrary commutative rings following the ideas in [10]. The result will be applied in the next section to Noetherian rings for computing the rank of a flat module. Local freeness of constant dimension and rank will be checked by applying the Fitting ideals of matrices and modules. We start by recalling this notion (see [3] and [10]).

DEFINITION 7. Let S be a commutative ring and F a matrix over S of size $n \times m$. For each integer r, the rth *Fitting ideal* of F, denoted by $F_r^S(F)$, is defined in the following way:

(i) $F_r^S(F)$ is the ideal of S generated by all minors of F of size $(n - r) \times (n - r)$ if $1 \leq n - r \leq \min\{n, m\}$.
(ii) $F_r^S(F) = S$ if $n - r \leq 0$.
(iii) $F_r^S(F) = 0$ if $n - r > \min\{n, m\}$.

Fitting ideals of modules are defined as follows.

Definition 8. Let S be a commutative ring and M a finitely presented S-module with presentation

$$S^m \xrightarrow{F_1} S^n \xrightarrow{F_0} M \to 0.$$

For each integer r, the rth *Fitting ideal* of M, denoted by $F_r^S(M)$, is defined by $F_r^S(M) = F_r^S(F_1)$.

Since M is a finitely generated module, we can change the system of generators of M obtaining a different presentation of M. Also, if we change the bases of S^m and S^n we get a new matrix F_1' in the presentation of M. The following lemma shows that the definition of $F_r^S(M)$ is independent of these changes.

Lemma 9 ([10]). *Let S be a commutative ring and M a finitely presented S-module. Then*:

(i) *$F_r^S(M)$ is independent of the choice of bases in S^m and S^n.*
(ii) *$F_r^S(M)$ is independent of the presentation of M.*
(iii) *Let B be an S-algebra. Then $F_r^B(M \otimes_S B) = \langle F_r^S(M)\rangle$. In particular, $F_r^{S_P}(M_P) = F_r^S(M)_P$ for each ideal $P \in \operatorname{Spec}(S)$ ($\langle F_r^S(M)\rangle$ is the ideal of B generated by $F_r^S(M)$).*

For local commutative rings, Fitting ideals can be used to check freeness.

Lemma 10 ([10]). *Let S be a local ring and M a finitely presented S-module. Then the following conditions are equivalent*:

(i) *M is free of dimension $r \geq 0$.*
(ii) *$F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$ for some $r \geq 0$.*

Local freeness of constant dimension could be checked for any commutative ring using Fitting ideals.

Theorem 11. *Let S be a commutative ring and M a finitely presented S-module. Then the following conditions are equivalent*:

(i) *M is locally free of constant dimension $r \geq 0$.*
(ii) *$F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$ for some $r \geq 0$.*

Proof. (i)⇒(ii). Let $P \in \operatorname{Spec}(S)$. Since M_P is free of dimension r, from Lemmas 9 and 10 we have $F_r^{S_P}(M_P) = F_r^S(M)_P = S_P$ and $F_{r-1}^{S_P}(M_P) = F_{r-1}^S(M)_P = 0$, i.e., $F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$.

(ii)⇒(i). If $F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$ for some $r \geq 0$, then $F_r^{S_P}(M_P) = F_r^S(M)_P = S_P$ and $F_{r-1}^{S_P}(M_P) = F_{r-1}^S(M)_P = 0$, hence M_P is free of dimension r for each $P \in \operatorname{Spec}(S)$, i.e., M is locally free of constant dimension $r \geq 0$. ∎

The above theorem supports the following algorithm.

TEST FOR LOCAL FREENESS OF CONSTANT DIMENSION

```
Input: M a finitely presented S-module with presentation
            S^{s_1} --G_1--> S^s --G_0--> M ---> 0
Output: TRUE if M is locally free of constant dimension,
     FALSE otherwise
Initialization: i := s - 1
while i >= -1 do
            Compute F_i^S(G_1)
            if F_i^S(G_1) != S then
                 if F_i^S(G_1) = 0 then
                      return TRUE
                 else
                      return FALSE
            else
                 i := i - 1
```

EXAMPLE 12. Let $N = \langle \boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3 \rangle \subseteq (\mathbb{Z}_{10}[x,y])^2$, where $\boldsymbol{g}_1 = (0,x)$, $\boldsymbol{g}_2 = (y,x)$ and $\boldsymbol{g}_3 = (2x,x)$. In [15] we computed a free resolution of N:

$$\cdots \to A^2 \xrightarrow{\begin{bmatrix} 5 & y \\ 0 & 8 \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} 2 & y \\ 0 & 5 \end{bmatrix}} A^2 \xrightarrow{\begin{bmatrix} 5 & 2x+9y \\ 0 & 8x \\ 5 & y \end{bmatrix}} A^3 \xrightarrow{\begin{bmatrix} 0 & y & 2x \\ x & x & x \end{bmatrix}} N \to 0,$$

hence a finite presentation of N is

$$A^2 \xrightarrow{\begin{bmatrix} 5 & 2x+9y \\ 0 & 8x \\ 5 & y \end{bmatrix}} A^3 \xrightarrow{\begin{bmatrix} 0 & y & 2x \\ x & x & x \end{bmatrix}} N \to 0.$$

In the notation of the algorithm, $s = 3$ and $s_1 = 2$, so $i = s - 1 = 2$ and we compute $F_2^A(G_1)$, the ideal of A generated by all entries of G_1. Thus $F_2^A(G_1) = \langle 5, 2x+9y, 8x, 5, y \rangle = \langle 5, 8x, y \rangle \neq A, 0$; so by the algorithm, N is not locally free of constant dimension.

Later we exhibit a nontrivial locally free module of constant dimension.

4. Rank of a flat module. Now we will present an algorithm for computing the rank of flat submodules of A^m. We need to recall some definitions and preliminary results.

Definition 13. Let S be a commutative ring.

(i) The *set of associated primes* of the ring S is defined by
$$\mathrm{Ass}(S) = \{P \in \mathrm{Spec}(S) \mid P = \mathrm{Ann}_S(x) \text{ for some } x \in S,\, x \neq 0\},$$
where $\mathrm{Ann}_S(x) = \{s \in S \mid sx = 0\}$.

(ii) If S_0 is the set of non-zero divisors of S then
$$Q = \{a/t \mid a \in S,\, t \in S_0\},$$
is the *total quotient ring* of S.

(iii) Let M be a finitely generated S-module. We say that M has rank $r \geq 0$, and write $\mathrm{rank}(M) = r$, if $M \otimes_R Q$ is a free Q-module of dimension r. If $M \otimes_R Q$ is not a free Q-module, then we write $\mathrm{rank}(M) = -1$.

Some well known properties of associated primes are summarized in the following proposition.

Proposition 14 ([10]). *Let S be a commutative ring. Then*:

(i) $\mathrm{Ass}(Q) = \{PQ \mid P \in \mathrm{Ass}(S)\}$, *where PQ is the extension of P in Q, i.e., PQ is the ideal of Q generated by P. Moreover*, $Q_{PQ} \cong S_P$.

(ii) *If S is Noetherian then* $\mathrm{Ass}(S)$ *is finite. Moreover, given* $P \in \mathrm{Spec}(S)$, *there exists* $P' \in \mathrm{Ass}(S)$ *such that* $P' \subseteq P$ *and* $(S_P)_{P'S_P} \cong S_{P'}$.

(iii) *If S is Noetherian and $D_0 = S - S_0 = \{s \in S \mid s$ is a zero divisor of $S\}$, then $D_0 = \bigcup_{i=1}^{r} P_i'$, where* $\mathrm{Ass}(S) = \{P_1', \ldots, P_r'\}$.

(iv) *If S is Noetherian, then* $\mathrm{Max}(Q) \subseteq \mathrm{Ass}(Q)$.

Proposition 15. *Let S be a commutative ring and M a finitely generated S-module. Then M is locally free of constant dimension $r \geq 0$ if and only if M_P is S_P-free of dimension r for each* $P \in \mathrm{Max}(S)$.

Proof. $\Rightarrow$ This is evident since $\mathrm{Max}(S) \subseteq \mathrm{Spec}(S)$.

$\Leftarrow$ Let $P' \in \mathrm{Spec}(S)$. Then there exists $P \in \mathrm{Max}(S)$ such that $P' \subseteq P$ and $(S_P)_{P'S_P} \cong S_{P'}$, where the isomorphism is defined by
$$\frac{z/r}{u/s} \mapsto \frac{zs}{ru},$$
with $z/r \in S_P$ and $u/s \notin P'S_P$. We have $M_P \cong (S_P)^r$ and hence $M_P \otimes_{S_p} (S_P)_{P'S_P} \cong (S_P)^r \otimes_{S_P} (S_P)_{P'S_P}$. Then $(M \otimes_S S_P) \otimes_{S_p} (S_P)_{P'S_P} \cong (S_P)^r \otimes_{S_P} (S_P)_{P'S_P}$, i.e., $M \otimes_S (S_P \otimes_{S_p} (S_P)_{P'S_P}) \cong (S_P \otimes_{S_P} (S_P)_{P'S_P})^r$. Thus, $M \otimes_S S_{P'} \cong (S_{P'})^r$, and hence $M_{P'}$ is $S_{P'}$-free of dimension r for each $P' \in \mathrm{Spec}(S)$, i.e., M is locally free of constant dimension r. ∎

Lemma 16. *Let S be a Noetherian commutative ring and M a finitely generated S-module such that M_P is an S_P-free module of finite dimension $r_P \geq 0$ for each prime $P \in \mathrm{Spec}(S)$. Then the following conditions are equivalent*:

(i) *M has rank $r \geq 0$.*
(ii) *For each $P \in \mathrm{Ass}(S)$, $r_P = r$.*
(iii) *For each $P \in \mathrm{Spec}(S)$, $r_P = r$.*

Proof. (i)⇒(ii). Let $P \in \mathrm{Ass}(S)$. By Proposition 14, $S_P \cong Q_{PQ}$ and so

$$\begin{aligned} M_P &\cong M \otimes_S S_P \cong M \otimes_S Q_{PQ} \cong M \otimes_S (Q \otimes_Q Q_{PQ}) \\ &\cong (M \otimes_S Q) \otimes_Q Q_{PQ} \cong Q^r \otimes_Q Q_{PQ} \cong (Q_{PQ})^r \cong (S_P)^r, \end{aligned}$$

i.e., M_P is S_P-free of dimension r. Thus $r_P = r$.

(ii)⇒(iii). Let $P \in \mathrm{Spec}(S)$. Then by Proposition 14, there exists $P' \in \mathrm{Ass}(S)$ such that $P' \subseteq P$ and $S_{P'} \cong (S_P)_{P'S_P}$. Since $(S_P)^{r_P} \cong M_P$, we have $(S_P)^{r_P} \otimes_{S_P} (S_P)_{P'S_P} \cong M_P \otimes_{S_P} (S_P)_{P'S_P}$, and hence $(S_P \otimes_{S_P} (S_P)_{P'S_P})^{r_P} \cong M \otimes_S S_P \otimes_{S_P} (S_P)_{P'S_P}$, i.e., $((S_P)_{P'S_P})^{r_P} \cong M \otimes_S (S_P)_{P'S_P}$. We get $(S_{P'})^{r_P} \cong M \otimes_S S_{P'} \cong M_{P'} \cong (S_{P'})^r$, and hence $r_P = r$.

(iii)⇒(i). If $r = 0$, then $M_P = 0$ for each $P \in \mathrm{Spec}(S)$, i.e., $M = 0$ and $M \otimes_S Q = 0$, hence M has rank 0.

Thus, we can assume that $r \geq 1$. Let $N = M \otimes_S Q$. We must prove that N is Q-free of dimension r.

We first prove that N_U is Q_U-free of dimension r for each $U \in \mathrm{Spec}(Q)$. Let $U \in \mathrm{Ass}(Q)$. Then by Proposition 14, $U = PQ$ with $P \in \mathrm{Ass}(S)$, hence

$$\begin{aligned} N_U &\cong N \otimes_Q Q_U \cong (M \otimes_S Q) \otimes_Q Q_U \cong M \otimes_S (Q \otimes_Q Q_U) \cong M \otimes_S Q_U \\ &= M \otimes_S Q_{PQ} \cong M \otimes_S S_P \cong M_P \cong (S_P)^r \cong (Q_{PQ})^r = (Q_U)^r. \end{aligned}$$

Proposition 14 implies that N_U is Q_U-free of dimension r for each $U \in \mathrm{Max}(Q)$, and by Proposition 15, N_U is Q_U-free of dimension r for each $U \in \mathrm{Spec}(Q)$.

By Proposition 14, $\mathrm{Max}(Q)$ is finite, say $\mathrm{Max}(Q) = \{U_1, \ldots, U_n\}$. For each $1 \leq i \leq n$, $U_1 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_n \nsubseteq U_i$ since otherwise $U_1 \cdots U_{i-1} U_{i+1} \cdots U_n \subseteq U_1 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_n$ so $U_j \subseteq U_i$ for some $j \neq i$, but this is impossible since U_j is maximal. Thus, for each $1 \leq i \leq n$ there exists $s_i \in U_1 \cap \cdots \cap U_{i-1} \cap U_{i+1} \cap \cdots \cap U_n$ such that $s_i \notin U_i$. Moreover, since we are assuming that $r > 0$, it follows that $N_{U_i} \neq 0$ and hence there exists $x_i \in N$ such that $x_i/1 \notin U_i N_{U_i}$. In fact, if $N \subseteq U_i N_{U_i}$ then $N_{U_i} = U_i N_{U_i}$ and by the Nakayama Lemma $N_{U_i} = 0$. We define $x = s_1 x_1 + \cdots + s_n x_n \in N$. Then we observe that $x/1 \notin U_i N_{U_i}$ for each $1 \leq i \leq n$. In fact, if $x/1 \in U_i N_{U_i}$ for some i, then $x_i/1 \in U_i N_{U_i}$ since $s_i/1$ is invertible Q_{U_i} and $s_j x_j/1 \in U_i N_{U_i}$ for each $j \neq i$.

Now we consider, for each $1 \leq i \leq n$, the vector space $N_{U_i}/U_i N_{U_i}$ over the field $Q_{U_i}/U_i Q_{U_i}$. Then $\overline{x}/1 \neq \overline{0}$ in $N_{U_i}/U_i N_{U_i}$. Thus, for each $1 \leq i \leq n$ there exists a basis $\overline{X}_i$ of $N_{U_i}/U_i N_{U_i}$ such that $\overline{x}/1 \in \overline{X}_i$, and by Proposition 3, X_i is a basis of N_{U_i}. Hence, we have constructed an element $x \in N$ such that $x/1$ is in some basis X_i of N_{U_i} for each $1 \leq i \leq n$.

Now we can conclude the proof of the lemma by showing that N is Q-free of dimension r. We prove this by induction on r. Consider the quotient module $N/\langle x\rangle$. For each $U \in \mathrm{Max}(Q)$ we have $(N/\langle x\rangle)_U = N_U/\langle x/1\rangle$. If $r = 1$, then $N_U = \langle x/1\rangle$ and hence $(N/\langle x\rangle)_U = 0$ for each $U \in \mathrm{Max}(Q)$, i.e., $N/\langle x\rangle = 0$. Thus, $N = \langle x\rangle$. We note that if $qx = 0$ for some $q \in Q$, then $\frac{q}{1}\frac{x}{1} = 0$ in N_U for each $U \in \mathrm{Max}(Q)$; but $x/1$ is linearly independent, so $q/1 = 0$ and hence $q = 0$. This means that N is Q-free of dimension 1. In the general case, $(N/\langle x\rangle)_U$ is free of dimension $r-1$ for each $U \in \mathrm{Max}(Q)$, and by induction we find that $N/\langle x\rangle$ is Q-free of dimension $r-1$. We have the exact sequence

$$0 \to \langle x\rangle \to N \to Q^{r-1} \to 0,$$

since Q^{r-1} is projective, so $N \cong \langle x\rangle \oplus Q^{r-1}$, and hence $N \cong Q^r$. ∎

Now we are able to prove the main theorem of this section.

THEOREM 17. *Let S be a Noetherian commutative ring and M a finitely presented S-module. Let $F_r(M)$ be the rth Fitting ideal of M. Then M is flat and has rank $r \geq 0$ if and only if $F_r(M) = S$ and $F_{r-1}(M) = 0$.*

Proof. $\Rightarrow$ Since M is flat, M_P is also flat for any $P \in \mathrm{Spec}(S)$ (see [10, Proposition 7.4.1]). Moreover, M_P is finitely generated because M is finitely presented. By Proposition 3, M_P is free of finite dimension $r_P \geq 0$. Since M has rank r, Lemma 16 shows that $r_P = r$ for each $P \in \mathrm{Spec}(S)$. Thus, by Theorem 11, $F_r(M) = S$ and $F_{r-1}(M) = 0$.

$\Leftarrow$ If $F_r(M) = S$ and $F_{r-1} = 0$ for some $r \geq 0$, then by Theorem 11, M_P is free of dimension r for any $P \in \mathrm{Spec}(S)$. Thus, M_P is flat for any prime ideal P of S, and hence M is flat. Moreover, Lemma 16 guarantees that $M \otimes_S Q$ is free of dimension r, i.e., M has rank r. ∎

COROLLARY 18. *Let S be a Noetherian commutative ring and M an S-module of finite presentation. Let $F_r(M)$ be the rth Fitting ideal of M. Then the following conditions are equivalent*:

(i) *M is flat and has rank $r \geq 0$.*
(ii) *$F_r(M) = S$ and $F_{r-1}(M) = 0$.*
(iii) *M is locally free of constant dimension $r \geq 0$.*
(iv) *M is projective and has rank $r \geq 0$.*

Proof. (i)$\Leftrightarrow$(ii). This is the content of Theorem 17.

(ii)$\Leftrightarrow$(iii). This is the content of Theorem 11.

(i)$\Leftrightarrow$(iv). This equivalence is evident as we saw in the proof of Theorem 4. ∎

We conclude this section with an algorithm that computes the rank of a flat submodule of A^m.

RANK OF A FLAT MODULE

`Input`: $M = \langle \boldsymbol{f}_1, \dots, \boldsymbol{f}_s \rangle \subseteq A^m$ a flat module, with
$\boldsymbol{f}_k \neq 0$ $(1 \leq k \leq s)$
`Output`: $\operatorname{rank}(M)$
`Initialization`: Compute a matrix presentation G_1 of M

$$A^{s_1} \xrightarrow{G_1} A^s \xrightarrow{G_0} M \to 0$$

$i := s - 1$
`while` $i \geq -1$ `do`
  Compute $F_i^A(G_1)$
  `if` $F_i^A(G_1) \neq A$ `then`
    `if` $F_i^A(G_1) = 0$ `then`
      $\operatorname{rank}(M) = i + 1$
    `else`
      $\operatorname{rank}(M) = -1$
  `else`
    $i := i - 1$

The following example illustrates the above algorithm.

EXAMPLE 19. We consider again the module M of Example 6. We know that M is flat and we have the presentation

$$A \xrightarrow{\begin{bmatrix} 0 \\ x \\ 1 \end{bmatrix}} A^3 \xrightarrow{\begin{bmatrix} 1 & xy + x + y & -x^2y - x^2 - xy \\ x & -y^2 & xy^2 \\ -1 & 0 & 0 \\ 0 & -x - y & x^2 + xy \end{bmatrix}} M \to 0.$$

In the notation of the algorithm, $s = 3$ and $s_1 = 1$, so $i = s - 1 = 2$ and we compute $F_2^A(G_1)$, which is the ideal of A generated by all entries of G_1, thus $F_2^A(G_1) = A$. We set $i := i - 1 = 1$. By Definition 7, $F_1^A(G_1) = 0$, so by the algorithm, $\operatorname{rank}(M) = 2$. We conclude that M is a flat module of rank 2, or equivalently, M is a projective module of rank 2, or equivalently, M is a locally free module of constant dimension 2.

5. Projective dimension. In this section we present an algorithm that computes the projective dimension of a submodule M of A^m. Since A is a Noetherian ring, the flat dimension of M coincides with the projective dimension (see [18, Theorem 9.22]), and hence the algorithm also gives the flat dimension of M. A similar algorithm is presented in [9], but assuming that we know how to compute finite free resolutions of M. Our algorithm only requires the computation of arbitrary free resolutions as is shown in [15]. We recall that if S is a commutative ring and M is an S-module, then the *projective dimension* of M, denoted by $\mathrm{pd}(M)$, is the minimum of all lengths of projective resolutions of M. A *projective resolution* of M is an exact sequence

$$\cdots \xrightarrow{f_{r+1}} P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0,$$

where P_i is a projective S-module for each $i \geq 0$; $\ker(f_i)$ is named the ith *syzygy* of M (we observe that if P_i is free of finite dimension, then each f_i can be represented by a matrix F_i and $\ker(F_i) = \mathrm{Syz}(F_i) =$ syzygy module of columns of F_i). The *global dimension* of the ring S is denoted by $D(S)$ and defined by

$$D(S) = \sup\{\mathrm{pd}(M) \mid M \text{ is an } S\text{-module}\}.$$

The Hilbert Syzygy Theorem says that

$$D(S[x_1, \ldots, x_n]) = D(S) + n$$

for any commutative ring S (see [18, Theorem 9.34]). Thus, in our particular situation, $D(A) = D(R) + n$, where $A = R[x_1, \ldots, x_n]$ and R is a Noetherian commutative ring. We will assume that $D(R)$ is finite, and hence any A-module M has finite projective dimension,

$$\mathrm{pd}(M) \leq D(R) + n.$$

Our algorithm is supported by the Hilbert Syzygy Theorem and the following result (see [8] and [18]).

Theorem 20. *Let S be a commutative ring and M an S-module. Let*

$$\cdots \xrightarrow{f_{r+1}} P_r \xrightarrow{f_r} P_{r-1} \xrightarrow{f_{r-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0 \tag{5.1}$$

be a projective resolution of M. Let r be the smallest integer such that $\mathrm{Im}(f_r)$ *is projective. Then r does not depend on the resolution and* $\mathrm{pd}(M) = r$.

The above theorem is valid for any projective resolution of M, thus we can consider a free resolution $\{F_i\}_{i\geq 0}$ computed as in [15]. Hence we obtain the following algorithm that computes the projective dimension of $M \subseteq A^m$. We will also denote by F_i the module generated by the columns of the matrix F_i.

PROJECTIVE DIMENSION OF A MODULE

`Input`: $D(R) < \infty, M = \langle \boldsymbol{f}_1, \ldots, \boldsymbol{f}_s \rangle \subseteq (R[x_1, \ldots, x_n])^m$
with $\boldsymbol{f}_k \neq 0\,(1 \leq k \leq s)$
`Output`: $\mathrm{pd}(M)$
`Initialization`: Compute a free resolution $\{F_i\}_{i \geq 0}$ of M
$i := 0$
`while` $i \leq D(R) + n$ `do`
`if` F_i is projective `then` $\mathrm{pd}(M) := i$
`else` $i := i + 1$

EXAMPLE 21. We saw that the module M in Example 5 is not projective, and hence $\mathrm{pd}(M) \geq 1$. We know that $D(\mathbb{Z}_{10}) = 0 < \infty$, by the Hilbert Syzygy Theorem, and $1 \leq \mathrm{pd}(M) \leq 2$. We will show with the above algorithm that $\mathrm{pd}(M) = 1$. Let $\{F_i\}_{i \geq 0}$ be the free resolution of M as in Example 5; we see that $F_0 = M$ is not projective. Let $i = 1$. We will prove that

$$F_1 = \begin{bmatrix} 5y \\ 5x \end{bmatrix} = \langle (5y, 5x) \rangle$$

is projective. We need to compute $\mathrm{Ext}_A^1(F_1, F_2)$ with $F_2 = [2]$. Presentations for F_1 and F_2 are

$$\begin{aligned} \langle (5y, 5x) \rangle \cong A/\mathrm{Syz}(\langle (5y, 5x) \rangle) = A/\langle 2 \rangle, \\ \langle 2 \rangle \cong A/\mathrm{Syz}(2) = A/\langle 5 \rangle. \end{aligned}$$

According to Example 5 a free resolution $\{H_i\}_{i \geq 0}$ of F_1 is

$$\cdots \longrightarrow A \xrightarrow{[5]} A \xrightarrow{[2]} A \xrightarrow{\begin{bmatrix} 5y \\ 5x \end{bmatrix}} F_1 \rightarrow 0,$$

in particular, $H_1 = [2]$ and $H_2 = [5]$; according to [15] we must compute $\mathrm{Syz}[\mathrm{Syz}[I_t \otimes H_2^T] | I_t \otimes H_1^T]$. In this case $t = 1$ and p_1 is the number of generators of the module $\mathrm{Syz}[I_t \otimes H_2^T] = \mathrm{Syz}[1 \otimes [5]] = \mathrm{Syz}[5] = [2]$. Thus, $p_1 = 1$ and since

$$\mathrm{Syz}[\mathrm{Syz}[I_t \otimes H_2^T] | I_t \otimes H_1^T] = \mathrm{Syz}[2 \;\; 2] = \begin{bmatrix} 5 & 1 & 0 \\ 0 & -1 & 5 \end{bmatrix},$$

we have $\mathrm{Ext}_A^1(F_1, F_2) \cong A/\langle 5, 1, 0 \rangle = A/A = 0$ and F_1 is projective.

EXAMPLE 22. Let $F_1 = \langle (5y, 5x) \rangle \subseteq (\mathbb{Z}_{10}[x, y])^2$ be as in the previous example. We saw that F_1 is projective, and by Theorem 4, F_1 is locally free. However, the algorithm of Section 3 shows that F_1 is not locally free of constant dimension. Hence, we have here another example that shows that locally free is not the same as locally free of constant dimension. Moreover, the algorithm of Section 4 shows that $\operatorname{rank}(F_1) = -1$. This example also shows that if M is a finitely presented module over a Noetherian commutative ring S and there is no integer $r \geq 0$ such that $F_r^S(M) = S$ and $F_{r-1}^S(M) = 0$, then we cannot conclude that M is not projective (cf. Corollary 18). In conclusion, the algorithm for checking local freeness of constant dimension is unsuitable for checking local freeness, or equivalently, it could not be used to check if a given module is projective. Compare with the comment at the end of [16], and see also the comment in [13] after Theorem 1.1.

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REFERENCES

[1] W. W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Grad. Stud. Math. 3, Amer. Math. Soc., 1994.

[2] T. Becker and V. Weispfenning, *Gröbner Bases*, Grad. Texts in Math. 141, Springer, 1993.

[3] W. Brown, *Matrices over Commutative Rings*, Dekker, 1993.

[4] B. Buchberger, *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal*, Ph.D. Thesis, Inst. Math., Univ. of Innsbruck, Innsbruck, 1965.

[5] B. Buchberger and F. Winkler, *Gröbner Bases and Applications*, London Math. Soc. Lecture Note Ser. 251, Cambridge Univ. Press, 1998.

[6] F. Chyzak, A. Quadrat and D. Robertz, *Effective algorithms for parametrizing linear control systems over Ore algebras*, Appl. Algebra Engrg. Comm. Comput. 16 (2005), 319–376.

[7] D. Cox, J. Little and D. O'Shea, *Ideals, Varieties and Algorithms*, Springer, 2006.

[8] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Grad. Texts in Math. 150, Springer, 1995.

[9] J. Gago-Vargas, *Bases for projective modules in $A_n(k)$*, J. Symbolic Comput. 36 (2003), 845–853.

[10] G. Greuel and G. Pfister, *A Singular Introduction to Commutative Algebra*, Springer, 2002.

[11] M. Kreuzer and L. Robbiano, *Computational Commutative Algebra 1*, Springer, 2000.

[12] —, —, *Computational Commutative Algebra 2*, Springer, 2005.

[13] R. Laubenbacher and K. Schlauch, *An algorithm for the Quillen–Suslin theorem for quotients of polynomial rings by monomial ideals*, J. Symbolic Comput. 30 (2000), 555–571.

[14] O. Lezama, *Gröbner bases for modules over Noetherian polynomial commutative rings*, Georgian Math. J. 15 (2008), 121–137.

[15] —, *Some applications of Gröbner bases in homological algebra*, São Paulo J. Math. Sci., to appear.

[16] A. Logar and B. Sturmfels, *Algorithms for the Quillen–Suslin theorem*, J. Algebra 145 (1992), 231–239.

[17] A. Quadrat and J. F. Pommaret, *Localization and parametrization of linear multidimensional control systems*, Systems Control Lett. 37 (1999), 247–260.

[18] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.

[19] L. R. Vermani, *An Elementary Approach to Homological Algebra*, Chapman & Hall/CRC Monogr. Surveys Pure Appl. Math. 130, 2003.

Departamento de Matemáticas
Universidad Nacional de Colombia
Bogotá, Colombia
E-mail: jolezamas@unal.edu.co

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