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EXPLICIT FUNDAMENTAL SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL OPERATORS ON HEISENBERG GROUPS

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Abstract. Let p, q, n be natural numbers such that $p+q=n$. Let $\mathbb{F}$ be either $\mathbb{C}$, the complex numbers field, or $\mathbb{H}$, the quaternionic division algebra. We consider the Heisenberg group $N(p,q,\mathbb{F})$ defined $\mathbb{F}^n \times \mathfrak{Im}\,\mathbb{F}$, with group law given by

$$(v,\zeta)(v',\zeta') = \left(v+v', \zeta+\zeta' - \frac{1}{2}\,\mathfrak{Im}\, B(v,v')\right),$$

where $B(v,w) = \sum_{j=1}^{p} v_j \overline{w_j} - \sum_{j=p+1}^{n} v_j \overline{w_j}$. Let $U(p,q,\mathbb{F})$ be the group of $n\times n$ matrices with coefficients in $\mathbb{F}$ that leave the form B invariant. We compute explicit fundamental solutions of some second order differential operators on $N(p,q,\mathbb{F})$ which are canonically associated to the action of $U(p,q,\mathbb{F})$.

1. Introduction. In [M-R2] the authors exhaustively discussed the problem of invertibility for the class of second order, homogeneous left invariant differential operators on the Heisenberg group, which in addition are formally selfadjoint, modulo a derivative in the central direction.

The best known examples of this class are of the form $L + i\alpha U$, where L is the sublaplacian, U generates the centre of the Lie algebra, and α is a complex number. For $\alpha \neq 2k+n$, k a nonnegative integer, an explicit fundamental solution was given in [F-S]. It is also mentioned in [M-R2] that these operators are essentially the only ones, in the class considered, which admit simple expressions for their fundamental solutions.

Moreover, in [K] the groups of Heisenberg type were introduced with the purpose, in part, of giving explicit fundamental solutions for some second order differential operators on two-step nilpotent Lie groups.

In [B-D-R] the authors considered the Heisenberg group under the action of $U(n)$, and used the spherical analysis of the associated Gelfand pair in order to obtain a fundamental solution for any power of the sublaplacian. Inspired by this work, the same was done in [G-S2] for a second order homogeneous differential operator canonically associated to the action of $U(p,q)$. The computation used the spherical distributions of the corresponding generalized Gelfand pair.

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The aim of this paper is to continue this research. More precisely, let p, q, n be natural numbers such that $p+q=n$. Let $\mathbb{F}$ be either $\mathbb{C}$, the complex field, or $\mathbb{H}$, the quaternionic division algebra. We consider the Heisenberg group $N(p,q,\mathbb{F}) = \mathbb{F}^n \times \mathfrak{Im}\,\mathbb{F}$, with group law given by

$$(v,\zeta)(v',\zeta') = \left(v+v', \zeta+\zeta' - \frac{1}{2}\,\mathfrak{Im}\,B(v,v')\right),$$

where $B(v,w) = \sum_{j=1}^{p} v_j\overline{w_j} - \sum_{j=p+1}^{n} v_j\overline{w_j}$. The associated Lie algebra is $\eta(p,q,\mathbb{F}) = \mathbb{F}^n \oplus \mathfrak{Im}(\mathbb{F})$, with Lie bracket given by

$$[(v,\zeta),(v',\zeta')] = (0, -\,\mathfrak{Im}\,B(v,v')).$$

Let $U(p,q,\mathbb{H})$ be the group of $n\times n$ matrices with coefficients in $\mathbb{F}$ that leave the form B invariant. Then $U(p,q,\mathbb{F})$ acts by automorphisms on $N(p,q,\mathbb{F})$ by

$$g\cdot(v,\zeta) = (gv,\zeta).$$

In [D-M] it is proved that $(U(p,q,\mathbb{F}) \ltimes N(p,q,\mathbb{F}), N(p,q,\mathbb{F}))$, where $\ltimes$ denotes semidirect product, is a generalized Gelfand pair, and thus the algebra $\mathcal{D}(N(p,q,\mathbb{F}))$ of left invariant and $U(p,q,\mathbb{F})$-invariant differential operators on $N(p,q,\mathbb{F})$ is commutative (see [D]).

In this paper we obtain explicit fundamental solutions for some generators of this algebra. Recall that a *fundamental solution* for a differential operator $\mathcal{L}$ is a distribution Φ such that for all test functions f, we have $\mathcal{L}(f*\Phi) = (\mathcal{L}f)*\Phi = f*\mathcal{L}(\Phi) = f$. So the operator K defined by $Kf = f*\Phi$ satisfies $K\circ\mathcal{L}f = \mathcal{L}\circ Kf = f$.

If $\mathbb{F}=\mathbb{C}$ and $\{X_1,\dots,X_n,Y_1,\dots,Y_n,U\}$ denotes the standard basis of the Heisenberg Lie algebra with $[X_i,Y_j] = \delta_{ij}U$ and all the other brackets zero, then $\mathcal{D}(N(p,q,\mathbb{C}))$ is generated by U and

$$L = \sum_{j=1}^{p}(X_j^2+Y_j^2) - \sum_{j=p+1}^{n}(X_j^2+Y_j^2).$$

A complete description of the spherical distributions associated to this pair is given in [D-M] and [G-S1]. Moreover, for $\lambda\in\mathbb{R}$, $\lambda\neq 0$ and $k\in\mathbb{Z}$, there exists a $U(p,q,\mathbb{C})$-invariant tempered distribution $S_{\lambda,k}$ on $N(p,q,\mathbb{C})$ satisfying

$$LS_{\lambda,k} = -|\lambda|(2k+p-q)S_{\lambda,k}, \qquad iUS_{\lambda,k} = \lambda S_{\lambda,k}. \tag{1.1}$$

Let us consider the operator $\mathcal{L}_\alpha = L + i\alpha U$, where α is a noninteger complex number. To obtain a fundamental solution Φ_α for $\mathcal{L}_\alpha$ we will strongly use the expression of the *inversion formula* for Schwartz functions f on the Heisenberg group, which is given by

$$f(z,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f*S_{\lambda,k}|\lambda|^n\,d\lambda, \qquad (z,t)\in N(p,q,\mathbb{C}). \tag{1.2}$$

Because of (1.1) and (1.2) it is natural to propose as a fundamental solution of $\mathcal{L}_\alpha$,

$$\langle \Phi_\alpha, f\rangle = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{-|\lambda|(2k+p-q-\alpha\operatorname{sgn}\lambda)} \langle S_{\lambda,k}, f\rangle |\lambda|^n \, d\lambda. \tag{1.3}$$

We will see in Theorem 3.1 that Φ_α is a tempered distribution and its expression is obtained in Theorem 3.9. The strategy for the computation is the use of explicit formulas for $S_{\lambda,k}$.

If $\mathbb{F} = \mathbb{H}$ we take $\{X_1^0, X_1^1, X_1^2, X_1^3, \ldots, X_n^0, X_n^1, X_n^2, X_n^3, Z_1, Z_2, Z_3\}$ the canonical basis for the Lie algebra, where Z_1, Z_2, Z_3 generate the center of $\eta(p, q, \mathbb{H})$. Here, the operators

$$L = \sum_{r=1}^{p}\sum_{l=0}^{3} (X_r^l)^2 - \sum_{r=p+1}^{n}\sum_{l=0}^{3} (X_r^l)^2, \qquad U = \sum_{l=1}^{3} Z_l^2,$$

generate the algebra $\mathcal{D}(N(p,q,\mathbb{H}))$.

In this case, the spherical distributions $\varphi_{w,k}$, $w \in \mathbb{R}^3$, $k \in \mathbb{Z}$, were computed in [V] and they satisfy

$$L\varphi_{w,k} = -|w|(2k+2(p-q))\varphi_{w,k}, \qquad U\varphi_{w,k} = -\lambda^2\varphi_{w,k}. \tag{1.4}$$

Since L has a nontrivial kernel, we can only hope to find a relative fundamental solution for L. We recall that if π denotes the orthogonal projection onto the kernel of a differential operator $\mathcal{L}$, a *relative fundamental solution* for $\mathcal{L}$ is a distribution Φ such that

$$\mathcal{L}(f * \Phi) = (\mathcal{L}f) * \Phi = f * \mathcal{L}(\Phi) = f - \pi(f)$$

for all test functions f.

In order to obtain a (relative) fundamental solution Φ for the operator L we will use the fact that the family $\{\varphi_{w,k}\}$ also provides an inversion formula (see [R]): for all $f \in \mathcal{S}(N(p,q,\mathbb{H}))$ we have

$$f(\alpha, z) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^3} (f * \varphi_{w,k})(\alpha, z)|w|^{2n} \, dw, \qquad (\alpha, z) \in N(p,q,\mathbb{H}). \tag{1.5}$$

Because of (1.4) and (1.5) we propose as a relative fundamental solution of L,

$$\langle \Phi, f\rangle = \sum_{k\in\mathbb{Z},\, k\neq(q-p)} \int_{\mathbb{R}^3} \frac{1}{-|w|(2k+2(p-q))} \langle \varphi_{w,k}, f\rangle |w|^{2n} \, dw. \tag{1.6}$$

The explicit form of Φ is given in Theorem 4.1, and for its computation we use the Radon transform in order to reduce this case to the classical one.

We remark that for $q = 0$, $\mathbb{F} = \mathbb{C}$ we recover the fundamental solution for the operator $\mathcal{L}_\alpha$ given in [F-S], and for $q = 0$, $\mathbb{F} = \mathbb{H}$ we recover Kaplan's fundamental solution for the operator L given in [K]. The case $q \neq 0$, $\alpha = 0$ was obtained in [G-S2].

2. Preliminaries. In order to describe both families of eigendistributions $\{S_{\lambda,k}\}$ and $\{\varphi_{w,k}\}$ we need to adapt a result by Tengstrand [T]. We describe the elements for $\mathbb{F}=\mathbb{C}$, the other case being similar. First of all we take bipolar coordinates on $\mathbb{C}^n$: for $(x_1,y_1,\dots,x_n,y_n)$ we set

$$\tau=\sum_{j=1}^{p}(x_j^2+y_j^2)-\sum_{j=p+1}^{n}(x_j^2+y_j^2),\qquad \rho=\sum_{j=1}^{n}(x_j^2+y_j^2),$$
$$u=(x_1,y_1,\dots,x_p,y_p),\qquad v=(x_{p+1},y_{p+1},\dots,x_n,y_n).$$

Hence $u=\left(\frac{\rho+\tau}{2}\right)^{1/2}\omega_u$ with $\omega_u\in S^{2p-1}$, and $v=\left(\frac{\rho-\tau}{2}\right)^{1/2}\omega_v$ with $\omega_v\in S^{2q-1}$. It is easy to see by changing variables that

$$\int_{\mathbb{C}^n} f(z)\,dz=\int_{-\infty}^{\infty}\int_{\rho>|\tau|}\int_{S^{2p-q}\times S^{2q-1}} f\left(\left(\frac{\rho+\tau}{2}\right)^{1/2}\omega_u,\left(\frac{\rho-\tau}{2}\right)^{1/2}\omega_v\right)d\omega_u\,d\omega_v \times(\rho+\tau)^{p-1}(\rho-\tau)^{q-1}\,d\rho\,d\tau.$$

Then for $f\in\mathcal{S}(\mathbb{R}^{2n})$ we define

$$Mf(\rho,\tau)=\int_{S^{2p-1}\times S^{2q-1}} f\left(\left(\frac{\rho+\tau}{2}\right)^{1/2}\omega_u,\left(\frac{\rho-\tau}{2}\right)^{1/2}\omega_v\right)d\omega_u\,d\omega_v,$$

and also

$$Nf(\tau)=\int_{|\tau|}^{\infty} Mf(\rho,\tau)(\rho+\tau)^{p-1}(\rho-\tau)^{q-1}\,d\rho.$$

Let us now define $\mathcal{H}_n$ to be the space of functions $\varphi:\mathbb{R}\to\mathbb{C}$ such that $\varphi(\tau)=\varphi_1(\tau)+\tau^{n-1}\varphi_2(\tau)H(\tau)$ for $\varphi_1,\varphi_2\in\mathcal{S}(\mathbb{R})$, where H denotes the Heaviside function. In [T] it is proved that $\mathcal{H}_n$ with a suitable topology is a Fréchet space, and following the same lines we can see that the linear maps $N:\mathcal{S}(\mathbb{R}^{2n}-\{0\})\to\mathcal{S}(\mathbb{R})$ and $N:\mathcal{S}(\mathbb{R}^{2n})\to\mathcal{H}$ are continuous and surjective.

Let us now consider $\mu\in\mathcal{S}'(\mathbb{R}^{2n})^{U(p,q)}$; then it is easy to see that there exists a unique $T\in\mathcal{S}'(\mathbb{R})$ such that $\langle\mu,f\rangle=\langle T,Nf\rangle$ for all $f\in\mathcal{S}(\mathbb{R}^{2n}-\{0\})$. Moreover, if $N':\mathcal{H}'\to\mathcal{S}'(\mathbb{R}^{2n})$ is the adjoint map, by following again the arguments of [T] we can see that N' is a homeomorphism. Finally, for a function $f\in\mathcal{S}(N(p,q,\mathbb{C}))$, we write $Nf(\tau,t)$ for $N(f(\cdot,t))(\tau)$.

The distributions $S_{\lambda,k}$ are defined as follows:

$$S_{\lambda,k}=\sum_{m\in\mathbb{N}_0^n,\,B(m)=k}E_\lambda(h_m,h_m),\tag{2.1}$$

where $B(m)=\sum_{j=1}^{p}m_j-\sum_{j=p+1}^{n}m_j$, the set of functions $\{h_m\}\subset L^2(\mathbb{R}^n)$ is the normalized Hermite basis, and $E_\lambda(h,h')(z,t)=\langle\pi_\lambda(z,t)h,h'\rangle$ are the matrix entries of the Schrödinger representation π_λ. Also, $S_{\lambda,k}=e^{-i\lambda t}\otimes F_{\lambda,k}$, where each $F_{\lambda,k}\in\mathcal{S}'(\mathbb{C}^n)^{U(p,q)}$ is a tempered distribution defined in terms of the Laguerre polynomials L_k^m and the map N as follows: for $g \in \mathcal{S}(\mathbb{C}^n)$, $\lambda \neq 0$, and $k \in \mathbb{Z}$, if $k \geq 0$ then

$$\langle F_{\lambda,k}, g\rangle = \Big\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|}e^{-\tau/2}Ng\Big(\frac{2}{|\lambda|}\tau\Big)\Big\rangle, \tag{2.2}$$

and if $k < 0$ then

$$\langle F_{\lambda,k}, g\rangle = \Big\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto \frac{2}{|\lambda|}e^{-\tau/2}Ng\Big(-\frac{2}{|\lambda|}\tau\Big)\Big\rangle. \tag{2.3}$$

For the quaternionic case we consider the Schrödinger representation π_w as given in [R] (see also [K-R]):

$$\pi_w(\alpha, z) = \pi_{|w|}(\alpha, \langle z, w/|w|\rangle), \tag{2.4}$$

where $\pi_{|w|}$ is the Schrödinger representation for the classical Heisenberg group $N(2p, 2q, \mathbb{C})$. Analogously, the distributions $\varphi_{w,k}$ are defined by

$$\varphi_{w,k} = \sum_{m\in\mathbb{N}_0^{2n},\, B(m)=k} E_w(h_m, h_m), \tag{2.5}$$

where $B(m) = \sum_{j=1}^{2p} m_j - \sum_{j=2p+1}^{2n} m_j$, and $E_w(h, h')(\alpha, z) = \langle \pi_w(\alpha, z)h, h'\rangle$ are the matrix entries of the Schrödinger representation π_w. Moreover, we have $\varphi_{w,k} = e^{i\langle w,z\rangle} \otimes \theta_{w,k}$, where $\theta_{w,k}$ is a tempered distribution such that $\theta_{w,k} = N'T_{|w|,k}$, where if we set $\lambda = |w|$, we have $T_{|w|,k} = F_{\lambda,k}$, replacing n, p, q by $2n, 2p, 2q$, respectively, in (2.2) and (2.3). Observe that if we define

$$\varphi_{\lambda,k}(\alpha, z) = \int_{S^2} e^{i\langle z,\lambda\xi\rangle}\, d\xi\, \theta_{\lambda,k}(\alpha), \tag{2.6}$$

these distributions are $\mathrm{Spin}(3) \otimes U(p, q, \mathbb{H})$-invariant.

3. A fundamental solution for the operator $\mathcal{L}_\alpha$. We know that Φ_α defined as in (1.3) is a well defined tempered distribution, and a fundamental solution for $\mathcal{L}_\alpha$. We include the proof since a misprint in Lemma 1 of [M-R1] is used in the proof of Lemma 2.10 of [B-D-R].

We will consider $\alpha \in \mathbb{C}$ such that $2k + p - q \pm \alpha \neq 0$ for all $k \in \mathbb{Z}$.

THEOREM 3.1. *Φ_α defined as in (1.3) is a well defined tempered distribution and it is a fundamental solution for the operator $\mathcal{L}_\alpha$.*

Proof. From (1.3) and (2.1) we can write

$$\begin{aligned}|\langle \Phi_\alpha, f\rangle| &\leq \sum_{k\in\mathbb{Z}} \int_0^\infty \bigg(\bigg|\frac{\langle S_{-\lambda,k}, f\rangle}{2k+p-q+\alpha}\bigg| + \bigg|\frac{\langle S_{\lambda,k}, f\rangle}{2k+p-q-\alpha}\bigg|\bigg)|\lambda|^{n-1}\, d\lambda \\ &\leq \sum_{k\in\mathbb{Z}} \int_0^\infty \sum_{\substack{\beta\in\mathbb{N}_0^n \\ B(\beta)=k}} \bigg(\bigg|\frac{\langle E_{-\lambda}(h_\beta, h_\beta), f\rangle}{2k+p-q+\alpha}\bigg| + \bigg|\frac{\langle E_{\lambda}(h_\beta, h_\beta), f\rangle}{2k+p-q-\alpha}\bigg|\bigg)|\lambda|^{n-1}\, d\lambda.\end{aligned}$$

From the known facts that

$$\sum_{k\in\mathbb{Z}}\sum_{\substack{\beta\in\mathbb{N}_0^n\\ B(\beta)=k}} p(\beta)=\sum_{k\geq 0}\binom{k+n-1}{n-1}p(k),$$

$$|\langle E_\lambda(h_\beta,h_\beta),f\rangle|=|\langle\pi_\lambda(f)h_\beta,h_\beta\rangle|\leq\|f\|_{L^1(N(p,q,\mathbb{C}))},$$

and that for $m\in\mathbb{N}$,

$$\pi_\lambda(f)h_\beta=\frac{1}{(-1)^m|\lambda|^m(2B(\beta)+p-q+\alpha\operatorname{sgn}(\lambda))^m}\pi_\lambda(L^m f)h_\beta,$$

we get

$$\begin{aligned}|\langle\Phi_\alpha,f\rangle|&\leq\|L^m f\|_{L^1(N(p,q,\mathbb{C}))}\\ &\times\sum_{k\geq 0}\int_0^\infty\binom{k+n-1}{k}\Big(\frac{|\lambda|^{n-1-m}}{|2k+p-q+\alpha|^{m+1}}+\frac{|\lambda|^{n-1-m}}{|2k+p-q-\alpha|^{m+1}}\Big)\,d\lambda.\end{aligned}$$

Let us consider the first term, the second one being analogous. We split the integral between $|\lambda|\,|2k+p-q+\alpha|\geq 1$ and $0\leq|\lambda|\,|2k+p-q+\alpha|\leq 1$. Now

$$\sum_{k\geq 0}\binom{k+n-1}{k}\int\limits_{|\lambda|\,|2k+p-q+\alpha|\geq 1}\frac{1}{|2k+p-q+\alpha|^{m+1}}|\lambda|^{n-1-m}\,d\lambda$$

is finite if we take $m>n$, and

$$\sum_{k\geq 0}\binom{k+n-1}{k}\int\limits_{0\leq|\lambda|\,|2k+p-q+\alpha|\leq 1}\frac{1}{|2k+p-q+\alpha|^{m+1}}|\lambda|^{n-1-m}\,d\lambda$$

is finite for any natural number m. From the above computations it also follows that Φ_α is a tempered distribution. Next we see that it is a fundamental solution by writing $L=L_0+L_1$, where in coordinates

$$\begin{aligned}L_0&=\frac{1}{4}\Big(\sum_{j=1}^{p}(x_j^2+y_j^2)-\sum_{j=p+1}^{n}(x_j^2+y_j^2)\Big)\frac{\partial^2}{\partial t^2}\\ &\quad+\sum_{j=1}^{p}\Big(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\Big)-\sum_{j=p+1}^{n}\Big(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2}\Big),\\ L_1&=\frac{\partial}{\partial t}\sum_{j=1}^{n}\Big(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\Big).\end{aligned}$$

Then, as L_0,L_1 and T commute with left translations and also $L_0(g^\vee)=(L_0g)^\vee$, $L_1(g^\vee)=-(L_1g)^\vee$ and $T(g^\vee)=-(Tg)^\vee$, we get

$$(\mathcal{L}f*\Phi_\alpha)(z,t)=\langle\Phi_\alpha,(L_{(z,t)^{-1}}\mathcal{L}f)^\vee\rangle=\langle\Phi_\alpha,(L_0-i\alpha)(L_{(z,t)^{-1}}f)^\vee\rangle,$$

because $L_1\Phi_\alpha = 0$. Hence,

$$(\mathcal{L}_\alpha f * \Phi)(z,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\langle S_{\lambda,k}, (L_0 - i\alpha T)(L_{(z,t)^{-1}} f)^\vee\rangle}{-|\lambda|(2k+p-q-\alpha\,\mathrm{sgn}\,\lambda)} |\lambda|^{n-1}\, d\lambda$$
$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \frac{\langle (L_0 + i\alpha T)S_{\lambda,k}, (L_{(z,t)^{-1}} f)^\vee\rangle}{-|\lambda|(2k+p-q-\alpha\,\mathrm{sgn}\,\lambda)} |\lambda|^{n-1}\, d\lambda$$
$$= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \langle S_{\lambda,k}, (L_{(z,t)^{-1}} f)^\vee\rangle |\lambda|^{n-1}\, d\lambda = f(z,t),$$

by the inversion formula. The other equality, $f * \mathcal{L}_\alpha(f) = f$, is immediate. ∎

Now we proceed with the computation of Φ_α. Since the series (1.3) defining Φ_α converges absolutely, we can split the sum over $k \in \mathbb{Z}$ into the sums for $k \ge q$, for $k \le -p$ and for $-p < k < q$. In the first case we change the summation index writing $k = k' + q$, and in the second we write $k = k' - p$. So we get

$$\langle \Phi_\alpha, f\rangle = (-1) \sum_{k'\ge 0} \frac{1}{2k'+n-\alpha} \int_0^\infty [\langle S_{\lambda,k'+q}, f\rangle - \langle S_{\lambda,-k'-p}, f\rangle] |\lambda|^{n-1}\, d\lambda$$
$$+ (-1) \sum_{k'\ge 0} \frac{1}{2k'+n+\alpha} \int_0^\infty [\langle S_{-\lambda,k'+q}, f\rangle - \langle S_{-\lambda,-k'-p}, f\rangle] |\lambda|^{n-1}\, d\lambda$$
$$+ (-1) \sum_{-p<k<q} \int_0^\infty \left(\frac{\langle S_{-\lambda,k}, f\rangle}{2k+p-q+\alpha} + \frac{\langle S_{\lambda,k}, f\rangle}{2k+p-q-\alpha} \right) |\lambda|^{n-1}\, d\lambda.$$

By Abel's Lemma and the Lebesgue Dominated Convergence Theorem we can write $\Phi_\alpha = \Phi_1 + \Phi_2$ where

$$\langle \Phi_1, f\rangle = \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k'\ge 0} \frac{r^{2k'+n-\alpha}}{2k'+n-\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \times [\langle S_{\lambda,k'+q}, f\rangle - \langle S_{\lambda,-k'-p}, f\rangle] |\lambda|^{n-1}\, d\lambda$$
$$+ \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} (-1) \sum_{k'\ge 0} \frac{r^{2k'+n+\alpha}}{2k'+n+\alpha} \int_0^\infty e^{-\epsilon|\lambda|} \times [\langle S_{-\lambda,k'+q}, f\rangle - \langle S_{-\lambda,-k'-p}, f\rangle] |\lambda|^{n-1}\, d\lambda, \tag{3.1}$$

$$\langle \Phi_2, f\rangle = \lim_{\epsilon\to 0^+} (-1) \sum_{-p<k<q} \int_0^\infty e^{-\epsilon|\lambda|} \times \left(\frac{\langle S_{-\lambda,k}, f\rangle}{2k+p-q+\alpha} + \frac{\langle S_{\lambda,k}, f\rangle}{2k+p-q-\alpha} \right) |\lambda|^{n-1}\, d\lambda. \tag{3.2}$$

Using that $S_{\lambda,k} = e^{-i\lambda t} \otimes F_{\lambda,k}$ and the computations from [G-S2, (2.6) to (2.9)], we get

$$\begin{aligned}\langle \Phi_1, f\rangle = &\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}(-1)\sum_{k\geq 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty e^{-\epsilon|\lambda|}\int_{-\infty}^\infty e^{-i\lambda t}\\ &\times\left\langle (L^0_{k+n-1}H)^{(n-1)}, \frac{2}{|\lambda|}e^{-\tau/2}\left[Nf\left(\frac{2}{|\lambda|}\tau, t\right) - Nf\left(-\frac{2}{|\lambda|}\tau, t\right)\right]\right\rangle dt\, d\lambda\\ &+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}(-1)\sum_{k\geq 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty e^{-\epsilon|\lambda|}\int_{-\infty}^\infty e^{i\lambda t}\\ &\times\left\langle (L^0_{k+n-1}H)^{(n-1)}, \frac{2}{|\lambda|}e^{-\tau/2}\left[Nf\left(\frac{2}{|\lambda|}\tau, t\right) - Nf\left(-\frac{2}{|\lambda|}\tau, t\right)\right]\right\rangle dt\, d\lambda.\end{aligned}$$

Thus setting

$$b_{k,l} = \sum_{j=l}^{n-2}\binom{j}{l}\left(\frac{1}{2}\right)^{2-l}(-1)^{n-j}\binom{k+n-1}{n-j-2}, \tag{3.3}$$

we have

$$\begin{aligned}\langle \Phi_1, f\rangle = &\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty e^{-\epsilon|\lambda|}\int_{-\infty}^\infty e^{-i\lambda t}\\ &\times\Bigg[(-1)^n\int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|s|\right)e^{-\frac{|\lambda|}{4}|s|}\operatorname{sgn}(s)Nf(s,t)\, ds\\ &- 2\sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{n-2}\left(\frac{2}{|\lambda|}\right)^{l+1} b_{k,l}\frac{\partial^l Nf}{\partial\tau^l}(0,t)\Bigg]\, dt\, d\lambda\\ &+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\geq 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty e^{-\epsilon|\lambda|}\int_{-\infty}^\infty e^{i\lambda t}\\ &\times\Bigg[(-1)^n\int_{-\infty}^\infty L_k^{n-1}\left(\frac{|\lambda|}{2}|s|\right)e^{-\frac{|\lambda|}{4}|s|}\operatorname{sgn}(s)Nf(s,t)\, ds\\ &- 2\sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{n-2}\left(\frac{2}{|\lambda|}\right)^{l+1} b_{k,l}\frac{\partial^l Nf}{\partial\tau^l}(0,t)\Bigg]\, dt\, d\lambda.\end{aligned}$$

Now we define

$$G_f(\tau, t) = Nf(\tau, t) - \sum_{j=0}^{n-2}\frac{\partial^j Nf}{\partial\tau^j}(0,t)\frac{\tau^j}{j!}, \tag{3.4}$$

and we split $\Phi_1 = \Phi_{11} + \Phi_{12}$, where

(3.5)
$$\langle \Phi_{11}, f\rangle = \lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\ge 0}(-1)^n \frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}|\lambda|^{n-1} \times \int_{-\infty}^\infty L_k^{n-1}\Big(\frac{|\lambda|}{2}|\tau|\Big)e^{-\frac{|\lambda|}{4}|\tau|}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt\,d\lambda$$
$$+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\ge 0}(-1)^n \frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{i\lambda t}|\lambda|^{n-1} \times \int_{-\infty}^\infty L_k^{n-1}\Big(\frac{|\lambda|}{2}|\tau|\Big)e^{-\frac{|\lambda|}{4}|\tau|}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt\,d\lambda,$$

and

(3.6)
$$\langle \Phi_{12}, f\rangle = \lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\ge 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}|\lambda|^{n-1} \times 2\sum_{\substack{l=0\\ l\,\text{odd}}}^{n-2}\Big(\frac{2}{|\lambda|}\Big)^{l+1}(a_{k,l}+b_{k,l})\frac{\partial^l Nf}{\partial\tau^l}(0,t)\,dt\,d\lambda$$
$$+\lim_{r\to 1^-}\lim_{\epsilon\to 0^+}\sum_{k\ge 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{i\lambda t}|\lambda|^{n-1} \times 2\sum_{\substack{l=0\\ l\,\text{odd}}}^{n-2}\Big(\frac{2}{|\lambda|}\Big)^{l+1}(a_{k,l}+b_{k,l})\frac{\partial^l Nf}{\partial\tau^l}(0,t)\,dt\,d\lambda,$$

with

(3.7)
$$a_{k,l} = (-1)^n\frac{1}{l!}\int_0^\infty L_k^{n-1}(s)e^{-s/2}s^l\,ds.$$

We will show that Φ_{11} is well defined. We have proved that the series (1.3) defining Φ_α converges and, as Φ_2 is a finite sum, we will deduce that Φ_{12} is also well defined.

Proposition 3.2. *The following identities hold:*

(i)
$$\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}L_k^{n-1}\Big(\frac{|\lambda|}{2}|\tau|\Big)e^{-\frac{|\lambda|}{4}|\tau|}|\lambda|^{n-1}\,d\lambda = 4^n(n-1)!(-1)^n\binom{k+n-1}{k}\frac{(|\tau|-4\epsilon-4it)^k}{(|\tau|+4\epsilon+4it)^{k+n}};$$

(ii)
$$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}\Big(\frac{(|\tau|-4it-4\epsilon)^k}{(|\tau|+4it+4\epsilon)^{k+n}}\Big)\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt = \int_{\mathbb{R}^2}\frac{1}{(|\tau|-4it)^{n/2-\alpha/2}}\,\frac{1}{(|\tau|+4it)^{n/2+\alpha/2}}\Big(\frac{|\tau|-4it}{\tau^2+16t^2}\Big)^{2k+n-\alpha} \times\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt;$$

(iii) $$\lim_{\epsilon\to 0^+}\int_{\mathbb{R}^2}\left(\frac{(|\tau|+4it-4\epsilon)^k}{(|\tau|-4it+4\epsilon)^{k+n}}\right)\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt$$
$$=\int_{\mathbb{R}^2}\frac{1}{(|\tau|-4it)^{n/2-\alpha/2}}\,\frac{1}{(|\tau|+4it)^{n/2+\alpha/2}}\left(\frac{|\tau|-4it}{\tau^2+16t^2}\right)^{2k+n+\alpha}$$
$$\times\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt.$$

Proof. From (4.9) of [G-S2] we deduce that (i) follows from the generating identity for the Laguerre polynomials,

$$\sum_{k\geq 0}L_k^{n-1}(t)z^k=\frac{1}{(1-z)^n}e^{-\frac{zt}{1-z}}. \tag{3.8}$$

From Lemma 2.2 of [G-S2], which states that the function $\frac{G_f(\tau,t)}{(\tau^2+16t^2)^{n/2}}$ is integrable in $\mathbb{R}^2$, and from the fact that

$$\left|\frac{1}{(|\tau|-4it)^{-\alpha/2}}\right|\left|\frac{1}{(|\tau|+4it)^{\alpha/2}}\right|=1,$$

it follows that the function

$$\frac{1}{(|\tau|-4it)^{n/2-\alpha/2}}\,\frac{1}{(|\tau|+4it)^{n/2+\alpha/2}}G_f(\tau,t)$$

is integrable in $\mathbb{R}^2$. So we get (ii). For (iii) we just change $e^{-i\lambda t}$ to $e^{i\lambda t}$ and argue as for (ii). ∎

Then, by Proposition 3.2, we obtain

$$\langle\Phi_{11},f\rangle=\beta_n\lim_{r\to 1^-}\sum_{k\geq 0}\frac{r^{2k+n-\alpha}}{2k+n-\alpha}\alpha_k$$
$$\times\int_{\mathbb{R}^2}\left(\frac{|\tau|-4it}{\tau^2+16t^2}\right)^{2k+n-\alpha}\frac{\operatorname{sgn}(\tau)G_f(\tau,t)}{(|\tau|-4it)^{n/2-\alpha/2}(|\tau|+4it)^{n/2+\alpha/2}}\,d\tau\,dt$$
$$+\beta_n\lim_{r\to 1^-}\sum_{k\geq 0}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}\alpha_k$$
$$\times\int_{\mathbb{R}^2}\left(\frac{|\tau|+4it}{\tau^2+16t^2}\right)^{2k+n+\alpha}\frac{\operatorname{sgn}(\tau)G_f(\tau,t)}{(|\tau|+4it)^{n/2+\alpha/2}(|\tau|-4it)^{n/2-\alpha/2}}\,d\tau\,dt,$$

where $\beta_n=4^n(n-1)!(-1)^n$ and $\alpha_k=\binom{k+n-1}{k}(-1)^k$.

To study $\langle\Phi_{11},f\rangle$ we split each integral into integrals over the left and right halfplanes and take polar coordinates $\tau-4it=\rho e^{i\theta}$ to obtain

$$\begin{aligned}
\langle \Phi_{11}, f\rangle &= \beta_n \lim_{r\to 1^-} \sum_{k\ge 0} \alpha_k \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \\
&\quad\times \int_0^\infty \Bigg[\int_{-\pi/2}^{\pi/2} e^{i(2k+n-\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha\theta} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\Big)\, d\theta \\
&\qquad + \int_{\pi/2}^{3\pi/2} \frac{e^{-i(2k+n-\alpha)\theta} e^{-i\alpha\theta}}{(-1)^n 4\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\Big)\, d\theta\Bigg] d\rho \\
&\quad + \beta_n \lim_{r\to 1^-} \sum_{k\ge 0} \alpha_k \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \\
&\quad\times \int_0^\infty \Bigg[\int_{-\pi/2}^{\pi/2} e^{-i(2k+n+\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha\theta} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\Big)\, d\theta \\
&\qquad + \int_{\pi/2}^{3\pi/2} \frac{e^{i(2k+n+\alpha)\theta} e^{-i\alpha\theta}}{(-1)^n 4\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\Big)\, d\theta\Bigg] d\rho.
\end{aligned}$$

Now we change variables in the second and fourth terms via $\theta \leftrightarrow -\theta$. Then, in the fourth term we change variables again according to $\theta \leftrightarrow \theta + 2\pi$. By Proposition 3.2 we can change the integration order, so we can write

$$\begin{aligned}
\langle \Phi_{11}, f\rangle &= \beta_n \lim_{r\to 1^-} \int_0^\infty \int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} \\
&\quad\times \Bigg[\sum_{k\ge 0} \alpha_k \Big(\frac{r^{2k+n-\alpha}}{2k+n-\alpha} e^{i(2k+n-\alpha)\theta} + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}\Big)\Bigg] \\
&\qquad\times \frac{1}{\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\Big)\, d\theta\, d\rho \\
&\quad + \frac{(-1)^n}{4} \beta_n \lim_{r\to 1^-} \int_0^\infty \int_{\pi/2}^{3\pi/2} e^{i\alpha\theta} \\
&\quad\times \Bigg[\sum_{k\ge 0} \alpha_k \Big(\frac{r^{2k+n-\alpha}}{2k+n-\alpha} e^{i(2k+n-\alpha)\theta} + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}\Big)\Bigg] \\
&\qquad\times \frac{1}{\rho^{n-1}} \operatorname{sgn}(\cos\theta) G_f\Big(\rho\cos\theta, \frac{\rho}{4}\sin\theta\Big)\, d\theta\, d\rho.
\end{aligned}$$

Let I denote the real interval $[-\pi/2, \pi/2]$. Consider the vector space

$$\mathcal{X} = \{g \in C^{n-2}(I) : g^{(j)}(\pm\pi/2) = 0,\ 0 \le j \le n-2,\ g^{(n-1)} \in L^\infty(I)\}.$$

We identify each function $g \in \mathcal{X}$ with a function $\widetilde{g}$ on $S^1 = \mathbb{R}/\mathbb{Z}$, defined to be equal to 0 outside $\operatorname{supp}(g)$, and we make no distinction between g and $\widetilde{g}$. Thus, if $g \in \mathcal{X}$ then $g \in C^{n-2}(S^1)$ with $g^{(n-1)} \in L^\infty(S^1)$. Observe that if $g \in \mathcal{X}$, then also $e^{i\alpha\theta} g \in \mathcal{X}$. The topology on $\mathcal{X}$ is given by $\|g\|_{\mathcal{X}} = \max_{0 \le j \le n-1} \|g^{(j)}\|_\infty$.

For $k \in \mathbb{Z}$ we set $\alpha_k = \binom{k+n-1}{k}(-1)^k$. Now let us define

$$\Psi_{r,\alpha}(\theta) = \sum_{k \ge 0} \alpha_k \left(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right), \tag{3.9}$$

$$\langle \Psi_\alpha, g \rangle = \left\langle \sum_{k \ge 0} \alpha_k \left(\frac{e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right), g \right\rangle. \tag{3.10}$$

We prove that $\Psi_\alpha \in \mathcal{X}'$, the dual space of $\mathcal{X}$. Indeed,

$$|\langle \Psi_\alpha, g \rangle| \le |e^{i\alpha\theta}| \sum_{k \ge 0} \binom{k+n-1}{k} \left(\frac{|\langle e^{i(2k+n)\theta}, g \rangle|}{|2k+n-\alpha|} + \frac{\langle e^{-i(2k+n)\theta}, g \rangle|}{|2k+n+\alpha|} \right). \tag{3.11}$$

If $\widehat{g}(n) = \langle g, e^{in\theta} \rangle$ denotes the nth Fourier coefficient of g, then

$$\begin{aligned} |\langle \Psi_\alpha, g \rangle| &\le c \sum_{k \ge 0} \frac{k^{n-1}}{|2k+n|^{n-1}} \left(\frac{|\widehat{g^{(n-1)}}(2k+n)|}{|2k+n-\alpha|} + \frac{|\widehat{g^{(n-1)}}(-2k-n)|}{|2k+n+\alpha|} \right) \\ &\le c \sum_{k \ge 0} \frac{1}{k} |\widehat{g^{(n-1)}}(2k+n)| + \frac{1}{k} |\widehat{g^{(n-1)}}(-2k-n)| \\ &\le c \left(\sum_{k \ge 0} \frac{1}{k^2} \right)^{1/2} \|\widehat{g^{(n-1)}}\|_{L^2}, \end{aligned}$$

by the Cauchy–Schwarz inequality. Observe that the constants c are not the same in each expression. By Abel's Lemma, $\lim_{r \to 1^-} \Psi_{r,\alpha} = \Psi_\alpha$ in $\mathcal{X}'$, that is, with respect to the weak convergence topology. Similarly, if J denotes the real interval $[\pi/2, 3\pi/2]$, we define the space

$$\mathcal{Y} = \{ g \in C^{n-2}(J) : g^{(j)}(\pi/2) = g^{(j)}(3\pi/2) = 0,\ 0 \le j \le n-2, \\ g^{(n-1)} \in L^\infty(J) \},$$

and find that Ψ_α is well defined in $\mathcal{Y}'$ and $\lim_{r \to 1^-} \Psi_{r,\alpha} = \Psi_\alpha$ in $\mathcal{Y}'$.

Our aim now is to compute Ψ_α. From Proposition 3.7 of [G-S2] we know that if $\Theta \in \mathcal{D}'(S^1)$ is defined by

$$\Theta(\theta) = i \sum_{k \ge 0} \binom{k+n-1}{k} (-1)^k e^{i(2k+n)\theta}, \tag{3.12}$$

then for n even we have

$$\mathfrak{Re}\,\Theta(\theta) = \frac{d}{d\theta}Q_{n-2}\left(\frac{d}{d\theta}\right)(\delta_{\pi/2} + \delta_{-\pi/2}) = \sum_{j=0}^{n-2} c_j(\delta_{\pi/2}^{(j+1)} + \delta_{-\pi/2}^{(j+1)}), \tag{3.13}$$

where Q_{n-2} is a polynomial of degree $n-2$; and for n odd we have

$$\begin{aligned}\mathfrak{Re}\,\Theta(\theta) &= d_0\frac{d}{d\theta}\widetilde{H} + \frac{d}{d\theta}Q_{n-2}\left(\frac{d}{d\theta}\right)(\delta_{\pi/2} - \delta_{-\pi/2}) \\ &= d_0(\delta_{-\pi/2} - \delta_{\pi/2}) + \sum_{j=0}^{n-2} c_j(\delta_{\pi/2}^{(j+1)} - \delta_{-\pi/2}^{(j+1)}),\end{aligned} \tag{3.14}$$

where Q_{n-2} is a polynomial of degree $n-2$, and $\widetilde{H}(\theta) = H(\cos\theta)$. Let us recall the generating identity for the Laguerre polynomials (3.8), and take $t = 0$ and $z = -r^2e^{2i\theta}$. We get

$$\sum_{k\geq 0}\binom{k+n-1}{k}(-1)^k r^{2k+n}e^{i(2k+n)\theta} = \left(\frac{re^{i\theta}}{1+r^2e^{2i\theta}}\right)^n. \tag{3.15}$$

We also need a couple of results:

LEMMA 3.3. *For a fixed $r > 1$ the functions $\alpha \mapsto \Psi_{r,\alpha}(0)$ and $\alpha \mapsto \lim_{r\to 1^-}\Psi_{r,\alpha}(0)$ are analytic on $\Omega = \mathbb{C}\setminus F$, where $F = \{2k+n : k \in \mathbb{Z}\}$.*

Proof. Let $K \subset \Omega$ be a compact set. It is easy to see that for fixed r the series (3.9) converges uniformly, since

$$|\Psi_{r,\alpha}(0)| \leq \max_{\alpha\in K}|r^\alpha|\left(\frac{r}{1+r^2}\right)^n d(K,F).$$

Also, for $\alpha \in \Omega$ the limit $\lim_{r\to 1^-}\Psi_{r,\alpha}(0)$ exists. Indeed, if $0 \leq r_1 < r < r_2 < 1$, from the Mean Value Theorem we deduce that for some $\xi \in (r_1, r_2)$,

$$\begin{aligned}\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0) &= \frac{d}{dr}\Psi_{\xi,\alpha}(0)(r_2 - r_1) \\ &= (\xi^{-\alpha-1} + \xi^{\alpha-1})\sum_{k\geq 0}\alpha_k\xi^{2k+n}(r_2 - r_1) \\ &= (\xi^{-\alpha-1} + \xi^{\alpha-1})\left(\frac{\xi}{1+\xi^2}\right)^n(r_2 - r_1),\end{aligned}$$

where the last equality holds by (3.15). Hence

$$|\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0)| \leq c(\xi)|r_2 - r_1|,$$

where $c(\xi)$ is a constant which depends on ξ. Moreover, for $\alpha \in K$ and $\xi \in [1/2, 1]$, $\xi^{n-\alpha-1} + \xi^{n+\alpha-1}$ is bounded in $K \times [1/2, 1]$, so the convergence is uniform, hence $\alpha \mapsto \lim_{r\to 1^-}\Psi_{r,\alpha}(0)$ is an analytic function. ∎

Lemma 3.4. *Let* $0<\delta<\pi/4$. *For* $0<r<1$ *and* $0\le|\theta|<\delta$ *we have*

$$|\Psi_{r,\alpha}(\theta)-\Psi_{r,\alpha}(0)|\le\Big(\max_{0\le|\theta|<\delta}e^{|\mathfrak{Im}\,\alpha|\,|\theta|}\Big)(a|r^{-\alpha}-r^{\alpha}|+b|r^{\alpha}|(1-r))|\theta|,$$

with a,b *positive constants. Also for* $0\le|\theta-\pi|<\delta<\pi/4$,

$$|\Psi_{r,\alpha}(\theta)-\Psi_{r,\alpha}(\pi)|\le\Big(\max_{0\le|\theta-\pi|<\delta}e^{|\mathfrak{Im}\,\alpha|\,|\theta|}\Big)(a|r^{-\alpha}-r^{\alpha}|+b|r^{\alpha}|(1-r))|\theta-\pi|,$$

with a,b *positive constants.*

Proof. We will estimate $|\Psi_{r,\alpha}(\theta)-\Psi_{r,\alpha}(0)|$ for $0<|\theta|<\delta<\pi/4$, the other case being similar. We have

$$\begin{aligned}\frac{d}{d\theta}&\Psi_{r,\alpha}(\theta)\\&=ie^{-i\alpha\theta}\sum_{k\ge0}\alpha_k r^{2k+n}\big((r^{-\alpha}-r^{\alpha})e^{i(2k+n)\theta}+(e^{i(2k+n)\theta}-e^{-i(2k+n)\theta})r^{\alpha}\big)\\&=ie^{-i\alpha\theta}\bigg((r^{-\alpha}-r^{\alpha})\Big(\frac{re^{i\theta}}{1+r^2e^{2i\theta}}\Big)^n+2ir^{\alpha}\,\mathfrak{Im}\Big(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\Big)^n\bigg),\end{aligned}$$

because of (3.15). We have

$$\begin{aligned}(3.16)\qquad&\Big|\frac{d}{d\theta}\Psi_{r,\alpha}(\theta)\Big|\\&\le e^{|\mathfrak{Im}\,\alpha|\,|\theta|}\bigg(|r^{-\alpha}-r^{\alpha}|\,\bigg|\Big(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\Big)^n\bigg|+2|r^{\alpha}|\,\bigg|\mathfrak{Im}\Big(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\Big)^n\bigg|\bigg).\end{aligned}$$

From Proposition 3.1 of [G-S2] we know that

$$\bigg|\mathfrak{Im}\Big(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\Big)^n\bigg|\to0\quad\text{as } r\to1^-,$$

uniformly for $|\theta|<\pi/4$, $|\theta-\pi|<\pi/4$. Also,

$$\bigg|\Big(\frac{re^{i\theta}}{1+r^2e^{i2\theta}}\Big)^n\bigg|\le c$$

for a constant c. Then $\big|\frac{d}{d\theta}\Psi_{r,\alpha}(\theta)\big|\to0$ uniformly on $|\theta|<\pi/4$ as $r\to1^-$, and we get the desired inequality by applying the Mean Value Theorem around 0. ■

Now we can state the following

Proposition 3.5. *For* $f\in\mathcal{X}$ *we have*

$$\langle\Psi_\alpha,f\rangle=C_\alpha\langle1,f\rangle,\quad\textit{where}\quad C_\alpha=\frac{\Gamma\big(\frac{n+\alpha}{2}\big)\Gamma\big(\frac{n-\alpha}{2}\big)}{(n-1)!};$$

and for $f\in\mathcal{Y}$ *we have*

$$\langle\Psi_\alpha,f\rangle=\widetilde{C}_\alpha\langle1,f\rangle,\quad\textit{where}\quad\widetilde{C}_\alpha=(-1)^ne^{-i\alpha\pi}C_\alpha.$$

Proof. First we consider $f \in \mathcal{X}$ such that $\int_{-\pi/2}^{\pi/2} f(t)\,dt = 0$ and we define $F(\theta) = \int_{-\pi/2}^{\theta} f(t)\,dt$. It is easy to see that $F \in \mathcal{X}$ and $F' = f$. By integration by parts,

$$\begin{aligned}\langle \Psi_\alpha, f\rangle = \langle \Psi_\alpha, F'\rangle &= \int_{-\pi/2}^{\pi/2} \sum_{k\geq 0} \alpha_k \Big(\frac{e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha}\Big) F'(\theta)\,d\theta \\ &= -\langle \Theta, e^{-i\alpha\theta}F\rangle - \langle \overline{\Theta}, e^{-i\alpha\theta}F\rangle,\end{aligned}$$

where $\overline{\Theta} = \sum_{k\geq 0}\binom{k+n-1}{k}(-1)^k e^{-i(2k+n)\theta}$. So, if n is even, from (3.13) we get

$$\langle \Psi_\alpha, f\rangle = -\sum_{j=0}^{n-2} c_j \langle \delta_{\pi/2}^{(j+1)} + \delta_{-\pi/2}^{(j+1)}, e^{-i\alpha\theta}F\rangle - \sum_{j=0}^{n-2} \overline{c_j}\langle \overline{\delta_{\pi/2}^{(j+1)}} + \overline{\delta_{-\pi/2}^{(j+1)}}, e^{-i\alpha\theta}F\rangle,$$

and because $\langle \delta_{\pm\pi/2}^{(j+1)}, e^{-i\alpha\theta}F\rangle = 0$ we conclude that $\langle \Psi_\alpha, f\rangle = 0$. If n is odd we use (3.14) to conclude that $\langle \Psi_\alpha, f\rangle = 0$. For a general $f \in \mathcal{X}$ we consider $h \in \mathcal{X}$ such that $\int_{-\pi/2}^{\pi/2} h(t)\,dt = 1$ and define

$$g(\theta) = f(\theta) - \Big(\int_{-\pi/2}^{\pi/2} f(t)\,dt\Big) h(\theta).$$

So we can apply the above result to g and get $\langle \Psi_\alpha, g\rangle = 0$. Then

$$\langle \Psi_\alpha, f\rangle = \langle \Psi_\alpha, g\rangle + \langle \Psi_\alpha, h\rangle\langle 1, f\rangle = \langle \Psi_\alpha, h\rangle\langle 1, f\rangle.$$

Let $C_\alpha = \langle \Psi_\alpha, h\rangle$. In order to compute C_α, consider $g \in \mathcal{X}$ such that $\operatorname{supp}(g) \subset (-\pi/4, \pi/4)$, $\int_{-\pi/4}^{\pi/4} g(t)\,dt = 1$ and $g \geq 0$. We have

$$\langle e^{i\alpha\theta}\Psi_\alpha, g\rangle = C_\alpha \int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} g(\theta)\,d\theta,$$

and also

$$\begin{aligned}&\langle e^{i\alpha\theta}\Psi_\alpha, g\rangle \\ &\quad = \lim_{r\to 1^-}\Big(\int_{-\pi/2}^{\pi/2} (\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)) e^{i\alpha\theta} g(\theta)\,d\theta + \Psi_{r,\alpha}(0)\int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} g(\theta)\,d\theta\Big).\end{aligned}$$

From Lemmas 3.3 and 3.4 we deduce that

$$C_\alpha = \lim_{r\to 1^-} \Psi_{r,\alpha}(0)$$

and also that C_α is an analytic function of α. Since $\Psi_{0,\alpha}(0) = 0$, we can write

$$C_\alpha = \lim_{r\to 1^-} \Psi_{r,\alpha}(0) = \Psi_{1,\alpha}(0) - \Psi_{0,\alpha}(0) = \int_0^1 w'_\alpha(s)\,ds,$$

where

$$w_\alpha(r) = \Psi_{r,\alpha}(0) = r^{-\alpha}\sum_{k\geq 0}\alpha_k \frac{r^{2k+n}}{2k+n-\alpha} + r^{\alpha}\sum_{k\geq 0}\alpha_k \frac{r^{2k+n}}{2k+n+\alpha}.$$

Applying (3.8) with $\theta = 0$ we obtain

$$w'_\alpha(r) = (r^{-\alpha-1} + r^{\alpha-1})\sum_{k\geq 0}\alpha_k r^{2k+n} = (r^{-\alpha-1} + r^{\alpha-1})\left(\frac{r}{1+r^2}\right)^n,$$

and we can compute the integral for $\mathfrak{Re}(n+\alpha) > 0$, $\mathfrak{Re}(n-\alpha) > 0$, obtaining

$$C_\alpha = B\left(\frac{n+\alpha}{2}, \frac{n-\alpha}{2}\right) = \frac{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{(n-1)!}, \tag{3.17}$$

where B is the Beta function and Γ is the Gamma function. By Lemma 3.3, (3.17) holds for $\alpha \in \Omega$ by analytic continuation. In a completely analogous way we conclude that $\widetilde{C}_\alpha = (-1)^n e^{-i\alpha\pi} C_\alpha$. ■

Let us now define

$$K_{1f}(\rho,\theta) = \frac{1}{\rho^{n-1}}\operatorname{sgn}(\cos\theta) G_f\left(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\right) \tag{3.18}$$

for $\theta \in [-\pi/2, \pi/2]$, $0 < \rho < \infty$, where G_f is the function defined in (3.4); and

$$K_{2f}(\rho,\theta) = \frac{1}{\rho^{n-1}}\operatorname{sgn}(\cos\theta) G_f\left(\rho\cos\theta, \frac{\rho}{4}\sin\theta\right) \tag{3.19}$$

for $\theta \in [\pi/2, 3\pi/2]$, $0 < \rho < \infty$.

It is easy to check that $K_{1f}(\rho,\cdot) \in \mathcal{X}$. Recall that we replaced $\tau - 4it$ with $\rho e^{i\theta}$. Since $Nf \in \mathcal{H}_n$, there exists a positive constant c such that

$$\sup_{\tau\neq 0,\, t\in\mathbb{R}} |(\tau^2 + 16t^2) Nf(\tau,t)| \leq c,$$

that is,

$$\left| Nf\left(\rho\cos\theta, -\frac{\rho}{4}\sin\theta\right)\right| \leq \frac{c}{\rho^2}.$$

Also, since $Nf(0,\cdot) \in \mathcal{S}(\mathbb{R})$, there exists a positive constant c_N such that for $t \in \mathbb{R}$,

$$\left| t^N \sum_{j=0}^{n-2} \frac{\partial^j}{\partial\tau^j} Nf(0,t)\frac{\tau^j}{j!}\right| \leq c_N |\tau|^{n-2}.$$

Thus, for $N \in \mathbb{N}$ there exists c_N such that

$$|K_{1f}(\rho,\theta)| \leq \frac{a}{\rho^{n+1}} + \frac{b}{\rho^{N+1}} \frac{|\cos\theta|^{n-2}}{|\sin\theta|^{N}}. \tag{3.20}$$

Analogous observations are also true for K_{2f}.

Proposition 3.6. *Let C_α and $\widetilde{C}_\alpha$ be the constants obtained in* (3.17). *Let K_{1f} and K_{2f} be defined by* (3.18) *and* (3.19), *and $\alpha_k = \binom{k+n-1}{k}(-1)^k$. Then*

$$\lim_{r\to 1^-} \int_0^\infty \int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} \sum_{k\geq 0} \alpha_k \bigg(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \bigg) \times K_{1f}(\rho,\theta)\, d\theta\, d\rho$$

$$= 4^{n-1}(n-1)! C_\alpha \int_{\mathbb{R}} \int_{\tau>0} \frac{1}{(\tau-4it)^{(n-\alpha)/2}} \frac{1}{(\tau+4it)^{(n+\alpha)/2}} \operatorname{sgn}(\tau) G_f(\tau,t)\, d\tau\, dt,$$

and

$$\lim_{r\to 1^-} \int_0^\infty \int_{\pi/2}^{3\pi/2} e^{i\alpha\theta} \sum_{k\geq 0} \alpha_k \bigg(\frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \bigg) \times K_{2f}(\rho,\theta)\, d\theta\, d\rho$$

$$= 4^{n-1}(n-1)! \widetilde{C}_\alpha \int_{\mathbb{R}} \int_{\tau<0} \frac{1}{(\tau-4it)^{(n-\alpha)/2}} \frac{1}{(\tau+4it)^{(n+\alpha)/2}} \operatorname{sgn}(\tau) G_f(\tau,t)\, d\tau\, dt.$$

Proof. The proof follows the same lines of Proposition 4.2 of [G-S2]. We sketch it for the sake of completeness.

Taking polar coordinates $\tau - 4it = \rho e^{i\theta}$ we only need to show that

$$\lim_{r\to 1^-} \int_0^\infty \langle \Psi_{r,\alpha}, e^{i\alpha\theta} K_{1f}(\rho,\theta)\rangle\, d\rho = \int_0^\infty \langle C_\alpha, e^{i\alpha\theta} K_{1f}(\rho,\theta)\rangle\, d\rho. \tag{3.21}$$

In order to do this we split the integral into integrals over $0<\rho<1$ and $1<\rho<\infty$.

We consider first the case $1<\rho<\infty$. For $|\theta| \leq \delta < \pi/4$, set

$$I = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)) K_{1f}(\rho,\theta)\, d\theta\, d\rho$$

$$II = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(0) - C_\alpha) K_{1f}(\rho,\theta)\, d\theta\, d\rho.$$

We bound I close to 0 by applying Lemma 3.4 and taking $N=1$ in (3.20). For II we just take $N = 1/2$ in (3.20). To analyze the case $\delta \leq |\theta| \leq \pi/2$, we observe that the function $K^*_{1f}(\theta) = \int_1^\infty K_{1f}(\rho,\theta)\, d\rho$ defined for $\theta \in$

$[-\pi/2,-\delta]\cup[\delta,\pi/2]$ can be extended to an element of $\mathcal{X}$ that we still denote by K^*_{1f}. Then

$$\int_1^\infty \int_{\delta<|\theta|<\pi/2} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta)-C_\alpha)K_{1f}(\rho,\theta)\,d\theta\,d\rho$$
$$=\int_{-\pi/2}^{\pi/2} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta)-C_\alpha)K^*_{1f}(\theta)\,d\theta-\int_{|\theta|<\delta} e^{i\alpha\theta}(\Psi_{r,\alpha}(\theta)-C_\alpha)K^*_{1f}(\theta)\,d\theta.$$

The first term converges to zero as $r\to 1^-$ since $\Psi_{r,\alpha}\to C_\alpha$ as $r\to 1^-$ in $\mathcal{X}'$. For the second term we argue as above.

Finally, for the case $0<\rho<1$ we apply the same arguments to the function $K^{**}_{1f}(\theta)=\int_0^1 K_{1f}(\rho,\theta)\,d\rho$. ■

COROLLARY 3.7. *$\langle\Phi_{11},f\rangle$ is well defined for $f\in\mathcal{S}(\mathbb{H}_n)$, and*

$$\langle\Phi_{11},f\rangle$$
$$=4^{n-1}(n-1)!C_\alpha\int_{\mathbb{R}}\int_{\tau>0}\frac{1}{(\tau-4it)^{(n-\alpha)/2}}\frac{1}{(\tau+4it)^{(n+\alpha)/2}}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt$$
$$+4^{n-1}(n-1)!\widetilde{C}_\alpha\int_{\mathbb{R}}\int_{\tau<0}\frac{1}{(\tau-4it)^{(n-\alpha)/2}}\frac{1}{(\tau+4it)^{(n+\alpha)/2}}\operatorname{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt.$$

From the corollary we also infer that $\langle\Phi_{12},f\rangle$ is well defined. In order to explicitly compute it, we define, for $0\le l\le n-2$, $\epsilon>0$ and $f\in\mathcal{S}(\mathbb{H}_n)$,

$$d^-_{\epsilon,l,f}=\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{-i\lambda t}|\lambda|^{n-l-2}\frac{\partial^l}{\partial\tau^l}Nf(0,t)\,dt\,d\lambda, \tag{3.22}$$

$$d^+_{\epsilon,l,f}=\int_0^\infty\int_{-\infty}^\infty e^{-\epsilon|\lambda|}e^{i\lambda t}|\lambda|^{n-l-2}\frac{\partial^l}{\partial\tau^l}Nf(0,t)\,dt\,d\lambda. \tag{3.23}$$

Then we can write (3.6) as

$$\langle\Phi_{12},f\rangle$$
$$=\lim_{r\to1^-}\lim_{\epsilon\to0^+}\sum_{k\ge0}\sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{n-2}2^{l+2}(a_{kl}+b_{kl})\left[\frac{r^{2k+n-\alpha}}{2k+n-\alpha}d^-_{\epsilon,l,f}+\frac{r^{2k+n+\alpha}}{2k+n+\alpha}d^+_{\epsilon,l,f}\right].$$

From Lemma 4.4 in [G-S2] we deduce that

$$a_{kl}+b_{kl}=(-1)^k\sum_{j=1}^{l+1}\frac{1}{2^{n-l-j-1}}\binom{n-j-1}{l-j+1}\binom{j+k-1}{k}.$$

We also have the following

LEMMA 3.8. *If* $0 \le l \le n-2$, $\epsilon > 0$ *and* $f \in \mathcal{S}(\mathbb{H}_n)$, *then*

$$\lim_{\epsilon\to 0^+} d^-_{\epsilon,l,f} = \frac{1}{i^{n-l-2}}\Big\langle \frac{\pi}{2}\delta - i\,\mathrm{p.v.}\Big(\frac{1}{\lambda}\Big), \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\Big\rangle,$$
$$\lim_{\epsilon\to 0^+} d^+_{\epsilon,l,f} = i^{n-l-2}\Big\langle \frac{\pi}{2}\delta + i\,\mathrm{p.v.}\Big(\frac{1}{\lambda}\Big), \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\Big\rangle.$$

Proof. Let us consider $g(\lambda) = e^{-\epsilon|\lambda|}|\lambda|^{n-l-2}$ and $h(t) = \frac{\partial^l}{\partial\tau^l}Nf(0,t)$, and observe that $\int_{-\infty}^{\infty} e^{-i\lambda t}h(t)\,dt = \hat{h}(\lambda)$. Then just by using the properties of the Fourier transform we get

$$d^-_{\epsilon,l,f} = \int_0^\infty \int_{-\infty}^\infty g(\lambda)e^{-i\lambda t}h(t)\,dt\,d\lambda = \int_0^\infty g(\lambda)\hat{h}(\lambda)\,d\lambda$$
$$= \frac{1}{i^{n-l-2}}\int_{-\infty}^\infty \frac{1}{\epsilon+i\lambda}h^{(n-l-2)}(\lambda)\,d\lambda.$$

For each $\epsilon > 0$, $\frac{1}{\epsilon+i\lambda}$ is a distribution such that the limit $\lim_{\epsilon\to0^+}\frac{1}{\epsilon+i\lambda}$ exists in $\mathcal{S}'(\mathbb{R})$. Moreover, it is easy to check that

$$\lim_{\epsilon\to 0^+}\frac{1}{\epsilon+i\lambda} = \frac{\pi}{2}\delta - i\,\mathrm{p.v.}\Big(\frac{1}{\lambda}\Big).$$

Thus the desired equality follows. For $d^+_{\epsilon,l,f}$ we need to change variables according to $\lambda \leftrightarrow -\lambda$ after considering the Fourier transform of h. ■

For $j \in \mathbb{N}$, $0 < j < n-1$, we define functions of r, with $0 \le r < 1$, by

$$w_j^-(r) = \sum_{k\ge0}(-1)^k\binom{j+k-1}{k}\frac{r^{2k+n-\alpha}}{2k+n-\alpha},$$
$$w_j^+(r) = \sum_{k\ge0}(-1)^k\binom{j+k-1}{k}\frac{r^{2k+n+\alpha}}{2k+n+\alpha}.$$

We can see, in a completely analogous way to the computations made for C_α and $\widetilde{C}_\alpha$, that these functions are well defined and that

$$\begin{aligned} c_j^- &:= \lim_{r\to1^-} w_j^-(r) = \frac{1}{2}B_{1/2}\Big(\frac{n-\alpha}{2}, j - \frac{n-\alpha}{2}\Big), \\ c_j^+ &:= \lim_{r\to1^-} w_j^+(r) = \frac{1}{2}B_{1/2}\Big(\frac{n+\alpha}{2}, j - \frac{n+\alpha}{2}\Big), \end{aligned} \tag{3.24}$$

where $B_{1/2}$ is another special function called the *incomplete Beta function.*

We now combine all of these definitions and results together to finally obtain an expression for Φ_{12}:

$$\langle \Phi_{12}, f\rangle = \sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{n-2}\sum_{j=1}^{l+1} 2^{2l-n+j+3}\binom{n-j-1}{l-j+1}\left[\left(\frac{1}{i^{n-l-2}}c_j^- + i^{n-l-2}c_j^+\right)\frac{\pi}{2}\right] \times \left\langle \delta, \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\right\rangle$$
$$+(-1)\sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{n-2}\sum_{j=1}^{l+1} 2^{2l-n+j+3}\binom{n-j-1}{l-j+1}\left(\frac{1}{i^{n-l+1}}c_j^- + i^{n-l+1}c_j^+\right) \times\left\langle \mathrm{p.v.}\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\right\rangle.$$

All we need to do now is to use again Lemma 3.8 to get an expression for Φ_2. Thus, we have proved the following

THEOREM 3.9. *Let C_α and $\widetilde{C}_\alpha$ be the constants defined as in* (3.17). *Then there exist constants C_l and $\widetilde{C}_l$, $l = 0,\dots,n-2$, such that*

$$\begin{aligned}&\langle \Phi_\alpha, f\rangle\\ &= 4^{n-1}(n-1)!C_\alpha \int_{-\infty}^{\infty}\int_{\tau>0}\frac{1}{(\tau-4it)^{(n-\alpha)/2}}\,\frac{1}{(\tau+4it)^{(n+\alpha)/2}}\,\mathrm{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt\\ &+ 4^{n-1}(n-1)!\widetilde{C}_\alpha \int_{-\infty}^{\infty}\int_{\tau<0}\frac{1}{(\tau-4it)^{(n-\alpha)/2}}\,\frac{1}{(\tau+4it)^{(n+\alpha)/2}}\,\mathrm{sgn}(\tau)G_f(\tau,t)\,d\tau\,dt\\ &+\sum_{l=0}^{n-2} C_l\left\langle \delta, \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\right\rangle\\ &+\sum_{l=0}^{n-2} \widetilde{C}_l\left\langle \mathrm{p.v.}\left(\frac{1}{\lambda}\right), \frac{\partial^{n-2}}{\partial\lambda^{n-l-2}\partial\tau^l}Nf(0,\cdot)\right\rangle.\end{aligned}$$

The constants C_l and $\widetilde{C}_l$ follow from the expressions obtained for Φ_{12} and Φ_2.

4. A fundamental solution for L. As in the classical case, the distribution Φ defined in (1.6) is a well defined tempered distribution and it is a fundamental solution for the operator L. The proof is identical to the one of Theorem 3.1.

We will compute the fundamental solution Φ by means of the Radon transform and the fundamental solution of the operator L in the classical case $N(2p, 2q, \mathbb{C})$.

Let $F \in \mathcal{S}(\mathbb{R}^3)$. We assign to F a function $\mathcal{R}F : \mathbb{R} \times S^2 \to \mathbb{R}$ defined by

$$\mathcal{R}F(t,\xi) = \int_{\mathbb{R}^2} F(t\xi + u_1 e_1 + u_2 e_2)\, du_1\, du_2,$$

where $\{\xi, e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^3$. It is easy to see that this definition does not depend on the choice of the basis. In order to recover F from $\mathcal{R}F$, we consider the space $\mathcal{S}(\mathbb{R}\times S^2)$ of continuous functions $G : \mathbb{R}\times S^2 \to \mathbb{R}$ that are infinitely differentiable in t and satisfy, for every $m, k \in \mathbb{N}_0$,

$$\sup_{t\in\mathbb{R},\, \xi\in S^2} \left| t^m \frac{\partial^k}{\partial t^k} G(t,\xi) \right| < \infty.$$

Now for $G \in \mathcal{S}(\mathbb{R} \times S^2)$ we define a function $\mathcal{R}^*G : \mathbb{R}^3 \to \mathbb{R}$ by

$$\mathcal{R}^*G(z) = \int_{S^2} G(\langle z,\xi\rangle, \xi)\, d\xi.$$

Both assignments are well defined. The map $\mathcal{R} : \mathcal{S}(\mathbb{R}^3) \to \mathcal{S}(\mathbb{R} \times S^2)$ is the *Radon transform*, $\mathcal{R}^* : \mathcal{S}(\mathbb{R}\times S^2) \to \mathcal{S}(\mathbb{R}^3)$ is the *dual Radon transform* and they satisfy, for every $F \in \mathcal{S}(\mathbb{R}^3)$,

$$-2\pi F = \Delta \mathcal{R}^*\mathcal{R}F, \tag{4.1}$$

where $\Delta = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \partial^2/\partial z_3^2$ is the $\mathbb{R}^3$-Laplacian (see for Example [S-Sh]).

Now, let us consider the function ϕ defined for a fixed $\tau \neq 0$ by

$$\phi(\tau, z) = \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16|z|^2)^{n+1}},$$

where $c_0 = -\int_0^1 \sigma^{2n-1}(1+\sigma^2)^{2n}\, d\sigma$. The function $\phi(\tau,\cdot)$ is not a Schwartz function on $\mathbb{R}^3$, but we have $(1+\Delta)^k\phi(\tau,\cdot) \in L^1(\mathbb{R}^3)$ for all k in $\mathbb{N}$, hence $(1+|\xi|^2)^k\widehat{\phi(\tau,\cdot)}(\xi) \in L^\infty(\mathbb{R}^3)$. With these properties the inversion formula for the Radon transform (4.1) still holds. The proof follows straightforwardly from Theorem 5.4 of [S-Sh].

Let us now compute the Radon transform of the function ϕ:

$$\begin{aligned}
\mathcal{R}\phi(\tau,t,\xi) &= \int_{\mathbb{R}^2} \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16(t^2 + u_1^2 + u_2^2))^{n+1}}\, du_1\, du_2 \\
&= \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{16^{n+1}} \int_{\mathbb{R}^2} \frac{1}{(\tau^2/16 + t^2 + (u_1^2+u_2^2))^{n+1}}\, du_1\, du_2 \\
&= \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{16^{n+1}} \int_{-\pi/2}^{3\pi/2} \int_0^\infty \frac{\rho}{(\tau^2/16 + t^2 + \rho^2)^{n+1}}\, d\rho\, d\theta \\
&= \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16|z|^2)^n},
\end{aligned}$$

where $z = t\xi$. Let

$$\varphi(\tau, z) = \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16|z|^2)^n}.$$

Now from the expression of the fundamental solution of L in the classical case (see for example 4.3 of [G-S2]) we know that

$$\varphi(\tau, t\xi) = \sum_{k\geq 0} \frac{(-1)}{2k+2n} \int_{-\infty}^{\infty} e^{i\lambda t} L_k^{2n-1}\left(\frac{\lambda}{2}|\tau|\right) e^{-\frac{\lambda}{4}|\tau|} |\lambda|^{2n-1}\, d\lambda.$$

We observe that the operator L has a nontrivial kernel, and define, for $f \in \mathcal{S}(N(p, q, \mathbb{H}))$,

$$\mathcal{P}f = \int_{\mathbb{R}^3} f * \varphi_{w,q-p} |w|^{2n}\, dw.$$

Then $L\mathcal{P}f = 0$.

To compute Φ we express the integral in (1.6) in polar coordinates:

$$\begin{aligned}\langle \Phi, f\rangle &= \sum_{k\in\mathbb{Z},\, k\neq q-p} \int_{\mathbb{R}^3} \frac{1}{-|\lambda|(2k+2(p-q))} \langle \varphi_{w,k}, f\rangle |w|^{2n}\, dw \\ &= \sum_{k\in\mathbb{Z},\, k\neq q-p} \int_{S^2}\int_0^{\infty} \frac{1}{-|\lambda|(2k+2(p-q))} \langle \varphi_{\lambda\xi,k}, f\rangle |\lambda|^{2n+2}\, d\lambda\, d\xi.\end{aligned}$$

By the absolute convergence of (1.6) we can interchange the summation with the integral over S^2. Since $\Delta e^{i\lambda\langle\xi,z\rangle} = -|\lambda|^2 e^{i\lambda\langle\xi,z\rangle}$, integrating by parts, we obtain

$$\begin{aligned}\langle \Phi, f\rangle &= \int_{S^2} \sum_{k\in\mathbb{Z},\, k\neq q-p} \frac{(-1)}{(2k+2(p-q))} \int_0^{\infty} \int_{N(p,q,\mathbb{H})} e^{i\lambda\langle\xi,z\rangle} \theta_{\lambda,k}(\alpha) f(\alpha, z)\, d\alpha\, dz \\ &\qquad \times |\lambda|^{2n+1}\, d\lambda\, d\xi \\ &= \int_{S^2} \sum_{k\in\mathbb{Z},\, k\neq q-p} \frac{1}{(2k+2(p-q))} \int_0^{\infty} \int_{N(p,q,\mathbb{H})} \Delta e^{i\lambda\langle\xi,z\rangle} \theta_{\lambda,k}(\alpha) f(\alpha, z)\, d\alpha\, dz \\ &\qquad \times |\lambda|^{2n-1}\, d\lambda\, d\xi \\ &= \int_{S^2} \sum_{k\in\mathbb{Z},\, k\neq q-p} \frac{1}{(2k+2(p-q))} \int_0^{\infty} \langle \varphi_{\lambda\xi,k}, \Delta f\rangle |\lambda|^{2n-1}\, d\lambda\, d\xi.\end{aligned}$$

Next we break the summation range into three parts, for $k \geq 2q$, $k \leq -2p$ and $-2p < k < 2q$, to get the splitting $\langle \Phi, f\rangle = \langle \Phi_1, f\rangle + \langle \Phi_2, f\rangle$, and as in Section 3 we change the summation index to make the series start from $k = 0$. Using the explicit definition of $\varphi_{\lambda\xi,k}$ we can write

$$\begin{aligned}\langle \Phi_1, f\rangle = \int_{S^2} \sum_{k\geq 0} \frac{1}{2k+2n} \int_0^{\infty} \int_{\mathbb{R}^3} & e^{i\lambda\langle\xi,z\rangle} \\ & \times \langle T_{\lambda,k+2q} - T_{\lambda,-k-2p}, N\Delta f(\cdot, z)\rangle\, dz\, |\lambda|^{2n-1}\, d\lambda\, d\xi,\end{aligned}$$

where $T_{\lambda,k} = F_{\lambda,k}$ is defined by equations (2.2) and (2.3). By performing similar computations to those in Section 3 and introducing the function

$$G_f(\tau,z) = Nf(\tau,z) - \sum_{j=0}^{2n-2} \frac{\partial^j Nf}{\partial \tau^j}(0,z)\frac{\tau^j}{j!},$$

we obtain the splitting

$$\langle \Phi_1, f\rangle = \langle \Phi_{11}, f\rangle + \langle \Phi_{12}, f\rangle,$$

where

$$\begin{aligned}(4.2)\qquad \langle \Phi_{11}, f\rangle = \int_{S^2} \sum_{k\geq 0} \frac{(-1)}{2k+2n} \int_0^\infty \int_{\mathbb{R}^3} \int_{-\infty}^{\infty} e^{i\lambda\langle \xi,z\rangle} \\ \times \operatorname{sgn}(\tau) L_k^{2n-1}\Big(\frac{2}{\lambda}|\tau|\Big) e^{-\lambda/4|\tau|} \Delta G_f(\tau,z)\, d\tau\, dz\, |\lambda|^{2n-1}\, d\lambda\, d\xi,\end{aligned}$$

$$\begin{aligned}(4.3)\qquad \langle \Phi_{12}, f\rangle = 2\int_{S^2} \sum_{k\geq 0} \frac{1}{2k+2n} \int_0^\infty \int_{\mathbb{R}^3} e^{i\lambda\langle \xi,z\rangle} \\ \times \sum_{\substack{l=0\\ l\,\text{odd}}}^{2n-2} \Big(\frac{2}{\lambda}\Big)^{l+1} (a_{k,l}+b_{k,l}) \langle \delta^{(l)}, \Delta Nf(\cdot,z)\rangle\, dz\, |\lambda|^{2n-1}\, d\xi,\end{aligned}$$

and a_{kl}, b_{kl} are the same constants defined in (3.7) and (3.3), respectively. Now we recall that

$$\int_{S^2} \int_0^\infty e^{i\lambda\langle \xi,z\rangle} F(|\lambda|)\, d\lambda\, d\xi = \frac{1}{2} \int_{S^2} \int_{-\infty}^{\infty} e^{i\lambda\langle \xi,z\rangle} F(|\lambda|)\, d\lambda\, d\xi,$$

and apply the dual Radon transform to (4.2).

Observe now that

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^3} \frac{\operatorname{sgn}(\tau) G_f(\tau,z)}{(1+16|z|^2)^{n+1}}\, dz\, d\tau$$

converges, which can be seen by changing to polar coordinates in $\mathbb{R}^3$ and arguing as in Lemma 2.2 of [G-S2].

We finally get

$$\begin{aligned}\langle \Phi_{11}, f\rangle &= \frac{1}{2}\Big\langle -2\pi \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{(\tau^2+16|z|^2)^{n+1}}, \operatorname{sgn}(\tau) G_f(\tau,z)\Big\rangle \\ &= -4^{2n+2} n(2n-1)!c_0 \Big\langle \frac{1}{(\tau^2+16|z|^2)^{n+1}}, \operatorname{sgn}(\tau) G_f(\tau,z)\Big\rangle.\end{aligned}$$

We have thus proven that the expression defining Φ_{11} is finite. Then the expression defining Φ_{12} is also finite, and by Abel's Lemma we can write

$$\langle \Phi_{12}, f\rangle = 2 \lim_{r\to 1^-} \lim_{\epsilon\to 0^+} \sum_{k\geq 0} \sum_{\substack{l=0\\ l\,\text{odd}}}^{2n-2} 2^{l+1}(a_{k,l}+b_{k,l}) \frac{r^{2k+2n}}{2k+2n} d_{\epsilon,l,f},$$

where

$$d_{\epsilon,l,f} = \int_{S^2}\int_0^\infty \int_{\mathbb{R}^3} e^{-\epsilon\lambda} e^{i\lambda\langle \xi, z\rangle} |\lambda|^{2n-l-2} \langle \delta^{(l)}, \Delta N f(\cdot, z)\rangle \, dz\, d\lambda\, d\xi. \tag{4.4}$$

We need to compute $\lim_{\epsilon\to 0^+} d_{\epsilon,l,f}$. Observing that $\Delta e^{i\lambda\langle\xi,z\rangle} = -|\lambda|^2 e^{i\lambda\langle\xi,z\rangle}$, we have

$$\begin{aligned} d_{\epsilon,l,f} &= (-1)^{l+1} \int_{S^2}\int_0^\infty \int_{\mathbb{R}^3} e^{-\epsilon\lambda} e^{i\lambda\langle\xi,z\rangle} |\lambda|^{2n-l} \frac{\partial^l}{\partial \tau^l} Nf(0,z)\, dz\, d\lambda\, d\xi \\ &= (-1)^{l+1} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\epsilon|x|} |x|^{2n-l-2} e^{i\langle x,z\rangle} \frac{\partial^l}{\partial\tau^l} Nf(0,z)\, dz\, dx, \end{aligned}$$

where we have changed to cartesian coordinates in $\mathbb{R}^3$. To compute this integral let us observe that

$$(-1)^{2n-l-2} e^{-\epsilon|x|}|x|^{2n-l-2} = \left(\frac{\partial^{2n-l-2}}{\partial\epsilon^{2n-l-2}}\right)^{\wedge} P_\epsilon(x),$$

where $P_\epsilon(x)$ is the Poisson kernel and $^\wedge$ denotes the Fourier transform. Let us write

$$\begin{aligned} d_{\epsilon,l,f} &= (-1)^l \int_{\mathbb{R}^3} \left(\frac{\partial^{2n-l-2}}{\partial\epsilon^{2n-l-2}}\right)^{\wedge} P_\epsilon(x) \left(\frac{\partial^l}{\partial\tau^l} Nf(0,\cdot)\right)^{\wedge}(x)\, dx \\ &= (-1)^l \frac{\partial^{2n-l-2}}{\partial\epsilon^{2n-l-2}} (P_\epsilon * h)(0). \end{aligned}$$

Taking the limit as $\epsilon\to 0^+$ we obtain

$$\lim_{\epsilon\to 0^+} = (-1)(-\Delta)^{(2n-l-2)/2} \frac{\partial^l}{\partial\tau^l} Nf(0,0),$$

where $(-\Delta)^{(2n-l-2)/2}$ is a fractional power of the Laplacian (see for example [S-Sh]), which is the operator defined for $g\in\mathcal{S}(\mathbb{R}^3)$ by

$$(-\Delta)^{(2n-l-2)/2} g(x) = \int_{\mathbb{R}^3} |\omega|^{2n-l-2} \widehat{g}(\omega) e^{i\langle\omega,z\rangle}\, d\omega.$$

By this computation together with Proposition 4.8 of [G-S2] we write

$$\langle \Phi_{12}, f\rangle = \sum_{\substack{l=0\\ l\,\mathrm{odd}}}^{2n-2} \sum_{j=1}^{l+1} \frac{1}{2^{2n-2l-j-3}} c_j \binom{2n-j-1}{l-j-1} (-1)(-\Delta)^{(2n-l-2)/2} \frac{\partial^l}{\partial \tau^l} Nf(0,0),$$

where each c_j is the constant defined in Remark 4.7 of [G-S2] as follows:

$$c_j = \int_0^1 \frac{r^{j-1}}{(1+r^2)^j}\, dr.$$

After performing the usual computations for Φ_2 we will have proved the main theorem of this section:

THEOREM 4.1. *Let c_0 be the constant defined above. Then there exist constants $c_l(k)$, $l = 0,\ldots,2n-2$ and $-2p < k < 2q$, such that*

$$\langle \Phi, f\rangle = -4^{2n+2} n(2n-1)! c_0 \left\langle \frac{1}{(\tau^2+16|z|^2)^{n+1}}, \operatorname{sgn}(\tau) G_f(\tau,z)\right\rangle + \sum_{\substack{-2p<k<2q\\ k\neq q-p}} \sum_{l=0}^{2n-2} c_l(k)(-\Delta)^{(2n-l-2)/2} \frac{\partial^l}{\partial\tau^l} Nf(0,0).$$

REMARK 4.2. Let N be a group of Heisenberg type and let η be its Lie algebra. So $\eta = V \oplus \mathfrak{z}$, with $\dim V = 2m$ and $\dim \mathfrak{z} = k$. Let $U(V)$ be the unitary group acting on V. Then it is known ([R]) that $(N \ltimes U(V), U(V))$ is a Gelfand pair. In [R] the spherical functions were computed. We fix an orthonormal basis of V, $\{X_1,\ldots,X_{2m}\}$, and consider the operator

$$L = \sum_{j=1}^{2m} X_j^2.$$

With the same arguments as above, using the Radon transform in $\mathbb{R}^k$ and the fundamental solution of L in the classical $2m+1$-dimensional Heisenberg group, we can recover the fundamental solution of L (see [K], [R]).

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