COLLOQUIUM MATHEMATICUM

VOL. 106 2006 NO. 2

THE THEORY OF REPRODUCING SYSTEMS ON LOCALLY COMPACT ABELIAN GROUPS

BY

GITTA KUTYNIOK (Giessen) and DEMETRIO LABATE (Raleigh, NC)

Abstract. A *reproducing system* is a countable collection of functions $\{\phi_j : j \in \mathcal{J}\}$ such that a general function f can be decomposed as $f = \sum_{j\in\mathcal{J}} c_j(f)\,\phi_j$, with some control on the analyzing coefficients $c_j(f)$. Several such systems have been introduced very successfully in mathematics and its applications. We present a unified viewpoint in the study of reproducing systems on locally compact abelian groups G. This approach gives a novel characterization of the Parseval frame generators for a very general class of reproducing systems on $L^2(G)$. As an application, we obtain a new characterization of Parseval frame generators for Gabor and affine systems on $L^2(G)$.

1. Introduction. The term *reproducing system* is applied to any of several methods that decompose a general function f in terms of a countable system of functions $\{\phi_j : j \in \mathcal{J}\}$ so that

$$f = \sum_{j\in\mathcal{J}} c_j(f)\,\phi_j,$$

where the $c_j(f)$ are appropriate coefficient functionals, and the norm of f is equivalent to the norm of the coefficients $\{c_j(f) : j \in \mathcal{J}\}$. A variety of such systems have been used very successfully in both pure and applied mathematics. They are generated by a single function or a finite collection of functions, by applying to these functions a countable family of operators. These operators involve typically two of the following three actions: dilations, modulations, and translations. The *Gabor systems*, for example, have the form

$$\mathcal{G}_B(\Psi) = \{M_{Bm}T_k\psi^l : m,k \in \mathbb{Z}^n,\ l = 1,\dots,L\}, \tag{1}$$

where $\Psi = (\psi^1,\dots,\psi^L) \subset L^2(\mathbb{R}^n)$, $B \in \mathrm{GL}_n(\mathbb{R})$, T_k are the *translations*, defined by $T_kf(x) = f(x-k)$, and M_y are the *modulations*, defined by $M_yf(x) = e^{2\pi i\langle y,x\rangle}f(x)$. The *affine systems* (which generate a variety of wavelets), on the other hand, have the form

$$\mathcal{W}_A(\Psi) = \{D_{A^j}T_k\psi^l : j \in \mathbb{Z},\ k \in \mathbb{Z}^n,\ l = 1,\dots,L\}, \tag{2}$$

2000 *Mathematics Subject Classification*: 43A70, 43A40, 43A15.

Key words and phrases: affine systems, frames, Gabor systems, locally compact groups, wavelets.

The first author acknowledges support by DFG research fellowship KU 1446/5.

where $A \in \mathrm{GL}_n(\mathbb{R})$ and D_A are the *dilations*, that are defined by $D_A f(x) = |\det A|^{-1/2} f(A^{-1}x)$. By choosing Ψ, B, and A appropriately, one can make $\mathcal{G}_B(\Psi)$ and $\mathcal{W}_A(\Psi)$ an orthonormal basis or, more generally, a Parseval frame for $L^2(\mathbb{R}^n)$ (defined below).

While the theory of Gabor and affine systems has usually been developed on $\mathbb{R}^n$, there is an increasing interest in the study of these systems in other settings (for example, [1, 8, 12, 16]). Indeed, discrete signal processing applications, as well as numerical implementations of these theories, require the construction of reproducing systems on $\mathbb{Z}^n$ or finite abelian groups. Moreover, in several applied problems, as for example in the numerical solution of PDE, one has to deal with bounded domains, and the Gabor or affine systems on $\mathbb{R}^n$ are unable to handle effectively the boundary conditions. Therefore it is quite useful to consider reproducing systems adapted to bounded domains.

One of the aims of this paper is to extend many ideas and constructions from the theory of Gabor and affine systems to the setting of locally compact abelian groups. One major result is a novel characterization of all those functions that form a Parseval frame for $L^2(G)$, where G is a locally compact abelian group. This allows us to handle several classes of reproducing systems on $L^2(G)$ in a unified manner. As an application of this approach, we are able to extend and generalize several results from the theory of Gabor and affine systems on $\mathbb{R}^n$ to the setting of general locally compact abelian groups.

The paper is organized as follows. We recall some basic facts about locally compact abelian groups and frame theory in Section 2. In Section 3 we present our general characterization result for Parseval frame generators. Finally, in Section 4 we describe several applications of this theorem, including the cases of Gabor and affine systems.

2. Preliminaries. Before embarking on our study, it is useful to establish some notation and recall some basic facts from the theory of locally compact abelian groups. More details can be found, for example, in the monographs [15, 20].

Let G denote a locally compact abelian group with unit element e. We will consider G to be equipped with a left-invariant Haar measure m_G, which is unique up to a constant multiple, and is finite if and only if G is compact. In addition, we will assume that G is σ-compact, i.e., G is a countable union of compact sets, and metrizable, i.e., there is a metric d on G. Any locally compact metrizable σ-compact abelian group will be called an *LCA group*.

The dual group of G, that we denote by $\widehat{G}$, is the set of all *characters*, i.e., all continuous homomorphisms from G into the torus $\mathbb{T}$. It turns out that $\widehat{G}$ also becomes an LCA group under the pointwise multiplication, with unit element 1, and thus possesses a Haar measure.

A discrete subgroup D of G with compact quotient group G/D will be called a *uniform lattice*. A *fundamental domain* for D is a measurable subset $F \subset G$ such that every $x \in G$ can be uniquely written in the form $x = fd$ for some $f \in F$ and $d \in D$. It was shown in [18, Lemma 2] that such a fundamental domain always exists. We define the *lattice size* of D to be $s(D) = m_G(F)$. It can be easily shown that this definition is independent of the particular choice of F. The *annihilator* of D in G, denoted by $D^\perp$, is defined by

$$D^\perp = \{\gamma \in \widehat{G} : \gamma(d) = 1 \text{ for all } d \in D\}.$$

Then $D^\perp$ is a uniform lattice in $\widehat{G}$, since $D^\perp$ is topologically isomorphic to $\widehat{G/D}$ and $\widehat{G}/D^\perp$ is topologically isomorphic to $\widehat{D}$ (via the restriction map $\omega D^\perp \to \omega|D$). The following lemma [12, Lemma 6.2.3(a)] will be useful.

Lemma 2.1. *If D is a uniform lattice in G, then*

$$s(D)s(D^\perp) = 1.$$

As usual, $L^2(G)$ is the space of square-integrable functions on G with respect to m_G, $L^1(G)$ is the space of integrable functions on G. Note that in the following we will just write $\int_G f(x)\,dx$ rather than $\int_G f(x)\,dm_G(x)$, and will always assume the Haar measure on the compact group G/D to be normalized.

Let D be a uniform lattice in G and let F be an associated fundamental domain. If we equip D with the counting measure, a relation between the Haar measures on G and G/D is given by the following special case of Weil's formula [20]. For $f \in L^1(G)$, we have $\sum_{d\in D} f(xd) \in L^1(G/D)$ and

$$\int_G f(x)\,dm_G(x) = s(D) \int_{G/D} \Big(\sum_{d\in D} f(xd)\Big)\,dm_{G/D}(\dot{x}), \tag{3}$$

where $\dot{x} = xD$ (later on, if the context is clear, we will write simply $d\dot{x}$ rather than $dm_{G/D}(\dot{x})$). The Fourier transform $\widehat{f}$ of any function $f \in L^1(G)$ is defined by

$$\widehat{f}(\omega) = \int_G f(t)\overline{\omega(t)}\,dm_G(t).$$

The transformation $f \mapsto \widehat{f}$, $L^1(G) \to C_0(\widehat{G})$, extends to a Hilbert space isomorphism of $L^2(G)$ onto $L^2(\widehat{G})$, the so-called *Plancherel isomorphism*. Subsequently, the Plancherel transform of a function $f \in L^2(G)$ will also be denoted by $\widehat{f}$. Throughout this paper, we will always assume that the Haar measure μ_G on $\widehat{G}$ is normalized so that the Plancherel formula holds, i.e., we have

$$\int_G |f(x)|^2\, dm_G(x) = \int_{\widehat{G}} |\widehat{f}(\omega)|^2\, d\mu_G(\omega)$$

for any $f \in L^2(G)$.

The following definitions will also be needed. A countable family $\{e_j : j \in \mathcal{J}\}$ of elements in a separable Hilbert space $\mathcal{H}$ (for example, $\mathcal{H} = L^2(G)$, where G is an LCA group) is a *frame* if there exist constants $0 < \alpha \le \beta < \infty$ satisfying

$$\alpha\|v\|^2 \le \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2 \le \beta\|v\|^2$$

for all $v \in \mathcal{H}$. If the right hand inequality, but not necessarily the left hand one, holds, we say that $\{e_j : j \in \mathcal{J}\}$ is a *Bessel system* with constant β. A frame is *tight* if α and β can be chosen so that $\alpha = \beta$, and is a *Parseval frame* if $\alpha = \beta = 1$. Thus, if $\{e_j : j \in \mathcal{J}\}$ is a Parseval frame in $\mathcal{H}$, then

$$\|v\|^2 = \sum_{j\in\mathcal{J}} |\langle v, e_j\rangle|^2$$

for each $v \in \mathcal{H}$. This is equivalent to the reproducing formula

$$v = \sum_{j\in\mathcal{J}} \langle v, e_j\rangle e_j \tag{4}$$

for all $v \in \mathcal{H}$, where the series in (4) converges in the norm of $\mathcal{H}$. We refer the reader to [9, 5] for the basic properties of frames.

3. Characterization of Parseval frame generators. It is well known that there are relatively simple equations that characterize those functions $\Psi = \{\psi^1, \dots, \psi^L\}$ for which a Gabor system $\mathcal{G}_B(\Psi)$, given by (1), or an affine system $\mathcal{W}_A(\Psi)$, given by (2), is a Parseval frame for $L^2(\mathbb{R}^n)$. Several papers have been devoted to the formulation and study of these characterizations, and they play a major role in the construction and study of Gabor and affine systems (for example, [3, 4, 6, 10, 11, 14, 17, 19, 21–23]). The approach that we develop in this paper adapts some ideas from [13, 19], where one of the present authors has developed a unified approach to Gabor systems and affine systems in $L^2(\mathbb{R}^n)$.

Let G be an LCA group, $\mathcal{P}$ a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. For $x \in G$, let the translation operator T_x on $L^2(G)$ be defined by $T_x f(t) = f(tx^{-1})$. We will consider families of the form

$$\Phi^{\{g_p\}}_{\{D_p\}} = \{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}. \tag{5}$$

In order to state our general characterization result, we introduce the following notation. Let $\Lambda = \bigcup_{p\in\mathcal{P}} D_p^{\perp}$, and, for each $\alpha \in \Lambda$, let $\mathcal{P}_\alpha = \{p \in \mathcal{P} : \alpha \in D_p^{\perp}\}$. We will also need the following definition.

DEFINITION 3.1. The system (5) satisfies the *local integrability condition* (*LIC*) if, for each compact subset K of $\widehat{G}$, we have

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p\in D_p^{\perp}} \Big(\int_{K\cap\gamma_p^{-1}K} |\widehat{g}_p(\omega)|^2\, d\omega\Big) < \infty. \tag{6}$$

Let

$$\mathcal{D} = \{f \in L^2(G) : \widehat{f} \in L^\infty(\widehat{G}) \text{ and } \operatorname{supp}\widehat{f} \text{ is compact}\}. \tag{7}$$

Observe that $\mathcal{D}$ is a dense subset of $L^2(G)$. If the system (5) satisfies the LIC, then it is clear that, for each $f \in \mathcal{D}$, we have

$$\sum_{p\in\mathcal{P}} \sum_{\gamma_p\in D_p^{\perp}} \Big(\int_{\operatorname{supp}\widehat{f}} |\widehat{f}(\omega\gamma_p)|^2 s(D_p)^{-1} |\widehat{g}_p(\omega)|^2\, d\omega\Big) < \infty. \tag{8}$$

In fact, one can show that this statement is equivalent to the LIC.

We can now state our general characterization result.

THEOREM 3.2. *Let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. Suppose that the set $\Phi^{\{g_p\}}_{\{D_p\}}$, given by* (5), *satisfies the LIC. Then the following conditions are equivalent*:

(i) *$\Phi^{\{g_p\}}_{\{D_p\}}$ is a Parseval frame for $L^2(G)$.*
(ii) *For each $\alpha \in \Lambda$, we have*

$$\sum_{p\in\mathcal{P}_\alpha} s(D_p)^{-1}\, \overline{\widehat{g}_p(\omega)}\, \widehat{g}_p(\omega\alpha) = \delta_{\alpha,1} \quad \textit{for a.e. } \omega \in \widehat{G}. \tag{9}$$

This general result will be later applied to several special families of functions $\Phi^{\{g_p\}}_{\{D_p\}}$. Observe that in many of these cases we will be able to remove the LIC hypothesis from the corresponding characterization theorem. The proof of Theorem 3.2 will adapt some ideas from the proof of [13, Theorem 2.1]. Before presenting this proof, we need several auxiliary results.

Let D be a uniform lattice in G and let $f, g \in L^2(G)$. The *D-bracket product* of f and g, which was originally introduced in [2] and extended in [13], is defined, in our setting, as

$$[f, g](x; D) = \sum_{d\in D} f(xd)\, \overline{g(xd)}. \tag{10}$$

The function $[f, g](x; D)$ in (10) is also called the *periodization of f and g with respect to D*. We establish the following useful lemmas.

LEMMA 3.3. *Let D be a uniform lattice. If $f \in \mathcal{D}$, where $\mathcal{D}$ is given by* (7), *and $g \in L^2(G)$, then*

$$\sum_{d\in D} |\langle f, T_d\, g\rangle|^2 = s(D)^{-2} \int_{\widehat{G}/D^\perp} |[\widehat{f},\widehat{g}](\omega; D^\perp)|^2 \, d\dot{\omega}.$$

Proof. Since $(T_d g)^\wedge(\omega) = \overline{\omega(d)}\,\widehat{g}(\omega)$, the Plancherel theorem implies

$$\sum_{d\in D} |\langle f, T_d\, g\rangle|^2 = \sum_{d\in D} \Big| \int_{\widehat{G}} \widehat{f}(\omega)\, \overline{\widehat{g}(\omega)}\omega(d)\, d\omega \Big|^2.$$

By applying Weil's formula (3) and Lemma 2.1, we obtain

$$\begin{aligned} \int_{\widehat{G}} \widehat{f}(\omega)\, \overline{\widehat{g}(\omega)}\, \omega(d)\, d\omega &= s(D^\perp) \int_{\widehat{G}/D^\perp} \Big(\sum_{\gamma\in D^\perp} \widehat{f}(\omega\gamma)\, \overline{\widehat{g}(\omega\gamma)}\, (\omega\gamma)(d) \Big)\, d\dot{\omega} \\ &= s(D)^{-1} \int_{\widehat{G}/D^\perp} [\widehat{f},\widehat{g}](\omega; D^\perp)\omega(d)\, d\dot{\omega}. \end{aligned}$$

Observe that $\widehat{G}/D^\perp$ is topologically isomorphic to $\widehat{D}$. Thus, by choosing the Haar measure on $\widehat{D}$ (via this isomorphism) and using once more the Plancherel theorem, we obtain

$$\begin{aligned} \sum_{d\in D} \Big| \int_{\widehat{G}} \widehat{f}(\omega)\, \overline{\widehat{g}(\omega)}\, \omega(d)\, d\omega \Big|^2 &= \sum_{d\in D} \Big| s(D)^{-1} \int_{\widehat{G}/D^\perp} [\widehat{f},\widehat{g}](\omega; D^\perp)\omega(d)\, d\dot{\omega} \Big|^2 \\ &= s(D)^{-2} \sum_{d\in D} \Big| \int_{\widehat{D}} [\widehat{f},\widehat{g}](\omega; D^\perp)\omega(d)\, d\omega \Big|^2 \\ &= s(D)^{-2} \int_{\widehat{D}} |[\widehat{f},\widehat{g}](\omega; D^\perp)|^2\, d\omega \\ &= s(D)^{-2} \int_{\widehat{G}/D^\perp} |[\widehat{f},\widehat{g}](\omega; D^\perp)|^2\, d\dot{\omega}. \ \blacksquare \end{aligned}$$

LEMMA 3.4. *Let D be a uniform lattice in G. For each $f \in \mathcal{D}$ and $g \in L^2(G)$, define the function H on G by*

$$H(x) = \sum_{d\in D} |\langle T_x f, T_d g\rangle|^2.$$

Then $H : G/D \to \mathbb{R}$ is the trigonometric polynomial

$$H(x) = \sum_{\gamma\in D^\perp} \Big(s(D)^{-1} \int_{\widehat{G}} \widehat{f}(\omega)\, \overline{\widehat{f}(\omega\gamma)}\, \overline{\widehat{g}(\omega)}\, \widehat{g}(\omega\gamma)\, d\omega \Big) \gamma(x).$$

Proof. By Lemma 3.3, we have

$$\begin{aligned}
s(D)^2 H(x) &= \int_{\widehat{G}/D^\perp} |[(T_x f)^\wedge, \widehat{g}](\omega; D^\perp)|^2 \, d\dot{\omega} \\
&= \int_{\widehat{G}/D^\perp} \Big| \overline{\omega(x)} \sum_{\gamma \in D^\perp} \overline{\gamma(x)} \widehat{f}(\omega\gamma) \, \overline{\widehat{g}(\omega\gamma)} \Big|^2 \, d\dot{\omega} \\
&= \int_{\widehat{G}/D^\perp} \sum_{\gamma \in D^\perp} \overline{\gamma(x)} \, \widehat{f}(\omega\gamma) \, \overline{\widehat{g}(\omega\gamma)} \sum_{\delta \in D^\perp} \delta(x) \, \overline{\widehat{f}(\omega\delta)} \, \widehat{g}(\omega\delta) \, d\dot{\omega}.
\end{aligned}$$

Next, we use the substitution $\delta = \gamma\eta$ and express the last integrand as a sum over γ and η; then, by applying Weil's formula and Lemma 2.1, we obtain

$$\begin{aligned}
s(D)^2 H(x) &= \sum_{\gamma \in D^\perp} \int_{\widehat{G}/D^\perp} \widehat{f}(\omega\gamma) \, \overline{\widehat{g}(\omega\gamma)} \sum_{\eta \in D^\perp} \eta(x) \, \overline{\widehat{f}(\omega\gamma\eta)} \, \widehat{g}(\omega\gamma\eta) \, d\omega \\
&= s(D^\perp)^{-1} \int_{\widehat{G}} \widehat{f}(\omega) \, \overline{\widehat{g}(\omega)} \sum_{\eta \in D^\perp} \eta(x) \, \overline{\widehat{f}(\omega\eta)} \, \widehat{g}(\omega\eta) \, d\omega \\
&= s(D) \sum_{\eta \in D^\perp} \Big(\int_{\widehat{G}} \widehat{f}(\omega) \, \overline{\widehat{f}(\omega\eta)} \, \overline{\widehat{g}(\omega)} \, \widehat{g}(\omega\eta) \, d\omega \Big) \eta(x).
\end{aligned}$$

Notice that all exchanges in the order of summations and integrations are justified since $f \in \mathcal{D}$. ∎

The following result will be the main tool in the proof of Theorem 3.2.

PROPOSITION 3.5. *Let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. Suppose that the collection $\Phi_{\{D_p\}}^{\{g_p\}}$, given by* (5), *satisfies the LIC. For each $f \in \mathcal{D}$, define the functional N^2 on $L^2(G)$ by*

$$N^2(f) = \sum_{p \in \mathcal{P}} \sum_{\lambda_p \in D_p} |\langle f, T_{\lambda_p} g_p \rangle|^2.$$

Then the function $w(x) = N^2(T_x f)$ is continuous and coincides pointwise with its absolutely convergent Fourier series $\sum_{\alpha \in \Lambda} \widehat{w}(\alpha)\alpha(x)$, where

$$\widehat{w}(\alpha) = \int_{\widehat{G}} \widehat{f}(\omega) \, \overline{\widehat{f}(\omega\alpha)} \sum_{p \in \mathcal{P}_\alpha} s(D_p)^{-1} \, \overline{\widehat{g}_p(\omega)} \, \widehat{g}_p(\omega\alpha) \, d\omega, \tag{11}$$

and the last integral converges absolutely.

Proof. We have

$$w(x) = \sum_{p \in \mathcal{P}} \sum_{\lambda_p \in D_p} |\langle f, T_{\lambda_p x^{-1}} g_p \rangle|^2.$$

Now for each $p \in \mathcal{P}$, define $w_p(x) = \sum_{\lambda_p \in D_p} |\langle f, T_{\lambda_p x^{-1}} g_p\rangle|^2$, so that $w(x) = \sum_{p\in\mathcal{P}} w_p(x)$. By Lemma 3.4, we can write w_p in the form

$$w_p(x) = \sum_{\gamma_p \in D_p^\perp} \widehat{w}_p(\gamma_p)\gamma_p(x),$$

where

$$\widehat{w}_p(\gamma_p) = s(D_p)^{-1} \int\limits_{K\cap\gamma_p^{-1}K} \widehat{f}(\omega)\, \overline{\widehat{f}(\omega\gamma_p)}\, \overline{\widehat{g}(\omega)}\, \widehat{g}(\omega\gamma_p)\, d\omega,$$

and $K = \operatorname{supp} \widehat{f}$ is a compact set (since $f \in \mathcal{D}$). We claim that $\{\widehat{w}_p(\gamma_p) : p \in \mathcal{P},\ \gamma_p \in D_p^\perp\}$ is in $l^1(\mathcal{P} \times D_p^\perp)$. To show that this is the case, we first apply Cauchy–Schwarz's inequality to the last expression and obtain

$$\begin{aligned} s(D_p)\widehat{w}_p(\gamma_p) &\le \Big(\int\limits_{\gamma_p^{-1}K} |\widehat{f}(\omega)\widehat{g}(\omega\gamma_p)|^2\, d\omega\Big)^{1/2}\Big(\int\limits_K |\widehat{f}(\omega\gamma_p)\widehat{g}(\omega)|^2\, d\omega\Big)^{1/2} \\ &= \Big(\int\limits_K |\widehat{f}(\omega\gamma_p^{-1})\widehat{g}(\omega)|^2\, d\omega\Big)^{1/2}\Big(\int\limits_K |\widehat{f}(\omega\gamma_p)\widehat{g}(\omega)|^2\, d\omega\Big)^{1/2}. \end{aligned}$$

Next, using the inequality $2|cd| \le |c|^2 + |d|^2$ together with (8) (since f satisfies the LIC), we have

$$\sum_{p\in\mathcal{P}} \sum_{\gamma_p\in D_p^\perp} |\widehat{w}_p(\gamma_p)| < \infty,$$

which proves the claim. This in turn implies that

$$w(x) = \sum_{p\in\mathcal{P}} w_p(x) = \sum_{p\in\mathcal{P}} \sum_{\gamma_p\in D_p^\perp} \widehat{w}_p(\gamma_p)\gamma(x),$$

where the convergence in the last sum is absolute and uniform. Finally, using the notation introduced at the beginning of this section, we can rewrite the last equality in the form

$$w(x) = \sum_{\alpha\in\Lambda} \widehat{w}(\alpha)\alpha(x),$$

where $\widehat{w}(\alpha)$ is given by (11). Observe that $\{\widehat{w}_p(\gamma_p) : p \in \mathcal{P},\ \gamma_p \in D_p^\perp\} \in l^1(\mathcal{P} \times D_p^\perp)$ implies $\{\widehat{w}(\alpha) : \alpha \in \Lambda\} \in l^1(\Lambda)$ and, thus, the Fourier series for w converges absolutely. ∎

Proof of Theorem 3.2. It suffices to prove the result for a dense subset of $L^2(G)$. The general case follows by a standard density argument.

We first prove that (ii) implies (i). By Proposition 3.5,

$$w(x) = \sum_{p\in\mathcal{P}} \sum_{\lambda_p\in D_p} |\langle T_x f, T_{\lambda_p} g_p\rangle|^2 = \sum_{\alpha\in\Lambda} \widehat{w}(\alpha)\alpha(x),$$

where the last series converges absolutely (thus, $w(x)$ is continuous). Applying condition (ii) to $\widehat{w}(\alpha)$, given by (11), we obtain

$$\widehat{w}(\alpha) = \Big(\int_{\widehat{G}} \widehat{f}(\omega)\,\overline{\widehat{f}(\omega\alpha)}\,d\omega\Big)\delta_{\alpha,1}$$

for each $f \in \mathcal{D}$, where $\delta_{\alpha,1}$ is the Kronecker delta. Then (i) follows by setting $x = e$ in the expression for $w(x)$.

To prove the converse implication, let us assume that $N^2(f) = \|f\|^2$ for all $f \in L^2(G)$. Consider the function $z(x) = w(x) - \|f\|^2$. By Proposition 3.5, if $f \in \mathcal{D}$ then the function z is continuous and equals an absolutely convergent (generalized) trigonometric series whose coefficients are

$$\widehat{z}(1) = \widehat{w}(1) - \|f\|^2, \qquad \widehat{z}(\alpha) = \widehat{w}(\alpha), \qquad \alpha \neq 1.$$

By hypothesis, $z(x) = 0$. Hence, applying [7, Theorem 7.12] (note that z is an almost periodic function) yields $\widehat{z}(\alpha) = 0$ for all $\alpha \in \Lambda$. Thus, for all $\alpha \in \Lambda$ and $f \in \mathcal{D}$, using (11) for the coefficients $\widehat{w}(\alpha)$, we have

$$\int_{\widehat{G}} \widehat{f}(\omega)\,\overline{\widehat{f}(\omega\alpha)}\Big(\sum_{p\in\mathcal{P}_\alpha} s(D_p)^{-1}\,\overline{\widehat{g}_p(\omega)}\,\widehat{g}_p(\omega\alpha)\Big)\,d\omega = \delta_{\alpha,1}\|f\|^2. \tag{12}$$

Observe that, by the LIC, the function h_α defined by

$$h_\alpha(\omega) = \sum_{p\in\mathcal{P}_\alpha} s(D_p)^{-1}\,\overline{\widehat{g}_p(\omega)}\,\widehat{g}_p(\omega\alpha)$$

with $\alpha \in \Lambda$ is locally integrable. In order to establish (ii), we need to show that $h_\alpha(\omega) = \delta_{\alpha,1}$ for a.e. $\omega \in \widehat{G}$.

Consider first the case $\alpha = 1$. Arguing by contradiction, assume that $h_1(\omega) > 1$ for $\omega \in E$, where $\mu(E) > 0$. Let $\widehat{f} = \chi_E$. Then

$$\int_{\widehat{G}} |\widehat{f}(\omega)|^2 h_1(\omega)\,d\omega = \int_E h_1(\omega)\,d\omega > \|f\|^2,$$

and this contradicts (12). A similar argument shows that one cannot have $h_1(\omega) < 1$ on any measurable set E of positive measure and, thus, $h_1(\omega) = 1$ for a.e. $\omega \in \widehat{G}$. Consider now the case $\alpha \neq 1$. Again, arguing by contradiction, assume that $h_\alpha(\omega) > 0$ for $\omega \in E$, where $\mu(E) > 0$. We can choose E small enough so that $E \cap E\alpha^{-1} = \emptyset$ for $\alpha \neq 1$. Let $\widehat{f} = \chi_E + \chi_{E\alpha^{-1}}$. Then

$$\int_{\widehat{G}} \widehat{f}(\omega)\,\overline{\widehat{f}(\omega\alpha)}\,h_\alpha(\omega)\,d\omega = \int_E h_\alpha(\omega)\,d\omega > 0,$$

and this contradicts (12). Thus $h_\alpha(\omega) = 0$ for a.e. $\omega \in \widehat{G}$, and this completes the proof. ∎

We can prove the following necessary condition for a family $\Phi_{\{D_p\}}^{\{g_p\}}$, given by (5), to form a Bessel system.

Proposition 3.6. *Let $\mathcal{P}$ be a countable set, $\{g_p\}_{p\in\mathcal{P}}$ a collection of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. If the system $\Phi_{\{D_p\}}^{\{g_p\}}$, given by* (5), *is Bessel with constant B, then*

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1}|\widehat{g}_p(\omega)|^2 \le B \quad \text{for a.a. } \omega \in \widehat{G}. \tag{13}$$

Proof. In all applications of this proposition that we will consider in this paper, $\mathcal{P}$ will be a subset of $\mathbb{Z}^r$ for some $r \in \mathbb{N}$. For simplicity we assume this to be the case here. However, the reader can easily check that this is not a loss of generality.

Since $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ is a Bessel sequence with constant B, for every $M \in \mathbb{N}$ we have

$$\sum_{p\in\mathcal{P},\, |p|\le M} \; \sum_{\lambda_p \in D_p} |\langle f, T_{\lambda_p} g_p\rangle|^2 \le B\|f\|_2^2 \tag{14}$$

for all $f \in L^2(G)$. Applying Lemma 3.4 to the left hand side of this inequality for each $p \in \mathcal{P}$ (letting $x = e$), we have

$$\sum_{p\in\mathcal{P},\, |p|\le M} \; \sum_{\gamma_p \in D_p^{\perp}} s(D_p)^{-1} \int_{\widehat{G}} \widehat{f}(\omega)\, \overline{\widehat{f}(\omega\gamma_p)}\, \overline{\widehat{g}_p(\omega)}\, \widehat{g}_p(\omega\gamma_p)\, d\omega \le B\|f\|_2^2 \tag{15}$$

for all $f \in \mathcal{D}$, $M \in \mathbb{N}$. Arguing by contradiction, let $\mathcal{B}(\omega_0, \delta)$ be a ball of radius $\delta > 0$ and center $\omega_0 \in \widehat{G}$, with respect to the metric d on $\widehat{G}$, and assume that

$$\sum_{p\in\mathcal{P},\, |p|\le M} s(D_p)^{-1}|\widehat{g}_p(\omega)|^2 > B \tag{16}$$

for a.e. $\omega \in \mathcal{B}(\omega_0, \delta)$, where δ is some positive constant. Next define f_ε by

$$\widehat{f}_\varepsilon(\omega) = \chi_{\mathcal{B}(\omega_0,\varepsilon)}(\omega),$$

where $\varepsilon < \min\{\delta, \delta_M/2\}$, and $\delta_M = \inf\{d(\gamma_p, 1) \in D_p^{\perp} \setminus \{1\} : |p| \le M\}$. Observe that $\delta_M > 0$ since there are only a finite number of elements $p \in \mathcal{P}$. Since $\varepsilon < \delta_M/2$, we see that $\widehat{f}_\varepsilon(\omega)$ and $\widehat{f}_\varepsilon(\omega\,\gamma_p)$ have disjoint supports for $\gamma_p \ne 1$. Using this observation, inequality (16) and the fact that $f_\varepsilon \in \mathcal{D}$, we deduce that

$$\begin{aligned}
&\sum_{p\in\mathcal{P},\, |p|\le M} \; \sum_{\gamma_p \in D_p^{\perp}} s(D_p)^{-1} \int_{\widehat{G}} \widehat{f}_\varepsilon(\omega)\, \overline{\widehat{f}_\varepsilon(\omega\gamma_p)}\, \overline{\widehat{g}_p(\omega)}\, \widehat{g}_p(\omega\gamma_p)\, d\omega \\
&\qquad = \int_{\mathcal{B}(\omega_0,\varepsilon)} |\widehat{f}_\varepsilon(\omega)|^2 \sum_{p\in\mathcal{P},\, |p|\le M} s(D_p)^{-1}|\widehat{g}_p(\omega)|^2\, d\omega > B\|f_\varepsilon\|^2,
\end{aligned}$$

and this contradicts (15). The proof is completed by letting $M \to \infty$ in (14). ∎

4. Applications of the general characterization theorem. In this section, we study several applications of Theorem 3.2. We begin by considering locally compact abelian groups with compact connected component, and show that in this situation the LIC is equivalent to a simpler condition or can even be removed. Next we apply Theorem 3.2 to obtain a new characterization of Parseval frame generators for Gabor and affine sytems.

Throughout this subsection, let $\mathcal{P}$ be a countable index set, $\{g_p : p \in \mathcal{P}\}$ a family of functions in $L^2(G)$, and $\{D_p : p \in \mathcal{P}\}$ a collection of uniform lattices in G. As before, we define $\Lambda = \bigcup_{p\in\mathcal{P}} D_p^\perp$. Furthermore, let $\mathcal{D}$ be given by (7).

We start with the following simple observation.

Lemma 4.1. *If $\mathcal{P}$ is finite, then the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}$ satisfies the LIC.*

Proof. Let K be a compact subset of $\widehat{G}$. Then, for each $p \in \mathcal{P}$, there only exist finitely many $\gamma_p \in D_p^\perp$ with $K \cap \gamma_p^{-1} K \neq \emptyset$. Since $\mathcal{P}$ is supposed to be finite, both sums in the LIC, given by (6), are finite and hence there exists some $M < \infty$ such that

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp} \int_{K\cap\gamma_p^{-1}K} |\widehat{g}_p(\omega)|^2 \, d\omega \le M \sum_{p\in\mathcal{P}} s(D_p)^{-1} \|\widehat{g}_p\|_2^2 < \infty. \ \blacksquare$$

4.1. *LCA groups with compact connected component.* We obtain the following general characterization of the LIC for LCA groups with compact connected component.

Proposition 4.2. *Let G be an LCA group with compact connected component and let H be some open compact subgroup of G. Then the following conditions are equivalent*:

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ satisfies the LIC.*
(ii) *We have*

$$\sum_{p\in\mathcal{P}} s_H(D_p \cap H)^{-1} \int_K |\widehat{g}_p(\omega)|^2 \, d\omega < \infty \quad \text{for all } K \subset \widehat{G} \text{ compact},$$

where s_H denotes the lattice size with respect to H and the Haar measure m_H on H induced by m_G.

Proof. Let D be some uniform lattice in G. We claim that

$$s(D)^{-1}(\#(D^\perp \cap H^\perp)) = s_H(D\cap H)^{-1}. \tag{17}$$

To prove this claim, we first choose a special fundamental domain S_H for D in G with respect to H. Since $D \cap H$ is a finite subgroup of H, there exists a fundamental domain $\widetilde{S}_H$ for $D \cap H$ in H. Moreover, we have $[G : HD] < \infty$, where $[G : HD]$ is the index of HD in G, that is, the cardinality of G/HD.

Thus we can choose a finite representative system $\{y_i : 1 \le i \le [G : HD]\}$ for the HD-cosets in G. Then we define the fundamental domain S_H by

$$S_H = \bigcup_{i=1}^{[G:HD]} y_i \widetilde{S}_H.$$

Notice that this union is disjoint. It is straightforward that S_H is indeed a fundamental domain for D in G. Since the lattice size does not depend on the fundamental domain we choose, we obtain

$$s(D) = m_G(S_H) = \sum_{i=1}^{[G:HD]} m_G(\widetilde{S}_H) = [G : HD] s_H(D \cap H). \tag{18}$$

Secondly, we have $D^\perp \cap H^\perp = (HD)^\perp$, hence

$$\#(D^\perp \cap H^\perp) = \#((HD)^\perp) = [G : HD], \tag{19}$$

where the second equality follows from the fact that the dual group of a finite group is the group itself. Obviously, equations (18) and (19) yield our claim in (17).

Now we suppose that (i) holds. To show that this implies (ii), let K be a compact subset of $\widehat{G}$. Then there exist $\tau_1, \ldots, \tau_n \in \widehat{G}$ and corresponding $C_i \subseteq H^\perp$, $i = 1, \ldots, n$, such that $K = \bigcup_{i=1}^n \tau_i C_i$. Thus it suffices to prove

$$\sum_{p \in \mathcal{P}} s_H(D_p \cap H)^{-1} \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2 \, d\omega < \infty. \tag{20}$$

By [15, Remark 23.24(d)], $H^\perp$ is also compact, hence $\bigcup_{i=1}^n \tau_i H^\perp$ is compact. Since the LIC is satisfied, using (17) we have

$$\begin{aligned}
\infty &> \sum_{p \in \mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp} \Bigg(\int\limits_{\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1} \tau_j H^\perp} |\widehat{g}_p(\omega)|^2 \, d\omega \Bigg) \\
&\ge \sum_{p \in \mathcal{P}} s(D_p)^{-1} \sum_{\gamma_p \in D_p^\perp \cap H^\perp} \Bigg(\int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2 \, d\omega \Bigg) \\
&= \sum_{p \in \mathcal{P}} s(D_p)^{-1} \#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2 \, d\omega \\
&= \sum_{p \in \mathcal{P}} s_H(D \cap H)^{-1} \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2 \, d\omega.
\end{aligned}$$

This shows (i)$\Rightarrow$(ii).

To prove the converse implication, suppose that (ii) holds. Let K be a compact subset of $\widehat{G}$. As above, there exist $\tau_1, \ldots, \tau_n \in \widehat{G}$ and $C_i \subset H^\perp$, $i = 1, \ldots, n$, such that $K = \bigcup_{i=1}^n \tau_i C_i$. Since $\bigcup_{i=1}^n \tau_i H^\perp$ is a compact set and (ii) holds, we have

$$\begin{aligned}\infty &> \sum_{p\in\mathcal{P}} s_H(D\cap H)^{-1} \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2\,d\omega \\ &= \sum_{p\in\mathcal{P}} s(D_p)^{-1}\#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2\,d\omega.\end{aligned}$$

Now we claim that

$$\#\Big(\mathcal{D}_p^\perp \cap \bigcup_{i,j=1}^n \tau_i^{-1}\tau_j H^\perp\Big) \le n^2\cdot \#(\mathcal{D}_p^\perp\cap H^\perp). \tag{21}$$

Once this is shown we can continue the above computation to obtain

$$\begin{aligned}\infty &> \sum_{p\in\mathcal{P}} s(D_p)^{-1}\#(D_p^\perp \cap H^\perp) \int\limits_{\bigcup_{i=1}^n \tau_i H^\perp} |\widehat{g}_p(\omega)|^2\,d\omega \\ &\ge n^{-2}\sum_{p\in\mathcal{P}} s(D_p)^{-1}\sum_{\gamma_p\in D_p^\perp}\Big(\int\limits_{\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1}\tau_j H^\perp} |\widehat{g}_p(\omega)|^2\,d\omega\Big),\end{aligned}$$

since $\bigcup_{i=1}^n \tau_i H^\perp \cap \bigcup_{j=1}^n \gamma_p^{-1}\tau_j H^\perp \neq \emptyset$ if and only if there exist $i,j\in\{1,\dots,n\}$ with $\gamma_p\tau_j^{-1}\tau_i \in H^\perp$. This proves that the LIC holds for K.

Hence it remains to prove (21). Notice that it suffices to replace $\bigcup_{i,j=1}^n \tau_i^{-1}\tau_j H^\perp$ by some disjoint union $\bigcup_{i=1}^{n^2} \alpha_i H^\perp$. Now we choose the coset representatives of $H^\perp$ in $\widehat{G}$ to be $D\cup R$, where $D\subset \mathcal{D}_p^\perp$ and $\mathcal{D}_p^\perp \cap \alpha H^\perp = \emptyset$ for all $\alpha\in R$. Then, if $\alpha\in R$, we have $\#(\mathcal{D}_p^\perp\cap\alpha H^\perp)=0$, and if $\alpha\in D$, we have

$$\#(\mathcal{D}_p^\perp\cap\alpha H^\perp) = \#(\alpha^{-1}\mathcal{D}_p^\perp \cap H^\perp) = \#(\mathcal{D}_p^\perp\cap H^\perp).$$

This proves (21) and hence (ii)⇒(i). ∎

In the special cases of compact and discrete abelian groups, the preceding proposition yields the following equivalent conditions for the LIC to hold.

COROLLARY 4.3. *Let G be a compact abelian group. Then the following conditions are equivalent*:

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p\in D_p,\ p\in\mathcal{P}\}$ satisfies the LIC.*
(ii) *For all $\omega\in\widehat{G}$, we have*

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1}|\widehat{g}_p(\omega)|^2 < \infty.$$

Proof. Choosing $H=G$ and $K=\{\gamma\}$ for some $\gamma\in\widehat{G}$ in Proposition 4.2, we obtain

$$\sum_{p\in\mathcal{P}} s_H(D_p\cap H)^{-1}\int_K |\widehat{g}_p(\omega)|^2\,d\omega = \sum_{p\in\mathcal{P}} s(D_p)^{-1}|\widehat{g}_p(\gamma)|^2.$$

Note that it suffices to consider $K = \{\gamma\}$, since each compact subset of $\widehat{G}$ contains only finitely many elements. ∎

Corollary 4.4. *Let G be a discrete abelian group. Then the following conditions are equivalent*:

(i) *The system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ satisfies the LIC.*
(ii) *We have*

$$\sum_{p\in\mathcal{P}} \int_{\widehat{G}} |\widehat{g}_p(\omega)|^2\, d\omega < \infty.$$

Proof. Choosing $H = \{e\}$ and $K = \widehat{G}$ in Proposition 4.2 yields

$$\sum_{p\in\mathcal{P}} s_H(D_p \cap H)^{-1} \int_K |\widehat{g}_p(\omega)|^2\, d\omega = \sum_{p\in\mathcal{P}} \int_{\widehat{G}} |\widehat{g}_p(\omega)|^2\, d\omega.$$

Notice that, as before, it suffices to consider only $K = \widehat{G}$. ∎

A natural question is the following. Let $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ be a system which satisfies Theorem 3.2(i) or (ii). Does this imply that this system then satisfies the LIC automatically, i.e., can we omit the hypothesis that the LIC has to be satisfied?

It will turn out that this is true in some cases, however it is not a necessary condition. The compact and discrete groups will turn out to be the extreme cases.

Lemma 4.5. *Let G be a compact abelian group and suppose that the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ satisfies Theorem* 3.2(i) *or* (ii). *Then it also satisfies the LIC.*

Proof. If $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\, p \in \mathcal{P}\}$ satisfies Theorem 3.2(i), it is, in particular, a Bessel system with constant B. Hence Proposition 3.6 shows that

$$\sum_{p\in\mathcal{P}} s(D_p)^{-1} |\widehat{g}_p(\omega)|^2 \le B \quad \text{for all } \omega \in \widehat{G}.$$

This inequality also follows by Theorem 3.2(ii) on choosing $\alpha = 1$. Now we can apply Corollary 4.3, which finishes the proof. ∎

Lemma 4.6. *Let G be a discrete abelian group and let $\{T_{\lambda_p} g_p : \lambda_p \in D_p,$ $p \in \mathcal{P}\}$ be a Parseval frame for $L^2(G)$. Then this system does not necessarily satisfy the LIC.*

Proof. Let $G = \mathbb{Z}$ and, for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, let $[m]_k$ denote the residue class of m modulo k, and let $\sqcup$ denote disjoint union. It is easy to see that, for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, we have

$$[m]_k = [m]_{2k} \sqcup [m+k]_{2k} \quad \text{and} \quad [m]_k = [m]_{2k} \sqcup [m-k]_{2k}. \tag{22}$$

Our aim is to write $\mathbb{Z}$ as a disjoint union of infinitely many pairwise different residue classes. For this, we start by observing that $\mathbb{Z} = [0]_2 \sqcup [1]_2$. Applying (22) we obtain $\mathbb{Z} = [0]_2 \sqcup [1]_4 \sqcup [-1]_4$. Now we use the other formula in (22), which yields $\mathbb{Z} = [0]_2 \sqcup [1]_4 \sqcup [-1]_8 \sqcup [3]_8$. Iterating this procedure, using a simple induction argument, we obtain

$$\mathbb{Z} = \bigsqcup_{j\in\mathbb{N}} [a_j]_{2^j}, \tag{23}$$

where $a_j = \sum_{k=0}^{j-2} (-1)^k 2^k$, $j \in \mathbb{N}$.

Now we define $\mathcal{P} = \mathbb{N}$, $g_p = \chi_{\{a_p\}}$, and $D_p = 2^p\mathbb{Z}$, $p \in \mathcal{P}$. An easy calculation shows that $\widehat{g}_p = e^{-2\pi i a_p \cdot}$. By (23), the family $\{T_{\lambda_p} g_p : \lambda_p \in D_p, p \in \mathcal{P}\}$ equals $\{\chi_{\{m\}} : m \in \mathbb{Z}\}$, and hence is an orthonormal basis for $L^2(\mathbb{Z})$.

But this system does not satisfy the LIC. To prove this, by Corollary 4.4, we only have to compute

$$\sum_{p\in\mathcal{P}} \int_{\widehat{G}} |\widehat{g}_p(\omega)|^2 \, d\omega = \sum_{p\in\mathbb{N}} \int_{\mathbb{T}} 1 \, d\omega,$$

which is not finite. ■

4.2. *Gabor systems.* Let M_ω be the modulation operator, defined by

$$M_\omega f(x) = \omega(x) f(x) \quad \text{for } x \in G,\ \omega \in \widehat{G},$$

and consider the *Gabor systems*

$$\mathcal{G}(\Psi) = \{T_\lambda M_\gamma \Psi : \lambda \in D,\ \gamma \in K\},$$

where $D \subset G$ is a uniform lattice, K is a discrete subset of $\widehat{G}$ and $\Psi = \{\psi^1, \ldots, \psi^L\} \subset L^2(G)$.

It is clear that we can write the system $\mathcal{G}(\Psi)$ in the form (5) by letting $\mathcal{P} = \{p = (\gamma, l) : \gamma \in K,\ l = 1, \ldots, L\}$, $D_p = D$ for each $p \in \mathcal{P}$ and $g_p = g_{(\gamma,l)} = M_\gamma \psi^l$.

We will deduce a characterization of all the functions Ψ such that $\mathcal{G}(\Psi)$ is a Parseval frame of $L^2(G)$. We start with the following observation.

LEMMA 4.7. *Let $\{g_p : p \in \mathcal{P}\}$ be a countable family of functions in $L^2(G)$. Assume that $D_p = D$ for each $p \in \mathcal{P}$. If there is a constant C such that $\sum_{p\in\mathcal{P}} |\widehat{g}_p(\omega)|^2 < C$ for a.e. $\omega \in \widehat{G}$, then the system $\{T_\lambda g_p : \lambda \in D, p \in \mathcal{P}\}$ satisfies the LIC.*

Proof. Since K is compact, there only exist finitely many $\gamma \in D^\perp$ with $K \cap \gamma^{-1}K \neq \emptyset$, say M of them. A direct computation of the left hand side of (6) shows that

$$\sum_{p\in\mathcal{P}} s(D)^{-1} \sum_{\gamma\in D^\perp} \int_{K\cap\gamma^{-1}K} |\widehat{g}_p(\omega)|^2 \, d\omega \le s(D)^{-1} C \sum_{\gamma\in D^\perp} \int_{K\cap\gamma^{-1}K} d\omega$$
$$\le s(D)^{-1} C M \mu(K) < \infty. \ ■$$

We now obtain the following characterization result.

THEOREM 4.8. *$\mathcal{G}(\Psi)$ is a Parseval frame for $L^2(G)$ if and only if, for each $\lambda \in D^\perp$,*

$$\sum_{\gamma\in K}\sum_{l=1}^{L} \widehat{\psi}^l(\omega\gamma^{-1})\,\overline{\widehat{\psi}^l(\omega\gamma^{-1}\lambda)} = s(D)\delta_{1,\lambda} \quad \textit{for a.e. } \omega \in \widehat{G}. \tag{24}$$

Proof. As we described before, the Gabor system $\mathcal{G}(\Psi)$ can be represented in the form (5). Using the notation of Theorem 3.2 for the system $\mathcal{G}(\Psi)$, we have $\mathcal{P}_\alpha = \mathcal{P}$, $\Lambda = D^\perp$, and $\widehat{g}_p = \widehat{g}_{(\gamma,l)} = T_\gamma\widehat{\psi}^l$. Also observe that, if $\mathcal{G}(\Psi)$ is a Parseval frame, then, by Proposition 3.6, for any $f \in \mathcal{D}$ there is a constant $B < \infty$ such that

$$\sum_{\gamma\in K}\sum_{l=1}^{L} s(D)^{-1}|\widehat{\psi}^l(\omega\gamma^{-1})|^2 < B \quad \text{for a.e. } \omega \in \widehat{G}.$$

This inequality also holds if equation (9) is satisfied (use $\alpha = 1$). Thus, by Lemma 4.7, we do not need to assume the LIC in Theorem 3.2. Finally, equation (24) follows from (9). ∎

This theorem generalizes similar results in $L^2(\mathbb{R}^n)$, that one can find, for example, in [17, 19, 22].

4.3. *Affine systems.* In the theory of affine systems on $\mathbb{R}^n$, the elements of the family $\mathcal{W}_A(\psi)$, given by (2), are obtained under the action of translations and dilations on $\mathbb{R}^n$. These operations can be defined on an LCA group G by identifying the translations with the group action, and the dilations with the action of a group automorphism A on G (see [8] for a similar approach).

Let d be a metric on $\widehat{G}$. Without loss of generality (see [15, Vol. I, Sec. 8]), the metric can be chosen to be translation-invariant, that is,

$$d(\alpha\omega, \beta\omega) = d(\alpha, \beta) \quad \text{for all } \alpha, \beta, \omega \in \widehat{G}.$$

Let A be a group automorphism on $\widehat{G}$ which is expanding in the sense that

$$d(A(\alpha), A(\beta)) \geq cd(\alpha, \beta) \quad \text{for some } c > 1 \text{ and } \alpha \neq \beta \in \widehat{G}. \tag{25}$$

We use the notation $A^2(\omega) = A(A(\omega))$, $A^0(\omega) = \omega$. Observe that A^{-1} is a contraction, since it follows from (25) that

$$d(A^{-1}(\alpha), A^{-1}(\beta)) \leq c^{-1}\, d(\alpha, \beta) \quad \text{for some } c > 1 \text{ and } \alpha \neq \beta \in \widehat{G}.$$

Throughout this section we fix a uniform lattice D in G and normalize the Haar measure on $\widehat{G}$ so that $s(D^\perp) = 1$.

Now let us consider the families $\Phi^{\{g_p\}}_{\{D_p\}}$ given by (5), where

$$\begin{aligned}(26)\qquad &\mathcal{P} = \{(j,l) : j \in \mathbb{Z},\, l = 1,\dots,L\},\ D_p^{\perp} = D_{(j,l)}^{\perp} = D_j^{\perp} = A^j(D^{\perp}),\\ &\widehat{g}_p(\omega) = \widehat{g}_{(j,l)}(\omega) = \nu_j^{-1/2}\widehat{\psi}^l(A^{-j}(\omega)),\end{aligned}$$

and the constant ν_j is chosen as $\nu_j := s(A^j(D^{\perp}))$, $j \in \mathbb{Z}$. Any family of this form will be called an *affine system* on $L^2(G)$ with respect to the automorphism A on $\widehat{G}$. The connection with the usual affine systems on $L^2(\mathbb{R}^n)$, related to the theory of wavelets, will be clarified later.

By Weil's formula (3) and the particular normalization of the Haar measure on $\widehat{G}$, it follows that, for each $l = 1,\dots,L$,

$$\int_{\widehat{G}} \nu_j^{-1}|\widehat{\psi}^l(A^{-j}(\omega))|^2\,d\omega = \int_{\widehat{G}} |\widehat{\psi}^l(\omega)|^2\,d\omega,$$

and this shows that the automorphism A, with the appropriate normalization, acts in a way similar to the unitary dilations in the case of classical wavelets (we will show in Example 4.13 that the unitary dilations are automorphisms on $\mathbb{R}^n$).

Under these assumptions, the LIC, given by (6) is

$$(27)\qquad L = \sum_{j\in\mathbb{Z}}\sum_{l=1}^{L}\sum_{\gamma\in D}\Big(\int_{K\cap A^{-j}(\gamma)K} |\widehat{\psi}^l(A^{-j}(\omega))|^2\,d\omega\Big) < \infty$$

for each compact subset K of $\widehat{G}$. The following proposition shows that the LIC is satisfied if A is an expanding automorphism on $\widehat{G}$.

Proposition 4.9. *Let G be an LCA group, and A be an expanding automorphism on $\widehat{G}$. Let $\mathcal{P}$, g_p and D_p be given by* (26)*. If there is a constant C such that*

$$(28)\qquad \sum_{p\in\mathcal{P}} s(D_p^{\perp})|\widehat{g}_p(\omega)|^2 = \sum_{j\in\mathbb{Z}}\sum_{l=1}^{L} |\widehat{\psi}^l(A^{-j}(\omega))|^2 < C \quad \textit{for a.e. } \omega \in \widehat{G},$$

then the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}$ satisfies the LIC (27)*.*

Before proving Proposition 4.9, we need some construction.

Given any $r > 0$, we use the notation $B(r) = \{g \in G : d(g,e) < r\}$ and $\widetilde{B}(r) = \{g \in G : 1/r < d(g,e) < r\}$. We obtain the following two lemmas.

Lemma 4.10. *Let G be an LCA group, and A be an expanding automorphism on G. Then there is a number $N = N(A,r) \geq 0$ such that*

$$\#\{j \in \mathbb{Z} : A^j(g) \in \widetilde{B}(r)\} \leq N \quad \textit{for all } g \in G.$$

Proof. If $g = e$, then $A^j(e) = e \notin \widetilde{B}(r)$ for any $j \in \mathbb{Z}$ and, thus, we can choose $N = 1$.

If $g \neq e$, let $j_0 = j_0(g)$ be the smallest integer such that $d(A^{j_0}(g), e) > 1/r$. This is possible since A is expanding, and there is a $c > 1$ such that $d(A^{-j}(g), e) \leq c^{-j}\, d(g, e)$ for all $j > 0$. Thus, $A^j(g) \notin \widetilde{B}(r)$ for all $j < j_0$. Next, choose $N_0 > 0$ such that $c^{N_0} > r^2$. Then, if $j \geq N_0$,

$$d(A^{j+j_0}(g), e) \geq c^j d(A^{j_0}(g), e) \geq c^{N_0}\, 1/r > r.$$

Thus, if $j \geq N_0$, then $A^{j+j_0}(g) \notin \widetilde{B}(r)$. It follows that

$$\{j \in \mathbb{Z} : A^j(g) \in \widetilde{B}(r)\} \subset \{j_0, j_0 + 1, \ldots, j_0 + N_0 - 1\}.$$

The proof is completed by taking $N = N_0$ (observe that N_0 does not depend on j_0 and, in particular, does not depend on $g \in G$). ∎

Lemma 4.11. *Let G be an LCA group, $r > 0$, and A be an expanding automorphism on G. Then there is a constant $K = K(D, r) \geq 0$ such that*

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \leq K s(A^j(D))^{-1} s(D)^j.$$

Proof. Let $S = \inf\{d(g, e) : g \in (D \setminus \{e\}) \cap B(r)\} > 0$. Since A is expanding, for any $r > 0$, there is a positive integer j such that

$$d(A^j(g), e) \geq c^j d(g, e) > c^j S > r$$

for all $g \in D \setminus \{e\}$. Let j_1 be the smallest positive integer for which this holds. Thus, for all $j \geq j_1$,

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} = 0.$$

Next consider the case $j < j_1$. Let $g \in D \setminus \{e\}$ with $A^j(g) \in B(r)$ and let $h \in F$, where F denotes a fundamental domain of D. We set $T := \sup\{d(h, e) : h \in F\}$. By [18, Lemma 2], the fundamental domain can be chosen to be relatively compact, hence $T < \infty$. Since A is expanding and $j < j_1$, we have

$$\begin{aligned} d(A^{-j_1+j}(gh), e) &= d(A^{-j_1+j}(g) A^{-j_1+j}(h), e) \\ &\leq d(A^{-j_1+j}(g), e) + d(A^{-j_1+j}(h), e) \\ &< c^{-j_1} r + d(h, e) \leq c^{-j_1} r + T =: R. \end{aligned}$$

Thus,

$$\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \subset \{g \in D : gF \subset A^{j_1-j}(B(R))\} =: \mathcal{M}_R^j.$$

Since the sets gF, $g \in D$, are disjoint, we obtain

$$\begin{aligned} &\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \\ &\qquad \leq \#\mathcal{M}_R^j = \frac{m_G(\bigcup_{g \in \mathcal{M}_R^j} gF)}{s(D)} \leq \frac{m_G(A^{j_1-j}(B(R)))}{s(D)}. \end{aligned}$$

For each measurable $Q \subset G$, Weil's formula implies

$$m_G(A(Q)) = \frac{s(A(D))}{s(D)}\, m_G(Q). \tag{29}$$

Applying this equation to the automorphisms A^{j_1} and A^{-j} yields

$$\#\{g \in D \setminus \{e\} : A^j(g) \in B(r)\} \leq \frac{s(A^{j_1}(D))s(A^{-j}(D))}{s(D)^{j_1-j-1}}\, m_G(B(R)).$$

It remains to prove that $s(A^{-j}(D)) = cs(A^j(D))^{-1}$ for some constant c. Let Q be a fundamental domain for $A^{-1}(D)$ in G. Using the definition of a fundamental domain, it follows that $A^2(Q)$ is a fundamental domain for $A(D)$ in G. Then, applying (29) twice, we obtain

$$\begin{aligned} s(A(D)) = m_G(A^2(Q)) &= \frac{s(A(D))^2}{s(D)^2}\, m_G(Q) \\ &= \frac{s(A(D))^2}{s(D)^2}\, s(A^{-1}(D)). \end{aligned}$$

This implies

$$s(A^{-1}(D)) = s(A(D))^{-1}s(D)^2.$$

Now our claim follows from

$$s(A^{-j}(D)) = s((A^j)^{-1}(D)) = s(A^j(D))^{-1}s(D)^2.$$

Setting

$$K := \frac{s(A^{j_1}(D))}{s(D)^{j_1-3}} m_G(B(R))$$

proves the lemma. ■

Proof of Proposition 4.9. We write $L = L_1 + L_2$, where L_1 is the sum in (27) corresponding to $\gamma = 1$ and L_2 is the sum corresponding to $\gamma \in D^\perp \setminus \{1\}$.

It is clear that

$$L_1 = \sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \int_K |\widehat{\psi}^l(A^{-j}(\omega))|^2\, d\omega < \int_K C\, d\omega < \infty.$$

Consider now L_2. Choose $r > 0$ such that $K \subset \widetilde{B}(r)$. Using the change of variables $\eta = A^{-j}(\omega)$ (observe that $d(A^j(\eta)) = \nu_j d(\eta)$) we have

$$\begin{aligned} L_2 &\leq \sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \sum_{\gamma\in D^\perp\setminus\{1\}} \int_{A^j(\eta)A^j(\gamma)\in\widetilde{B}(r)} |\widehat{\psi}^l(\eta)|^2\, d(A^j(\eta)) \\ &= \sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \sum_{\gamma\in D^\perp\setminus\{1\}} \int_{A^j(\eta)A^j(\gamma)\in\widetilde{B}(r)} s(A^j(D^\perp))|\widehat{\psi}^l(\eta)|^2\, d\eta. \end{aligned}$$

Observe that (use the triangle inequality for the metric d)

$$\begin{aligned} &\{\gamma \in D^\perp \setminus \{1\} : A^j(\eta) \in B(r) \text{ and } A^j(\eta)A^j(\gamma) \in B(r)\} \\ &\qquad\qquad \subseteq \{\gamma \in D^\perp : A^j(\gamma) \in B(2r)\} \end{aligned}$$

and, by Lemma 4.11,

$$\#\{\gamma \in D^\perp : A^j(\gamma) \in B(r)\} \le K(D,r) s(A^j(D^\perp))^{-1},$$

since the Haar measure on $\widehat{G}$ is normalized so that $s(D^\perp) = 1$. This implies that

$$L_2 \le K(D,r) \sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} \int_{A^j(\eta)\in\widetilde{B}(r)} |\widehat{\psi}^l(\eta)|^2 \, d\eta.$$

Finally, using Lemma 4.10, we find that

$$L_2 \le K(D,r) N(A,r) \sum_{l=1}^{L} \|\psi^l\|_2^2 < \infty.$$

Thus, $L_1 + L_2 < \infty$ and this completes the proof. ∎

Using Proposition 4.9, we can now state the following characterization result for the affine systems. Notice that we do not need to assume the LIC in this theorem.

THEOREM 4.12. *Let G be an LCA group, and A be an expanding automorphism on $\widehat{G}$. Then the affine system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}$, where $\mathcal{P}$, g_p and D_p are given by* (26), *is a Parseval frame for $L^2(G)$ if and only if, for each $\alpha \in \bigcup_{j\in\mathbb{Z}} A^j(D^\perp)$, we have*

$$\sum_{(j,l)\in\mathcal{P}_\alpha} \overline{\widehat{\psi}^l(A^{-j}(\omega))}\, \widehat{\psi}^l(A^{-j}(\omega)\alpha) = \delta_{\alpha,1} \quad \text{for a.e. } \omega \in \widehat{G}, \tag{30}$$

where $\mathcal{P}_\alpha = \{(j,l) \in \mathcal{P} : \alpha \in A^j(D^\perp)\}$.

Proof. We apply Theorem 3.2, where $\mathcal{P}$, g_p and D_p are given by (26). Equation (30) follows directly from (9). Thus, we only have to show that the LIC is satisfied. In order to do that, observe that if the affine system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}$ is a Parseval frame, then, by Proposition 4.9, for any $f \in \mathcal{D}$ there is a constant $C < \infty$ such that

$$\sum_{j\in\mathbb{Z}} \sum_{l=1}^{L} |\widehat{\psi}^l(A^{-j}(\omega))|^2 < C \quad \text{for a.e. } \omega \in \widehat{G}.$$

This inequality also holds if equation (30) is satisfied (use $\alpha = 1$). Thus, by Proposition 28, we do not need to assume the LIC in Theorem 3.2, and this completes the proof. ∎

In the following, we apply Theorem 4.12 to some special LCA groups. As a first application, we consider the case $G = \mathbb{R}^n$, and we show that the usual affine systems on $L^2(\mathbb{R}^n)$ are easily described within our framework.

EXAMPLE 4.13. Let $G = \mathbb{R}^n$ and $D = \mathbb{Z}^n$. Then $\widehat{G} = \mathbb{R}^n$, with the usual Euclidean metric, and $D^\perp = \mathbb{Z}^n$. The matrix $A \in \mathrm{GL}_n(\mathbb{R})$, where all eigenvalues λ of A satisfy $|\lambda| > 1$, is an expanding group automorphism on $\widehat{G}$. Under these assumptions, from definitions (26), for $p \in \mathcal{P} = \{(j,l) : j \in \mathbb{Z},\ l = 1,\dots,L\}$ we have $\widehat{g}_p(\omega) = |\det A|^{-j/2}\,\widehat{\psi}^l(A^{-j}\omega)$ and $D_p^\perp = A^j\,D^\perp = A^j\,\mathbb{Z}^n$. It follows that $g_p(x) = |\det B|^{j/2}\,\psi^l(B^j x)$, where $B = A^{\mathrm{t}}$. Thus, the system $\{T_{\lambda_p} g_p : \lambda_p \in D_p,\ p \in \mathcal{P}\}$ is the usual affine system on $L^2(\mathbb{R}^n)$:

$$\{T_{B^{-j}k}\, g_{(j,l)}(x) = |\det B|^{j/2}\psi^l(B^j x - k) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1,\dots,L\}. \tag{31}$$

From Theorem 4.12, we deduce that the affine system (31) is a Parseval frame for $L^2(\mathbb{R}^n)$ if and only if, for all $\alpha \in \Lambda = \bigcup_{j\in\mathbb{Z}} A^j\mathbb{Z}^n$,

$$\sum_{(j,l)\in\mathcal{P}_\alpha} \widehat{\psi}^l(A^{-j}\xi)\,\overline{\widehat{\psi}^l(A^{-j}(\xi+\alpha))} = \delta_{\alpha,0} \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{32}$$

where $\mathcal{P}_\alpha = \{(j,l) \in \mathcal{P} : \alpha \in A^j\mathbb{Z}^n\}$ for $\alpha \in \Lambda$. This result recovers Theorem 5.9 in [13] and, as shown in that paper, it generalizes and contains all classical characterization results about affine systems, including those in [4, 6, 11, 23]. We refer to the same paper for more details about the motivation and history of these and similar characterization equations for the affine systems in $L^2(\mathbb{R}^n)$.

EXAMPLE 4.14. A *radial function* on $\mathbb{R}^n$ is a function $f : \mathbb{R}^n \to \mathbb{C}$ which satisfies $f(\|x\|) = f(\|y\|)$ for all $x, y \in \mathbb{R}^n$ with $\|x\| = \|y\|$. Radial functions occur in a natural way both in mathematics and applications, including, for example, the study of radar signals. In order to introduce an affine system that can be used to decompose and analyze these types of signals, we define a new function $\psi : \mathbb{R}^+ \to \mathbb{C}$ by $\psi(r) := f(\|x\|)$ for all $r \in \mathbb{R}^+$, where $x \in \mathbb{R}^n$ is chosen such that $\|x\| = r$. Notice that the radial function f is uniquely determined by ψ and $f(0)$.

Now we can apply our general method to the group $G = \mathbb{R}^+$. This is a locally compact abelian group with dual group $\mathbb{R}$. The character of G associated with some $y \in \mathbb{R}$ is the function $x \mapsto e^{2\pi i y \ln x}$. As a uniform lattice in G we choose $D = \{2^n : n \in \mathbb{Z}\}$. A simple calculation shows that $D^\perp = (\ln 2)^{-1}\mathbb{Z}$. Now let $\mathcal{P} = \mathbb{Z}$ and let $A : \mathbb{R} \to \mathbb{R}$ be the expansive automorphism defined by $A(y) = (\ln 2)y$. Then $\nu_p = s(A^p(D^\perp)) = (\ln 2)^{p-1}$ for all $p \in \mathbb{Z}$. Further we define the functions g_p by

$$\widehat{g}_p(y) = (\ln 2)^{(1-p)/2}\,\widehat{\psi}((\ln 2)^{-p} y).$$

Then it follows that $g_p(x) = (\ln 2)^{(p+1)/2}\psi(x^{(\ln 2)^p})$. Observing that

$$T_{2^n} g_p(x) = (\ln 2)^{(1-p)/2}\psi((2^{-n}x)^{(\ln 2)^p}) = (\ln 2)^{(1-p)/2}\psi(e^{-n(\ln 2)^{p+1}}x^{(\ln 2)^p}),$$

we obtain the affine system

$$\Phi(\psi) = \{(\ln 2)^{(p+1)/2}\psi(e^{-n(\ln 2)^{p+1}}x^{(\ln 2)^p}) : p, n \in \mathbb{Z}\}.$$

Thus we can apply Theorem 4.12, which shows that $\Phi(\psi)$ is a Parseval frame for $L^2(\mathbb{R}^+)$ if and only if, for each $p \in \mathbb{Z}$ and $a \in A^p(D^\perp) = (\ln 2)^{p-1}\mathbb{Z}$, $a \neq 0$, we have

$$(\ln 2)^{2(1-p)}\,\overline{\widehat{\psi}((\ln 2)^{-p}y)}\,\widehat{\psi}((\ln 2)^{-p}(y+a)) = 0 \qquad \text{for a.e. } y \in \mathbb{R}$$

and

$$\sum_{p\in\mathbb{Z}} (\ln 2)^{2(1-p)}|\widehat{\psi}((\ln 2)^{-p}y)|^2 = 1 \qquad \text{for a.e. } y \in \mathbb{R},$$

since $\mathcal{P}_a = \{q \in \mathbb{Z} : a \in (\ln 2)^{q-1}\mathbb{Z}\} = \{p\}$ if $a \in (\ln 2)^{p-1}\mathbb{Z}$, $a \neq 0$ and $\mathcal{P}_0 = \mathcal{P} = \mathbb{Z}$.

Example 4.15. Let us consider the subgroup of upper triangular matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y \in \mathbb{R}.$$

We can identify this group with $G = \mathbb{R}^2$ equipped with the group multiplication given by

$$(x_1, y_1)(x_2, y_2) = (x_1 + x_2, y_1 + y_2 - x_1 x_2).$$

This is a locally compact abelian group with dual group $\mathbb{R}^2$. The character of G associated with some $z \in \mathbb{R}^2$ is the function $(x, y) \mapsto e^{2\pi i\langle z,(x,y+x^2/2)\rangle}$. As a uniform lattice in G we choose $D = \mathbb{Z}^2$. A simple calculation shows that $D^\perp = \mathbb{Z} \times 2\mathbb{Z}$. Now let $\mathcal{P} = \mathbb{Z}$ and let $A : \mathbb{R}^2 \to \mathbb{R}^2$ be the expansive automorphism defined by $A(x, y) = B(x, y)^{\mathrm{t}}$, where

$$B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Then $\nu_p = s(A^p(D^\perp)) = 2^{2p+1}$ for all $p \in \mathbb{Z}$. Further we define g_p by

$$\widehat{g}_p(x, y) = 2^{-p-1/2}\widehat{\psi}(B^{-p}(x, y)^{\mathrm{t}}).$$

It follows that $g_p(x, y) = 2^{p-1/2}\psi(2^p x, 2^{p-1}(2y + (1 - 2^p)x^2))$. By the definition of the group multiplication we have $(m, n)^{-1} = (-m, -n - m^2)$. Observing that $T_{(m,n)}g_p(x, y) = \psi_{p,m,n}(x, y)$, where

$$\psi_{p,m,n}(x, y) = 2^{p-1/2}\psi(2^p(x - m), 2^{p-1}(2(y - n - m^2) + (1 - 2^p)(x - m)^2)),$$

we obtain the affine system

$$\Phi(\psi) = \{\psi_{p,m,n}(x, y) : p, m, n \in \mathbb{Z}\}.$$

Thus, we can apply Theorem 4.12, which shows that $\Phi(\psi)$ is a Parseval frame for $L^2(G)$ if and only if, for each $a \in \bigcup_{p\in\mathcal{P}} A^p(D^\perp) = \bigcup_{p\in\mathbb{Z}} B^p(\mathbb{Z}\times 2\mathbb{Z})$, we have

$$\sum_{p\in\mathcal{P}_a} 2^{-4p-2}\,\overline{\widehat{\psi}(B^{-p}(x,y)^{\mathrm{t}})}\,\widehat{\psi}(B^{-p}((x,y)+a)^{\mathrm{t}}) = \delta_{a,0} \quad \text{for a.e. } (x,y)\in\mathbb{R}^2,$$

with $\mathcal{P}_a = \{p\in\mathbb{Z} : a \in B^p(\mathbb{Z}\times 2\mathbb{Z})\}$. Observe that one can deduce several variants of this construction for more general matrices $B \in \mathrm{GL}_2(\mathbb{R})$.

Acknowledgments. The authors wish to thank G. Weiss for his encouragement for this project and for making this collaboration possible, and I. Krishtal, J. Soria and E. Wilson for several helpful discussions. Most of this paper was written while the first author was visiting the Department of Mathematics at Washington University in St. Louis. This author thanks the Department for its hospitality and support during this visit.

REFERENCES

[1] J. J. Benedetto and R. L. Benedetto, *A wavelet theory for local fields and related groups*, J. Geom. Anal. 14 (2004), 423–456.

[2] C. de Boor, R. A. DeVore and A. Ron, *The structure of finitely generated shift-invariant spaces in* $L_2(\mathbb{R}^d)$, J. Funct. Anal. 119 (1994), 37–78.

[3] M. Bownik, *On characterizations of multiwavelets in* $L^2(\mathbb{R}^n)$, Proc. Amer. Math. Soc. 129 (2001), 3265–3274.

[4] A. Calogero, *A characterization of wavelets on general lattices*, J. Geom. Anal. 10 (2001), 597–622.

[5] P. G. Casazza, *The art of frame theory*, Taiwanese J. Math. 4 (2000), 129–201.

[6] C. Chui and X. Shi, *Orthonormal wavelets and tight frames with arbitrary real dilations*, Appl. Comput. Harmon. Anal. 9 (2000), 243–264.

[7] C. Corduneanu, *Almost Periodic Functions*, Interscience, New York, 1968.

[8] S. Dahlke, *Multiresolution analysis and wavelets on locally compact Abelian groups*, in: Wavelets, Images, and Surface Fitting, P.-J. Laurent, A. Le Méhauté and L. L. Schumaker (eds.), A K Peters, Wellesley, MA, 1994, 141–156.

[9] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341–366.

[10] M. Frazier, G. Garrigós, K. Wang and G. Weiss, *A characterization of functions that generate wavelets and related expansions*, J. Fourier Anal. Appl. 3 (1997), special issue, 883–906.

[11] G. Gripenberg, *A necessary and sufficient condition for the existence of a father wavelet*, Studia Math. 114 (1995), 207–226.

[12] K. Gröchenig, *Aspects of Gabor analysis on locally compact abelian groups*, in: Gabor Analysis and Algorithms, Birkhäuser, Boston, MA, 1998, 211–231.

[13] E. Hernández, D. Labate and G. Weiss, *A unified characterization of reproducing systems generated by a finite family, II*, J. Geom. Anal. 12 (2002), 615–662.

[14] E. Hernández and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, 1996.

[15] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I, II*, Springer, Berlin, 1963/1970.

[16] M. Holschneider, *Wavelet analysis over abelian groups*, Appl. Comput. Harmon. Anal. 2 (1995), 52–60.

[17] A. J. E. M. Janssen, *Signal analytic proofs of two basic results on lattice expansions*, ibid. 1 (1994), 350–354.

[18] E. Kaniuth and G. Kutyniok, *Zeros of the Zak transform on locally compact abelian groups*, Proc. Amer. Math. Soc. 126 (1998), 3561–3569.

[19] D. Labate, *A unified characterization of reproducing systems generated by a finite family*, J. Geom. Anal. 12 (2002), 469–491.

[20] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, Oxford, 1968.

[21] A. Ron and Z. Shen, *Frames and stable bases for shift-invariant subspaces of* $L_2(\mathbb{R}^d)$, Canad. J. Math. 47 (1995), 1051–1094.

[22] —, —, *Weyl–Heisenberg frames and Riesz bases in* $L_2(\mathbb{R}^d)$, Duke Math. J. 89 (1997), 237–282.

[23] —, —, *Affine systems in* $L_2(\mathbb{R}^d)$*: the analysis of the analysis operator*, J. Funct. Anal. 148 (1997), 408–447.

Institute of Mathematics
Justus-Liebig-University Giessen
Arndtstr. 2
35392 Giessen, Germany
E-mail: gitta.kutyniok@math.uni-giessen.de

Department of Mathematics
North Carolina State University
Campus Box 8205
Raleigh, NC 27695, U.S.A.
E-mail: dlabate@math.ncsu.edu

Received 8 January 2005 (4551)