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CYCLE-FINITE ALGEBRAS OF SEMIREGULAR TYPE

BY

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Abstract. We describe the structure of artin algebras for which all cycles of indecomposable finitely generated modules are finite and all Auslander–Reiten components are semiregular.

1. Introduction and the main results. Throughout the paper, by an *algebra* we mean a basic indecomposable artin algebra over a commutative artin ring K. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, E)$, where E is a minimal injective cogenerator in $\operatorname{mod} K$.

The Jacobson radical rad_A of $\operatorname{mod} A$ is the ideal generated by all non-invertible homomorphisms between modules in $\operatorname{ind} A$, and the infinite radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a result due to M. Auslander [4], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\operatorname{ind} A$ admits only a finite number of pairwise non-isomorphic modules. On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a result proved in [6].

We denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We do not distinguish between an indecomposable module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. By a *component* of Γ_A we mean a connected component of the translation quiver Γ_A. A component $\mathcal{C}$ of Γ_A is called *regular* if $\mathcal{C}$ contains neither a projective module nor an injective module, and *semiregular* if $\mathcal{C}$ does not contain both a projective module and an injective module. The shapes of regular and semiregular components of the Auslander–Reiten quivers Γ_A of algebras A have been described by S. Liu

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in [16], [17] and Y. Zhang (regular components) in [41]. An algebra A is said to be of *semiregular type* if all components in Γ_A are semiregular.

In the paper we are concerned with the problem of describing the algebras A of semiregular type. This class of algebras contains: the hereditary algebras of infinite representation type [8], [26], the tilted algebras with semiregular connecting components [10], [18], [28], the canonical algebras [27], [29], and the quasitilted algebras of canonical type [7], [15]. We also note that every algebra A with Γ_A having all components semiregular is of infinite representation type.

A prominent role in the representation theory of algebras is played by cycles of modules (see [22], [33]). Recall that a *cycle* in the module category $\operatorname{mod} A$ of an algebra A is a sequence

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r = X_0$$

of non-zero non-isomorphisms in $\operatorname{ind} A$, and such a cycle is said to be *finite* if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\operatorname{rad}_A^\infty$. We mention that the Auslander–Reiten quiver Γ_A admits at most finitely many τ_A-orbits containing indecomposable modules not lying on cycles in $\operatorname{mod} A$ (directing modules) [35]. Following [3] an algebra A is said to be *cycle-finite* if all cycles in $\operatorname{mod} A$ are finite. The class of cycle-finite algebras contains: the algebras of finite representation type, the tame tilted algebras [12], [27], the tame double tilted algebras [24], the tame generalized double tilted algebras [25], the tubular algebras [27], [29], the iterated tubular algebras [23], the tame quasitilted algebras [15], [38], the tame generalized multicoil algebras [21], the algebras with cycle-finite derived categories [2], and the strongly simply connected algebras of polynomial growth [36]. The representation theory of arbitrary cycle-finite algebras is still only emerging. We refer to the survey article [19] for some general results on the structure of finite-dimensional cycle-finite algebras over an algebraically closed field, and their module categories.

In Section 3 we introduce the concept of a coherent sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of tame quasitilted algebras of canonical type and the associated algebra $A(\mathbb{B})$, being a pushout glueing of the algebras $B_1, \ldots, B_n$.

The main aim of the paper is to prove the following theorem.

Theorem 1.1. *Let A be an algebra. The following statements are equivalent:*

(i) *A is cycle-finite of semiregular type.*
(ii) *A is isomorphic to the algebra $A(\mathbb{B})$ associated to a coherent sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of tame quasitilted algebras of canonical type.*

As a direct consequence of the above theorem and Theorem 3.5 we obtain the following description of components in the Auslander–Reiten quivers of cycle-finite algebras of semiregular type.

COROLLARY 1.2. *Let A be a cycle-finite algebra of semiregular type. Then the Auslander–Reiten quiver $\varGamma_A$ of A consists of one postprojective component, one preinjective component, and infinitely many semiregular tubes.*

Following [33], the *component quiver* $\varSigma_A$ of an algebra A has the components of $\varGamma_A$ as vertices, and two components $\mathcal{C}$ and $\mathcal{D}$ are linked in $\varSigma_A$ by an arrow $\mathcal{C} \to \mathcal{D}$ if $\operatorname{rad}_A^{\infty}(X, Y) \neq 0$ for some modules X in $\mathcal{C}$ and Y in $\mathcal{D}$. Then we obtain the following consequence of Theorems 1.1 and 3.5.

COROLLARY 1.3. *Let A be a cycle-finite algebra of semiregular type. Then the component quiver $\varSigma_A$ of A is acyclic.*

A crucial rôle in the proof of Theorem 1.1 is played by the following structure results.

THEOREM 1.4. *Let A be a cycle-finite algebra of semiregular type. Then A admits a tame concealed convex subcategory C such that all but finitely many stable tubes of $\varGamma_C$ are stable tubes of $\varGamma_A$.*

THEOREM 1.5. *Let A be a cycle-finite algebra of semiregular type, C a tame concealed convex subcategory of A and $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda\in\varLambda}$ the family of all stable tubes of $\varGamma_C$. The following statements hold:*

- (i) *For each $\lambda \in \varLambda$, $\varGamma_A$ contains a unique semiregular tube $\mathcal{T}_\lambda^A(C)$ containing all modules of $\mathcal{T}_\lambda^C$.*
- (ii) *The support $B(C) = \operatorname{supp}(\mathcal{T}^A(C))$ of the family $\mathcal{T}^A(C) = (\mathcal{T}_\lambda^A(C))_{\lambda\in\varLambda}$ is a tame quasitilted algebra of canonical type and a convex subcategory of A.*
- (iii) *$B(C)$ is a tame semiregular branch enlargement of C.*

COROLLARY 1.6. *Let A be a cycle-finite algebra of semiregular type and $\mathcal{C}$ a component of $\varGamma_A$. Then there exists a tame concealed convex subcategory C of A such that $\mathcal{C}$ is a component of $\varGamma_{B(C)}$.*

For basic background on the relevant representation theory we refer to the books [1], [5], [27], [30], [31], [40].

2. Preliminaries. We recall some notation, concepts and results on algebras and modules needed in our further considerations.

Let A be an algebra (basic, indecomposable) and $e_1, \dots, e_n$ be a set of pairwise orthogonal primitive idempotents of A with $1_A = e_1 + \dots + e_n$. Then

- $P_i = e_iA$, $i \in \{1, \dots, n\}$, is a complete set of pairwise non-isomorphic indecomposable projective modules in $\operatorname{mod} A$;
- $I_i = D(Ae_i)$, $i \in \{1, \dots, n\}$, is a complete set of pairwise non-isomorphic indecomposable injective modules in $\operatorname{mod} A$;

- $S_i = \mathrm{top}(P_i) = e_iA/e_i \,\mathrm{rad}\, A$, $i \in \{1,\dots,n\}$, is a complete set of pairwise non-isomorphic simple modules in $\mathrm{mod}\, A$;
- $S_i = \mathrm{soc}(I_i)$ for any $i \in \{1,\dots,n\}$.

Moreover, $F_i = \mathrm{End}_A(S_i) \cong e_iAe_i/e_i(\mathrm{rad}\, A)e_i$, for $i \in \{1,\dots,n\}$, are division algebras. The *quiver* Q_A of A is the valued quiver defined as follows:

- the vertices of Q_A are the indices $1,\dots,n$ of the chosen set $e_1,\dots,e_n$ of primitive idempotents of A;
- for two vertices i and j in Q_A, there is an arrow $i \to j$ from i to j in Q_A if and only if $e_i(\mathrm{rad}\, A)e_j/e_i(\mathrm{rad}\, A)^2e_j \neq 0$. Moreover, one associates to an arrow $i \to j$ in Q_A the valuation (d_{ij}, d'_{ij}), so we have in Q_A the valued arrow
$$i \xrightarrow{(d_{ij},d'_{ij})} j,$$
where the valuation numbers are $d_{ij} = \dim_{F_j} e_i(\mathrm{rad}\, A)e_j/e_i(\mathrm{rad}\, A)^2e_j$ and $d'_{ij} = \dim_{F_i} e_i(\mathrm{rad}\, A)e_j/e_i(\mathrm{rad}\, A)^2e_j$.

It is known that Q_A coincides with the Ext-quiver of A. Namely, Q_A contains a valued arrow $i \xrightarrow{(d_{ij},d'_{ij})} j$ if and only if $\mathrm{Ext}^1_A(S_i,S_j) \neq 0$ and $d_{ij} = \dim_{F_j} \mathrm{Ext}^1_A(S_i,S_j)$, $d'_{ij} = \dim_{F_i} \mathrm{Ext}^1_A(S_i,S_j)$. An algebra A is called *triangular* provided its quiver Q_A is acyclic (has no oriented cycle). We shall identify an algebra A with the associated category A^* whose objects are the vertices $1,\dots,n$ of Q_A, $\mathrm{Hom}_{A^*}(i,j) = e_jAe_i$ for any objects i and j of A^*, and the composition of morphisms in A^* is given by multiplication in A. For a module M in $\mathrm{mod}\, A$, we denote by $\mathrm{supp}(M)$ the full subcategory of $A = A^*$ given by all objects i such that $Me_i \neq 0$, and call it the *support* of M. More generally, for a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A, we denote by $\mathrm{supp}(\mathcal{C})$ the full subcategory of A given by all objects i such that $Xe_i \neq 0$ for some indecomposable module X in $\mathcal{C}$, and call it the *support* of $\mathcal{C}$. Then a module M in $\mathrm{mod}\, A$ (respectively, a family $\mathcal{C}$ of components in Γ_A) is said to be *sincere* if $\mathrm{supp}(M) = A$ (respectively, $\mathrm{supp}(\mathcal{C}) = A$). Finally, a full subcategory B of A is said to be a *convex subcategory* of A if every path in Q_A with source and target in B has all vertices in B. Observe that, for a convex subcategory B of A, there is a fully faithful embedding of $\mathrm{mod}\, B$ into $\mathrm{mod}\, A$ such that $\mathrm{mod}\, B$ is the full subcategory of $\mathrm{mod}\, A$ consisting of the modules M with $Me_i = 0$ for all objects i of A which are not objects of B.

For algebras A, B and C such that C^* is a common full subcategory of A^* and B^*, we may consider the pushout category
$$D^* = A^* \bigsqcup_{C^*} B^*$$
of A^* and B^* over C^*, defined as follows:

- the objects of D^* are the objects of A^* and of B^*, where the common objects from C^* are counted only once;
- $\mathrm{Hom}_{D^*}(x,y) = \mathrm{Hom}_{A^*}(x,y)$ for objects x, y in A^*;
- $\mathrm{Hom}_{D^*}(x,y) = \mathrm{Hom}_{B^*}(x,y)$ for objects x, y in B^*;
- $\mathrm{Hom}_{D^*}(x,y) = 0$ and $\mathrm{Hom}_{D^*}(y,x) = 0$ for any objects x in A^* but not in B^* and y in B^* but not in A^*.

We may also consider the associated algebra

$$D = A \sqcup_{C} B,$$

with $(A \sqcup_C B)^* = A^* \sqcup_{C^*} B^*$, called the *pushout algebra* of A and B over C. Note that the algebra C can be viewed as $C = eAe = eBe$ for a common idempotent e of A and B, the pushout algebra D is (as a K-module) the pushout $(A \oplus B)/\Delta(C)$ of the K-modules A and B over C, with $\Delta(C) = \{(c, -c) \in A \oplus B \mid c \in C\}$, multiplication in D is given by

$$\big((a_1, b_1) + \Delta(C)\big)\big((a_2, b_2) + \Delta(C)\big) = (a_1a_2, b_1b_2) + \Delta(C)$$

for $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and $1_D = (1_A, 1_B) + \Delta(C)$ is the identity of D.

More generally, for a family of algebras $A_1, \dots, A_n$ and $C_1, \dots, C_{n-1}$, with $n \geq 3$, such that C_i^* is a common full subcategory of A_i^* and A_{i+1}^*, for any $i \in \{1, \dots, n-1\}$, we define the pushout category

$$A_1^* \sqcup_{C_1^*} \cdots \sqcup_{C_{n-1}^*} A_n^*$$

of $A_1^*, \dots, A_n^*$ over $C_1^*, \dots, C_{n-1}^*$, and the associated pushout algebra

$$A_1 \sqcup_{C_1} \cdots \sqcup_{C_{n-1}} A_n$$

of $A_1, \dots, A_n$ over $C_1, \dots, C_n$ such that

$$(A_1 \sqcup_{C_1} \cdots \sqcup_{C_{n-1}} A_n)^* = A_1^* \sqcup_{C_1^*} \cdots \sqcup_{C_{n-1}^*} A_n^*.$$

Let A be an algebra and $\mathcal{C}$ be a component of Γ_A. Then $\mathcal{C}$ is said to be *postprojective* if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of a projective module. Dually, $\mathcal{C}$ is said to be *preinjective* if $\mathcal{C}$ is acyclic and each module in $\mathcal{C}$ belongs to the τ_A-orbit of an injective module. Moreover, $\mathcal{C}$ is called a *postprojective component of Euclidean type* (respectively, *preinjective component of Euclidean type*) if $\mathcal{C}$ is a semiregular postprojective component (respectively, a semiregular preinjective component) and admits a Euclidean section. Further, a *stable tube* of Γ_A is a component $\mathcal{T}$ of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, for some positive integer r called the *rank* of $\mathcal{T}$. A *ray tube* (respectively, a *coray tube*) of Γ_A is a component $\mathcal{C}$ obtained from a stable tube by a finite number (possibly zero) of ray insertions (respectively, coray insertions) [25], [31]. By a *semiregular tube* of Γ_A we mean a ray tube or a coray tube of Γ_A. Following [32], a component $\mathcal{C}$ of Γ_A is said to be *generalized standard* if

$\mathrm{rad}_A^\infty(X,Y)=0$ for all modules X and Y in $\mathcal{C}$. Two components $\mathcal{C}$ and $\mathcal{D}$ of Γ_A are said to be *orthogonal* if $\mathrm{Hom}_A(X,Y)=0$ and $\mathrm{Hom}_A(Y,X)=0$ for all modules X in $\mathcal{C}$ and Y in $\mathcal{D}$.

Let A be an algebra and X an indecomposable module in $\mathrm{mod}\,A$. Then X is said to be *acyclic* if X does not lie on an oriented cycle in Γ_A. Following [20], the *cyclic part* ${}_c\Gamma_A$ of Γ_A is the translation quiver obtained by removing all acyclic modules and the arrows attached to them. The connected components of ${}_c\Gamma_A$ are called *cyclic components* of Γ_A. It has been proved in [20, Proposition 5.1] that two indecomposable modules X and Y belong to one cyclic component of Γ_A if and only if there is an oriented cycle in Γ_A passing through X and Y. We note that the cyclic part ${}_c\mathcal{T}$ of a semiregular tube $\mathcal{T}$ of Γ_A is a cyclic component of Γ_A containing all but finitely many modules of $\mathcal{T}$.

The following result on the structure of semiregular components of the Auslander–Reiten quivers of cycle-finite algebras was proved in [37, Proposition 3.3].

PROPOSITION 2.1. *Let A be a cycle-finite algebra and $\mathcal{C}$ be a semiregular component of Γ_A. Then $\mathcal{C}$ is a generalized standard component, and has one of the following forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube, or a coray tube.*

This leads to the following fact proved in [37, Corollary 3.4].

PROPOSITION 2.2. *Let A be a cycle-finite algebra of semiregular type. Then A is a triangular algebra.*

We also need the following lemma.

LEMMA 2.3. *Let A be a cycle-finite algebra and $\mathcal{C}$ a semiregular tube of Γ_A. Then* $\mathrm{supp}(\mathcal{C})$ *is a convex subcategory of A.*

Proof. Let $C=\mathrm{supp}(\mathcal{C})$. Assume to the contrary that C is not a convex subcategory of A. Then Q_A contains a path

$$i=i_0\xrightarrow{(d_{i_0i_1},d'_{i_0i_1})} i_1\xrightarrow{(d_{i_1i_2},d'_{i_1i_2})} i_2\to\cdots\to i_{s-1}\xrightarrow{(d_{i_{s-1}i_s},d'_{i_{s-1}i_s})} i_s=j,$$

with $s\ge 2$, i,j in C and $i_1,\dots,i_{s-1}$ not in C. Since Q_A coincides with the Ext-quiver of A, we have $\mathrm{Ext}^1_A(S_{i_{t-1}},S_{i_t})\ne 0$ for $t\in\{1,\dots,s\}$. Then there exist in $\mathrm{mod}\,A$ non-split exact sequences

$$0\to S_{i_t}\to L_t\to S_{i_{t-1}}\to 0$$

for all $t\in\{1,\dots,s\}$. Clearly, $L_1,\dots,L_s$ are indecomposable modules in $\mathrm{mod}\,A$ of length 2. In particular, we obtain non-zero non-isomorphisms $f_r: L_r\to L_{r-1}$ with $\mathrm{Im}\,f_r=S_{i_{r-1}}$ for $r\in\{2,\dots,s\}$.

Consider now the ideal J in A of the form

$$J=Ae_i(\mathrm{rad}\,A)e_{i_1}(\mathrm{rad}\,A)+(\mathrm{rad}\,A)e_{i_{s-1}}(\mathrm{rad}\,A)e_jA$$

and the quotient algebra $B = A/J$. Since i_1 and i_{s-1} do not belong to $C = \operatorname{supp}(\mathcal{C})$, for any module M in $\mathcal{C}$ we have $Me_{i_1} = 0$ and $Me_{i_{s-1}} = 0$, and consequently $MJ = 0$. This shows that $\mathcal{C}$ is a stable tube of Γ_B. Moreover, it follows from the definition of J that S_{i_1} is a direct summand of the radical $\operatorname{rad} P_i^*$ of the projective cover $P_i^* = e_iB$ of S_i in $\operatorname{mod} B$ and $S_{i_{s-1}}$ is a direct summand of the socle factor I_j^*/S_j of the injective envelope $I_j^* = D(Be_j)$ of S_j in $\operatorname{mod} B$. Further, since i and j are in C, there exist indecomposable modules X and Y in the cyclic part ${}_c\mathcal{C}$ of $\mathcal{C}$ such that S_i is a composition factor of X and S_j is a composition factor of Y. Then we infer that $\operatorname{Hom}_B(P_i^*, X) \neq 0$ and $\operatorname{Hom}_B(Y, I_j^*) \neq 0$, because $\mathcal{C}$ is a component of Γ_B. Observe that we have in $\mathcal{C}$ a path from X to Y, because X and Y are in ${}_c\mathcal{C}$. Therefore, we obtain in $\operatorname{mod} A$ a cycle of the form

$$X \to \cdots \to Y \to I_j^* \to S_{i_{s-1}} \to L_{s-1} \to \cdots \to L_2 \to S_{i_1} \to P_i^* \to X,$$

which is an infinite cycle, because X and Y belong to $\mathcal{C}$ but S_{i_1} and $S_{i_{s-1}}$ are not in $\mathcal{C}$. This contradicts the cycle-finiteness of A. Hence $C = \operatorname{supp}(\mathcal{C})$ is indeed a convex subcategory of A. ∎

We also recall the following concept. For an algebra A, a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to be a *separating family* in $\operatorname{mod} A$ if the components in Γ_A split into three disjoint families, $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$, such that the following conditions are satisfied:

- (S1) $\mathcal{C}^A$ is a sincere family of pairwise orthogonal generalized standard components;
- (S2) $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\operatorname{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\operatorname{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;
- (S3) every homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through $\operatorname{add}(\mathcal{C}^A)$.

Moreover, if (S1), (S2) and the condition

- (S3*) every homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ in $\operatorname{mod} A$ factors through $\operatorname{add}(\mathcal{C}_i)$ for any $i \in I$

are satisfied, then $\mathcal{C}$ is said to be a *strongly separating family* in $\operatorname{mod} A$ (see [21], [22], [27]). We then say that $\mathcal{C}^A$ separates (respectively, strongly separates) $\mathcal{P}^A$ from $\mathcal{Q}^A$.

We shall also use the following lemmas on almost split sequences over triangular matrix algebras (see [27, (2.5)], [39, Lemma 5.6]).

LEMMA 2.4. *Let R and S be algebras, M an S-R-bimodule and $\Lambda = \left[\begin{smallmatrix} S & M \\ 0 & R \end{smallmatrix}\right]$ the matrix algebra defined by the bimodule ${}_SM_R$. Then an almost split sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\operatorname{mod} R$ *is almost split in* $\operatorname{mod} \Lambda$ *if and only if* $\operatorname{Hom}_R(M, X) = 0$.

LEMMA 2.5. *Let R and S be algebras, N an S-R-bimodule and $\Gamma = \begin{bmatrix} R & D(N) \\ 0 & S \end{bmatrix}$ be the matrix algebra defined by the dual R-S-bimodule $D(N) = \operatorname{Hom}_K(N, E)$. Then an almost split sequence*

$$0 \to X \to Y \to Z \to 0$$

in $\operatorname{mod} R$ *is almost split in* $\operatorname{mod} \Gamma$ *if and only if* $\operatorname{Hom}_R(Z, N) = 0$.

3. Tame quasitilted algebras of canonical type. In this section we recall the structure of the Auslander–Reiten quivers of representation-infinite tilted algebras of Euclidean type and tubular algebras, and then describe the structure of the Auslander–Reiten quivers of tame quasitilted algebras of canonical type.

By a *tame concealed algebra* we mean a tilted algebra $C = \operatorname{End}_H(T)$, where H is a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_{11}$, $\widetilde{\mathbb{A}}_{12}$, $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{B}}_n$, $\widetilde{\mathbb{C}}_n$, $\widetilde{\mathbb{BC}}_n$, $\widetilde{\mathbb{BD}}_n$, $\widetilde{\mathbb{CD}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, $\widetilde{\mathbb{F}}_{41}$, $\widetilde{\mathbb{F}}_{42}$, $\widetilde{\mathbb{G}}_{21}$, or $\widetilde{\mathbb{G}}_{22}$ (see [8]), and T is a (multiplicity-free) tilting H-module from the additive category of the postprojective component of Γ_H. The Auslander–Reiten quiver Γ_C of a tame concealed algebra C is of the form

$$\Gamma_C = \mathcal{P}^C \cup \mathcal{T}^C \cup \mathcal{Q}^C,$$

where $\mathcal{P}^C$ is a postprojective component of Euclidean type containing all indecomposable projective C-modules, $\mathcal{Q}^C$ is a preinjective component of Euclidean type containing all indecomposable injective C-modules, and $\mathcal{T}^C$ is an infinite family of pairwise orthogonal generalized standard stable tubes strongly separating $\mathcal{P}^C$ from $\mathcal{Q}^C$.

More generally, by a *tilted algebra of Euclidean type* we mean a tilted algebra $B = \operatorname{End}_H(T)$, where H is a hereditary algebra of Euclidean type and T is a (multiplicity-free) tilting module in $\operatorname{mod} H$. Assume B is a representation-infinite tilted algebra of Euclidean type. Then one of the following holds:

(1) B is a *domestic tubular* (*branch*) *extension* of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B = \mathcal{P}^C$ is the postprojective component of Γ_C, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes, obtained from the family $\mathcal{T}^C$ of stable tubes of Γ_C by ray insertions, $\mathcal{Q}^B$ is a preinjective component of Euclidean type containing all indecomposable injective B-modules, and $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$;

(2) B is a *domestic tubular* (*branch*) *coextension* of a tame concealed algebra C and

$$\Gamma_B = \mathcal{P}^B \cup \mathcal{T}^B \cup \mathcal{Q}^B,$$

where $\mathcal{P}^B$ is a postprojective component of Euclidean type containing all indecomposable projective B-modules, $\mathcal{T}^B$ is an infinite family of pairwise orthogonal generalized standard coray tubes, obtained from the family $\mathcal{T}^C$ of stable tubes of Γ_C by coray insertions, $\mathcal{Q}^B = \mathcal{Q}^C$ is the preinjective component of Γ_C, and $\mathcal{T}^B$ strongly separates $\mathcal{P}^B$ from $\mathcal{Q}^B$.

By a *tubular algebra* we mean a tubular (branch) extension (equivalently, tubular (branch) coextension) of a tame concealed algebra with the Euler quadratic form positive semidefinite of corank 2 (see [13], [14], [27], [29]). By general theory, a tubular algebra B admits two different tame concealed convex subcategories C_0 and C_∞ such that B is a tubular (branch) extension of C_0 and a tubular (branch) coextension of C_∞, and the Auslander–Reiten quiver Γ_B is of the form

$$\Gamma_B = \mathcal{P}_0^B \cup \mathcal{T}_0^B \cup \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^B\Big) \cup \mathcal{T}_\infty^B \cup \mathcal{Q}_\infty^B,$$

where $\mathcal{P}_0^B = \mathcal{P}^{C_0}$ is the postprojective component of Γ_{C_0}, $\mathcal{T}_0^B$ is an infinite family of pairwise orthogonal generalized standard ray tubes with at least one projective module, obtained from the family $\mathcal{T}^{C_0}$ of stable tubes of Γ_{C_0} by ray insertions, $\mathcal{Q}_\infty^B = \mathcal{Q}^{C_\infty}$ is the preinjective component of Γ_{C_∞}, $\mathcal{T}_\infty^B$ is an infinite family of pairwise orthogonal generalized standard coray tubes with at least one injective module, obtained from the family $\mathcal{T}^{C_\infty}$ of stable tubes of Γ_{C_∞} by coray insertions, and, for each $q \in \mathbb{Q}^+$ (the set of positive rational numbers), $\mathcal{T}_q^B$ is an infinite family of pairwise orthogonal generalized standard stable tubes. Moreover, for any $q \in \mathbb{Q}^+ \cup \{0,\infty\}$, the family $\mathcal{T}_q^B$ strongly separates $\mathcal{P}^B \cup (\bigcup_{p<q} \mathcal{T}_p^B)$ from $(\bigcup_{p>q} \mathcal{T}_p^B) \cup \mathcal{Q}^B$.

The following characterizations of tame concealed and tubular algebras have been established in [37, Theorem 4.1].

THEOREM 3.1. *Let A be an algebra. The following statements are equivalent:*

(i) *A is cycle-finite and Γ_A admits a sincere stable tube;*
(ii) *A is either tame concealed or tubular.*

An algebra is said to be *minimal representation-infinite* if A is of infinite representation type and, for every non-zero two-sided ideal I of A, A/I is of finite representation type. Then we have the following characterization of representation-infinite cycle-finite algebras, which is a consequence of a more general result proved in [34, Theorem 4.1].

THEOREM 3.2. *Let A be an algebra. The following statements are equivalent:*

(i) *A is a minimal representation-infinite and cycle-finite algebra;*
(ii) *A is a tame concealed algebra.*

Our next aim is to describe the tame quasitilted algebras of canonical type and their Auslander–Reiten quivers.

Let C be a tame concealed algebra and $\mathcal{T}^C$ the family of all stable tubes in Γ_C. By a *semiregular branch enlargement* of C we mean an algebra of the form

$$B = \begin{bmatrix} D & M & 0 \\ 0 & C & D(N) \\ 0 & 0 & H \end{bmatrix},$$

where

$$B^{(r)} = \begin{bmatrix} D & M \\ 0 & C \end{bmatrix} \quad \text{and} \quad B^{(l)} = \begin{bmatrix} C & D(N) \\ 0 & H \end{bmatrix}$$

are respectively a tubular extension of C and a tubular coextension of C in the sense of [27, (4.7)] (see also [31, Chapter XV]), and no tube in $\mathcal{T}^C$ admits both a direct summand of M and a direct summand of N (see [15], [38]). Then B is a quasitilted algebra of canonical type, and $B^{(r)}$ and $B^{(l)}$ are called the *right part* and the *left part* of B, respectively. Moreover, following [38], B is said to be a *tame semiregular branch enlargement of* C if $B^{(r)}$ and $B^{(l)}$ are tilted algebras of Euclidean type or tubular algebras. Finally, by a *tame quasitilted algebra of canonical type* we mean a tame semiregular branch enlargement of a tame concealed algebra. We note that tame quasitilted algebras of canonical type are quasitilted algebras in the sense of [9], that is, algebras A of global dimension at most 2 and with every indecomposable module in $\operatorname{mod} A$ of projective or injective dimension at most 1.

The following characterization of tame quasitilted algebras of canonical type follows from [15, Theorem 3.4] and [38, Theorem A].

THEOREM 3.3. *Let A be an algebra. The following statements are equivalent:*

- (i) *A is a tame quasitilted algebra of canonical type;*
- (ii) *A is a cycle-finite quasitilted algebra of canonical type;*
- (iii) *A is cycle-finite and Γ_A admits a separating family of semiregular tubes;*
- (iv) *A is cycle-finite and Γ_A admits a strongly separating family of semiregular tubes.*

In particular, we obtain the following theorem on the structure of the Auslander–Reiten quiver of a tame quasitilted algebra of canonical type.

THEOREM 3.4. *Let B be a tame quasitilted algebra of canonical type. Then the Auslander–Reiten quiver Γ_B of B has a disjoint union decomposition*

$$\Gamma_B = \mathcal{P}^B \vee \mathcal{T}^B \vee \mathcal{Q}^B,$$

where

(i) $\mathcal{T}^B$ *is a sincere family of pairwise orthogonal generalized standard semiregular tubes strongly separating* $\mathcal{P}^B$ *from* $\mathcal{Q}^B$*;*
(ii) *if* $B^{(l)}$ *is a tilted algebra of Euclidean type, then* $\mathcal{P}^B$ *is the unique postprojective component* $\mathcal{P}^{B^{(l)}}$ *of* $\Gamma_{B^{(l)}}$*, and contains all indecomposable projective* $B^{(l)}$*-modules;*
(iii) *if* $B^{(l)}$ *is a tubular algebra, then*

$$\mathcal{P}^B = \mathcal{P}_0^{B^{(l)}} \cup \mathcal{T}_0^{B^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(l)}} \Big),$$

and $\mathcal{P}_0^{B^{(l)}} \cup \mathcal{T}_0^{B^{(l)}}$ *contains all indecomposable projective* $B^{(l)}$*-modules;*
(iv) *if* $B^{(r)}$ *is a tilted algebra of Euclidean type, then* $\mathcal{Q}^B$ *is the unique preinjective component* $\mathcal{Q}^{B^{(r)}}$ *of* $\Gamma_{B^{(r)}}$*, and contains all indecomposable injective* $B^{(r)}$*-modules;*
(v) *if* $B^{(r)}$ *is a tubular algebra, then*

$$\mathcal{Q}^B = \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{B^{(r)}} \Big) \cup \mathcal{T}_\infty^{B^{(r)}} \cup \mathcal{Q}_\infty^{B^{(r)}},$$

and $\mathcal{T}_\infty^{B^{(r)}} \cup \mathcal{Q}_\infty^{B^{(r)}}$ *contains all indecomposable injective* $B^{(r)}$*-modules;*
(vi) *every indecomposable projective* B*-module belongs to* $\mathcal{P}^B \cup \mathcal{T}^B$*;*
(vii) *every indecomposable injective* B*-module belongs to* $\mathcal{T}^B \cup \mathcal{Q}^B$*.*

A sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of algebras is said to be a *coherent sequence of tame quasitilted algebras of canonical type* if the following conditions are satisfied:

(1) $B_1, \ldots, B_n$ are tame quasitilted algebras of canonical type,
(2) for $n \geq 2$ and $i \in \{1, \ldots, n-1\}$, $B_i^{(r)} = B_{i+1}^{(l)}$ and it is a tubular algebra.

For a coherent sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of tame quasitilted algebras of canonical type, we define the algebra $A(\mathbb{B})$ in the following way: $A(\mathbb{B}) = B_1$ for $n = 1$, and $A(\mathbb{B})$ is the pushout algebra

$$B_1 \underset{B_1^{(r)}}{\sqcup} \cdots \underset{B_{n-1}^{(r)}}{\sqcup} B_n = B_1 \underset{B_2^{(l)}}{\sqcup} \cdots \underset{B_n^{(l)}}{\sqcup} B_n,$$

for $n \geq 2$. We note that each B_i, for $i \in \{1, \ldots, n\}$, is a convex subcategory of $A(\mathbb{B})$. We have the following consequence of Theorem 3.4.

THEOREM 3.5. *Let* $\mathbb{B} = (B_1, \ldots, B_n)$ *be a coherent sequence of tame quasitilted algebras of canonical type and* $A = A(\mathbb{B})$ *the associated algebra. Then the following statements hold:*

(i) *A is a cycle-finite algebra of semiregular type.*

(ii) *The Auslander–Reiten quiver Γ_A of A has a disjoint union decomposition*

$$\Gamma_A = \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_n^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}$$

where $\bar{\mathbb{Q}}_n^1 = \mathbb{Q} \cap [1, n]$, and the following statements hold:

(a) *If $B_1^{(l)}$ is a tilted algebra of Euclidean type, then $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ is a unique postprojective component of Γ_A.*

(b) *If $B_1^{(l)}$ is a tubular algebra, then*

$$\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}} = \mathcal{P}_0^{B_1^{(l)}} \cup \mathcal{T}_0^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{\mathbb{B}_1^{(l)}} \Big),$$

and $\mathcal{P}^{B_1^{(l)}}$ is a unique postprojective component of Γ_A.

(c) *If $B_n^{(r)}$ is a tilted algebra of Euclidean type, then $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$ is a unique preinjective component of Γ_A.*

(d) *If $B_n^{(r)}$ is a tubular algebra, then*

$$\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}} = \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{\mathbb{B}_n^{(r)}} \Big) \cup \mathcal{T}_\infty^{B_n^{(r)}} \cup \mathcal{Q}_\infty^{B_n^{(r)}},$$

and $\mathcal{Q}^{B_n^{(r)}}$ is a unique preinjective component of Γ_A.

(e) *For each $r \in \{1, \ldots, n\}$, $\mathcal{T}_r^{\mathbb{B}} = \mathcal{T}^{B_r}$ is a family $(\mathcal{T}_{r,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_r}$ of pairwise orthogonal generalized standard semiregular tubes.*

(f) *For each $q \in \bar{\mathbb{Q}}_n^1 \setminus \{1, \ldots, n\}$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_q}$ of pairwise orthogonal generalized standard stable tubes.*

(g) *For each $q \in \bar{\mathbb{Q}}_n^1$, we have*

$$\operatorname{Hom}_A\Big(\Big(\bigcup_{p > q} \mathcal{T}_p^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}, \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{p < q} \mathcal{T}_p^{\mathbb{B}} \Big) \Big) = 0.$$

(h) *For each $q \in \bar{\mathbb{Q}}_n^1$, every homomorphism from $\mathcal{P}^{\mathbb{B}} \cup (\bigcup_{p<q} \mathcal{T}_p^{\mathbb{B}})$ to $(\bigcup_{p>q} \mathcal{T}_p^{\mathbb{B}}) \cup \mathcal{Q}^{\mathbb{B}}$ factors through $\operatorname{add}(\mathcal{T}_{q,\lambda}^{\mathbb{B}})$ for any $\lambda \in \Lambda_q$.*

Proof. The statement (i) is a direct consequence of (ii). Therefore we will prove that Γ_A has the structure and properties described in (ii).

For $n = 1$, the statement (ii) follows from Theorem 3.4, because then $A(\mathbb{B}) = B_1$ is a tame quasitilted algebra of canonical type.

Assume $n \geq 2$. For a positive integer i, we set

$$\mathbb{Q}_{i+1}^i = \mathbb{Q} \cap (i, i+1) \quad \text{and} \quad \bar{\mathbb{Q}}_{i+1}^i = \mathbb{Q} \cap [i, i+1].$$

Observe that there are order-preserving bijections of sets

$$\mathbb{Q}_{i+1}^i \to \mathbb{Q}^+ \quad \text{and} \quad \bar{\mathbb{Q}}_{i+1}^i \to \{0\} \cup \mathbb{Q}^+ \cup \{\infty\}.$$

Applying Theorem 3.4, we may describe the Auslander–Reiten quivers $\varGamma_{B_i}$ of the algebras B_i, $i \in \{1, \dots, n\}$, as follows:

- $\varGamma_{B_1}$ has the form
$$\varGamma_{B_1} = \mathcal{P}^{B_1^{(l)}} \cup \mathcal{T}^{B_1} \cup \Big(\bigcup_{q \in \mathbb{Q}_2^1} \mathcal{T}_q^{B_1^{(r)}}\Big) \cup \mathcal{T}_\infty^{B_1^{(r)}} \cup \mathcal{Q}_\infty^{B_1^{(r)}},$$
because $B_1^{(r)}$ is a tubular algebra, where $\mathcal{P}^{B_1^{(l)}}$ is a postprojective component of Euclidean type if $B_1^{(l)}$ is a tilted algebra of Euclidean type, and $\mathcal{P}^{B_1^{(l)}}$ is of the form
$$\mathcal{P}^{B_1^{(l)}} = \mathcal{P}_0^{B_1^{(l)}} \cup \mathcal{T}_0^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{\mathbb{B}_1^{(l)}}\Big),$$
if $B_1^{(l)}$ is a tubular algebra;
- if $n \geq 3$ and $i \in \{2, \dots, n-1\}$, then $\varGamma_{B_i}$ is of the form
$$\varGamma_{B_i} = \mathcal{P}_0^{B_i^{(l)}} \cup \mathcal{T}_0^{B_i^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}_i^{i-1}} \mathcal{T}_q^{B_i^{(l)}}\Big) \cup \mathcal{T}^{B_i} \cup \Big(\bigcup_{q \in \mathbb{Q}_{i+1}^{i}} \mathcal{T}_q^{B_i^{(r)}}\Big) \cup \mathcal{T}_\infty^{B_i^{(r)}} \cup \mathcal{Q}_\infty^{B_i^{(r)}},$$
because $B_i^{(l)}$ and $B_i^{(r)}$ are tubular algebras;
- $\varGamma_{B_n}$ has the form
$$\varGamma_{B_n} = \mathcal{P}_0^{B_n^{(l)}} \cup \mathcal{T}_0^{B_n^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q}_n^{n-1}} \mathcal{T}_q^{B_n^{(l)}}\Big) \cup \mathcal{T}^{B_n} \cup \mathcal{Q}^{B_n^{(r)}},$$
because $B_n^{(l)}$ is a tubular algebra, where $\mathcal{Q}^{B_n^{(r)}}$ is a preinjective component of Euclidean type if $B_n^{(r)}$ is a tilted algebra of Euclidean type, and $\mathcal{Q}^{B_n^{(r)}}$ is of the form
$$\mathcal{Q}^{B_n^{(r)}} = \Big(\bigcup_{q \in \mathbb{Q}^+} \mathcal{T}_q^{\mathbb{B}_n^{(r)}}\Big) \cup \mathcal{T}_\infty^{B_n^{(r)}} \cup \mathcal{Q}_\infty^{B_n^{(r)}}$$
if $B_n^{(r)}$ is a tubular algebra.

For each $r \in \{1, \dots, n\}$, we define $\mathcal{T}_r^{\mathbb{B}} = \mathcal{T}^{B_r}$. Observe that $\mathcal{T}_r^{\mathbb{B}}$ is a family $\mathcal{T}_{r,\lambda}^{\mathbb{B}}$, $\lambda \in \varLambda_r$, of pairwise orthogonal generalized standard semiregular tubes of $\varGamma_{B_r}$. For $n \geq 3$ and $i \in \{1, \dots, n-1\}$, we have $B_i^{(r)} = B_{i+1}^{(l)}$, and hence we may define $\mathcal{T}_q^{\mathbb{B}} = \mathcal{T}_q^{B_i^{(r)}} = \mathcal{T}_q^{B_{i+1}^{(l)}}$ for any $q \in \mathbb{Q}_{i+1}^i$. We note that, for each $q \in \mathbb{Q}_{i+1}^i$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $\mathcal{T}_{q,\lambda}^{\mathbb{B}}$, $\lambda \in \varLambda_q$, of pairwise orthogonal generalized standard stable tubes of $\varGamma_{B_i}$ and $\varGamma_{B_{i+1}}$.

Now, consider the algebras
$$A(\mathbb{B})^{(i)} = B_1 \mathop{\sqcup}_{B_1^{(r)}} \cdots \mathop{\sqcup}_{B_{i-1}^{(r)}} B_i = B_1 \mathop{\sqcup}_{B_2^{(l)}} \cdots \mathop{\sqcup}_{B_i^{(l)}} B_i$$

for $i \in \{2, \dots, n\}$. Observe that $A(\mathbb{B})^{(2)}$ is a tubular extension of B_1 using modules from stable tubes of the family $\mathcal{T}_\infty^{B_1^{(r)}}$, and consequently the Auslander–Reiten quiver $\Gamma_{A(\mathbb{B})^{(2)}}$ of $A(\mathbb{B})^{(2)}$ has a disjoint union decomposition

$$\Gamma_{A(\mathbb{B})^{(2)}} = \mathcal{P}^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_2^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{B_2^{(r)}}$$

if $n = 2$, and

$$\Gamma_{A(\mathbb{B})^{(2)}} = \mathcal{P}^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q} \cap [1,3)} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{T}_\infty^{B_2^{(r)}} \cup \mathcal{Q}_\infty^{B_2^{(r)}}$$

if $n \geq 3$. In particular, if $n = 2$, then $A(\mathbb{B})^{(2)} = A(\mathbb{B}) = A$ and Γ_A has the required disjoint union decomposition with $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ and $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_2^{(r)}}$.

Assume now that $n \geq 3$, $i \in \{1, \dots, n-1\}$, and $\Gamma_{A(\mathbb{B})^{(i)}}$ has the disjoint union decomposition

$$\Gamma_{A(\mathbb{B})^{(i)}} = \mathcal{P}^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q} \cap [1,i+1)} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{T}_\infty^{B_i^{(r)}} \cup \mathcal{Q}_\infty^{B_i^{(r)}}.$$

We note that $A(\mathbb{B})^{(i+1)}$ is a tubular extension of $A(\mathbb{B})^{(i)}$ using modules from stable tubes of the family $\mathcal{T}_\infty^{B_i^{(r)}}$. Then the Auslander–Reiten quiver $\Gamma_{A(\mathbb{B})^{(i+1)}}$ of $A(\mathbb{B})^{(i+1)}$ has a disjoint union form

$$\Gamma_{A(\mathbb{B})^{(i+1)}} = \mathcal{P}^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_{i+1}^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{B_{i+1}^{(r)}}$$

if $i = n - 1$, and

$$\Gamma_{A(\mathbb{B})^{(i+1)}} = \mathcal{P}^{B_1^{(l)}} \cup \Big(\bigcup_{q \in \mathbb{Q} \cap [1,i+2)} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{T}_\infty^{B_{i+1}^{(r)}} \cup \mathcal{Q}_\infty^{B_{i+1}^{(r)}}$$

if $i < n - 1$. Hence, it follows by induction on i that Γ_A has the required disjoint union decomposition

$$\Gamma_A = \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_n^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}$$

with $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ and $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$, and the families of tubes $\mathcal{T}_q^{\mathbb{B}}$, $q \in \bar{\mathbb{Q}}_n^1 = \mathbb{Q} \cap [1, n]$, described above. Consequently, we have proved that the conditions (a)–(f) are satisfied.

The statements (g) and (h) follow from the fact that

- for any $r \in \{1, \dots, n\}$, $\mathcal{T}_r^{\mathbb{B}} = \mathcal{T}^{B_r}$ is a strongly separating family of semiregular tubes of Γ_{B_r},

- for any $q \in \mathbb{Q}^i_{i+1}$ with $i \in \{1, \dots, n-1\}$ and $n \geq 3$, $\mathcal{T}^{\mathbb{B}}_q = \mathcal{T}^{B^{(r)}_i}_q = \mathcal{T}^{B^{(l)}_{i+1}}_q$ is a strongly separating family of stable tubes of $\Gamma_{B^{(r)}_i} = \Gamma_{B^{(l)}_{i+1}}$. ■

4. Proof of Theorem 1.4. Let A be a cycle-finite algebra of semiregular type. Then A is of infinite representation type and it follows from Theorem 3.2 that there is an ideal I in A such that $C = A/I$ is a tame concealed algebra. Let

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$$

be the disjoint union decomposition of Γ_C, where $\mathcal{P}^C$ is a postprojective component containing all indecomposable projective C-modules, $\mathcal{Q}^C$ is a preinjective component containing all indecomposable injective C-modules, and $\mathcal{T}^C$ is an infinite family of pairwise orthogonal generalized standard stable tubes strongly separating $\mathcal{P}^C$ from $\mathcal{Q}^C$. Then Theorem 1.4 follows from Theorem 3.2 and the following theorem.

THEOREM 4.1. *Let A be a cycle-finite algebra of semiregular type, C a tame concealed quotient algebra of A, and $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda \in \Lambda}$ the family of all stable tubes of Γ_C. Then the following statements hold:*

(i) *For any $\lambda \in \Lambda$, Γ_A contains a semiregular tube $\mathcal{T}^A_\lambda(C)$ containing all modules of $\mathcal{T}^C_\lambda$.*
(ii) *$\mathcal{T}^A_\lambda(C) \neq \mathcal{T}^A_\mu(C)$ for any $\lambda \neq \mu$ in Λ.*
(iii) *For all but finitely many $\lambda \in \Lambda$, we have $\mathcal{T}^A_\lambda(C) = \mathcal{T}^C_\lambda$.*
(iv) *C is a convex subcategory of A.*

Proof. (i) Let $\lambda \in \Lambda$. Then, for any two indecomposable C-modules X and Y lying in $\mathcal{T}^C_\lambda$, there exists a cycle

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} X_r = X$$

of irreducible homomorphisms in $\operatorname{mod} C$ between indecomposable modules from $\mathcal{T}^C_\lambda$ and with $X_s = Y$ for some $s \in \{1, \dots, r-1\}$. Since C is a quotient algebra of A, this cycle is also a cycle in $\operatorname{mod} A$, and hence $f_1, \dots, f_r$ do not belong to $\operatorname{rad}^\infty_A$, by the assumption on A. Then it follows that there is a cycle of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ passing through the modules $X_0, X_1, \dots, X_r$. In particular, the modules $X = X_0$ and $Y = X_s$ lie in the same component of Γ_A. Therefore, there exists a component $\mathcal{T}^A_\lambda(C)$ in Γ_A containing all modules of the stable tube $\mathcal{T}^C_\lambda$. Observe also that $\mathcal{T}^A_\lambda(C)$ contains oriented cycles and is semiregular, because all components in Γ_A are assumed to be semiregular. Applying now Proposition 2.1 we conclude that $\mathcal{T}^A_\lambda(C)$ is a semiregular tube.

(ii) Take $\lambda \neq \mu$ in Λ. Assume to the contrary that $\mathcal{T}^A_\lambda(C) = \mathcal{T}^A_\mu(C)$. Since $\mathcal{T}^A_\lambda(C) = \mathcal{T}^A_\mu(C)$ is a semiregular tube containing all indecomposable modules of $\mathcal{T}^C_\lambda$ and $\mathcal{T}^C_\mu$, we conclude that there are indecomposable modules $U \in \mathcal{T}^C_\lambda$ and $W \in \mathcal{T}^C_\mu$, and sectional paths of irreducible homomorphisms in $\operatorname{mod} A$ between indecomposable modules in $\mathcal{T}^A_\lambda(C)$ of the forms

$$U = U_0 \xrightarrow{g_1} U_1 \xrightarrow{g_2} \cdots \xrightarrow{g_s} U_s = V,$$

corresponding to arrows of $\mathcal{T}^A_\lambda(C)$ pointing to the mouth,

$$V = V_0 \xrightarrow{h_1} V_1 \xrightarrow{h_2} \cdots \xrightarrow{h_t} V_t = W,$$

corresponding to arrows of $\mathcal{T}^A_\lambda(C)$ pointing to infinity, and with $U_{s-1} = \tau_A V_1$. Moreover, $\mathcal{T}^A_\lambda(C)$ admits full translation subquivers

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & Z^{(j-1)}_{m+1} & \longrightarrow & Z^{(j-1)}_{m} & \longrightarrow \cdots \longrightarrow & Z^{(j-1)}_{1} & \longrightarrow & Z^{(j-1)}_{0} = V_{j-1} \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longrightarrow & Z^{(j)}_{m+1} & \longrightarrow & Z^{(j)}_{m} & \longrightarrow \cdots \longrightarrow & Z^{(j)}_{1} & \longrightarrow & Z^{(j)}_{0} = V_{j}
\end{array}$$

for $j \in \{1, \dots, t\}$, formed by parallel infinite sectional paths. Then it follows from [16, Corollary 1.6] that the irreducible homomorphisms $h_1, \dots, h_t$ are of infinite left degree. Further, by [11, Theorem 13.3], we have $g_s \dots g_1 \in \operatorname{rad}^s_A(U,V) \setminus \operatorname{rad}^{s+1}_A(U,V)$. Hence we conclude that $h_t \dots h_1 g_s \dots g_1$ belongs to $\operatorname{rad}^{s+t}_A(U,W) \setminus \operatorname{rad}^{s+t+1}_A(U,W)$, and consequently $\operatorname{Hom}_A(U,W) \neq 0$. But then $\operatorname{Hom}_C(U,W) = \operatorname{Hom}_A(U,W) \neq 0$, which contradicts the orthogonality of $\mathcal{T}^C_\lambda$ and $\mathcal{T}^C_\mu$ in $\operatorname{mod} C$, because $\lambda \neq \mu$. Summing up, we have proved that Γ_A contains a family $\mathcal{T}^A(C) = (\mathcal{T}^A_\lambda(C))_{\lambda \in \Lambda}$ of semiregular tubes such that $\mathcal{T}^A_\lambda(C)$ contains all modules of $\mathcal{T}^C_\lambda$, for any $\lambda \in \Lambda$.

(iii) Since Γ_A admits only finitely many components containing projective or injective modules, we conclude that $\mathcal{T}^A_\lambda(C)$ is a stable tube for all but finitely many $\lambda \in \Lambda$. Take $\lambda \in \Lambda$ such that $\mathcal{T}^A_\lambda(C)$ is a stable tube of Γ_A. We claim that then $\mathcal{T}^A_\lambda(C) = \mathcal{T}^C_\lambda$. We know from (i) that $\mathcal{T}^A_\lambda(C)$ contains all modules of $\mathcal{T}^C_\lambda$, and hence infinitely many indecomposable C-modules. Take an indecomposable module M in $\mathcal{T}^A_\lambda(C)$. Then there exist in $\operatorname{mod} A$ a sectional path of irreducible monomorphisms in $\operatorname{mod} A$

$$M = M_0 \xrightarrow{\phi_1} M_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_l} M_l = Z$$

and a sectional path of irreducible epimorphisms in $\operatorname{mod} A$

$$N = N_0 \xrightarrow{\psi_1} N_1 \xrightarrow{\psi_2} \cdots \xrightarrow{\psi_l} N_m = Z$$

with N an indecomposable C-module from $\mathcal{T}^C_\lambda$. Hence Z is a quotient module of N and M is isomorphic to a submodule of Z, and consequently M is a C-module. This shows that $\mathcal{T}^A_\lambda(C)$ consists of C-modules, and then $\mathcal{T}^A_\lambda(C) = \mathcal{T}^C_\lambda$.

(iv) Since Λ is infinite, we may choose $\lambda \in \Lambda$ such that $\mathcal{T}_\lambda^A(C)$ is a stable tube, and consequently $\mathcal{T}_\lambda^A(C) = \mathcal{T}_\lambda^C$. We note that $C = \operatorname{supp}(\mathcal{T}_\lambda^C)$, because $\mathcal{T}_\lambda^C$ belongs to the strongly separating family $\mathcal{T}^C$ of stable tubes of Γ_C. Finally, it follows from Lemma 2.3 that the support $C = \operatorname{supp}(\mathcal{T}_\lambda^C) = \operatorname{supp}(\mathcal{T}_\lambda^A(C))$ of the stable tube $\mathcal{T}_\lambda^A(C)$ of the cycle-finite algebra A is a convex subcategory of A. ■

5. Proof of Theorem 1.5. Let A be a cycle-finite algebra of semiregular type, C a tame concealed convex subcategory of A, and $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda\in\Lambda}$ the family of all stable tubes of Γ_C. Since C is a tame concealed quotient algebra of A, it follows from Theorem 4.1 that Γ_A contains a family $\mathcal{T}^A(C) = (\mathcal{T}_\lambda^A(C))_{\lambda\in\Lambda}$ of semiregular tubes such that $\mathcal{T}_\lambda^A(C)$ contains all modules of $\mathcal{T}_\lambda^C$, for any $\lambda \in \Lambda$. Moreover, $\mathcal{T}_\lambda^A(C) \neq \mathcal{T}_\mu^A(C)$ for $\lambda \neq \mu$ in Λ. This proves (i). We will prove that (ii) and (iii) hold.

Consider the family $\mathcal{T}^A(C)^{(r)}$ of all ray tubes in $\mathcal{T}^A(C)$ and the family $\mathcal{T}^A(C)^{(l)}$ of all coray tubes in $\mathcal{T}^A(C)$, and their support categories

$$B(C)^{(r)} = \operatorname{supp}(\mathcal{T}^A(C)^{(r)}) \quad \text{and} \quad B(C)^{(l)} = \operatorname{supp}(\mathcal{T}^A(C)^{(l)}).$$

We note that, for all but finitely many $\lambda \in \Lambda$, $\mathcal{T}_\lambda^A(C) = \mathcal{T}_\lambda^C$ is a stable tube and belongs to both $\mathcal{T}^A(C)^{(r)}$ and $\mathcal{T}^A(C)^{(l)}$, and hence C is a convex subcategory of $B(C)^{(r)}$ and a convex subcategory of $B(C)^{(l)}$.

Assume that $B(C) = \operatorname{supp}(\mathcal{T}^A(C))$ is not a convex subcategory of A. Then Q_A contains a path

$$(*) \qquad i = i_0 \xrightarrow{(d_{i_0i_1},d'_{i_0i_1})} i_1 \xrightarrow{(d_{i_1i_2},d'_{i_1i_2})} \cdots \xrightarrow{(d_{i_{s-1}i_s},d'_{i_{s-1}i_s})} i_s = j$$

with $s \geq 2$, i,j in $B(C)$ and i_t not in $B(C)$ for any $t \in \{1,\ldots,s-1\}$. Then we have a path in $\operatorname{mod} A$ of the form

$$P_j = P_{i_s} \xrightarrow{f_s} \cdots \xrightarrow{f_1} P_{i_0} = P_i,$$

where $P_{i_t} = e_{i_t}A$ are the indecomposable projective modules in $\operatorname{mod} A$ given by the vertices i_t, for $t \in \{0,1,\ldots,s\}$, and the homomorphisms $f_k : P_{i_k} \to P_{i_{k-1}}$ are given by elements $a_k \in e_{i_{k-1}}(\operatorname{rad} A)e_{i_k} \setminus e_{i_{k-1}}(\operatorname{rad} A)^2 e_{i_k}$ for $k \in \{1,\ldots,s\}$.

Since C is a convex subcategory of A, we have $i \notin Q_C$ or $j \notin Q_C$. We first prove that, if i belongs to $B(C)^{(r)}$, then $i \in Q_C$ and j is not in $B(C)^{(r)}$.

Assume that i belongs to $B(C)^{(r)}$. Suppose to the contrary that $i \notin Q_C$. Then P_i is a projective module of a ray tube $\mathcal{T}_\lambda^A(C)$. Moreover, $\operatorname{rad} P_i$ is a direct sum of indecomposable modules lying in $\mathcal{T}_\lambda^A(C)$, and hence the projective cover $P(\operatorname{rad} P_i)$ of $\operatorname{rad} P_i$ in $\operatorname{mod} A$ is a direct sum of indecomposable projective modules P_l with l in $B(C)^{(r)}$. On the other hand, we have in mod A a commutative diagram of the form

$$\begin{array}{ccc} & & P_{i_1} \\ & \overset{g_1}{\swarrow} & \downarrow{\scriptstyle f_1} \\ P(\operatorname{rad} P_i) & \xrightarrow{p} & \operatorname{rad} P_i \end{array}$$

because $\operatorname{Im} f_1$ is contained in $\operatorname{rad} P_i = \operatorname{rad} P_{i_0}$. Since i_1 is not in $B(C)^{(r)}$, we see that $g_1 \in \operatorname{rad}_A(P_{i_1}, P(\operatorname{rad} P_i))$. But this leads to a contradiction because f_1 is given by an element $a_1 \in e_i(\operatorname{rad} A)e_{i_1} \setminus e_i(\operatorname{rad} A)^2 e_{i_1}$. Therefore, indeed $i \in Q_C$.

We now show that j is not in $B(C)^{(r)}$. Assume to the contrary that j is an object of $B(C)^{(r)}$. Observe that $i \in Q_C$ forces $j \notin Q_C$. Hence P_j lies in a ray tube $\mathcal{T}_\mu^A(C)$ of $\mathcal{T}^A(C)$. Since i_{s-1} is not in $B(C)$, we conclude that $P_{i_{s-1}}$ is not in $\mathcal{T}_\mu^A(C)$, and so f_s is a non-zero homomorphism in $\operatorname{rad}_A^\infty(P_j, P_{i_{s-1}})$. Then there exists an infinite path in $\mathcal{T}_\mu^A(C)$ of the form

$$P_j = Z_0 \to Z_1 \to \cdots \to Z_m \to \cdots$$

such that $\operatorname{rad}_A(Z_m, P_{i_{s-1}}) = \operatorname{Hom}_A(Z_m, P_{i_{s-1}}) \neq 0$ for any $m \in \mathbb{N}$. Since $i \in Q_C$ and $\mathcal{T}_\mu^C$ is a sincere stable tube of Γ_C, there exists an indecomposable module M in $\mathcal{T}_\mu^C$ such that $\operatorname{rad}_A^\infty(P_i, M) = \operatorname{rad}_A(P_i, M) = \operatorname{Hom}_A(P_i, M) \neq 0$. Moreover every module of $\mathcal{T}_\mu^C$ belongs to the cyclic part of $\mathcal{T}_\mu^A(C)$. Further, there exists a positive integer m_0 such that all modules Z_m with $m \geq m_0$ belong to the cyclic part of $\mathcal{T}_\mu^A(C)$, because the ray tube $\mathcal{T}_\mu^A(C)$ may contain only finitely many acyclic (directing) indecomposable modules. In particular, we conclude that there is a path in $\mathcal{T}_\mu^A(C)$ from M to Z_{m_0}. Summing up, we obtain in mod A a cycle of the form

$$P_i \to M \to \cdots \to Z_{m_0} \to P_{i_{s-1}} \to \cdots \to P_{i_1} \to P_{i_0} = P_i,$$

which is not a finite cycle in mod A, because $\operatorname{Hom}_A(P_i, M) = \operatorname{rad}_A^\infty(P_i, M)$, a contradiction with the cycle-finiteness of A. Therefore, j is not in $B(C)^{(r)}$. Observe that this also shows that $B(C)^{(r)}$ is a convex subcategory of A.

Further, it follows from [37, Proposition 2.3] that, for any ray tube $\mathcal{T}_\xi^A(C)$ of $\mathcal{T}^A(C)$ containing at least one projective module, all rays of $\mathcal{T}_\xi^C$ are complete rays of $\mathcal{T}_\xi^A(C)$. Since all tubes in $\mathcal{T}^A(C)$ are pairwise orthogonal and generalized standard, we conclude that $B(C)^{(r)}$ is a tubular (branch) extension of the tame concealed algebra C and $\Gamma_{B(C)^{(r)}}$ admits a strongly separating family $\mathcal{T}^{B(C)^{(r)}} = (\mathcal{T}_\lambda^{B(C)^{(r)}})_{\lambda \in \Lambda}$ of ray tubes, obtained from the strongly separating family $\mathcal{T}^C = (\mathcal{T}_\lambda^C)_{\lambda \in \Lambda}$ of stable tubes of Γ_C by the corresponding ray insertions. Clearly, $B(C)^{(r)}$ is cycle-finite as a convex subcategory of the cycle-finite algebra A. In particular, Theorems 3.3 and 3.4 imply that $B(C)^{(r)}$ is either a tilted algebra of Euclidean type with all indecomposable injective modules lying in the preinjective component, or a tubular algebra.

The given path $(*)$ in Q_A also induces a path in $\operatorname{mod} A^{\mathrm{op}}$,

$$Ae_i = Ae_{i_0} \xrightarrow{g_1} Ae_{i_1} \xrightarrow{g_2} \cdots \xrightarrow{g_s} Ae_{i_s} = Ae_j$$

between indecomposable projective modules in $\operatorname{mod} A^{\mathrm{op}}$ with homomorphisms $g_k : Ae_{i_{k-1}} \to Ae_{i_k}$ given by $a_k \in e_{i_{k-1}}(\operatorname{rad} A)e_{i_k} \setminus e_{i_{k-1}}(\operatorname{rad} A)^2 e_{i_k}$, for $k \in \{1, \dots, s\}$, and consequently a path in $\operatorname{mod} A$ of the form

$$I_j = I_{i_s} \xrightarrow{h_s} I_{i_{s-1}} \xrightarrow{h_{s-1}} \cdots \xrightarrow{h_1} I_{i_0} = I_i$$

with $h_k = D(g_k)$ for any $k \in \{1, \dots, s\}$. Then, applying dual arguments, we prove that, if j belongs to $B(C)^{(l)}$ then $j \in Q_C$ and i is not in $B(C)^{(l)}$. In particular, $B(C)^{(l)}$ is also a convex subcategory of A.

Further, it follows from [37, Proposition 2.2] that, for any coray tube $\mathcal{T}^A_\eta(C)$ of $\mathcal{T}^A(C)$ containing at least one injective module, all corays of $\mathcal{T}^C_\eta$ are complete corays of $\mathcal{T}^A_\eta(C)$. Since all tubes in $\mathcal{T}^A(C)$ are pairwise orthogonal and generalized standard, we conclude that $B(C)^{(l)}$ is a tubular (branch) coextension of the tame concealed algebra C, and $\Gamma_{B(C)^{(l)}}$ admits a strongly separating family $\mathcal{T}^{B(C)^{(l)}} = (\mathcal{T}^{B(C)^{(l)}}_\lambda)_{\lambda \in \Lambda}$ of coray tubes, obtained from the strongly separating family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda\in\Lambda}$ of stable tubes of Γ_C by the corresponding coray insertions. Obviously, $B(C)^{(l)}$ is cycle-finite as a convex subcategory of the cycle-finite algebra A. In particular, Theorems 3.3 and 3.4 imply $B(C)^{(l)}$ is either a tilted algebra of Euclidean type with all indecomposable projective modules lying in the postprojective component, or a tubular algebra.

It follows from the above discussion that i belongs to $B(C)^{(l)}$ but not to C, and j belongs to $B(C)^{(r)}$ but not to C. In particular, P_i is not in $\mathcal{T}^A(C)$ and P_j is in $\mathcal{T}^A(C)$. Moreover, either P_i lies in the unique postprojective component of $\Gamma_{B(C)^{(l)}}$, or $B(C)^{(l)}$ is a tubular algebra and P_i lies in the family $\mathcal{T}^{B(C)^{(l)}}_0$ of ray tubes of $\Gamma_{B(C)^{(l)}}$ containing the projective modules not lying in the postprojective component, and all coray tubes with injective modules in the family $\mathcal{T}^{B(C)^{(l)}}_\infty$ are coray tubes of $\mathcal{T}^A(C)^{(l)}$. Then we conclude that there is in $\operatorname{mod} A$ a path from P_i to a module N in the ray tube $\mathcal{T}^A_\mu(C)$ containing P_j (see Theorem 3.4). But then we obtain in $\operatorname{mod} A$ an infinite cycle of the form

$$P_i \to \cdots \to N \to \cdots \to Z_{m_0} \to P_{i_{s-1}} \to \cdots \to P_{i_1} \to P_{i_0} = P_i,$$

because $\operatorname{rad}^\infty_A(Z_{m_0}, P_{i_{s-1}}) = \operatorname{Hom}_A(Z_{m_0}, P_{i_{s-1}})$ for the module Z_{m_0} in $\mathcal{T}^A_\mu(C)$ described above.

Summing up, we have proved that $B(C)$ is a convex subcategory of A and a semiregular branch enlargement of C. Moreover, $B(C)$ is cycle-finite.

Hence it follows from Theorem 3.3 that $B(C)$ is a tame quasitilted algebra of canonical type.

6. Proof of Theorem 1.1. The implication (ii)⇒(i) follows directly from Theorem 3.5. We will prove that (i) implies (ii).

Let A be a cycle-finite algebra of semiregular type. Then it follows from Theorem 1.4 that A admits a tame concealed convex subcategory C. Applying now Theorem 1.5 we conclude that there exists a convex subcategory $B(C)$ of A such that $B(C)$ is a tame quasitilted algebra of canonical type, and a tame semiregular branch enlargement of C. Further, Γ_A admits a family $\mathcal{T}^A(C) = (\mathcal{T}^A_\lambda(C))_{\lambda\in\Lambda}$ of semiregular tubes such that $\mathcal{T}^A(C)$ is a strongly separating family of semiregular tubes in $\Gamma_{B(C)}$ and, for any $\lambda \in \Lambda$, $\mathcal{T}^A_\lambda(C)$ contains all modules of the stable tube $\mathcal{T}^C_\lambda$ of the family $\mathcal{T}^C = (\mathcal{T}^C_\lambda)_{\lambda\in\Lambda}$ of all stable tubes of Γ_C. Moreover, $\mathcal{T}^A_\lambda(C) = \mathcal{T}^C_\lambda$ if $\mathcal{T}^A_\lambda(C)$ is a stable tube. The Auslander–Reiten quiver of $\Gamma_{B(C)}$ has, by Theorem 3.4, the disjoint union decomposition

$$\Gamma_{B(C)} = \mathcal{P}^{B(C)} \cup \mathcal{T}^{B(C)} \cup \mathcal{Q}^{B(C)},$$

where $\mathcal{T}^{B(C)} = \mathcal{T}^A(C)$, and $\mathcal{P}^{B(C)}$ and $\mathcal{Q}^{B(C)}$ are of the following forms:

- If $B(C)^{(l)}$ is a tilted algebra of Euclidean type, then $\mathcal{P}^{B(C)}$ is the unique postprojective component $\mathcal{P}^{B(C)^{(l)}}$ of $\Gamma_{B(C)^{(l)}}$, containing all indecomposable projective $B(C)^{(l)}$-modules.
- If $B(C)^{(l)}$ is a tubular algebra, then

$$\mathcal{P}^{B(C)} = \mathcal{P}_0^{B(C)^{(l)}} \cup \mathcal{T}_0^{B(C)^{(l)}} \cup \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^{B(C)^{(l)}}\Big),$$

  where $\mathcal{P}_0^{B(C)^{(l)}}$ is the unique postprojective component of $\Gamma_{B(C)^{(l)}}$, $\mathcal{T}_0^{B(C)^{(l)}}$ is a strongly separating family of ray tubes of $\Gamma_{B(C)^{(l)}}$ having at least one projective module, and, for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q^{B(C)^{(l)}}$ is a strongly separating family of stable tubes in $\Gamma_{B(C)^{(l)}}$, and hence $\mathcal{P}_0^{B(C)^{(l)}} \cup \mathcal{T}_0^{B(C)^{(l)}}$ contains all indecomposable projective $B(C)^{(l)}$-modules.
- If $B(C)^{(r)}$ is a tilted algebra of Euclidean type, then $\mathcal{Q}^{B(C)}$ is the unique preinjective component $\mathcal{P}^{B(C)^{(r)}}$ of $\Gamma_{B(C)^{(r)}}$, containing all indecomposable injective $B(C)^{(r)}$-modules.
- If $B(C)^{(r)}$ is a tubular algebra, then

$$\mathcal{Q}^{B(C)} = \Big(\bigcup_{q\in\mathbb{Q}^+} \mathcal{T}_q^{B(C)^{(r)}}\Big) \cup \mathcal{T}_\infty^{B(C)^{(r)}} \cup \mathcal{Q}_\infty^{B(C)^{(r)}},$$

where $\mathcal{Q}_\infty^{B(C)^{(r)}}$ is the unique preinjective component of $\Gamma_{B(C)^{(r)}}$, $\mathcal{T}_\infty^{B(C)^{(r)}}$ is a strongly separating family of coray tubes of $\Gamma_{B(C)^{(r)}}$ having at least one injective module, and, for each $q \in \mathbb{Q}^+$, $\mathcal{T}_q^{B(C)^{(r)}}$ is a strongly separating family of stable tubes in $\Gamma_{B(C)^{(r)}}$, and hence $\mathcal{T}_\infty^{B(C)^{(r)}} \cup \mathcal{Q}_\infty^{B(C)^{(r)}}$ contains all indecomposable injective $B(C)^{(r)}$-modules.

We will prove that there exists a coherent sequence $\mathbb{B} = (B_1, \ldots, B_n)$ of tame quasitilted algebras of canonical type such that $A(\mathbb{B})$ is a convex subcategory of A and, for the canonical decomposition

$$\Gamma_{A(\mathbb{B})} = \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}_n^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}$$

of $\Gamma_{A(\mathbb{B})}$ with $\bar{\mathbb{Q}}_n^1 = \mathbb{Q} \cap [1, n]$, we have:

- $\mathcal{P}^{\mathbb{B}}$ is a postprojective component of $\Gamma_{A(\mathbb{B})}$;
- $\mathcal{Q}^{\mathbb{B}}$ is a preinjective component of $\Gamma_{A(\mathbb{B})}$;
- $\bigcup_{q \in \bar{\mathbb{Q}}_n^1} \mathcal{T}_q^{\mathbb{B}}$ is a family of components of Γ_A.

This implies that $B_1^{(l)}$ and $B_n^{(r)}$ are tilted algebras of Euclidean type and the following statements hold:

- $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ is a unique postprojective component of $\Gamma_{B_1^{(l)}}$.
- $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$ is a unique preinjective component of $\Gamma_{B_n^{(r)}}$.
- For each $r \in \{1, \ldots, n\}$, $\mathcal{T}_r^{\mathbb{B}} = \mathcal{T}^{B_r}$ is a family $(\mathcal{T}_{r,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_r}$ of pairwise orthogonal generalized standard semiregular tubes.
- For each $q \in \bar{\mathbb{Q}}_n^1 \setminus \{1, \ldots, n\}$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \Lambda_q}$ of pairwise orthogonal generalized standard stable tubes.
- For each $q \in \bar{\mathbb{Q}}_n^1$, we have

$$\mathrm{Hom}_A\Big(\Big(\bigcup_{p > q} \mathcal{T}_p^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}}, \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{p < q} \mathcal{T}_p^{\mathbb{B}} \Big) \Big) = 0.$$

- For each $q \in \bar{\mathbb{Q}}_n^1$, every homomorphism from $\mathcal{P}^{\mathbb{B}} \cup (\bigcup_{p<q} \mathcal{T}_p^{\mathbb{B}})$ to $(\bigcup_{p>q} \mathcal{T}_p^{\mathbb{B}}) \cup \mathcal{Q}^{\mathbb{B}}$ factors through $\mathrm{add}(\mathcal{T}_{q,\lambda}^{\mathbb{B}})$ for any $\lambda \in \Lambda_q$.

Moreover, for any $i \in \{1, \ldots, n\}$, B_i is a maximal semiregular branch enlargement of a tame concealed convex subcategory C_i inside A. Further, if $n \geq 2$, then $B_i^{(r)} = B_{i+1}^{(l)}$, for $i \in \{1, \ldots, n\}$, are tubular algebras.

We have two cases to consider. Recall that it follows from Lemma 2.3 and Theorem 3.1 that the support category $\mathrm{supp}(\mathcal{T})$ of a stable tube $\mathcal{T}$ of Γ_A is either a tame concealed or tubular convex subcategory of A.

Assume that A does not contain a tubular convex subcategory. Let C be a tame concealed convex subcategory of A. Then, for the tame semiregular branch enlargement $B(C)$ of C, $B(C)^{(l)}$ and $B(C)^{(r)}$ are tilted algebras of Euclidean type, and hence the one-element sequence $\mathbb{B} = (B_1)$ with $B_1 = B(C)$ has the required properties, because $A(\mathbb{B}) = B(C)$ is a convex subcategory of A and $\mathcal{T}_1^{\mathbb{B}} = \mathcal{T}^{B(C)}$ is a family of semiregular tubes of Γ_A.

Assume now that A contains a convex tubular subcategory B. Observe that then B is a tubular extension $B^{(r)} = B$ of a tame concealed convex subcategory C_0 of A and a tubular coextension $B^{(l)} = B$ of a tame concealed convex subcategory C_∞ of A, and we have the coherent sequence $(B^{(r)}, B^{(l)})$ of tame quasitilted algebras of canonical type. Hence we may choose a coherent sequence $\bar{\mathbb{B}} = (\bar{B}_1, \ldots, \bar{B}_n)$ of tame quasitilted algebras of canonical type with $\bar{B}_1^{(r)} = \bar{B}_2^{(l)}, \ldots, \bar{B}_{n-1}^{(r)} = \bar{B}_n^{(l)}$ tubular algebras, $n \geq 2$ maximal, and such that $A(\bar{\mathbb{B}})$ is a convex subcategory of A. Then, there exist tame concealed convex subcategories $C_1, \ldots, C_n$ of A such that, for any $i \in \{1, \ldots, n-1\}$, $\bar{B}_i^{(r)}$ is a maximal tubular extension of C_i and $\bar{B}_{i+1}^{(l)}$ is a maximal tubular coextension of C_{i+1} inside A. This implies that, if $n \geq 3$, then for any $r \in \{2, \ldots, n-1\}$, we have $\bar{B}_r = B(C_r)$, and hence $\mathcal{T}_r^{\bar{\mathbb{B}}} = \mathcal{T}_r^{\bar{B}_r} = \mathcal{T}^{B(C_r)}$ is a family of semiregular tubes of Γ_A. Take now $q \in \mathbb{Q}_{i+1}^i = \mathbb{Q} \cap (i, i+1)$ for some $i \in \{1, \ldots, n-1\}$. Then $\mathcal{T}_q^{\bar{\mathbb{B}}} = \mathcal{T}_q^{\bar{B}_i^{(r)}} = (\mathcal{T}_{q,\lambda}^{\bar{B}_i^{(r)}})_{\lambda \in \Lambda_q}$ is a family of pairwise orthogonal generalized standard stable tubes in the Auslander–Reiten quiver $\Gamma_{\bar{B}_i^{(r)}}$ of the tubular algebra $\bar{B}_i^{(r)}$. We claim that $\mathcal{T}_q^{\bar{\mathbb{B}}}$ is a family of semiregular tubes in Γ_A. Indeed, since A is a cycle-finite algebra of semiregular type, for any $\lambda \in \Lambda_q$ there exists a semiregular tube $\mathcal{T}_{q,\lambda}^A$ in Γ_A containing all modules of $\mathcal{T}_{q,\lambda}^{\bar{B}_i^{(r)}}$. Assume $\mathcal{T}_{q,\lambda}^A \neq \mathcal{T}_{q,\lambda}^{\bar{B}_i^{(r)}}$ for some $\lambda \in \Lambda_q$. Then there is inside A a semiregular branch enlargement D of $\bar{B}_i^{(r)}$ using the strongly separating family $\mathcal{T}_q^{\bar{B}_i^{(r)}}$ of stable tubes of $\Gamma_{\bar{B}_i^{(r)}}$, and D is a quasitilted algebra of wild canonical type (see [15, Theorem 3.4]). Moreover, the Auslander–Reiten quiver Γ_D contains acyclic components of the form $\mathbb{Z}\mathbb{A}_\infty$ (see [15, Theorem 4.3]), and these components consist of modules lying on infinite cycles, by [35, Corollary 2]. This contradicts the cycle-finiteness of A. Summing up, we have proved that

$$\bigcup_{q \in \mathbb{Q} \cap (1,n)} \mathcal{T}_q^{\bar{\mathbb{B}}}$$

is a family of components of Γ_A.

Applying Theorem 1.5, we conclude that there exists a convex subcategory $B(C_1)$ of A which is a tame semiregular branch enlargement of C_1 inside A such that Γ_A admits a family $\mathcal{T}^A(C_1) = (\mathcal{T}^A_\lambda(C_1))_{\lambda\in\Lambda_1}$ of semiregular tubes, which is a strongly separating family of semiregular tubes in $\Gamma_{B(C_1)}$. Moreover, for any $\lambda \in \Lambda_1$, $\mathcal{T}^A_\lambda(C_1)$ contains all modules of the stable tube $\mathcal{T}^{C_1}_\lambda$ of the family $\mathcal{T}^{C_1} = (\mathcal{T}^{C_1}_\lambda)_{\lambda\in\Lambda_1}$ of all stable tubes of Γ_{C_1}. Observe also that $B(C_1)^{(r)} = \bar{B}_1^{(r)}$, because $\bar{B}_1$ is a tubular algebra, and hence a maximal tubular extension of C_1 inside A. Similarly, applying Theorem 1.5 again, we conclude that there exists a convex subcategory $B(C_n)$ of A which is a semiregular branch enlargement of C_n inside A such that Γ_A admits a family $\mathcal{T}^A(C_n) = (\mathcal{T}^A_\lambda(C_n))_{\lambda\in\Lambda_n}$ of semiregular tubes, which is a strongly separating family of semiregular tubes in $\Gamma_{B(C_n)}$. Moreover, for any $\lambda \in \Lambda_n$, $\mathcal{T}^A_\lambda(C_n)$ contains all modules of the stable tube $\mathcal{T}^{C_n}_\lambda$ of the family $\mathcal{T}^{C_n} = (\mathcal{T}^{C_n}_\lambda)_{\lambda\in\Lambda_n}$ of all stable tubes of Γ_{C_n}. Observe also that $B(C_n)^{(l)} = \bar{B}_n^{(l)}$, because $\bar{B}_n$ is a tubular algebra, and hence a maximal tubular coextension of C_n inside A. We define

$$\mathbb{B} = (B_1, \dots, B_n),$$

where $B_1 = B(C_1)$, $B_n = B(C_n)$, and $B_i = \bar{B}_i$ for $i \in \{2, \dots, n-1\}$ if $n \geq 3$. Clearly, $\mathbb{B}$ is a coherent sequence of tame quasitilted algebras of canonical type. We claim that $A(\mathbb{B})$ is a convex subcategory of A.

Consider the coherent sequences of tame quasitilted algebras of canonical type $\mathbb{B}^{(l)} = (B_1, \bar{B}_2, \dots, \bar{B}_n)$ and $\mathbb{B}^{(r)} = (\bar{B}_1, \dots, \bar{B}_{n-1}, B_n)$, and the associated algebras $A(\mathbb{B}^{(l)})$ and $A(\mathbb{B}^{(r)})$. Observe that $A(\bar{\mathbb{B}})$ is a common convex subcategory of $A(\mathbb{B}^{(l)})$ and $A(\mathbb{B}^{(r)})$, and

$$A(\mathbb{B}) = A(\mathbb{B}^{(l)}) \underset{A(\bar{\mathbb{B}})}{\sqcup} A(\mathbb{B}^{(r)}).$$

Assume that $A(\mathbb{B})$ is not a convex subcategory of A. Then Q_A contains a path

$$i = i_0 \xrightarrow{(d_{i_0i_1}, d'_{i_0i_1})} i_1 \xrightarrow{(d_{i_1i_2}, d'_{i_1i_2})} i_2 \to \cdots \to i_{s-1} \xrightarrow{(d_{i_{s-1}i_s}, d'_{i_{s-1}i_s})} i_s = j$$

with $s \geq 2$, $i, j \in A(\mathbb{B})$ and i_t not in $A(\mathbb{B})$ for any $t \in \{1, \dots, s-1\}$. Then there exist elements $a_k \in e_{i_{k-1}}(\operatorname{rad} A)e_{i_k} \setminus e_{i_{k-1}}(\operatorname{rad} A)^2 e_{i_k}$ for $k \in \{1, \dots, s\}$. Hence we have a path in $\operatorname{mod} A$ of the form

$$P_j = P_{i_s} \xrightarrow{f_s} P_{i_{s-1}} \xrightarrow{f_{s-1}} \cdots \xrightarrow{f_1} P_{i_0} = P_i,$$

with $P_{i_t} = e_{i_t}A$ the indecomposable projective modules in $\operatorname{mod} A$ given by the vertices i_t for $t \in \{0, 1, \dots, s\}$, and the homomorphisms $f_k : P_{i_k} \to P_{i_{k-1}}$ given by the elements a_k for $k \in \{1, \dots, s\}$. Similarly, we have in $\operatorname{mod} A$ a path of the form

$$I_j = I_{i_s} \xrightarrow{h_s} I_{i_{s-1}} \xrightarrow{h_{s-1}} \cdots \xrightarrow{h_1} I_{i_0} = I_i$$

with $I_{i_t} = D(Ae_{i_s})$ the indecomposable injective modules in $\operatorname{mod} A$ given by the vertices i_t for $t \in \{0, 1, \dots, s\}$, and the homomorphisms $h_k = D(g_k) : I_{i_k} \to I_{i_{k-1}}$ with $g_k : Ae_{i_{k-1}} \to Ae_{i_k}$ given by the elements a_k for $k \in \{1, \dots, s\}$. Applying arguments as in the proof of Theorem 1.5 we conclude that i belongs to $A(\mathbb{B}^{(l)})$ but not to $A(\bar{\mathbb{B}})$ and j belongs to $A(\mathbb{B}^{(r)})$ but not to $A(\bar{\mathbb{B}})$. In particular, we have $B_1 \neq \bar{B}_1$ and $B_n \neq \bar{B}_n$. Observe that then either P_i lies in the unique postprojective component of $\Gamma_{B(C_1)}$, or $B(C_1)^{(l)}$ is a tubular algebra and P_i lies in the family $\mathcal{T}_0^{B(C_1)^{(l)}}$ of ray tubes of $\Gamma_{B(C_1)^{(l)}}$. On the other hand, P_j belongs to a ray tube $\mathcal{T}_\lambda^{B(C_n)^{(r)}}$ of the strongly separating family $\mathcal{T}^{B(C_n)^{(r)}} = (\mathcal{T}_\lambda^{B(C_n)^{(r)}})_{\lambda \in \Lambda_n}$ of ray tubes of $\Gamma_{B(C_n)^{(r)}}$. Then, using the structure of $\Gamma_{A(\mathbb{B})}$ described in Theorem 3.5, we conclude that we have in $\operatorname{mod} A$ an infinite cycle of the form

$$P_i \to \cdots \to Z \to P_{i_{s-1}} \to \cdots \to P_{i_1} \to P_{i_0} = P_i$$

with Z an indecomposable module in $\mathcal{T}_\lambda^{B(C_n)^{(r)}}$ such that $\operatorname{Hom}_A(P_j, Z) \neq 0$ and $\operatorname{Hom}_A(Z, P_{i_{s-1}}) = \operatorname{rad}_A^\infty(Z, P_{i_{s-1}}) \neq 0$. This contradicts the cycle-finiteness of A. Therefore, $A(\mathbb{B})$ is indeed a convex subcategory of A. Finally observe that, by the maximality of the number n in the chosen coherent sequence $\bar{B} = (\bar{B}_1, \dots, \bar{B}_n)$ of quasitilted algebras of canonical type, the algebras $B_1^{(l)}$ and $B_n^{(r)}$ are tilted algebras of Euclidean type. Indeed, if $B_1^{(l)}$ (respectively, $B_n^{(r)}$) is a tubular algebra, then we have the coherent sequence $\mathbb{B}' = (B_1^{(l)}, B_1, \dots, B_n)$ (respectively, $\mathbb{B}'' = (B_1, \dots, B_n, B_n^{(r)})$) of quasitilted algebras of canonical type, consisting of $n+1$ algebras, and with $A(\mathbb{B}') = A(\mathbb{B})$ (respectively, $A(\mathbb{B}'') = A(\mathbb{B})$) a convex subcategory of A.

Summing up, $\mathbb{B} = (B_1, \dots, B_n)$ is a coherent sequence of tame quasitilted algebras satisfying the required conditions.

We will show that $A = A(\mathbb{B})$. We know from Proposition 2.2 that A is a triangular algebra. In particular, for any indecomposable projective module P and indecomposable injective module I in $\operatorname{mod} A$, the endomorphism algebras $\operatorname{End}_A(P)$ and $\operatorname{End}_A(I)$ are division algebras. Assume to the contrary that $A \neq A(\mathbb{B})$. Then A can be obtained from its convex subcategory $A(\mathbb{B})$ by iterated one-point extensions and coextensions, starting from one-point extensions and one-point coextensions by modules in $\operatorname{mod} A(\mathbb{B})$. Suppose that there is inside A a one-point extension

$$A(\mathbb{B})[M] = \begin{bmatrix} F & 0 \\ M & A(\mathbb{B}) \end{bmatrix}$$

with M a module in $\operatorname{mod} A(\mathbb{B})$ and F a division algebra. Then $A(\mathbb{B})[M]$ is a quotient algebra of A, and hence $\mathcal{T}_q^{\mathbb{B}}$, $q \in \bar{\mathbb{Q}}_n^1$, are families of components in $\Gamma_{A(\mathbb{B})[M]}$. Therefore, applying Lemma 2.4, we conclude that $\operatorname{Hom}_{A(\mathbb{B})}(M, \mathcal{T}_q^{\mathbb{B}})$

$= 0$ for any $q \in \bar{\mathbb{Q}}_n^1$. Further, for any module X in the postprojective component $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$, there is a monomorphism $X \to Y$ for a module Y in $\mathrm{add}(\mathcal{T}_1^{\mathbb{B}})$, because there is a monomorphism $X \to I$ with I an injective module in $\mathrm{mod}\, A(\mathbb{B})$, $\mathcal{P}^{\mathbb{B}}$ does not contain injective modules, and every homomorphism from X to an injective module in $\mathcal{T}_p^{\mathbb{B}}$ with $p \in \{2, \ldots, n\}$ or in $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_n^{(r)}}$ factors through a module in $\mathrm{add}(\mathcal{T}_1^{\mathbb{B}})$. Then $\mathrm{Hom}_{A(\mathbb{B})}(M, \mathcal{T}_1^{\mathbb{B}}) = 0$ implies $\mathrm{Hom}_{A(\mathbb{B})}(M, X) = 0$, and consequently $\mathrm{Hom}_{A(\mathbb{B})}(M, \mathcal{P}^{\mathbb{B}}) = 0$. This shows that M belongs to the additive category $\mathrm{add}(\mathcal{Q}^{\mathbb{B}})$ of the preinjective component $\mathcal{Q}^{\mathbb{B}}$ of $\Gamma_{A(\mathbb{B})}$. Similarly, if there is inside A a one-point coextension

$$[N]A(\mathbb{B}) = \begin{bmatrix} A(\mathbb{B}) & D(N) \\ 0 & G \end{bmatrix}$$

with N a module in $\mathrm{mod}\, A(\mathbb{B})$ and G a division algebra, then applying Lemma 2.5, we conclude, as above, that N belongs to the additive category $\mathrm{add}(\mathcal{P}^{\mathbb{B}})$ of the postprojective component $\mathcal{P}^{\mathbb{B}}$ of $\Gamma_{A(\mathbb{B})}$. Summing up, applying Lemmas 2.4 and 2.5, we conclude that one of the following holds:

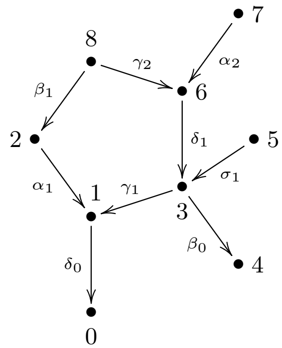
- the postprojective component $\mathcal{P}^{\mathbb{B}}$ of $\Gamma_{A(\mathbb{B})}$ contains a cofinite translation subquiver Σ, closed under successors, which is a full translation subquiver of a component $\mathcal{C}$ of Γ_A and is closed under successors in $\mathcal{C}$, and $\mathcal{C}$ contains an injective module,
- the preinjective component $\mathcal{Q}^{\mathbb{B}}$ of $\Gamma_{A(\mathbb{B})}$ contains a cofinite translation subquiver Ω, closed under predecessors, which is a full translation subquiver of a component $\mathcal{D}$ of Γ_A and is closed under predecessors in $\mathcal{D}$, and $\mathcal{D}$ contains a projective module.

On the other hand, it follows from Proposition 2.1 that every semiregular component of the cycle-finite algebra A is one of the following forms: a postprojective component of Euclidean type, a preinjective component of Euclidean type, a ray tube or a coray tube. Because the translation quivers Σ and Ω are acyclic, this implies that one of the components $\mathcal{C}$ or $\mathcal{D}$ is not semiregular, which contradicts the assumption on A. Therefore, $A = A(\mathbb{B})$.

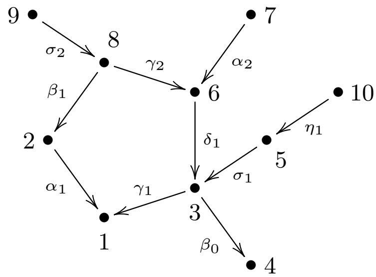
7. Examples. The aim of this section is to present some examples of cycle-finite algebras of semiregular type, illustrating the above considerations.

EXAMPLE 7.1. Let K be an algebraically closed field. Consider the bound quiver algebras

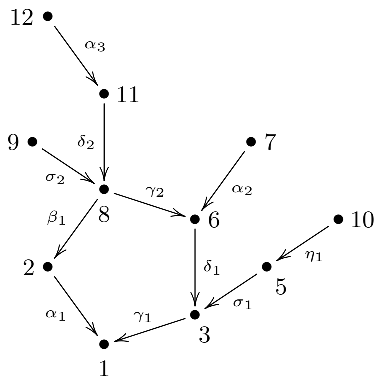
- $B_1 = KQ^{(1)}/I^{(1)}$ given by the quiver $Q^{(1)}$ of the form



  and the ideal $I^{(1)}$ in the path algebra $KQ^{(1)}$ of $Q^{(1)}$ generated by the elements $\beta_1\alpha_1 - \gamma_2\delta_1\gamma_1$, $\gamma_2\delta_1\beta_0$, $\delta_1\gamma_1\delta_0$, $\sigma_1\beta_0$;
- $B_2 = KQ^{(2)}/I^{(2)}$ given by the quiver $Q^{(2)}$ of the form

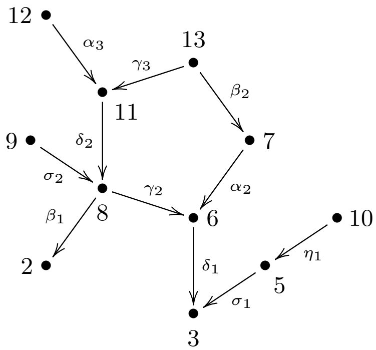


  and the ideal $I^{(2)}$ in the path algebra $KQ^{(2)}$ of $Q^{(2)}$ generated by the elements $\beta_1\alpha_1 - \gamma_2\delta_1\gamma_1$, $\gamma_2\delta_1\beta_0$, $\sigma_1\beta_0$, $\eta_1\sigma_1\gamma_1$, $\sigma_2\beta_1$;
- $B_3 = KQ^{(3)}/I^{(3)}$ given by the quiver $Q^{(3)}$ of the form

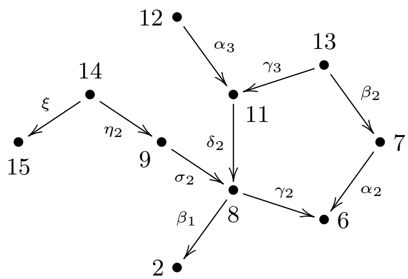


  and the ideal $I^{(3)}$ in the path algebra $KQ^{(3)}$ of $Q^{(3)}$ generated by the elements $\beta_1\alpha_1 - \gamma_2\delta_1\gamma_1$, $\eta_1\sigma_1\gamma_1$, $\sigma_2\beta_1$, $\delta_2\gamma_2\delta_1$;

- $B_4 = KQ^{(4)}/I^{(4)}$ given by the quiver $Q^{(4)}$ of the form



  and the ideal $I^{(4)}$ in the path algebra $KQ^{(4)}$ of $Q^{(4)}$ generated by the elements $\beta_2\alpha_2 - \gamma_3\delta_2\gamma_2$, $\sigma_2\beta_1$, $\delta_2\gamma_2\delta_1$;
- $B_5 = KQ^{(5)}/I^{(5)}$ given by the quiver $Q^{(5)}$ of the form



  and the ideal $I^{(5)}$ in the path algebra $KQ^{(5)}$ of $Q^{(5)}$ generated by the elements $\beta_2\alpha_2 - \gamma_3\delta_2\gamma_2$, $\sigma_2\beta_1$, $\eta_2\sigma_2\gamma_2$.

We will show that $\mathbb{B} = (B_1, B_2, B_3, B_4, B_5)$ is a coherent sequence of tame quasitilted algebras of canonical type. We refer to [27, Appendix A2] or [30, XIV.4] for a classification of tame concealed algebras of Euclidean types $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$.

(1) The algebra B_1 contains the convex subcategory C_1 given by all objects of B_1 except 0 and 8, and C_1 is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$. Further, the convex subcategory D_0 of B_1 given by all objects of B_1 except 8 is a one-point coextension of C_1 using an indecomposable C_1-module lying on the mouth of a stable tube of Γ_{C_1} of rank 3, and hence D_0 is a tilted algebra of of Euclidean type $\widetilde{\mathbb{E}}_7$. On the other hand, the convex subcategory D_1 of B_1 given by all objects of B_1 except 0 is a one-point extension of C_1 using an indecomposable C_1-module lying on the mouth of the unique stable tube of rank 2 in Γ_{C_1}, and hence D_1 is a tubular algebra of tubular type $(3, 3, 3)$. Therefore, B_1 is a tame quasitilted algebra of canonical type with $B_1^{(l)} = D_0$ and $B_1^{(r)} = D_1$.

(2) The algebra B_2 contains the convex subcategory C_2 given by all objects of B_2 except 4, 9 and 10, and C_2 is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$. Further, the convex subcategory D_2 of B_2 given by all objects of B_2 except 4 is a tubular extension of C_2 using two indecomposable C_2-modules lying on the mouth of two stable tubes of Γ_{C_2} of rank 3, creating the vertices 9 and 10, and hence D_2 is a tubular algebra of type $(2,4,4)$. On the other hand, the tubular algebra D_1 is a one-point coextension of C_2 by an indecomposable C_2-module lying on the mouth of the unique stable tube of Γ_{C_2} of rank 2, creating the vertex 4. Therefore, B_2 is a tame quasitilted algebra of canonical type with $B_2^{(l)} = D_1 = B_1^{(r)}$ a tubular algebra of type $(3,3,3)$ and $B_2^{(r)} = D_2$ a tubular algebra of type $(2,4,4)$.

(3) The algebra B_3 contains the convex subcategory C_3 given by all objects of B_3 except 1, 11 and 12, and C_3 is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_7$. Further, the convex subcategory D_3 of B_3 given by all objects of B_3 except 1 is a tubular extension of C_3 using an indecomposable C_3-module lying on the mouth of the unique stable tube of Γ_{C_3} of rank 4 and the branch $11 \xleftarrow{\alpha_3} 12$, and hence D_3 is a tubular algebra of type $(2,3,6)$. We also note that the tubular algebra D_2 is the one-point coextension of C_3 using an indecomposable C_3-module lying on the mouth of the unique stable tube of Γ_{C_3} of rank 3. Therefore, B_3 is a tame quasitilted algebra of canonical type with $B_3^{(l)} = D_2 = B_2^{(r)}$ a tubular algebra of type $(2,4,4)$ and $B_3^{(r)} = D_3$ a tubular algebra of type $(2,3,6)$.

(4) The algebra B_4 contains the convex subcategory C_4 given by all objects of B_4 except 3, 5, 10 and 13, which is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$. Further, the convex subcategory D_4 of B_4 formed by all objects of B_4 except 3, 5, 10 is the one-point extension of C_4 using an indecomposable C_4-module lying on the mouth of the unique stable tube of Γ_{C_4} of rank 2, and hence D_4 is a tubular algebra of type $(3,3,3)$. Observe also that the tubular algebra D_3 is a tubular coextension of C_4 using an indecomposable C_4-module lying on the mouth of a stable tube of Γ_{C_4} of rank 3 and the branch $3 \xleftarrow[\sigma_1]{} 5 \xleftarrow[\eta_1]{} 10$. Therefore, B_4 is a tame quasitilted algebra of canonical type with $B_4^{(l)} = D_3 = B_3^{(r)}$ a tubular algebra of type $(2,3,6)$ and $B_4^{(r)} = D_4$ a tubular algebra of type $(3,3,3)$.

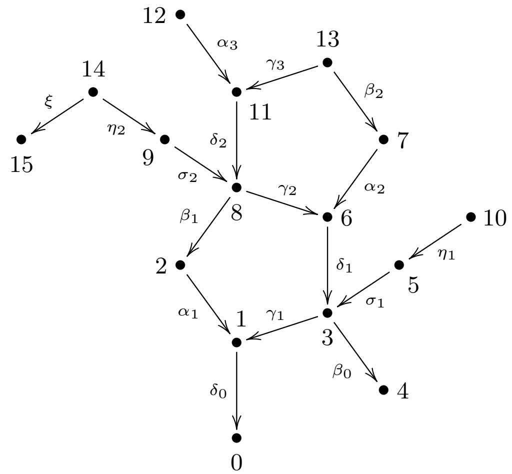
(5) The algebra B_5 contains the convex subcategory C_5 given by all objects of B_5 except 2, 14 and 15, which is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$. Further, the convex subcategory D_5 of B_5 formed by all objects of B_5 except 2 is a tubular extension of C_5 using an indecomposable C_5-module lying on the mouth of a stable tube of Γ_{C_5} of rank 3 and the branch $15 \xleftarrow{\xi} 14$, and hence D_5 is a tilted algebra of Euclidean type $\widetilde{\mathbb{E}}_8$.

Observe also that the tubular algebra D_4 is the one-point coextension of C_5 using an indecomposable C_5-module lying on the mouth of the stable tube of $\mathit{\Gamma}_{C_5}$ of rank 3 different from the stable tube of rank 3 used in the tubular extension of C_5 creating the vertices 14 and 15. Hence, B_5 is a tame quasitilted algebra of canonical type with $B_5^{(l)} = D_4 = B_4^{(r)}$ a tubular algebra of type $(3, 3, 3)$ and $B_5^{(r)} = D_5$ a tilted algebra of of Euclidean type $\widetilde{\mathbb{E}}_8$.

Therefore, indeed $\mathbb{B} = (B_1, B_2, B_3, B_4, B_5)$ is a coherent sequence of tame quasitilted algebras of canonical type. Moreover, the associated algebra

$$A(\mathbb{B}) = B_1 \underset{B_1^{(r)}}{\sqcup} B_2 \underset{B_2^{(r)}}{\sqcup} B_3 \underset{B_3^{(r)}}{\sqcup} B_4 \underset{B_4^{(r)}}{\sqcup} B_5 = B_1 \underset{B_2^{(l)}}{\sqcup} B_2 \underset{B_3^{(l)}}{\sqcup} B_3 \underset{B_4^{(l)}}{\sqcup} B_4 \underset{B_5^{(l)}}{\sqcup} B_5$$

is the bound quiver algebra KQ/I given by the quiver Q of the form



and the ideal I in the path algebra KQ of Q generated by the elements $\beta_1\alpha_1 - \gamma_2\delta_1\gamma_1$, $\beta_2\alpha_2 - \gamma_3\delta_2\gamma_2$, $\sigma_1\beta_0$, $\delta_1\gamma_1\delta_0$, $\gamma_2\delta_1\beta_0$, $\eta_1\sigma_1\gamma_1$, $\sigma_2\beta_1$, $\delta_2\gamma_2\delta_1$, $\eta_2\sigma_2\gamma_2$. It follows from Theorem 3.5 that $A(\mathbb{B})$ is a cycle-finite algebra of semiregular type and the Auslander–Reiten quiver $\mathit{\Gamma}_{A(\mathbb{B})}$ of $A(\mathbb{B})$ has a disjoint union decomposition

$$\mathit{\Gamma}_{A(\mathbb{B})} = \mathcal{P}^{\mathbb{B}} \cup \Big(\bigcup_{q \in \bar{Q}_5^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}^{\mathbb{B}},$$

where $\bar{Q}_5^1 = \mathbb{Q} \cap [1, 5]$, and

- $\mathcal{P}^{\mathbb{B}} = \mathcal{P}^{B_1^{(l)}}$ is a postprojective component of Euclidean type $\widetilde{\mathbb{E}}_7$, containing the indecomposable projective modules P_0, P_1, P_2, P_3, P_4, P_5, P_6, P_7,
- $\mathcal{Q}^{\mathbb{B}} = \mathcal{Q}^{B_5^{(r)}} = \mathcal{Q}^{C_5}$ is a preinjective component of Euclidean type $\widetilde{\mathbb{E}}_8$, containing the indecomposable injective modules I_6, I_7, I_8, I_9, I_{11}, I_{12}, I_{13}, I_{14}, I_{15},

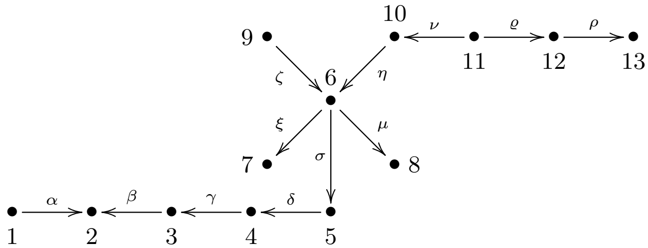
- $\mathcal{T}_1^{\mathbb{B}}$ is a family $(\mathcal{T}_{1,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube with one indecomposable injective module I_0, one ray tube with one indecomposable projective module P_8, one stable tube of rank 3, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_2^1 = \mathbb{Q} \cap (1,2)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(3,3,3)$,
- $\mathcal{T}_2^{\mathbb{B}}$ is a family $(\mathcal{T}_{2,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube with one indecomposable injective module I_4, a ray tube containing the indecomposable projective module P_9, a ray tube containing the indecomposable projective module P_{10}, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_3^2 = \mathbb{Q} \cap (2,3)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2,4,4)$,
- $\mathcal{T}_3^{\mathbb{B}}$ is a family $(\mathcal{T}_{3,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube containing the indecomposable injective module I_1, one ray tube containing the indecomposable projective modules P_{11} and P_{12}, one stable tube of rank 2, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_4^3 = \mathbb{Q} \cap (3,4)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2,3,6)$,
- $\mathcal{T}_4^{\mathbb{B}}$ is a family $(\mathcal{T}_{4,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube containing the indecomposable injective modules I_3, I_5, I_{10}, one ray tube containing the indecomposable projective module P_{13}, one stable tube of rank 3, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_5^4 = \mathbb{Q} \cap (4,5)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(3,3,3)$,
- $\mathcal{T}_5^{\mathbb{B}}$ is a family $(\mathcal{T}_{5,\lambda}^{\mathbb{B}})_{\lambda\in\mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube with one indecomposable injective module I_2, one ray tube with the indecomposable projective modules P_{14} and P_{15}, one stable tube of rank 3, and the remaining tubes being stable tubes of rank 1.

Observe also that $B_2^{(l)} = B_1^{(r)} = B_4^{(r)}$, after renaming the vertices and arrows of the quiver of $B_4^{(r)}$. Hence, we may define, for any positive integer m, the coherent sequence of tame quasitilted algebras of canonical type

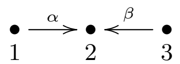
$$\mathbb{B}^{(m)} = (B_1, B_2, B_3, B_4, B_2, B_3, B_4, \dots, B_2, B_3, B_4, B_5),$$

having m triples B_2, B_3, B_4, and the cycle-finite algebra $A(\mathbb{B}^{(m)})$ of semiregular type. This shows that there are coherent sequences with large numbers of tame quasitilted algebras of canonical type, containing tubular algebras of different tubular types.

EXAMPLE 7.2. Let K be an algebraically closed field. Consider the bound quiver algebra $B = KQ/I$, where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by the elements $\zeta\xi$, $\eta\mu$, $\zeta\sigma\delta\gamma$, $\nu\eta\sigma\delta$. The algebra B contains the convex subcategory C given by the objects 4, 5, 6, 7, 8, 9, 10, and C is a tame concealed algebra of Euclidean type $\widetilde{\mathbb{E}}_6$. Further, the convex subcategory D of B given by the objects $i \in \{1, \dots, 10\}$ is a tubular coextension of C using an indecomposable C-module lying on the mouth of a stable tube $\mathcal{T}$ of Γ_C of rank 3 and the branch



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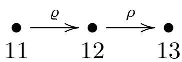
and hence D is a tubular algebra of type $(2, 3, 6)$. Similarly, the convex subcategory E of B formed by the objects $i \in \{4, \dots, 13\}$ is a tubular extension of C using an indecomposable C-module lying on the mouth of a stable tube $\mathcal{T}'$ of Γ_C of rank 3, different from $\mathcal{T}$, and the branch



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and hence E is a tubular algebra of type $(2, 3, 6)$. Therefore, B is a tame quasitilted algebra of canonical type with $B^{(l)} = D$ and $B^{(r)} = E$. We claim that $\mathbb{B} = (B)$ is a unique coherent sequence of tame quasitilted algebras of canonical type containing B.

Consider the convex subcategory C' of B given by the objects $i \in \{1, \dots, 8\}$ and 10, and the convex subcategory C'' of B given by the objects $j \in \{5, \dots, 13\}$. Then C' and C'' are tame concealed algebras of Euclidean type $\widetilde{\mathbb{E}}_8$. Moreover, the tubular algebra D is the one-point extension of C', with the extension vertex 9, using an indecomposable C'-module lying on the mouth of the unique stable tube of rank 5 in $\Gamma_{C'}$. Similarly, the tubular algebra E is the one-point coextension of C'', with the coextension vertex 4, using an indecomposable C''-module lying on the mouth of the unique stable tube of rank 5 in $\Gamma_{C''}$.

Let $\mathbb{B} = (B)$. It follows also from Theorem 3.4 that the Auslander–Reiten quiver $\Gamma_{A(\mathbb{B})} = \Gamma_B$ has a disjoint union decomposition

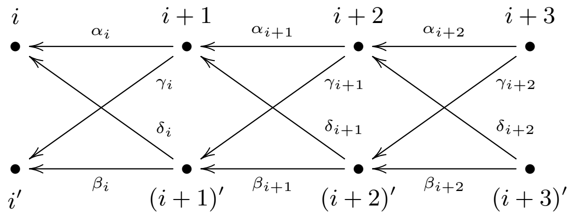
$$\Gamma_{A(\mathbb{B})} = \mathcal{P}(\mathbb{B}) \cup \Big(\bigcup_{q \in \bar{Q}_3^1} \mathcal{T}_q^{\mathbb{B}} \Big) \cup \mathcal{Q}(\mathbb{B}),$$

where $\bar{Q}_3^1 = \mathbb{Q} \cap [1, 3]$, and

- $\mathcal{P}(\mathbb{B}) = \mathcal{P}^{C'}$ is a postprojective component of Euclidean type $\widetilde{\mathbb{E}}_8$, containing the indecomposable projective modules P_i for $i \in \{1, \ldots, 8\} \cup \{10\}$,
- $\mathcal{Q}(\mathbb{B}) = \mathcal{Q}^{C''}$ is a preinjective component of Euclidean type $\widetilde{\mathbb{E}}_8$, containing the indecomposable injective modules I_j, for $j \in \{5, \ldots, 13\}$,
- $\mathcal{T}_1^{\mathbb{B}}$ is a family $(\mathcal{T}_{1,\lambda}^{\mathbb{B}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one ray tube with six rays and containing the indecomposable projective module P_9, one stable tube of rank 2, one stable tube of rank 3, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_2^1 = \mathbb{Q} \cap (1, 2)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2, 3, 6)$,
- $\mathcal{T}_2^{\mathbb{B}}$ is a family $(\mathcal{T}_{2,\lambda}^{\mathbb{B}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube with six corays and containing the indecomposable injective modules I_1, I_2, I_3, one ray tube with six rays and containing the indecomposable projective modules P_{11}, P_{12}, P_{13}, one stable tube of rank 2, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \mathbb{Q}_3^2 = \mathbb{Q} \cap (2, 3)$, $\mathcal{T}_q^{\mathbb{B}}$ is a family $(\mathcal{T}_{q,\lambda}^{\mathbb{B}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2, 3, 6)$,
- $\mathcal{T}_3^{\mathbb{B}}$ is a family $(\mathcal{T}_{3,\lambda}^{\mathbb{B}})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having one coray tube with six corays and containing the indecomposable injective module I_4, one stable tube of rank 2, one stable tube of rank 3, and the remaining tubes being stable tubes of rank 1.

Observe now that the family $\mathcal{T}^{C'} = (\mathcal{T}_\lambda^{C'})_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes in $\Gamma_{C'}$ is of tubular type $(2, 3, 5)$, and the unique stable tube of rank 5 in $\mathcal{T}^{C'}$ has been enlarged to the ray tube in $\mathcal{T}_1^{\mathbb{B}}$ containing the projective module P_9. Similarly, the family $\mathcal{T}^{C''} = (\mathcal{T}_\lambda^{C''})_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes in $\Gamma_{C''}$ is of tubular type $(2, 3, 5)$, and the unique stable tube of rank 5 in $\mathcal{T}^{C''}$ has been enlarged to the coray tube in $\mathcal{T}_3^{\mathbb{B}}$ containing the injective module I_4. This shows that there is no tame semiregular branch enlargement of C' having $B^{(l)}$ as a proper convex subcategory, and there is no tame semiregular branch enlargement of C'' having $B^{(r)}$ as a proper convex subcategory. Therefore, $\mathbb{B} = (B)$ is a unique coherent sequence of tame quasitilted algebras of canonical type containing the algebra B.

EXAMPLE 7.3. Let K be an algebraically closed field and $n \geq 1$ a natural number. We choose a family $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ of pairwise different elements in $K \setminus \{0, 1\}$. For each $i \in \{1, \dots, n\}$, consider the bound quiver algebra $B_i = KQ^{(i)}/I^{(i)}$, where $Q^{(i)}$ is the quiver



and $I^{(i)}$ is the ideal in the path algebra $KQ^{(i)}$ of $Q^{(i)}$ generated by the elements

$$\alpha_{i+1}\alpha_i - a_i\gamma_{i+1}\delta_i, \alpha_{i+1}\gamma_i - \gamma_{i+1}\beta_i, \delta_{i+1}\alpha_i - b_i\beta_{i+1}\delta_i,$$
$$\delta_{i+1}\gamma_i - \beta_{i+1}\beta_i, \alpha_{i+2}\alpha_{i+1} - a_{i+1}\gamma_{i+2}\delta_{i+1}, \alpha_{i+2}\gamma_{i+1} - \gamma_{i+2}\beta_{i+1},$$
$$\delta_{i+2}\alpha_{i+1} - b_{i+1}\beta_{i+2}\delta_{i+1}, \delta_{i+2}\gamma_{i+1} - \beta_{i+2}\beta_{i+1}.$$

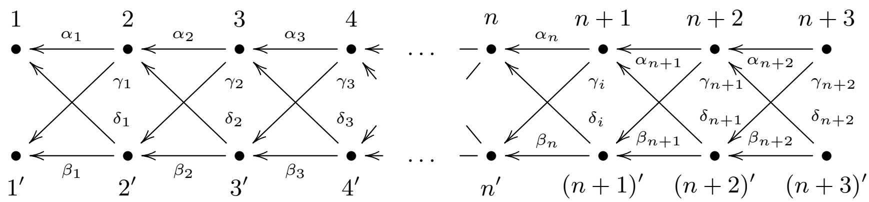
Then B_i contains the three tame concealed convex subcategories of Euclidean type $\widetilde{\mathbb{A}}_3$: C_{i-1} given by the objects $i, i', i+1$ and $(i+1)'$, C_i given by the objects $i+1, (i+1)', i+2$ and $(i+2)'$, and C_{i+1} given by the objects $i+2, (i+2)', i+3$ and $(i+3)'$. Further, B_i is a tame semiregular branch enlargement of the algebra C_i using four indecomposable C_i-modules lying in four pairwise different stable tubes of rank 1 in Γ_{C_i}, and hence B_i is a tame quasitilted algebra of canonical type. Moreover, $B_i^{(l)}$ is a tubular algebra of type $(2, 2, 2, 2)$, which is a tubular extension of C_{i-1} and a tubular coextension of C_i. Similarly, $B_i^{(r)}$ is a tubular algebra of type $(2, 2, 2, 2)$, which is a tubular extension of C_i and a tubular coextension of C_{i+1}. Therefore, we obtain the coherent sequence

$$\mathbb{B} = (B_1, \dots, B_n)$$

of tame quasitilted algebras of canonical type. The associated algebra

$$A(\mathbb{B}) = B_1 \sqcup_{B_1^{(r)}} \cdots \sqcup_{B_{n-1}^{(r)}} B_n = B_1 \sqcup_{B_2^{(l)}} \cdots \sqcup_{B_n^{(l)}} B_n$$

is the bound quiver algebra KQ/I, where Q is the quiver



and I is the ideal in the path algebra KQ of Q generated by the elements

$$\alpha_{i+1}\alpha_i - a_i\gamma_{i+1}\delta_i, \alpha_{i+1}\gamma_i - \gamma_{i+1}\beta_i, \delta_{i+1}\alpha_i - b_i\beta_{i+1}\delta_i, \delta_{i+1}\gamma_i - \beta_{i+1}\beta_i,$$

for all $i \in \{1, \dots, n+1\}$. It follows from Theorem 3.5 that the Auslander–Reiten quiver $\Gamma_{A(\mathbb{B})}$ of $A(\mathbb{B})$ has a disjoint union decomposition

$$\Gamma_{A(\mathbb{B})} = \mathcal{P}(\mathbb{B}) \cup \Big(\bigcup_{q \in \bar{\mathbb{Q}}^0_{n+1}} \mathcal{T}^{\mathbb{B}}_q \Big) \cup \mathcal{Q}(\mathbb{B}),$$

where $\bar{\mathbb{Q}}^0_{n+1} = \mathbb{Q} \cap [0, n+1]$, and

- $\mathcal{P}(\mathbb{B}) = \mathcal{P}^{C_0}$ is a postprojective component of Euclidean type $\widetilde{\mathbb{A}}_3$, containing the indecomposable projective modules $P_1, P_{1'}, P_2, P_{2'}$,
- $\mathcal{Q}(\mathbb{B}) = \mathcal{Q}^{C_{n+1}}$ is a preinjective component of Euclidean type $\widetilde{\mathbb{A}}_3$, containing the indecomposable injective modules $I_{n+2}, I_{(n+2)'}, I_{n+3}, I_{(n+3)'}$,
- $\mathcal{T}^{\mathbb{B}}_1$ is a family $(\mathcal{T}^{\mathbb{B}}_{0,\lambda})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having two ray tubes containing the indecomposable projective modules P_3 and P'_3, two stable tubes of rank 2, and the remaining tubes being stable tubes of rank 1,
- $\mathcal{T}^{\mathbb{B}}_{n+1}$ is a family $(\mathcal{T}^{\mathbb{B}}_{n+1,\lambda})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes, having two coray tubes containing the indecomposable injective modules I_{n+1} and $I_{(n+1)'}$, two stable tubes of rank 2, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \{1, \dots, n\}$, $\mathcal{T}^{\mathbb{B}}_q$ is a family $(\mathcal{T}^{\mathbb{B}}_{q,\lambda})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard semiregular tubes having two coray tubes containing the indecomposable injective modules I_q, and $I_{q'}$, two ray tubes containing the indecomposable projective modules P_{q+3}, $P_{(q+3)'}$, two stable tubes of rank 2, and the remaining tubes being stable tubes of rank 1,
- for each $q \in \bar{\mathbb{Q}}^0_{n+1} \setminus \{0, 1, \dots, n\}$, $\mathcal{T}^{\mathbb{B}}_q$ is a family $(\mathcal{T}^{\mathbb{B}}_{q,\lambda})_{\lambda \in \mathbb{P}_1(K)}$ of pairwise orthogonal generalized standard stable tubes of tubular type $(2, 2, 2, 2)$.

We would like to point that, for any fixed natural number $n \geq 1$, there are infinitely many pairwise non-isomorphic algebras $A(\mathbb{B})$ given by the coherent sequences $\mathbb{B} = (B_1, \dots, B_n)$ of quasitilted algebras of canonical type of the above form, created by different choices of elements $a_1, \dots, a_{n+1}, b_1, \dots, b_{n+1}$ in $K \setminus \{0, 1\}$. Moreover, we note that for all such sequences $\mathbb{B}$, $A(\mathbb{B})$ is of global dimension $n + 1$.

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