**EXPLICIT FUNDAMENTAL SOLUTIONS OF SOME SECOND ORDER DIFFERENTIAL OPERATORS ON HEISENBERG GROUPS**

BY

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**Abstract.** Let $p, q, n$ be natural numbers such that $p + q = n$. Let $\mathbb{F}$ be either $\mathbb{C}$, the complex numbers field, or $\mathbb{H}$, the quaternionic division algebra. We consider the Heisenberg group $N(p, q, \mathbb{F})$ defined $\mathbb{F}^n \times \text{Im} \mathbb{F}$, with group law given by

$$(v, \zeta)(v', \zeta') = \left(v + v', \zeta + \zeta' - \frac{1}{2}\text{Im} B(v, v')\right),$$

where $B(v, w) = \sum_{j=1}^p v_j w_j - \sum_{j=p+1}^n v_j w_j$. Let $U(p, q, \mathbb{F})$ be the group of $n \times n$ matrices with coefficients in $\mathbb{F}$ that leave the form $B$ invariant. We compute explicit fundamental solutions of some second order differential operators on $N(p, q, \mathbb{F})$ which are canonically associated to the action of $U(p, q, \mathbb{F})$.

1. Introduction. In [M-R2] the authors exhaustively discussed the problem of invertibility for the class of second order, homogeneous left invariant differential operators on the Heisenberg group, which in addition are formally selfadjoint, modulo a derivative in the central direction.

The best known examples of this class are of the form $L + i\alpha U$, where $L$ is the sublaplacian, $U$ generates the centre of the Lie algebra, and $\alpha$ is a complex number. For $\alpha \neq 2k + n$, $k$ a nonnegative integer, an explicit fundamental solution was given in [F-S]. It is also mentioned in [M-R2] that these operators are essentially the only ones, in the class considered, which admit simple expressions for their fundamental solutions.

Moreover, in [K] the groups of Heisenberg type were introduced with the purpose, in part, of giving explicit fundamental solutions for some second order differential operators on two-step nilpotent Lie groups.

In [B-D-R] the authors considered the Heisenberg group under the action of $U(n)$, and used the spherical analysis of the associated Gelfand pair in order to obtain a fundamental solution for any power of the sublaplacian. Inspired by this work, the same was done in [G-S2] for a second order homogeneous differential operator canonically associated to the action of $U(p, q)$. The computation used the spherical distributions of the corresponding generalized Gelfand pair.

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The aim of this paper is to continue this research. More precisely, let \( p, q, n \) be natural numbers such that \( p+q = n \). Let \( \mathbb{F} \) be either \( \mathbb{C} \), the complex field, or \( \mathbb{H} \), the quaternionic division algebra. We consider the Heisenberg group \( N(p, q, \mathbb{F}) = \mathbb{F}^n \times \mathfrak{Im} \mathbb{F} \), with group law given by

\[
(v, \zeta)(v', \zeta') = \left( v + v', \zeta + \zeta' - \frac{1}{2} \mathfrak{Im} B(v, v') \right),
\]

where \( B(v, w) = \sum_{j=1}^{p} v_j \overline{w_j} - \sum_{j=p+1}^{n} v_j \overline{w_j} \). The associated Lie algebra is \( \eta(p, q, \mathbb{F}) = \mathbb{F}^n \oplus \mathfrak{Im} \mathbb{F} \), with Lie bracket given by

\[
[(v, \zeta), (v', \zeta')] = (0, -\mathfrak{Im} B(v, v')).
\]

Let \( U(p, q, \mathbb{H}) \) be the group of \( n \times n \) matrices with coefficients in \( \mathbb{F} \) that leave the form \( B \) invariant. Then \( U(p, q, \mathbb{F}) \) acts by automorphisms on \( N(p, q, \mathbb{F}) \) by

\[
g \cdot (v, \zeta) = (gv, \zeta).
\]

In \[D-M\] it is proved that \((U(p, q, \mathbb{F}) \ltimes N(p, q, \mathbb{F}), N(p, q, \mathbb{F}))\), where \( \ltimes \) denotes semidirect product, is a generalized Gelfand pair, and thus the algebra \( \mathcal{D}(N(p, q, \mathbb{F})) \) of left invariant and \( U(p, q, \mathbb{F}) \)-invariant differential operators on \( N(p, q, \mathbb{F}) \) is commutative (see \[D\]).

In this paper we obtain explicit fundamental solutions for some generators of this algebra. Recall that a fundamental solution for a differential operator \( L \) is a distribution \( \Phi \) such that for all test functions \( f \), we have \( L(f * \Phi) = (L f) * \Phi = f * L(\Phi) = f \). So the operator \( K \) defined by \( K f = f * \Phi \) satisfies \( K \circ L f = L \circ K f = f \).

If \( \mathbb{F} = \mathbb{C} \) and \{\( X_1, \ldots, X_n, Y_1, \ldots, Y_n, U \)\} denotes the standard basis of the Heisenberg Lie algebra with \([X_i, Y_j] = \delta_{ij} U\) and all the other brackets zero, then \( \mathcal{D}(N(p, q, \mathbb{C})) \) is generated by \( U \) and

\[
L = \sum_{j=1}^{p} (X_j^2 + Y_j^2) - \sum_{j=p+1}^{n} (X_j^2 + Y_j^2).
\]

A complete description of the spherical distributions associated to this pair is given in \[D-M\] and \[G-S1\]. Moreover, for \( \lambda \in \mathbb{R}, \lambda \neq 0 \) and \( k \in \mathbb{Z} \), there exists a \( U(p, q, \mathbb{C}) \)-invariant tempered distribution \( S_{\lambda,k} \) on \( N(p, q, \mathbb{C}) \) satisfying

\[
(1.1) \quad LS_{\lambda,k} = -|\lambda|(2k + p - q)S_{\lambda,k}, \quad iUS_{\lambda,k} = \lambda S_{\lambda,k}.
\]

Let us consider the operator \( L_\alpha = L + i\alpha U \), where \( \alpha \) is a noninteger complex number. To obtain a fundamental solution \( \Phi_\alpha \) for \( L_\alpha \) we will strongly use the expression of the inversion formula for Schwartz functions \( f \) on the Heisenberg group, which is given by

\[
(1.2) \quad f(z, t) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k} |\lambda|^n d\lambda, \quad (z, t) \in N(p, q, \mathbb{C}).
\]
Because of (1.1) and (1.2) it is natural to propose as a fundamental solution of $L\alpha$,

$$\langle \Phi_\alpha, f \rangle = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{1}{-|\lambda|(2k + p - q - \alpha \text{sgn} \lambda)} \langle S_{\lambda,k}, f \rangle |\lambda|^n d\lambda.$$  

(1.3)

We will see in Theorem 3.1 that $\Phi_\alpha$ is a tempered distribution and its expression is obtained in Theorem 3.9. The strategy for the computation is the use of explicit formulas for $S_{\lambda,k}$.

If $F = \mathbb{H}$ we take $\{X_0^1, X_1^1, X_2^1, \ldots, X_0^n, X_1^n, X_2^n, X_3^n, Z_1, Z_2, Z_3\}$ the canonical basis for the Lie algebra, where $Z_1, Z_2, Z_3$ generate the center of $\eta(p,q,\mathbb{H})$. Here, the operators

$$L = \sum_{r=1}^{3} \sum_{l=0}^{3} (X_r^l)^2 - \sum_{r=p+1}^{n} \sum_{l=0}^{3} (X_r^l)^2, \quad U = \sum_{l=1}^{3} Z_l^2,$$

generate the algebra $\mathcal{D}(N(p,q,\mathbb{H}))$.

In this case, the spherical distributions $\varphi_{w,k}, w \in \mathbb{R}^3, k \in \mathbb{Z}$, were computed in [V] and they satisfy

$$L \varphi_{w,k} = -|w|(2k + 2(p - q)) \varphi_{w,k}, \quad U \varphi_{w,k} = -\lambda^2 \varphi_{w,k}.$$  

(1.4)

Since $L$ has a nontrivial kernel, we can only hope to find a relative fundamental solution for $L$. We recall that if $\pi$ denotes the orthogonal projection onto the kernel of a differential operator $L$, a relative fundamental solution for $L$ is a distribution $\Phi$ such that

$$L(f \ast \Phi) = (Lf) \ast \Phi = f \ast L(\Phi) = f - \pi(f)$$

for all test functions $f$.

In order to obtain a (relative) fundamental solution $\Phi$ for the operator $L$ we will use the fact that the family $\{\varphi_{w,k}\}$ also provides an inversion formula (see [R]): for all $f \in \mathcal{S}(N(p,q,\mathbb{H}))$ we have

$$f(\alpha, z) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^3} (f \ast \varphi_{w,k})(\alpha, z)|w|^{2n} dw, \quad (\alpha, z) \in N(p,q,\mathbb{H}).$$  

(1.5)

Because of (1.4) and (1.5) we propose as a relative fundamental solution of $L$,

$$\langle \Phi, f \rangle = \sum_{k \in \mathbb{Z}, k \neq (q-p) \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{1}{-|w|(2k + 2(p - q))} \langle \varphi_{w,k}, f \rangle |w|^{2n} dw.$$  

(1.6)

The explicit form of $\Phi$ is given in Theorem 4.1 and for its computation we use the Radon transform in order to reduce this case to the classical one.

We remark that for $q = 0, F = \mathbb{C}$ we recover the fundamental solution for the operator $L_\alpha$ given in [F-S], and for $q = 0, F = \mathbb{H}$ we recover Kaplan’s fundamental solution for the operator $L$ given in [K]. The case $q \neq 0, \alpha = 0$ was obtained in [G-S2].
2. Preliminaries. In order to describe both families of eigendistributions \( \{S_{\lambda,k}\} \) and \( \{\varphi_{w,k}\} \) we need to adapt a result by Tengstrand [T]. We describe the elements for \( \mathbb{F} = \mathbb{C} \), the other case being similar. First of all we take bipolar coordinates on \( \mathbb{C}^n \): for \( (x_1, y_1, \ldots, x_n, y_n) \) we set

\[
\tau = \sum_{j=1}^{p} (x_j^2 + y_j^2) - \sum_{j=p+1}^{n} (x_j^2 + y_j^2), \quad \rho = \sum_{j=1}^{n} (x_j^2 + y_j^2),
\]

\[
u = (x_1, y_1, \ldots, x_p, y_p), \quad v = (x_{p+1}, y_{p+1}, \ldots, x_n, y_n).
\]

Hence \( u = (\frac{\rho + \tau}{2})^{1/2} \omega_u \) with \( \omega_u \in S^{2p-1} \), and \( v = (\frac{\rho - \tau}{2})^{1/2} \omega_v \) with \( \omega_v \in S^{2q-1} \). It is easy to see by changing variables that

\[
\int_{\mathbb{C}^n} f(z) \, dz = \int_{-\infty}^{\infty} \int_{S^{2p-1} \times S^{2q-1}} f\left(\left(\frac{\rho + \tau}{2}\right)^{1/2} \omega_u, \left(\frac{\rho - \tau}{2}\right)^{1/2} \omega_v\right) \, d\omega_u \, d\omega_v \times (\rho + \tau)^{p-1}(\rho - \tau)^{q-1} \, d\rho \, d\tau.
\]

Then for \( f \in S(\mathbb{R}^{2n}) \) we define

\[
Mf(\rho, \tau) = \int_{S^{2p-1} \times S^{2q-1}} f\left(\left(\frac{\rho + \tau}{2}\right)^{1/2} \omega_u, \left(\frac{\rho - \tau}{2}\right)^{1/2} \omega_v\right) \, d\omega_u \, d\omega_v,
\]

and also

\[
Nf(\tau) = \int_{|\tau|}^{\infty} Mf(\rho, \tau)(\rho + \tau)^{p-1}(\rho - \tau)^{q-1} \, d\rho.
\]

Let us now define \( \mathcal{H}_n \) to be the space of functions \( \varphi : \mathbb{R} \to \mathbb{C} \) such that \( \varphi(\tau) = \varphi_1(\tau) + \tau^{-1}\varphi_2(\tau)H(\tau) \) for \( \varphi_1, \varphi_2 \in S(\mathbb{R}) \), where \( H \) denotes the Heaviside function. In [T] it is proved that \( \mathcal{H}_n \) with a suitable topology is a Fréchet space, and following the same lines we can see that the linear maps \( N : S(\mathbb{R}^{2n} - \{0\}) \to S(\mathbb{R}) \) and \( N : S(\mathbb{R}^{2n}) \to \mathcal{H} \) are continuous and surjective.

Let us now consider \( \mu \in S'(\mathbb{R}^{2n})^{U(p,q)} \); then it is easy to see that there exists a unique \( T \in S'(\mathbb{R}) \) such that \( \langle \mu, f \rangle = \langle T, Nf \rangle \) for all \( f \in S(\mathbb{R}^{2n} - \{0\}) \). Moreover, if \( N' : \mathcal{H} \to S'(\mathbb{R}^{2n}) \) is the adjoint map, by following again the arguments of [T] we can see that \( N' \) is a homeomorphism. Finally, for a function \( f \in S(N(p, q, \mathbb{C})) \), we write \( Nf(\tau, t) \) for \( N(f(\cdot, t))(\tau) \).

The distributions \( S_{\lambda,k} \) are defined as follows:

\[
S_{\lambda,k} = \sum_{m \in \mathbb{N}_0^n, B(m) = k} E_\lambda(h_m, h_m),
\]

where \( B(m) = \sum_{j=1}^{p} m_j - \sum_{j=p+1}^{n} m_j \), the set of functions \( \{h_m\} \subset L^2(\mathbb{R}^n) \) is the normalized Hermite basis, and \( E_\lambda(h, h')(z, t) = \langle \pi_\lambda(z, t)h, h' \rangle \) are the matrix entries of the Schrödinger representation \( \pi_\lambda \). Also, \( S_{\lambda,k} = e^{-i\lambda t} \otimes F_{\lambda,k} \), where each \( F_{\lambda,k} \in S'(\mathbb{C}^n)^{U(p,q)} \) is a tempered distribution defined in terms
of the Laguerre polynomials $L^m_k$ and the map $N$ as follows: for $g \in \mathcal{S}(\mathbb{C}^n)$, $\lambda \neq 0$, and $k \in \mathbb{Z}$, if $k \geq 0$ then
\begin{equation}
\langle F_{\lambda,k}, g \rangle = \left\langle (L^0_{k-q+n-1}H)^{(n-1)}, \tau \mapsto 2 \left\lvert \lambda \right\rvert e^{-\tau/2}Ng\left(\frac{2}{|\lambda|}\tau\right) \right\rangle,
\end{equation}
and if $k < 0$ then
\begin{equation}
\langle F_{\lambda,k}, g \rangle = \left\langle (L^0_{-k-p+n-1}H)^{(n-1)}, \tau \mapsto 2 \left\lvert \lambda \right\rvert e^{-\tau/2}Ng\left(-\frac{2}{|\lambda|}\tau\right) \right\rangle.
\end{equation}

For the quaternionic case we consider the Schrödinger representation $\pi_w$ as given in $\mathbb{R}$ (see also $[K-R]$):
\begin{equation}
\pi_w(\alpha, z) = \pi_{|w|}(\alpha, \langle z, w/|w| \rangle),
\end{equation}
where $\pi_{|w|}$ is the Schrödinger representation for the classical Heisenberg group $N(2p, 2q, \mathbb{C})$. Analogously, the distributions $\varphi_{w,k}$ are defined by
\begin{equation}
\varphi_{w,k}(\alpha, z) = \sum_{m \in \mathbb{N}^{2n}_0, B(m)=k} E_w(h_m, h_{m}),
\end{equation}
where $B(m) = \sum_{j=1}^{2p} m_j - \sum_{j=2p+1}^{2n} m_j$, and $E_w(h, h')(\alpha, z) = \langle \pi_w(\alpha, z) h, h' \rangle$ are the matrix entries of the Schrödinger representation $\pi_w$. Moreover, we have $\varphi_{w,k} = e^{i\langle w, z \rangle} \otimes \theta_{w,k}$, where $\theta_{w,k}$ is a tempered distribution such that $\theta_{w,k} = N'T_{|w|, k}$, where if we set $\lambda = |w|$, we have $T_{|w|, k} = F_{\lambda, k}$, replacing $n, p, q$ by $2n, 2p, 2q$, respectively, in (2.2) and (2.3). Observe that if we define
\begin{equation}
\varphi_{\lambda,k}(\alpha, z) = \int_{\mathbb{S}^2} e^{i\langle z, \lambda \xi \rangle} d\xi \theta_{\lambda,k}(\alpha),
\end{equation}
these distributions are $\text{Spin}(3) \otimes U(p, q, \mathbb{H})$-invariant.

3. A fundamental solution for the operator $L_\alpha$. We know that $\Phi_\alpha$ defined as in (1.3) is a well defined tempered distribution, and a fundamental solution for $L_\alpha$. We include the proof since a misprint in Lemma 1 of $[M-R1]$ is used in the proof of Lemma 2.10 of $[B-D-R]$.

We will consider $\alpha \in \mathbb{C}$ such that $2k + p - q \pm \alpha \neq 0$ for all $k \in \mathbb{Z}$.

**Theorem 3.1.** $\Phi_\alpha$ defined as in (1.3) is a well defined tempered distribution and it is a fundamental solution for the operator $L_\alpha$.

**Proof.** From (1.3) and (2.1) we can write
\begin{align*}
|\langle \Phi_\alpha, f \rangle| & \leq \sum_{k \in \mathbb{Z}} \int_0^\infty \left( \left| \langle S_{-\lambda,k}, f \rangle \right| \frac{2k + p - q + \alpha}{2k + p - q - \alpha} + \left| \langle S_{\lambda,k}, f \rangle \right| \frac{2k + p - q - \alpha}{2k + p - q + \alpha} \right) |\lambda|^{n-1} d\lambda \\
& \leq \sum_{k \in \mathbb{Z}} \int_0^\infty \sum_{\beta \in \mathbb{N}^n_0} \left( \left| \langle E_{-\lambda}(h_{\beta}, h_{\beta}), f \rangle \right| \frac{2k + p - q + \alpha}{2k + p - q - \alpha} + \left| \langle E_{\lambda}(h_{\beta}, h_{\beta}), f \rangle \right| \frac{2k + p - q - \alpha}{2k + p - q + \alpha} \right) |\lambda|^{n-1} d\lambda.
\end{align*}
From the known facts that
\[
\sum_{k \in \mathbb{Z}} \sum_{\beta \in \mathbb{N}_0} p(\beta) = \sum_{k \geq 0} \binom{k + n - 1}{n - 1} p(k),
\]
and that for \(m \in \mathbb{N}\),
\[
\pi_\lambda(f)h_\beta = \frac{1}{(-1)^m|\lambda|^m(2B(\beta) + p - q + \alpha \text{ sgn}(\lambda))} \pi_\lambda(L^m f)h_\beta,
\]
we get
\[
|\langle \Phi_\alpha, f \rangle| \leq \|L^m f\|_{L^1(N(p,q,\mathbb{C})}
\]
\[
\times \sum_{k \geq 0} \int_0^\infty \binom{k + n - 1}{k} \left( \frac{|\lambda|^{n-1-m}}{|2k + p - q + \alpha|^{m+1}} + \frac{|\lambda|^{n-1-m}}{|2k + p - q - \alpha|^{m+1}} \right) d\lambda.
\]
Let us consider the first term, the second one being analogous. We split the integral between \(|\lambda|\geq 1\) and \(0 \leq |\lambda| \leq 1\). Now
\[
\sum_{k \geq 0} \binom{k + n - 1}{k} \int_{|\lambda| \geq 1} \frac{1}{|2k + p - q + \alpha|^{m+1}|\lambda|^{n-1-m}} d\lambda
\]
is finite if we take \(m > n\), and
\[
\sum_{k \geq 0} \binom{k + n - 1}{k} \int_{0 \leq |\lambda| \leq 1} \frac{1}{|2k + p - q + \alpha|^{m+1}|\lambda|^{n-1-m}} d\lambda
\]
is finite for any natural number \(m\). From the above computations it also follows that \(\Phi_\alpha\) is a tempered distribution. Next we see that it is a fundamental solution by writing \(L = L_0 + L_1\), where in coordinates
\[
L_0 = \frac{1}{4} \left( \sum_{j=1}^p (x_j^2 + y_j^2) - \sum_{j=p+1}^n (x_j^2 + y_j^2) \right) \frac{\partial^2}{\partial t^2}
\]
\[
+ \sum_{j=1}^p \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) - \sum_{j=p+1}^n \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right),
\]
\[
L_1 = \frac{\partial}{\partial t} \sum_{j=1}^n \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).
\]
Then, as \(L_0, L_1\) and \(T\) commute with left translations and also \(L_0(g^\vee) = (L_0 g)^\vee, L_1(g^\vee) = -(L_1 g)^\vee\) and \(T(g^\vee) = -(T g)^\vee\), we get
\[
(Lf * \Phi_\alpha)(z,t) = \langle \Phi_\alpha, (L_{(z,t)})^{-1} L f \rangle = \langle \Phi_\alpha, (L_0 - i\alpha)(L_{(z,t)})^{-1} f \rangle,
\]
because \(L_1 \Phi_\alpha = 0\). Hence,
\[
(L_\alpha f \ast \Phi)(z, t) = \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{\langle S_{\lambda,k}, (L_0 - i\alpha T) (L(z,t))^{-1} f \rangle}{-|\lambda|(2k + p - q - \alpha \text{sgn } \lambda)} |\lambda|^{n-1} d\lambda
\]
\[
= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \frac{\langle (L_0 + i\alpha T) S_{\lambda,k}, (L(z,t))^{-1} f \rangle}{-|\lambda|(2k + p - q - \alpha \text{sgn } \lambda)} |\lambda|^{n-1} d\lambda
\]
\[
= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} \langle S_{\lambda,k}, (L(z,t))^{-1} f \rangle |\lambda|^{n-1} d\lambda = f(z, t),
\]
by the inversion formula. The other equality, \(f \ast L_\alpha(f) = f\), is immediate. ■

Now we proceed with the computation of \(\Phi_\alpha\). Since the series \((1.3)\) defining \(\Phi_\alpha\) converges absolutely, we can split the sum over \(k \in \mathbb{Z}\) into the sums for \(k \geq q\), for \(k \leq -p\) and for \(-p < k < q\). In the first case we change the summation index writing \(k = k' + q\), and in the second we write \(k = k' - p\). So we get
\[
\langle \Phi_\alpha, f \rangle = (-1) \sum_{k' \geq 0} \frac{1}{2k' + n - \alpha} \int_{0}^{\infty} \left( \langle S_{\lambda,k' + q}, f \rangle - \langle S_{\lambda,-k' - p}, f \rangle \right) |\lambda|^{n-1} d\lambda
\]
\[
+ (-1) \sum_{k' \geq 0} \frac{1}{2k' + n + \alpha} \int_{0}^{\infty} \left( \langle S_{\lambda,k' + q}, f \rangle - \langle S_{\lambda,-k' - p}, f \rangle \right) |\lambda|^{n-1} d\lambda
\]
\[
+ (-1) \sum_{-p < k < q} \int_{0}^{\infty} \left( \frac{\langle S_{\lambda,k}, f \rangle}{2k + p - q + \alpha} + \frac{\langle S_{\lambda,k}, f \rangle}{2k + p - q - \alpha} \right) |\lambda|^{n-1} d\lambda.
\]
By Abel’s Lemma and the Lebesgue Dominated Convergence Theorem we can write \(\Phi_\alpha = \Phi_1 + \Phi_2\) where
\[
(3.1) \quad \langle \Phi_1, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} (-1) \sum_{k' \geq 0} \frac{r^{2k' + n - \alpha}}{2k' + n - \alpha} \int_{0}^{\infty} e^{-|\lambda|} \times \left( \langle S_{\lambda,k' + q}, f \rangle - \langle S_{\lambda,-k' - p}, f \rangle \right) |\lambda|^{n-1} d\lambda
\]
\[
+ \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} (-1) \sum_{k' \geq 0} \frac{r^{2k' + n + \alpha}}{2k' + n + \alpha} \int_{0}^{\infty} e^{-|\lambda|} \times \left( \langle S_{\lambda,k' + q}, f \rangle - \langle S_{\lambda,-k' - p}, f \rangle \right) |\lambda|^{n-1} d\lambda,
\]
\[
(3.2) \quad \langle \Phi_2, f \rangle = \lim_{\epsilon \to 0^+} (-1) \sum_{-p < k < q} \int_{0}^{\infty} e^{-|\lambda|} \times \left( \frac{\langle S_{\lambda,k}, f \rangle}{2k + p - q + \alpha} + \frac{\langle S_{\lambda,k}, f \rangle}{2k + p - q - \alpha} \right) |\lambda|^{n-1} d\lambda.
\]
Using that $S_{\lambda,k} = e^{-i\lambda t} \otimes F_{\lambda,k}$ and the computations from [G-S2] (2.6) to (2.9), we get

$$\langle \Phi_1, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} (-1) \sum_{k \geq 0} \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} \int_{\lambda}^{\infty} e^{-i\lambda t}$$

$$\times \left\langle \left( L_{k+n-1}^0 H \right)^{(n-1)} \frac{2}{|\lambda|} e^{-\tau/2} \left[ Nf \left( \frac{2}{|\lambda|}, t \right) - Nf \left( -\frac{2}{|\lambda|}, t \right) \right] \right\rangle dt d\lambda$$

$$+ \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} (-1) \sum_{k \geq 0} \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} \int_{\lambda}^{\infty} e^{i\lambda t}$$

$$\times \left\langle \left( L_{k+n-1}^0 H \right)^{(n-1)} \frac{2}{|\lambda|} e^{-\tau/2} \left[ Nf \left( \frac{2}{|\lambda|}, t \right) - Nf \left( -\frac{2}{|\lambda|}, t \right) \right] \right\rangle dt d\lambda.$$ 

Thus setting

$$b_{k,l} = \sum_{j=l}^{n-2} \binom{j}{l} \left( \frac{1}{2} \right)^{2-l} (-1)^{n-j} \binom{k+n-1}{n-j-2},$$

we have

$$\langle \Phi_1, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} \int_{\lambda}^{\infty} e^{-i\lambda t}$$

$$\times \left[ (-1)^n \int_{-\infty}^{\infty} L_{k}^{n-1} \left( \frac{|\lambda|}{2}, s \right) e^{-\frac{|\lambda|}{4}|s|} \sgn(s) Nf(s, t) ds \right.$$ 

$$- 2 \sum_{l=0}^{n-2} \frac{2}{|\lambda|} b_{k,l} \frac{\partial^l Nf}{\partial \tau^l} (0, t) \right] dt d\lambda$$

$$+ \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} \int_{\lambda}^{\infty} e^{i\lambda t}$$

$$\times \left[ (-1)^n \int_{-\infty}^{\infty} L_{k}^{n-1} \left( \frac{|\lambda|}{2}, s \right) e^{-\frac{|\lambda|}{4}|s|} \sgn(s) Nf(s, t) ds \right.$$ 

$$- 2 \sum_{l=0}^{n-2} \frac{2}{|\lambda|} b_{k,l} \frac{\partial^l Nf}{\partial \tau^l} (0, t) \right] dt d\lambda.$$

Now we define

$$G_f(\tau, t) = Nf(\tau, t) - \sum_{j=0}^{n-2} \frac{\partial^j Nf}{\partial \tau^j} (0, t) \frac{\tau^j}{j!},$$

and we split $\Phi_1 = \Phi_{11} + \Phi_{12}$, where
\begin{equation}
\langle \Phi_{11}, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} (-1)^n \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} e^{-i\lambda t |\lambda|^{n-1}}
\end{equation}

\begin{equation}
\times \int_{-\infty}^{\infty} L_k^{n-1} \left( \frac{|\lambda|}{2 |\tau|} \right) e^{-|\lambda| |\tau|} \text{sgn}(\tau) G_f(\tau, t) d\tau dt d\lambda
\end{equation}

\begin{equation}
+ \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} (-1)^n \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} e^{i\lambda t |\lambda|^{n-1}}
\end{equation}

\begin{equation}
\times \int_{-\infty}^{\infty} L_k^{n-1} \left( \frac{|\lambda|}{2 |\tau|} \right) e^{-|\lambda| |\tau|} \text{sgn}(\tau) G_f(\tau, t) d\tau dt d\lambda,
\end{equation}

and

\begin{equation}
\langle \Phi_{12}, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \frac{r^{2k+n-\alpha}}{2k+n-\alpha} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} e^{-i\lambda t |\lambda|^{n-1}}
\end{equation}

\begin{equation}
\times 2 \sum_{l=0}^{n-2} \left( \frac{2}{|\lambda|} \right)^{l+1} (a_{k,l} + b_{k,l}) \frac{\partial^l N f}{\partial \tau^l} (0, t) dt d\lambda
\end{equation}

\begin{equation}
+ \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \frac{r^{2k+n+\alpha}}{2k+n+\alpha} \int_{0}^{\infty} \int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} e^{i\lambda t |\lambda|^{n-1}}
\end{equation}

\begin{equation}
\times 2 \sum_{l=0}^{n-2} \left( \frac{2}{|\lambda|} \right)^{l+1} (a_{k,l} + b_{k,l}) \frac{\partial^l N f}{\partial \tau^l} (0, t) dt d\lambda,
\end{equation}

with

\begin{equation}
a_{k,l} = (-1)^n \frac{1}{l!} \int_{0}^{\infty} L_k^{n-1} (s) e^{-s/2} s^l ds.
\end{equation}

We will show that \( \Phi_{11} \) is well defined. We have proved that the series defining \( \Phi_\alpha \) converges and, as \( \Phi_2 \) is a finite sum, we will deduce that \( \Phi_{12} \) is also well defined.

**Proposition 3.2.** The following identities hold:

(i) \begin{equation}
\int_{-\infty}^{\infty} e^{-\epsilon |\lambda|} e^{-i\lambda t} L_k^{n-1} \left( \frac{|\lambda|}{2 |\tau|} \right) e^{-|\lambda| |\tau|} |\lambda|^{n-1} d\lambda
\end{equation}

\begin{equation}
= 4^n (n-1)! (-1)^n \binom{k+n-1}{k} \frac{(1-4\epsilon - 4i\epsilon)^k}{(|\tau| + 4\epsilon + 4i\epsilon)^{k+n}};
\end{equation}

(ii) \begin{equation}
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \left( \frac{|\tau| - 4i\epsilon - 4\epsilon}{|\tau| + 4i\epsilon + 4\epsilon} \right)^{k+n} \text{sgn}(\tau) G_f(\tau, t) d\tau dt
\end{equation}

\begin{equation}
= \int_{\mathbb{R}^2} \frac{1}{(|\tau| - 4i\epsilon)^{n/2-\alpha/2}} \frac{1}{(|\tau| + 4i\epsilon)^{n/2+\alpha/2}} \frac{|\tau| - 4i\epsilon}{\tau^2 + 16\epsilon^2}^{2k+n-\alpha}
\end{equation}

\begin{equation}
\times \text{sgn}(\tau) G_f(\tau, t) d\tau dt;
\end{equation}
(iii) \[
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^2} \frac{(|\tau| + 4it - 4\epsilon)^k}{(|\tau| - 4it + 4\epsilon)^{k+n}} \text{sgn}(\tau)G_f(\tau, t) \, d\tau \, dt
\]
\[
= \int_{\mathbb{R}^2} \frac{1}{(|\tau| - 4it)^{n/2-\alpha/2}} \frac{1}{(|\tau| + 4it)^{n/2+\alpha/2}} \left(\frac{|\tau| - 4it}{\tau^2 + 16t^2}\right)^{2k+n+\alpha} \times \text{sgn}(\tau)G_f(\tau, t) \, d\tau \, dt.
\]

**Proof.** From (4.9) of [G-S2] we deduce that (i) follows from the generating identity for the Laguerre polynomials,

\[
\sum_{k \geq 0} L_{n-1}^k(t)z^k = \frac{1}{(1-z)^n}e^{-\frac{zt}{1-z}}.
\]

From Lemma 2.2 of [G-S2], which states that the function \(G_f(\tau,t)\) is integrable in \(\mathbb{R}^2\), and from the fact that \(\frac{1}{(|\tau| - 4it)^{n/2-\alpha/2}} \frac{1}{(|\tau| + 4it)^{n/2+\alpha/2}}G_f(\tau, t)\) is integrable in \(\mathbb{R}^2\). So we get (ii). For (iii) we just change \(e^{-i\lambda t}\) to \(e^{i\lambda t}\) and argue as for (ii).

Then, by Proposition 3.2, we obtain

\[
\langle \Phi_{11}, f \rangle = \beta_n \lim_{r \to 1^-} \sum_{k \geq 0} \frac{r^{2k+n-\alpha}}{2k + n - \alpha} \alpha_k \times \int_{\mathbb{R}^2} \left(\frac{|\tau| - 4it}{\tau^2 + 16t^2}\right)^{2k+n-\alpha} \text{sgn}(\tau)G_f(\tau, t) \left(\frac{|\tau| - 4it}{|\tau| + 4it}\right)^n d\tau \, dt
\]
\[
+ \beta_n \lim_{r \to 1^-} \sum_{k \geq 0} \frac{r^{2k+n+\alpha}}{2k + n + \alpha} \alpha_k \times \int_{\mathbb{R}^2} \left(\frac{|\tau| + 4it}{\tau^2 + 16t^2}\right)^{2k+n+\alpha} \text{sgn}(\tau)G_f(\tau, t) \left(\frac{|\tau| + 4it}{|\tau| - 4it}\right)^n d\tau \, dt,
\]

where \(\beta_n = 4^n(n-1)!(1)^n\) and \(\alpha_k = \binom{k+n-1}{k}(-1)^k\).

To study \(\langle \Phi_{11}, f \rangle\) we split each integral into integrals over the left and right halfplanes and take polar coordinates \(\tau - 4it = re^{i\theta}\) to obtain
\[
\langle \Phi_{11}, f \rangle = \beta_n \lim_{r \to 1^-} \sum_{k \geq 0} \alpha_k \frac{r^{2k+n-\alpha}}{2k + n - \alpha}
\]

\[
\times \int_0^{\pi/2} \left[ \int_{-\pi/2}^{\pi/2} e^{i(2k+n-\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha \theta} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) d\theta \right. \\
+ \left. \frac{3\pi/2}{\pi/2} \left( \frac{e^{-i(2k+n-\alpha)\theta}}{(-1)^n 4 \rho^{n-1}} e^{-i\alpha \theta} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) d\theta \right]\right] d\rho
\]

\[
+ \beta_n \lim_{r \to 1^-} \sum_{k \geq 0} \alpha_k \frac{r^{2k+n+\alpha}}{2k + n + \alpha}
\]

\[
\times \int_0^{\pi/2} \left[ \int_{-\pi/2}^{\pi/2} e^{-i(2k+n+\alpha)\theta} \frac{1}{4\rho^{n-1}} e^{i\alpha \theta} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) d\theta \right. \\
+ \left. \frac{3\pi/2}{\pi/2} \left( \frac{e^{i(2k+n+\alpha)\theta}}{(-1)^n 4 \rho^{n-1}} e^{-i\alpha \theta} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) d\theta \right]\right] d\rho.
\]

Now we change variables in the second and fourth terms via \(\theta \leftrightarrow -\theta\). Then, in the fourth term we change variables again according to \(\theta \leftrightarrow \theta + 2\pi\). By Proposition 3.2 we can change the integration order, so we can write

\[
\langle \Phi_{11}, f \rangle = \beta_n \lim_{r \to 1^-} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} \frac{e^{i\alpha \theta}}{\rho^{n-1}} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) d\theta d\rho
\]

\[
+ \frac{(-1)^n}{4} \beta_n \lim_{r \to 1^-} \int_0^{3\pi/2} \int_{\pi/2}^{\pi/2} e^{i\alpha \theta} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, \frac{\rho}{4} \sin \theta \right) d\theta d\rho.
\]

Let \(I\) denote the real interval \([-\pi/2, \pi/2]\). Consider the vector space

\[
X = \{ g \in C^{n-2}(I) : g^{(j)}(\pm \pi/2) = 0, 0 \leq j \leq n-2, g^{(n-1)} \in L^\infty(I) \}.
\]
We identify each function \( g \in \mathcal{X} \) with a function \( \tilde{g} \) on \( S^1 = \mathbb{R}/\mathbb{Z} \), defined to be equal to 0 outside \( \text{supp}(g) \), and we make no distinction between \( g \) and \( \tilde{g} \). Thus, if \( g \in \mathcal{X} \) then \( g \in C^{n-2}(S^1) \) with \( g^{(n-1)} \in L^\infty(S^1) \). Observe that if \( g \in \mathcal{X} \), then also \( e^{i\alpha \theta} g \in \mathcal{X} \). The topology on \( \mathcal{X} \) is given by \( \|g\|_\mathcal{X} = \max_{0 \leq j \leq n-1} \|g^{(j)}\|_\infty \).

For \( k \in \mathbb{Z} \) we set \( \alpha_k = \frac{(k+n-1)}{k} (-1)^k \). Now let us define

\[
\Psi_{r,\alpha}(\theta) = \sum_{k \geq 0} \alpha_k \left( \frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right),
\]

\[
\langle \Psi_\alpha, g \rangle = \left\langle \sum_{k \geq 0} \alpha_k \left( \frac{e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right), g \right\rangle.
\]

We prove that \( \Psi_\alpha \in \mathcal{X}' \), the dual space of \( \mathcal{X} \). Indeed,

\[
|\langle \Psi_\alpha, g \rangle| \leq |e^{i\alpha \theta}| \sum_{k \geq 0} \left( \frac{k+n-1}{k} \right) \left( \left| \frac{e^{i(2k+n)\theta}}{2k+n-\alpha} \right| + \left| \frac{e^{-i(2k+n)\theta}}{2k+n+\alpha} \right| \right).
\]

If \( \hat{g}(n) = \langle g, e^{in\theta} \rangle \) denotes the \( n \)th Fourier coefficient of \( g \), then

\[
|\langle \Psi_\alpha, g \rangle| \leq c \sum_{k \geq 0} \frac{k^{n-1}}{|2k+n|^{n-1}} \left( \frac{|g^{(n-1)}(2k+n)|}{|2k+n-\alpha|} + \frac{|g^{(n-1)}(-2k-n)|}{|2k+n+\alpha|} \right)
\]

\[
\leq c \sum_{k \geq 0} \frac{1}{k} |\hat{g}^{(n-1)}(2k+n)| + \frac{1}{k} |\hat{g}^{(n-1)}(-2k-n)|
\]

\[
\leq c \left( \sum_{k \geq 0} \frac{1}{k^2} \right)^{1/2} \|\hat{g}^{(n-1)}\|_{L^2},
\]

by the Cauchy–Schwarz inequality. Observe that the constants \( c \) are not the same in each expression. By Abel’s Lemma, \( \lim_{r \to 1-} \Psi_{r,\alpha} = \Psi_\alpha \) in \( \mathcal{X}' \), that is, with respect to the weak convergence topology. Similarly, if \( J \) denotes the real interval \([\pi/2, 3\pi/2]\), we define the space

\[
\mathcal{Y} = \{ g \in C^{n-2}(J) : g^{(j)}(\pi/2) = g^{(j)}(3\pi/2) = 0, 0 \leq j \leq n-2, g^{(n-1)} \in L^\infty(J) \},
\]

and find that \( \Psi_\alpha \) is well defined in \( \mathcal{Y}' \) and \( \lim_{r \to 1-} \Psi_{r,\alpha} = \Psi_\alpha \) in \( \mathcal{Y}' \).

Our aim now is to compute \( \Psi_\alpha \). From Proposition 3.7 of \[G-S2\] we know that if \( \Theta \in \mathcal{D}'(S^1) \) is defined by

\[
\Theta(\theta) = i \sum_{k \geq 0} \left( \frac{k+n-1}{k} \right) (-1)^k e^{i(2k+n)\theta},
\]
then for \( n \) even we have

\[
\Re \Theta(\theta) = \frac{d}{d\theta} Q_{n-2} \left( \frac{d}{d\theta} \right) (\delta_{\pi/2} + \delta_{-\pi/2}) = \sum_{j=0}^{n-2} c_j (\delta_{\pi/2}^{(j+1)} + \delta_{-\pi/2}^{(j+1)}),
\]

where \( Q_{n-2} \) is a polynomial of degree \( n - 2 \); and for \( n \) odd we have

\[
\Re \Theta(\theta) = d_0 \frac{d}{d\theta} \tilde{H} + \frac{d}{d\theta} Q_{n-2} \left( \frac{d}{d\theta} \right) (\delta_{\pi/2} - \delta_{-\pi/2}) = d_0 (\delta_{-\pi/2} - \delta_{\pi/2}) + \sum_{j=0}^{n-2} c_j (\delta_{\pi/2}^{(j+1)} - \delta_{-\pi/2}^{(j+1)}),
\]

where \( Q_{n-2} \) is a polynomial of degree \( n - 2 \), and \( \tilde{H}(\theta) = H(\cos \theta) \). Let us recall the generating identity for the Laguerre polynomials (3.8), and take \( t = 0 \) and \( z = -r^2 e^{2i\theta} \). We get

\[
\sum_{k \geq 0} \binom{k + n - 1}{k} (-1)^k r^{2k+n} e^{i(2k+n)\theta} = \left( \frac{r e^{i\theta}}{1 + r^2 e^{2i\theta}} \right)^n.
\]

We also need a couple of results:

**Lemma 3.3.** For a fixed \( r > 1 \) the functions \( \alpha \mapsto \Psi_{r,\alpha}(0) \) and \( \alpha \mapsto \lim_{r \to 1^-} \Psi_{r,\alpha}(0) \) are analytic on \( \Omega = \mathbb{C} \setminus F \), where \( F = \{ 2k + n : k \in \mathbb{Z} \} \).

**Proof.** Let \( K \subset \Omega \) be a compact set. It is easy to see that for fixed \( r \) the series (3.9) converges uniformly, since

\[
|\Psi_{r,\alpha}(0)| \leq \max_{\alpha \in K} |r^\alpha| \left( \frac{r}{1 + r^2} \right)^n d(K, F).
\]

Also, for \( \alpha \in \Omega \) the limit \( \lim_{r \to 1^-} \Psi_{r,\alpha}(0) \) exists. Indeed, if \( 0 \leq r_1 < r < r_2 < 1 \), from the Mean Value Theorem we deduce that for some \( \xi \in (r_1, r_2) \),

\[
\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0) = \frac{d}{dr} \Psi_{\xi,\alpha}(0)(r_2 - r_1) = (\xi^{-\alpha-1} + \xi^{\alpha-1}) \sum_{k \geq 0} \alpha_k \xi^{2k+n}(r_2 - r_1)
\]

\[
= (\xi^{-\alpha-1} + \xi^{\alpha-1}) \left( \frac{\xi}{1 + \xi^2} \right)^n (r_2 - r_1),
\]

where the last equality holds by (3.15). Hence

\[
|\Psi_{r_1,\alpha}(0) - \Psi_{r_2,\alpha}(0)| \leq c(\xi)|r_2 - r_1|,
\]

where \( c(\xi) \) is a constant which depends on \( \xi \). Moreover, for \( \alpha \in K \) and \( \xi \in [1/2, 1] \), \( \xi^{n-\alpha-1} + \xi^{n+\alpha-1} \) is bounded in \( K \times [1/2, 1] \), so the convergence is uniform, hence \( \alpha \mapsto \lim_{r \to 1^-} \Psi_{r,\alpha}(0) \) is an analytic function. ■
Lemma 3.4. Let $0 < \delta < \pi/4$. For $0 < r < 1$ and $0 \leq |\theta| < \delta$ we have
\[
|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)| \leq \left( \max_{0 \leq |\theta| < \delta} e^{\Im \alpha |\theta|} \right) (a |r^{-\alpha} - r^\alpha| + b |r^\alpha|(1 - r)) |\theta|,
\]
with $a, b$ positive constants. Also for $0 \leq |\theta - \pi| < \delta < \pi/4$,\[
|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(\pi)| \leq \left( \max_{0 \leq |\theta - \pi| < \delta} e^{\Im \alpha |\theta|} \right) (a |r^{-\alpha} - r^\alpha| + b |r^\alpha|(1 - r)) |\theta - \pi|,
\]
with $a, b$ positive constants.

Proof. We will estimate $|\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)|$ for $0 < |\theta| < \delta < \pi/4$, the other case being similar. We have\[
\frac{d}{d\theta} \Psi_{r,\alpha}(\theta) = ie^{-i\alpha \theta} \sum_{k \geq 0} \alpha_k r^{2k+n} \left( (r^{-\alpha} - r^\alpha) e^{i(2k+n)\theta} + (e^{i(2k+n)\theta} - e^{-i(2k+n)\theta}) r^\alpha \right)
\]
\[
= ie^{-i\alpha \theta} \left( (r^{-\alpha} - r^\alpha) \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n + 2i r^\alpha \Im \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n \right),
\]
because of (3.15). We have
\[
(3.16) \quad \left| \frac{d}{d\theta} \Psi_{r,\alpha}(\theta) \right| \leq e^{\Im \alpha |\theta|} \left( |r^{-\alpha} - r^\alpha| \left| \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n \right| + 2|r^\alpha| \left| \Im \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n \right| \right).
\]
From Proposition 3.1 of [G-S2] we know that
\[
\left| \Im \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n \right| \rightarrow 0 \quad \text{as } r \rightarrow 1^-, \quad \text{uniformly for } |\theta| < \pi/4, |\theta - \pi| < \pi/4.
\]
Also,\[
\left| \left( \frac{r e^{i\theta}}{1 + r^2 e^{i2\theta}} \right)^n \right| \leq c
\]
for a constant $c$. Then $\left| \frac{d}{d\theta} \Psi_{r,\alpha}(\theta) \right| \rightarrow 0$ uniformly on $|\theta| < \pi/4$ as $r \rightarrow 1^-$, and we get the desired inequality by applying the Mean Value Theorem around 0.

Now we can state the following

Proposition 3.5. For $f \in \mathcal{X}$ we have\[
\langle \Psi_{\alpha}, f \rangle = C_{\alpha} \langle 1, f \rangle, \quad \text{where } C_{\alpha} = \frac{\Gamma \left( \frac{n+\alpha}{2} \right) \Gamma \left( \frac{n-\alpha}{2} \right)}{(n-1)!};
\]
and for $f \in \mathcal{Y}$ we have\[
\langle \Psi_{\alpha}, f \rangle = \tilde{C}_{\alpha} \langle 1, f \rangle, \quad \text{where } \tilde{C}_{\alpha} = (-1)^n e^{-i\alpha \pi} C_{\alpha}.
Proof. First we consider \( f \in X \) such that \( \int_{-\pi/2}^{\pi/2} f(t) \, dt = 0 \) and we define \( F(\theta) = \int_{-\pi/2}^{\theta} f(t) \, dt \). It is easy to see that \( F \in X \) and \( F' = f \). By integration by parts,

\[
\langle \Psi_\alpha, f \rangle = \langle \Psi_\alpha, F' \rangle = \int_{-\pi/2}^{\pi/2} \sum_{k \geq 0} \alpha_k \left( \frac{e^{i(2k+n-\alpha)}\theta}{2k+n-\alpha} + \frac{e^{-i(2k+n+\alpha)}\theta}{2k+n+\alpha} \right) F'(\theta) \, d\theta
\]

\[
= -\langle \Theta, e^{-i\alpha\theta} F \rangle - \langle \bar{\Theta}, e^{-i\alpha\theta} F \rangle,
\]

where \( \bar{\Theta} = \sum_{k \geq 0} \binom{k+n-1}{k} (-1)^k e^{-i(2k+n)\theta} \). So, if \( n \) is even, from (3.13) we get

\[
\langle \Psi_\alpha, f \rangle = 0.
\]

If \( n \) is odd we use (3.14) to conclude that \( \langle \Psi_\alpha, f \rangle = 0 \). For a general \( f \in X \) we consider \( h \in X \) such that \( \int_{-\pi/2}^{\pi/2} h(t) \, dt = 1 \) and define

\[
g(\theta) = f(\theta) - \left( \int_{-\pi/2}^{\pi/2} f(t) \, dt \right) h(\theta).
\]

So we can apply the above result to \( g \) and get \( \langle \Psi_\alpha, g \rangle = 0 \). Then

\[
\langle \Psi_\alpha, f \rangle = \langle \Psi_\alpha, g \rangle + \langle \Psi_\alpha, h \rangle \langle 1, f \rangle = \langle \Psi_\alpha, h \rangle \langle 1, f \rangle.
\]

Let \( C_\alpha = \langle \Psi_\alpha, h \rangle \). In order to compute \( C_\alpha \), consider \( g \in X \) such that \( \text{supp}(g) \subset (-\pi/4, \pi/4) \), \( \int_{-\pi/4}^{\pi/4} g(t) \, dt = 1 \) and \( g \geq 0 \). We have

\[
\langle e^{i\alpha\theta} \Psi_\alpha, g \rangle = C_\alpha \int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} g(\theta) \, d\theta,
\]

and also

\[
\langle e^{i\alpha\theta} \Psi_\alpha, g \rangle
\]

\[
= \lim_{r \to 1^-} \int_{-\pi/2}^{\pi/2} \left( \langle \Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0), e^{i\alpha\theta} g(\theta) \rangle + \Psi_{r,\alpha}(0) \int_{-\pi/2}^{\pi/2} e^{i\alpha\theta} g(\theta) \, d\theta \right) \, d\theta.
\]

From Lemmas 3.3 and 3.4 we deduce that

\[
C_\alpha = \lim_{r \to 1^-} \Psi_{r,\alpha}(0)
\]

and also that \( C_\alpha \) is an analytic function of \( \alpha \). Since \( \Psi_{0,\alpha}(0) = 0 \), we can
write

\[ C_\alpha = \lim_{r \to 1^-} \Psi_{r,\alpha}(0) = \Psi_{1,\alpha}(0) - \Psi_{0,\alpha}(0) = \frac{1}{0} \int w'_\alpha(s) \, ds, \]

where

\[ w_\alpha(r) = \Psi_{r,\alpha}(0) = r^{-\alpha} \sum_{k \geq 0} \frac{\alpha_k r^{2k+n}}{2k+n-\alpha} + r^\alpha \sum_{k \geq 0} \frac{\alpha_k r^{2k+n}}{2k+n+\alpha}. \]

Applying (3.8) with \( \theta = 0 \) we obtain

\[ w'_\alpha(r) = (r^{-\alpha-1} + r^{\alpha-1}) \sum_{k \geq 0} \alpha_k r^{2k+n} = (r^{-\alpha-1} + r^{\alpha-1}) \left( \frac{r}{1 + r^2} \right)^n, \]

and we can compute the integral for \( \Re(n+\alpha) > 0, \Re(n-\alpha) > 0, \) obtaining

\[ C_\alpha = B \left( \frac{n + \alpha}{2}, \frac{n - \alpha}{2} \right) = \frac{\Gamma \left( \frac{n + \alpha}{2} \right) \Gamma \left( \frac{n - \alpha}{2} \right)}{(n-1)!}, \]

where \( B \) is the Beta function and \( \Gamma \) is the Gamma function. By Lemma 3.3, (3.17) holds for \( \alpha \in \Omega \) by analytic continuation. In a completely analogous way we conclude that \( \tilde{C}_\alpha = (-1)^n e^{-i \alpha \pi} C_\alpha. \)

Let us now define

\[ K_1 f(\rho, \theta) = \frac{1}{\rho^{n-1}} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) \]

for \( \theta \in [-\pi/2, \pi/2], \) \( 0 < \rho < \infty, \) where \( G_f \) is the function defined in (3.4); and

\[ K_2 f(\rho, \theta) = \frac{1}{\rho^{n-1}} \text{sgn}(\cos \theta) G_f \left( \rho \cos \theta, \frac{\rho}{4} \sin \theta \right) \]

for \( \theta \in [\pi/2, 3\pi/2], \) \( 0 < \rho < \infty. \)

It is easy to check that \( K_1 f(\rho, \cdot) \in \mathcal{X}. \) Recall that we replaced \( \tau - 4it \) with \( \rho e^{i\theta}. \) Since \( N f \in \mathcal{H}_n, \) there exists a positive constant \( c \) such that

\[ \sup_{\tau \neq 0, t \in \mathbb{R}} |(\tau^2 + 16t^2)N f(\tau, t)| \leq c, \]

that is,

\[ \left| N f \left( \rho \cos \theta, -\frac{\rho}{4} \sin \theta \right) \right| \leq \frac{c}{\rho^2}. \]

Also, since \( N f(0, \cdot) \in \mathcal{S}(\mathbb{R}), \) there exists a positive constant \( c_N \) such that for \( t \in \mathbb{R}, \)

\[ \left| t^N \sum_{j=0}^{n-2} \frac{\partial^j}{\partial \tau^j} N f(0, t) \frac{\tau^j}{j!} \right| \leq c_N |\tau|^{n-2}. \]
Thus, for \( N \in \mathbb{N} \) there exists \( c_N \) such that

\[
(3.20) \quad |K_{1f}(\rho, \theta)| \leq \frac{a}{\rho^{n+1}} + \frac{b}{\rho^{N+1}} \frac{|\cos \theta|^{n-2}}{|\sin \theta|^{N}}.
\]

Analogous observations are also true for \( K_{2f} \).

**Proposition 3.6.** Let \( C_\alpha \) and \( \tilde{C}_\alpha \) be the constants obtained in (3.17). Let \( K_{1f} \) and \( K_{2f} \) be defined by (3.18) and (3.19), and \( \alpha_k = \left( \frac{k+n-1}{k} \right)(-1)^k \). Then

\[
\lim_{r \to 1^-} \int_0^{\pi/2} \int_{-\pi/2}^{\pi/2} e^{i\alpha \theta} \sum_{k \geq 0} \alpha_k \left( \frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right) K_{1f}(\rho, \theta) d\theta d\rho
\]

\[
= 4^{n-1}(n-1)! C_\alpha \int_{\mathbb{R}} \int_{\tau>0} \frac{1}{(\tau-4it)^{(n-\alpha)/2}} \frac{1}{(\tau+4it)^{(n+\alpha)/2}} \text{sgn}(\tau) G_f(\tau, t) d\tau dt,
\]

and

\[
\lim_{r \to 1^-} \int_0^{3\pi/2} \int_{\pi/2}^{\pi/2} e^{i\alpha \theta} \sum_{k \geq 0} \alpha_k \left( \frac{r^{2k+n-\alpha} e^{i(2k+n-\alpha)\theta}}{2k+n-\alpha} + \frac{r^{2k+n+\alpha} e^{-i(2k+n+\alpha)\theta}}{2k+n+\alpha} \right) \times K_{2f}(\rho, \theta) d\theta d\rho
\]

\[
= 4^{n-1}(n-1)! \tilde{C}_\alpha \int_{\mathbb{R}} \int_{\tau<0} \frac{1}{(\tau-4it)^{(n-\alpha)/2}} \frac{1}{(\tau+4it)^{(n+\alpha)/2}} \text{sgn}(\tau) G_f(\tau, t) d\tau dt.
\]

**Proof.** The proof follows the same lines of Proposition 4.2 of [G-S2]. We sketch it for the sake of completeness.

Taking polar coordinates \( \tau - 4it = \rho e^{i\theta} \) we only need to show that

\[
(3.21) \quad \lim_{r \to 1^-} \int_0^\infty \langle \Psi_r, e^{i\alpha \theta} K_{1f}(\rho, \theta) \rangle d\rho = \int_0^\infty \langle C_\alpha, e^{i\alpha \theta} K_{1f}(\rho, \theta) \rangle d\rho.
\]

In order to do this we split the integral into integrals over \( 0 < \rho < 1 \) and \( 1 < \rho < \infty \).

We consider first the case \( 1 < \rho < \infty \). For \( |\theta| \leq \delta < \pi/4 \), set

\[
I = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha \theta} (\Psi_{r,\alpha}(\theta) - \Psi_{r,\alpha}(0)) K_{1f}(\rho, \theta) d\theta d\rho
\]

\[
II = \int_1^\infty \int_{|\theta|<\delta} e^{i\alpha \theta} (\Psi_{r,\alpha}(0) - C_\alpha) K_{1f}(\rho, \theta) d\theta d\rho.
\]

We bound \( I \) close to 0 by applying Lemma 3.4 and taking \( N = 1 \) in (3.20). For \( II \) we just take \( N = 1/2 \) in (3.20). To analyze the case \( \delta \leq |\theta| \leq \pi/2 \), we observe that the function \( K_{1f}^*(\theta) = \int_1^\infty K_{1f}(\rho, \theta) d\rho \) defined for \( \theta \in \)
$[-\pi/2, -\delta] \cup [\delta, \pi/2]$ can be extended to an element of $X$ that we still denote by $K_1^*f$. Then

$$\int_{\delta<|\theta|<\pi/2} e^{i\alpha\theta} (\Psi_{r,\alpha}(\theta) - C_\alpha) K_1 f(\rho, \theta) d\theta d\rho$$

$$= \int_{|\theta|<\delta} e^{i\alpha\theta} (\Psi_{r,\alpha}(\theta) - C_\alpha) K_1^* f(\theta) d\theta - \int_{\delta<|\theta|<\pi/2} e^{i\alpha\theta} (\Psi_{r,\alpha}(\theta) - C_\alpha) K_1^* f(\theta) d\theta.$$ 

The first term converges to zero as $r \to 1^-$ since $\Psi_{r,\alpha} \to C_\alpha$ as $r \to 1^-$ in $X'$. For the second term we argue as above.

Finally, for the case $0 < \rho < 1$ we apply the same arguments to the function $K_1^*\alpha f(\theta) = \int_0^1 K_1 f(\rho, \theta) d\rho$.

**Corollary 3.7.** $\langle \Phi_{11}, f \rangle$ is well defined for $f \in S(\mathbb{H}_n)$, and

$$\langle \Phi_{11}, f \rangle = 4^{n-1}(n-1)! C_\alpha \int_{\mathbb{R}} \int_{\tau > 0} \frac{1}{(\tau - 4it)^{(n-\alpha)/2}} \frac{1}{(\tau + 4it)^{(n+\alpha)/2}} \sgn(\tau) G_f(\tau, t) d\tau dt$$

$$+ 4^{n-1}(n-1)! \tilde{C}_\alpha \int_{\mathbb{R}} \int_{\tau < 0} \frac{1}{(\tau - 4it)^{(n-\alpha)/2}} \frac{1}{(\tau + 4it)^{(n+\alpha)/2}} \sgn(\tau) G_f(\tau, t) d\tau dt.$$ 

From the corollary we also infer that $\langle \Phi_{12}, f \rangle$ is well defined. In order to explicitly compute it, we define, for $0 \leq l \leq n - 2$, $\epsilon > 0$ and $f \in S(\mathbb{H}_n)$,

$$d_{\epsilon,l,f}^- = \int_{\infty}^{\infty} \int_{0}^{-\infty} e^{-\epsilon |\lambda|} e^{-i\lambda t} |\lambda|^{n-l-2} \frac{\partial^l}{\partial \tau^l} N f(0, t) dt d\lambda, $$

$$d_{\epsilon,l,f}^+ = \int_{\infty}^{\infty} \int_{0}^{-\infty} e^{-\epsilon |\lambda|} e^{i\lambda t} |\lambda|^{n-l-2} \frac{\partial^l}{\partial \tau^l} N f(0, t) dt d\lambda. $$

Then we can write (3.6) as

$$\langle \Phi_{12}, f \rangle = \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \sum_{l=0}^{n-2} 2^{l+2} (a_{kl} + b_{kl}) \left[ \frac{r^{2k+n-\alpha}}{2k+n-\alpha} d_{\epsilon,l,f}^- + \frac{r^{2k+n+\alpha}}{2k+n+\alpha} d_{\epsilon,l,f}^+ \right]. $$

From Lemma 4.4 in [G-S2] we deduce that

$$a_{kl} + b_{kl} = (-1)^k \sum_{j=1}^{l+1} \frac{1}{2^{n-l-j-1}} \binom{n-j-1}{l-j+1} \binom{j+k-1}{k}. $$

We also have the following
Lemma 3.8. If $0 \leq l \leq n - 2$, $\epsilon > 0$ and $f \in \mathcal{S}(\mathbb{H}_n)$, then

$$\lim_{\epsilon \to 0^+} d_{\epsilon, l, f}^- = \frac{1}{i^{n-l-2}} \left\langle \frac{\pi}{2} \delta - i \text{ p.v.} \left( \frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} Nf(0, \cdot) \right\rangle,$$

$$\lim_{\epsilon \to 0^+} d_{\epsilon, l, f}^+ = i^{n-l-2} \left\langle \frac{\pi}{2} \delta + i \text{ p.v.} \left( \frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} Nf(0, \cdot) \right\rangle,$$

Proof. Let us consider $g(\lambda) = e^{-\epsilon |\lambda| |\lambda|^{n-l-2}}$ and $h(t) = \frac{\partial^l}{\partial \tau^l} Nf(0, t)$, and observe that $\int_{-\infty}^{\infty} e^{-i\lambda t} h(t) \, dt = \hat{h}(\lambda)$. Then just by using the properties of the Fourier transform we get

$$d_{\epsilon, l, f}^- = \int_{-\infty}^{\infty} g(\lambda) e^{-i\lambda t} h(t) \, dt \, d\lambda = \int_{0}^{\infty} g(\lambda) \hat{h}(\lambda) \, d\lambda$$

$$= \frac{1}{i^{n-l-2}} \int_{-\infty}^{\infty} \frac{1}{\epsilon + i\lambda} h(n-l-2)(\lambda) \, d\lambda.$$

For each $\epsilon > 0$, $\frac{1}{\epsilon + i\lambda}$ is a distribution such that the limit $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon + i\lambda}$ exists in $\mathcal{S}'(\mathbb{R})$. Moreover, it is easy to check that

$$\lim_{\epsilon \to 0^+} \frac{1}{\epsilon + i\lambda} = \frac{\pi}{2} \delta - i \text{ p.v.} \left( \frac{1}{\lambda} \right).$$

Thus the desired equality follows. For $d_{\epsilon, l, f}^+$ we need to change variables according to $\lambda \leftrightarrow -\lambda$ after considering the Fourier transform of $h$. 

For $j \in \mathbb{N}$, $0 < j < n - 1$, we define functions of $r$, with $0 \leq r < 1$, by

$$w_j^-(r) = \sum_{k \geq 0} (-1)^k \binom{j + k - 1}{k} \frac{r^{2k+n-\alpha}}{2k + n - \alpha},$$

$$w_j^+(r) = \sum_{k \geq 0} (-1)^k \binom{j + k - 1}{k} \frac{r^{2k+n+\alpha}}{2k + n + \alpha}.$$

We can see, in a completely analogous way to the computations made for $C_{\alpha}$ and $\tilde{C}_{\alpha}$, that these functions are well defined and that

$$c_j^- := \lim_{r \to 1^-} w_j^-(r) = \frac{1}{2} B_{1/2} \left( \frac{n - \alpha}{2}, j - \frac{n - \alpha}{2} \right),$$

$$c_j^+ := \lim_{r \to 1^-} w_j^+(r) = \frac{1}{2} B_{1/2} \left( \frac{n + \alpha}{2}, j - \frac{n + \alpha}{2} \right),$$

where $B_{1/2}$ is another special function called the incomplete Beta function.
We now combine all of these definitions and results together to finally obtain an expression for \( \Phi_{12} \):

\[
\langle \Phi_{12}, f \rangle = \sum_{l=0}^{n-2} \sum_{l \text{ odd}}^{l+1} 2^{l-n+j+3} \left( \frac{n-j-1}{l-j+1} \right) \left[ \left( \frac{1}{i^{n-l-2}} c_j^- + i^{n-l-2} c_j^+ \right) \pi \right] \times \left\langle \delta, \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N_f(0, \cdot) \right\rangle 
\]

\[
+ (-1)^{l} \sum_{j=1}^{2^{l-n+j+3}} \left( \frac{n-j-1}{l-j+1} \right) \left( \frac{1}{i^{n-l+1}} c_j^- + i^{n-l+1} c_j^+ \right) \times \left\langle \text{p.v.}, \left( \frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N_f(0, \cdot) \right\rangle.
\]

All we need to do now is to use again Lemma 3.8 to get an expression for \( \Phi_2 \). Thus, we have proved the following

**Theorem 3.9.** Let \( C_\alpha \) and \( \tilde{C}_\alpha \) be the constants defined as in (3.17). Then there exist constants \( C_l \) and \( \tilde{C}_l \), \( l = 0, \ldots, n-2 \), such that

\[
\langle \Phi_\alpha, f \rangle = 4^{n-1}(n-1)!C_\alpha \int_{-\infty}^{\infty} \int_{\tau>0} \frac{1}{(\tau - 4it)^{(n-\alpha)/2}} \frac{1}{(\tau + 4it)^{(n+\alpha)/2}} \text{sgn}(\tau) G_f(\tau, t) \, d\tau \, dt 
\]

\[
+ 4^{n-1}(n-1)!\tilde{C}_\alpha \int_{-\infty}^{\infty} \int_{\tau<0} \frac{1}{(\tau - 4it)^{(n-\alpha)/2}} \frac{1}{(\tau + 4it)^{(n+\alpha)/2}} \text{sgn}(\tau) G_f(\tau, t) \, d\tau \, dt 
\]

\[
+ \sum_{l=0}^{n-2} C_l \left\langle \delta, \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N_f(0, \cdot) \right\rangle 
\]

\[
+ \sum_{l=0}^{n-2} \tilde{C}_l \left\langle \text{p.v.}, \left( \frac{1}{\lambda} \right), \frac{\partial^{n-2}}{\partial \lambda^{n-l-2} \partial \tau^l} N_f(0, \cdot) \right\rangle.
\]

The constants \( C_l \) and \( \tilde{C}_l \) follow from the expressions obtained for \( \Phi_{12} \) and \( \Phi_2 \).

**4. A fundamental solution for \( L \).** As in the classical case, the distribution \( \Phi \) defined in (1.6) is a well defined tempered distribution and it is a fundamental solution for the operator \( L \). The proof is identical to the one of Theorem 3.1.

We will compute the fundamental solution \( \Phi \) by means of the Radon transform and the fundamental solution of the operator \( L \) in the classical case \( N(2p, 2q, \mathbb{C}) \).
Let $F \in \mathcal{S}(\mathbb{R}^3)$. We assign to $F$ a function $\mathcal{R}F : \mathbb{R} \times S^2 \to \mathbb{R}$ defined by

$$\mathcal{R}F(t, \xi) = \int_{\mathbb{R}^2} F(t\xi + u_1e_1 + u_2e_2) \, du_1 \, du_2,$$

where $\{\xi, e_1, e_2\}$ is an orthonormal basis of $\mathbb{R}^3$. It is easy to see that this definition does not depend on the choice of the basis. In order to recover $F$ from $\mathcal{R}F$, we consider the space $\mathcal{S}(\mathbb{R} \times S^2)$ of continuous functions $G : \mathbb{R} \times S^2 \to \mathbb{R}$ that are infinitely differentiable in $t$ and satisfy, for every $m, k \in \mathbb{N}_0$,

$$\sup_{t \in \mathbb{R}, \xi \in S^2} \left| t^m \frac{\partial^k}{\partial t^k} G(t, \xi) \right| < \infty.$$ 

Now for $G \in \mathcal{S}(\mathbb{R} \times S^2)$ we define a function $\mathcal{R}^*G : \mathbb{R}^3 \to \mathbb{R}$ by

$$\mathcal{R}^*G(z) = \int_{S^2} G(\langle z, \xi \rangle, \xi) \, d\xi.$$ 

Both assignments are well defined. The map $\mathcal{R} : \mathcal{S}(\mathbb{R}^3) \to \mathcal{S}(\mathbb{R} \times S^2)$ is the Radon transform, $\mathcal{R}^* : \mathcal{S}(\mathbb{R} \times S^2) \to \mathcal{S}(\mathbb{R}^3)$ is the dual Radon transform and they satisfy, for every $F \in \mathcal{S}(\mathbb{R}^3)$,

(4.1) \quad -2\pi F = \Delta \mathcal{R}^* \mathcal{R} F,

where $\Delta = \partial^2/\partial z_1^2 + \partial^2/\partial z_2^2 + \partial^2/\partial z_3^2$ is the $\mathbb{R}^3$-Laplacian (see for Example [S-Sh]).

Now, let us consider the function $\phi$ defined for a fixed $\tau \neq 0$ by

$$\phi(\tau, z) = \frac{16n}{\pi} \cdot \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16|z|^2)^{n+1}},$$

where $c_0 = -\int_0^1 \sigma^{2n-1}(1 + \sigma^2)^{2n} \, d\sigma$. The function $\phi(\tau, \cdot)$ is not a Schwartz function on $\mathbb{R}^3$, but we have $(1 + \Delta)^k \phi(\tau, \cdot) \in L^1(\mathbb{R}^3)$ for all $k \in \mathbb{N}$, hence $(1 + |\xi|^2)^k \phi(\tau, \cdot)(\xi) \in L^\infty(\mathbb{R}^3)$. With these properties the inversion formula for the Radon transform (4.1) still holds. The proof follows straightforwardly from Theorem 5.4 of [S-Sh].

Let us now compute the Radon transform of the function $\phi$:

$$\mathcal{R}\phi(\tau, t, \xi) = \int_{\mathbb{R}^2} \frac{16n}{\pi} \cdot \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16(t^2 + u_1^2 + u_2^2))^{n+1}} \, du_1 \, du_2$$

$$= \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{16^{n+1}} \int_{\mathbb{R}^2} \frac{1}{(\tau^2 / 16 + t^2 + (u_1^2 + u_2^2))^{n+1}} \, du_1 \, du_2$$

$$= \frac{16n}{\pi} \frac{4^{2n}(2n-1)!c_0}{16^{n+1}} \int_{-\pi/2}^{\pi/2} \int_0^\infty \frac{\rho}{(\tau^2 / 16 + t^2 + \rho^2)^{n+1}} \, d\rho \, d\theta$$

$$= \frac{4^{2n}(2n-1)!c_0}{(\tau^2 + 16|z|^2)^n}.$$
where \( z = t\xi \). Let
\[
\varphi(\tau, z) = \frac{4^{2n}(2n - 1)!c_0}{(\tau^2 + 16|z|^2)^n}.
\]
Now from the expression of the fundamental solution of \( L \) in the classical case (see for example 4.3 of [G-S2]) we know that
\[
\varphi(\tau, t\xi) = \sum_{k \geq 0} \frac{(-1)^{2k + n}}{(2k + 2n - 1)!} c_0 \left( \frac{\lambda}{2} \right)^{|\tau|} e^{-\frac{1}{4}r|\lambda|^{2n-1}} d\lambda.
\]
We observe that the operator \( L \) has a nontrivial kernel, and define, for \( f \in S(N(p,q,\mathbb{H})) \),
\[
Pf = \int_{\mathbb{R}^3} f \ast \varphi_{w,q-p} \, |w|^{2n} \, dw.
\]
Then \( LPf = 0 \).

To compute \( \Phi \) we express the integral in (1.6) in polar coordinates:
\[
\langle \Phi, f \rangle = \sum_{k \in \mathbb{Z}, k \neq q-p} \int_{S^2} \frac{1}{-|\lambda|(2k + 2(p - q))} \langle \varphi_{w,k}, f \rangle |w|^{2n} \, dw
\]
\[
= \sum_{k \in \mathbb{Z}, k \neq q-p} \int_{S^2} \frac{1}{-|\lambda|(2k + 2(p - q))} \langle \varphi_{\lambda\xi,k}, f \rangle |\lambda|^{2n+2} \, d\lambda \, d\xi.
\]
By the absolute convergence of (1.6) we can interchange the summation with the integral over \( S^2 \). Since \( \Delta e^{i\lambda(\xi,z)} = -|\lambda|^2 e^{i\lambda(\xi,z)} \), integrating by parts, we obtain
\[
\langle \Phi, f \rangle = \int_{S^2} \sum_{k \in \mathbb{Z}, k \neq q-p} \frac{(-1)}{(2k + 2(p - q))} \int_0^{\infty} \int_{N(p,q,\mathbb{H})} e^{i\lambda(\xi,z)} \theta_{\lambda,k}(\alpha)f(\alpha, z) \, d\alpha \, dz
\]
\[
\times |\lambda|^{2n+1} \, d\lambda \, d\xi
\]
\[
= \int_{S^2} \sum_{k \in \mathbb{Z}, k \neq q-p} \frac{1}{(2k + 2(p - q))} \int_0^{\infty} \int_{N(p,q,\mathbb{H})} \Delta e^{i\lambda(\xi,z)} \theta_{\lambda,k}(\alpha)f(\alpha, z) \, d\alpha \, dz
\]
\[
\times |\lambda|^{2n-1} \, d\lambda \, d\xi
\]
\[
= \int_{S^2} \sum_{k \in \mathbb{Z}, k \neq q-p} \frac{1}{(2k + 2(p - q))} \int_0^{\infty} \int_{N(p,q,\mathbb{H})} \langle \varphi_{\lambda\xi,k}, \Delta f \rangle |\lambda|^{2n-1} \, d\lambda \, d\xi.
\]
Next we break the summation range into three parts, for \( k \geq 2q, k \leq -2p \) and \(-2p < k < 2q\), to get the splitting \( \langle \Phi, f \rangle = \langle \Phi_1, f \rangle + \langle \Phi_2, f \rangle \), and as in Section 3 we change the summation index to make the series start from \( k = 0 \). Using the explicit definition of \( \varphi_{\lambda\xi,k} \) we can write
\[
\langle \Phi_1, f \rangle = \sum_{k \geq 0} \frac{1}{2k + 2n} \int_{\mathbb{R}^3} e^{i\lambda(\xi,z)}
\]
\[
\times \langle T_{\lambda,k+2q} - T_{\lambda,-k-2p}, N\Delta f(\cdot, z) \rangle \, dz \, |\lambda|^{2n-1} \, d\lambda \, d\xi,
\]
where $T_{\lambda,k} = F_{\lambda,k}$ is defined by equations (2.2) and (2.3). By performing similar computations to those in Section 3 and introducing the function

$$G_f(\tau, z) = N_f(\tau, z) - \sum_{j=0}^{2n-2} \frac{\partial^j N_f}{\partial \tau^j}(0, z) \frac{\tau^j}{j!},$$

we obtain the splitting

$$\langle \Phi_1, f \rangle = \langle \Phi_{11}, f \rangle + \langle \Phi_{12}, f \rangle,$$

where

$$\langle \Phi_{11}, f \rangle = \int \sum_{S^2} \sum_{k \geq 0} \frac{(-1)}{2k + 2n} \int \int_{\mathbb{R}^3} e^{i\lambda(\xi, z)} \times \text{sgn}(\tau) L_{2n-1}^k \left( \frac{2}{\lambda} |\tau| \right) e^{-\lambda/4|\tau| \Delta G_f(\tau, z)} d\tau dz |\lambda|^{2n-1} d\lambda d\xi,$$

and $a_{kl}, b_{kl}$ are the same constants defined in (3.7) and (3.3), respectively. Now we recall that

$$\int \int_{S^2} e^{i\lambda(\xi, z)} F(|\lambda|) d\lambda d\xi = \frac{1}{2} \int \int_{S^2} e^{i\lambda(\xi, z)} F(|\lambda|) d\lambda d\xi,$$

and apply the dual Radon transform to (4.2).

Observe now that

$$\int \int_{\mathbb{R}^3} \text{sgn}(\tau) G_f(\tau, z) (1 + 16|z|^2)^{n+1} d\tau dz$$

converges, which can be seen by changing to polar coordinates in $\mathbb{R}^3$ and arguing as in Lemma 2.2 of [G-S2].

We finally get

$$\langle \Phi_{11}, f \rangle = \frac{1}{2} \left\langle -2\pi \frac{16n}{\pi} \frac{4^{2n}(2n - 1)!c_0}{(\tau^2 + 16|z|^2)^{n+1}}, \text{sgn}(\tau) G_f(\tau, z) \right\rangle$$

$$= -4^{2n+2}n(2n - 1)!c_0 \left\langle \frac{1}{(\tau^2 + 16|z|^2)^{n+1}}, \text{sgn}(\tau) G_f(\tau, z) \right\rangle.$$
We have thus proven that the expression defining $\Phi_{11}$ is finite. Then the expression defining $\Phi_{12}$ is also finite, and by Abel’s Lemma we can write

$$\langle \Phi_{12}, f \rangle = 2 \lim_{r \to 1^-} \lim_{\epsilon \to 0^+} \sum_{k \geq 0} \sum_{l=0}^{2n-2} 2^{l+1} (a_{k,l} + b_{k,l}) r^{2k+2n} \frac{2k+2n}{2k+2n} \sum_{l=0}^{2n-2} \sum_{l=0}^{2n-2} (a_{k,l} + b_{k,l}) r^{2k+2n} \frac{2k+2n}{2k+2n} \sum_{l=0}^{2n-2} \sum_{l=0}^{2n-2} (a_{k,l} + b_{k,l})$$

where

$$(4.4) \quad d_{\epsilon,l,f} = \int_{0}^{\infty} \int_{0}^{\infty} e^{-\epsilon |x|} |\lambda|^{2n-l-2} \langle \delta^{(l)} \Delta N f(\cdot, z) \rangle d\lambda d\xi.$$
By this computation together with Proposition 4.8 of \[\text{[G-S2]}\] we write
\[
\langle \Phi_{12}, f \rangle = \sum_{l=0}^{2n-2} \sum_{j=1}^{l+1} \frac{1}{2^{2n-2l-j-3}} c_j \binom{2n-j-1}{l-j-1} (-1)^{2n-l-2} \frac{\partial^l}{\partial \tau^l} N f(0,0),
\]
where each \(c_j\) is the constant defined in Remark 4.7 of \[\text{[G-S2]}\] as follows:
\[
c_j = \int_0^1 \frac{r^{j-1}}{(1+r^2)^j} dr.
\]

After performing the usual computations for \(\Phi_2\) we will have proved the main theorem of this section:

**Theorem 4.1.** Let \(c_0\) be the constant defined above. Then there exist constants \(c_l(k)\), \(l = 0, \ldots, 2n - 2\) and \(-2p < k < 2q\), such that
\[
\langle \Phi, f \rangle = -4^{2n+2} n(2n-1)! c_0 \frac{1}{(\tau^2 + 16|z|^2)^n + 1} \text{sgn}(\tau) G_f(\tau, z) + \sum_{-2p < k < 2q} \sum_{l=0}^{2n-2} c_l(k) (-\Delta)^{(2n-l-2)/2} \frac{\partial^l}{\partial \tau^l} N f(0,0).
\]

**Remark 4.2.** Let \(N\) be a group of Heisenberg type and let \(\eta\) be its Lie algebra. So \(\eta = V \oplus \mathfrak{z}\), with \(\dim V = 2m\) and \(\dim \mathfrak{z} = k\). Let \(U(V)\) be the unitary group acting on \(V\). Then it is known (\[\text{[R]}\]) that \((N \ltimes U(V), U(V))\) is a Gelfand pair. In \[\text{[R]}\] the spherical functions were computed. We fix an orthonormal basis of \(V\), \(\{X_1, \ldots, X_{2m}\}\), and consider the operator
\[
L = \sum_{j=1}^{2m} X_j^2.
\]

With the same arguments as above, using the Radon transform in \(\mathbb{R}^k\) and the fundamental solution of \(L\) in the classical \(2m+1\)-dimensional Heisenberg group, we can recover the fundamental solution of \(L\) (see \[\text{[K], [R]}\]).

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