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A MULTIPLIER THEOREM FOR FOURIER SERIES IN SEVERAL VARIABLES

BY

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Abstract. We define a new type of multiplier operators on $L^p(\mathbb{T}^N)$, where $\mathbb{T}^N$ is the N-dimensional torus, and use tangent sequences from probability theory to prove that the operator norms of these multipliers are independent of the dimension N. Our construction is motivated by the conjugate function operator on $L^p(\mathbb{T}^N)$, to which the theorem applies as a particular example.

1. Introduction. On the one-dimensional torus $\mathbb{T}$, we can define the conjugate function $\widetilde{f}$ of $f \in L^p(\mathbb{T})$ by the Fourier multiplier operator

$$\widehat{\widetilde{f}}(n) = -i\,\mathrm{sgn}(n)\widehat{f}(n) \qquad (n \in \mathbb{Z}),$$

where $\mathrm{sgn}(n) = 1,\ -1$, or 0, according as n is positive, negative or 0. Parseval's theorem implies that the operator $f \mapsto \widetilde{f}$ is bounded from $L^2(\mathbb{T})$ into $L^2(\mathbb{T})$ with norm equal to 1. The celebrated theorem of M. Riesz establishes the boundedness of this operator from $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$, where $1 < p < \infty$. M. Riesz's theorem plays an important role in harmonic analysis. It has been generalized in many directions (for a brief history of this theorem, including the extensions cited below, see [1] or [3]).

One version of the M. Riesz theorem on the N-dimensional torus, due to S. Bochner (1939), was extended by H. Helson to any compact (connected) abelian group G whose dual group Γ contains an order P. Recall that a subset P of Γ is called an *order* if it satisfies the following three axioms:

$$P \cap (-P) = \{0\}, \qquad P \cup (-P) = \Gamma, \qquad P + P = P.$$

Helson's definition of the conjugate function is as follows. Given an order $P \subset \Gamma$, we define a signum function with respect to P by $\mathrm{sgn}_P(\chi) = -1$, 0, or 1, according as $\chi \in (-P) \setminus \{0\}$, $\chi = 0$, or $\chi \in P \setminus \{0\}$. For $f \in L^2(G)$, define $\widetilde{f}$ by the Fourier multiplier

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$$\widehat{\widetilde{f}}(\chi) = -i \operatorname{sgn}_P(\chi) \widehat{f}(\chi) \quad (\chi \in \Gamma).$$

Helson proved that, for $1 < p < \infty$, this operator is bounded from $L^p(G)$ into $L^p(G)$, with norm that depends only on p and not on P or G. Indeed, Berkson and Gillepsie [4], using transference methods, showed that the norm on $L^p(G)$ is equal to the norm on $L^p(\mathbb{T})$. It is also clear from [4] (see also [2]) that, in order to study the conjugate function on an arbitrary group G, it is enough to consider the case $G = \mathbb{T}^N$, with a lexicographic order on $\Gamma = \mathbb{Z}^N$. This will be the setting of our main result.

2. The Fourier series of $E(f \mid \mathcal{F}_n)$. In this section we recall several well known properties and constructions of martingales on $\mathbb{T}^N$. We sketch some proofs as we establish the notation for this paper. For more details, we refer the reader to [5].

We define a partition $\mathcal{F}_n$ of $\mathbb{T}^N$ and show that for $1 \leq n \leq N$ the Fourier series of the conditional expectation $E(f \mid \mathcal{F}_n)$ is the projection of the Fourier series of f onto $\mathbb{Z}^n$. For $1 \leq n \leq N$, let $\theta_n : \mathbb{T}^N \to \mathbb{T}$ be given by $\theta_n(x_1, \ldots, x_N) = x_n$, and let $\mathcal{F}_n$ be the σ-algebra generated by $\{\theta_1, \ldots, \theta_n\}$. Let $\mathcal{F}_0$ be the σ-algebra containing only $\mathbb{T}^N$ and the empty set. Note that the sets in $\mathcal{F}_n$ are of the form $A \times [0, 2\pi)^{N-n}$, where $A \subseteq [0, 2\pi)^n$ is a Lebesgue measurable set. Furthermore, a function $g : \mathbb{T}^N \to \mathbb{R}$ is measurable with respect to $\mathcal{F}_n$ if and only if there is a measurable function $G : \mathbb{T}^n \to \mathbb{R}$ such that $g(x_1, \ldots, x_N) = G(x_1, \ldots, x_n)$, i.e. g does not depend on the last $N - n$ coordinates. Functions measurable with respect to $\mathcal{F}_0$ are constant on $\mathbb{T}^N$.

Let $g \in L^1(\mathbb{T}^N)$. Then the Fourier coefficients of g are given by

$$b_{\mathbf{k}} = \int_{\mathbb{T}^N} g(x) e^{-i\mathbf{k} \cdot \mathbf{x}} \, dx,$$

where dx denotes the normalized Lebesgue measure on $\mathbb{T}^N$. Suppose g is measurable with respect to the σ-algebra $\mathcal{F}_n$, and let $G : \mathbb{T}^n \to \mathbb{T}$ be such that $g(x_1, \ldots, x_N) = G(x_1, \ldots, x_n)$. Then the Fourier coefficients for g are given by

$$\begin{aligned} b_{\mathbf{k}} = \int_{\mathbb{T}^N} g(x) e^{-i\mathbf{k} \cdot \mathbf{x}} \, d\mathbf{x} &= \int_{\mathbb{T}^n} G(x_1, \ldots, x_n) e^{-i(k_1 x_1 + \cdots + k_n x_n)} \, dx_1 \ldots dx_n \\ &\quad \times \int_{\mathbb{T}^{N-n}} e^{-i(k_{n+1} x_{n+1} + \cdots + k_N x_N)} \, dx_{n+1} \ldots dx_N. \end{aligned}$$

But the last integral above equals 0 when $(k_{n+1}, \ldots, k_N) \neq (0, \ldots, 0)$, and equals 1 when $(k_{n+1}, \ldots, k_N) = (0, \ldots, 0)$. Thus we have

$$b_{\mathbf{k}} = \int_{\mathbb{T}^n} G(x_1, \ldots, x_n) e^{-i(k_1 x_1 + \cdots + k_n x_n)} \, dx_1 \ldots dx_n$$

when $\mathbf{k} \in \mathbb{Z}^n$, and $b_{\mathbf{k}} = 0$ when $\mathbf{k} \in \mathbb{Z}^N \setminus \mathbb{Z}^n$, where $\mathbb{Z}^0 = \{0\}$, and

$$\mathbb{Z}^n = \{(k_1, \dots, k_n, 0, \dots, 0) : k_j \in \mathbb{Z}\}.$$

If g is measurable with respect to the σ-algebra $\mathcal{F}_0$, then the Fourier coefficients are given by $b_{\mathbf{k}} = 0$ if $\mathbf{k} \neq \mathbf{0}$ and

$$b_{\mathbf{0}} = \int_{\mathbb{T}^N} f\, dx.$$

Now let $f \in L^1(\mathbb{T}^N)$, and let $E(f \,|\, \mathcal{F}_n)$ be the conditional expectation of f relative to the σ-algebra $\mathcal{F}_n$. Then $E(f \,|\, \mathcal{F}_n)$ is the unique function up to a set of measure zero that is measurable with respect to $\mathcal{F}_n$ such that for any set $A \times \mathbb{T}^{N-n} \in \mathcal{F}_n$,

$$\int_{A\times\mathbb{T}^{N-n}} E(f \,|\, \mathcal{F}_n)\, dx = \int_{A\times\mathbb{T}^{N-n}} f\, dx.$$

Consider the function

$$G(\mathbf{x}) = \int_{\mathbb{T}^{N-n}} f(x_1, \dots, x_n, s_{n+1}, \dots, s_N)\, ds_{n+1} \dots ds_N.$$

Then G is measurable with respect to $\mathcal{F}_n$ by Fubini's theorem. Furthermore, if $A \times \mathbb{T}^{N-n} \in \mathcal{F}_n$, then

$$\begin{aligned}\int_{A\times\mathbb{T}^{N-n}} f(\mathbf{x})\, dx &= \int_A \Big[\int_{\mathbb{T}^{N-n}} f(\mathbf{x})\, dx_{n+1} \dots dx_N \Big] dx_1 \dots dx_n \\ &= \int_A G(\mathbf{x})\, dx_1 \dots dx_n = \int_{A\times\mathbb{T}^{N-n}} G(\mathbf{x})\, d\mathbf{x}.\end{aligned}$$

By the uniqueness of the conditional expectation, we have $E(f \,|\, \mathcal{F}_n) = G$ a.e. In other words,

$$E(f \,|\, \mathcal{F}_n)(x_1, \dots, x_N) = \int_{\mathbb{T}^{N-n}} f(x_1, \dots, x_n, s_{n+1}, \dots, s_N)\, ds_{n+1} \dots ds_N.$$

We just calculated the Fourier coefficients of an $\mathcal{F}_n$-measurable function g in terms of the function G on $\mathbb{T}^n$ such that $g(x_1, \dots, x_N) = G(x_1, \dots, x_n)$. We will apply this to find the Fourier coefficients of $E(f \,|\, \mathcal{F}_n)$.

For $\mathbf{k} \in \mathbb{Z}^N$, denote the Fourier coefficients of $E(f \,|\, \mathcal{F}_n)$ by $b_{\mathbf{k}}$ and of f by $a_{\mathbf{k}}$. Let $1 \le n \le N$; since $E(f \,|\, \mathcal{F}_n)$ is measurable with respect to $\mathcal{F}_n$, we have $b_{\mathbf{k}} = 0$ when $\mathbf{k} \in \mathbb{Z}^N \setminus \mathbb{Z}^n$, and when $\mathbf{k} \in \mathbb{Z}^n$,

$$\begin{aligned}b_{\mathbf{k}} &= \int_{\mathbb{T}^n} G(x_1, \dots, x_n) e^{-i(k_1x_1+\dots+k_nx_n)}\, dx_1 \dots dx_n \\ &= \int_{\mathbb{T}^n} \Big[\int_{\mathbb{T}^{N-n}} f(x_1, \dots, x_n, s_{n+1}, \dots, s_N)\, ds_{n+1} \dots ds_N \Big] \\ &\quad \times e^{-i(k_1x_1+\dots+k_nx_n)}\, dx_1 \dots dx_n, \\ &= \int_{\mathbb{T}^N} e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x})\, dx = a_{\mathbf{k}},\end{aligned}$$

since $\mathbf{k} \in \mathbb{Z}^n$ implies $k_1 x_1 + \cdots + k_n x_n = \mathbf{k} \cdot \mathbf{x}$. Thus we have

$$b_{\mathbf{k}} = \begin{cases} a_{\mathbf{k}} & \text{if } \mathbf{k} \in \mathbb{Z}^n, \\ 0 & \text{if } \mathbf{k} \in \mathbb{Z}^N \setminus \mathbb{Z}^n. \end{cases}$$

It follows that for $1 \le n \le N$, $E(f \mid \mathcal{F}_n)$ is obtained from f by projecting the Fourier transform of f on $\mathbb{Z}^n$. Trivially, $E(f \mid \mathcal{F}_0)$ is obtained from f by projecting the Fourier transform of f onto $\{\mathbf{0}\} \subseteq \mathbb{Z}^N$.

3. Martingale difference series decomposition. The finite sequence $(\mathbb{E}(f \mid \mathcal{F}_n))_{n=1}^N$ forms a martingale relative to $(\mathcal{F}_n)_{n=1}^N$. (For details, see [3].) We define the martingale difference series decomposition of a function $f \in L^1(\mathbb{T}^N)$, using these conditional expectations. Let

$$d_0(f) = E(f \mid \mathcal{F}_0) = \int_{\mathbb{T}^N} f\,dx,$$

and for $j = 1, \ldots, N$, let

$$d_j(f) = E(f \mid \mathcal{F}_j) - E(f \mid \mathcal{F}_{j-1}).$$

Since f is measurable with respect to $\mathcal{F}_N$, we have

$$f = E(f \mid \mathcal{F}_N) = \sum_{n=0}^{N} d_n(f).$$

This is called the *martingale difference series decomposition* of f.

We will calculate the Fourier coefficients $a_{\mathbf{k}}^j$ of $d_j(f)$ in terms of the Fourier coefficients of f to show that $d_j(f)$ $(j = 1, \ldots, N)$ is obtained from f by projecting the Fourier transform of f onto the set differences $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$. Note that the sets $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$ for $j = 1, \ldots, N$ partition $\mathbb{Z}^N$.

Let $a_{\mathbf{k}}$ and $b_{\mathbf{k}}^j$ denote the $\mathbf{k}$th Fourier coefficient for f and for $E(f \mid \mathcal{F}_j)$, respectively. Then for $2 \le j \le N$,

$$b_{\mathbf{k}}^j = \begin{cases} a_{\mathbf{k}} & \text{if } \mathbf{k} \in \mathbb{Z}^j, \\ 0 & \text{if } \mathbf{k} \in \mathbb{Z}^N \setminus \mathbb{Z}^j. \end{cases}$$

For $j = 2, \ldots, N$, by linearity, the Fourier coefficient $a_{\mathbf{k}}^j$ for $d_j(f)$ is given by

$$a_{\mathbf{k}}^j = b_{\mathbf{k}}^j - b_{\mathbf{k}}^{j-1} = \begin{cases} 0 & \text{if } \mathbf{k} \in \mathbb{Z}^{j-1}, \\ a_{\mathbf{k}} & \text{if } \mathbf{k} \in \mathbb{Z}^j \setminus \mathbb{Z}^{j-1}, \\ 0 & \text{if } \mathbf{k} \in \mathbb{Z}^N \setminus \mathbb{Z}^j. \end{cases}$$

Furthermore, since $d_0(f)$ is constant the Fourier coefficients for $d_0(f)$ are given by

$$a_{\mathbf{k}}^0 = \left\{ \begin{array}{ll} \int_{\mathbb{T}^N} f\,dx & \text{if } \mathbf{k} = \mathbf{0} \\ 0 & \text{if } \mathbf{k} \in \mathbb{Z}^N \setminus \{\mathbf{0}\} \end{array} \right\} = \begin{cases} a_{\mathbf{0}} & \text{if } \mathbf{k} = \mathbf{0}, \\ 0 & \text{if } \mathbf{k} \in \mathbb{Z}^N \setminus \{\mathbf{0}\}. \end{cases}$$

Thus for $j = 1, \dots, N$, $d_j(f)$ is obtained from f by projecting the Fourier transform of f on $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$, and $d_0(f)$ is obtained from f by projecting the Fourier transform of f onto $\{\mathbf{0}\}$.

In our notation, we write the Fourier series for $d_j(f)$ as

$$d_j(f) = \sum_{\mathbf{k} \in \mathbb{Z}^N} a^j_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Since $a^j_{\mathbf{k}} = 0$ on the complement of $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$, it follows that

$$d_j(f) = \sum_{\mathbf{k} \in \mathbb{Z}^j \setminus \mathbb{Z}^{j-1}} a^j_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Letting $\mathbf{x} = (t_1, \dots, t_N)$ and $\mathbf{k} = (k_1, \dots, k_N)$, we have

$$a^j_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = e^{ik_jt_j}\big[a^j_{\mathbf{k}} e^{i(k_1t_1+\dots+k_{j-1}t_{j-1}+k_{j+1}t_{j+1}+\dots+k_Nt_N)}\big].$$

For $k \in \mathbb{Z}$ with $k \neq 0$, define

$$f_{j,k} = \sum_{\mathbf{k}} a^j_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

where the sum ranges over all $\mathbf{k} \in \mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$ of the form $\mathbf{k} = (k_1, \dots, k_{j-1}, k, 0, \dots, 0)$. Note that $a^j_{\mathbf{k}} = 0$ whenever $\mathbf{k}$ satisfies $k_i \neq 0$ for $i > j$ or $k_j = 0$. Thus $f_{j,k} = f_{j,k}(t_1, \dots, t_{j-1})$ is a function of the first $j-1$ coordinates of $\mathbf{x}$ only. As a result, we can express the Fourier series of $d_j(f)$ as

$$d_j(f) = \sum_{k=-\infty,\, k\neq 0}^{\infty} f_{j,k}(t_1, \dots, t_{j-1}) e^{ikt_j}.$$

Since the sets $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$ for $j = 1, \dots, N$ partition $\mathbb{Z}^N$, we get the *martingale difference series* decomposition of f:

$$f = \sum_{j=0}^{N} d_j(f). \tag{1}$$

4. Main theorem. With respect to the lexicographic order on $\mathbb{Z}^N$, the conjugate function operator on $L^2(\mathbb{T}^N)$ can be defined using the martingale difference decomposition given in (1) and the operator in one dimension, as follows. For $f \in L^2(\mathbb{T}^N)$, define the conjugate function of f by

$$\widetilde{f} = \sum_{j=0}^{N} d_j(f)^{\sim}, \tag{2}$$

where

$$d_j(f)^{\sim} = -i \sum_{k=-\infty,\, k\neq 0}^{\infty} \operatorname{sgn}(k) f_{j,k}(t_1, \dots, t_{j-1}) e^{ikt_j}.$$

If we let T_j denote the one-dimensional conjugate function operator applied to the jth coordinate, then (2) has the following expression:

$$\widetilde{f} = \sum_{j=0}^{N} T_j(d_j(f))(t_1, \ldots, t_{j-1},\, t_j),$$

where T_j is the conjugate function of the function $t_j \mapsto d_j(f)(t_1, \ldots, t_{j-1}, t_j)$. Thus, from [4], we have, for $1 < p < \infty$,

$$\Big\| \sum_{j=0}^{N} T_j(d_j(f)) \Big\|_p \leq A_p \|f\|_p, \tag{3}$$

where $A_p = \|T_j\|_p$ is the norm of the operator T_j on $L^p(\mathbb{T})$.

We are now ready to state the main result of our paper. For $j = 1, \ldots, N$, and $1 < p < \infty$, let T_j denote a bounded multiplier from $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$, with multiplier function m_j. Thus, for $f \in L^2(\mathbb{T})$,

$$\widehat{T_j(f)}(n) = m_j(n)\widehat{f}(n).$$

Define a multiplier T on $L^p(\mathbb{T}^N)$ by

$$T(f) = \sum_{j=0}^{N} T_j(d_j(f)), \tag{4}$$

where

$$T_j(d_j(f)) = \sum_{k=-\infty,\, k\neq 0}^{\infty} m_j(k) f_{j,k}(t_1, \ldots, t_{j-1}) e^{ikt_j}. \tag{5}$$

(We are abusing notation here for convenience, since T_j is an operator on $L^p(\mathbb{T})$ and $d_j f$ is a function in $L^p(\mathbb{T}^N)$.) Let $\|T_j\|_p$ denote the norm of the operator T_j from $L^p(\mathbb{T})$ into $L^p(\mathbb{T})$ and let $\|T\|_p$ denote the norm of the operator T from $L^p(\mathbb{T}^N)$ into $L^p(\mathbb{T}^N)$. The main result of this paper is the following.

THEOREM 1 (Main Theorem). *Given* $1 < p < \infty$, *there is a constant* $c_p > 0$, *depending only on* p, *such that*

$$\|T\|_p^p \leq c_p \max_{1\leq j\leq N} \{\|T_j\|_p^p, \|m_j\|_\infty^2\}.$$

REMARKS.

(a) The operator T is indeed a multiplier operator on $L^p(\mathbb{T}^N)$ with multiplier function

$$m(n_1, \ldots, n_N) = m_0(0)1_{\{0\}} + \sum_{j=1}^{N} 1_{\mathbb{Z}^j\setminus\mathbb{Z}^{j-1}} m_j(n_j),$$

where $1_{\mathbb{Z}^j\setminus\mathbb{Z}^{j-1}}$ is the indicator function of the set $\mathbb{Z}^j \setminus \mathbb{Z}^{j-1}$.

(b) We should emphasize that c_p is independent of N, and so a version of the theorem holds for infinite sequences of operators if both $\|T_j\|_p$ and $\|m_j\|_\infty$ are uniformly bounded for all j.
(c) When all the T_j's are equal to the conjugate function operator on $L^p(\mathbb{T})$, we know from the Berkson–Gillespie paper [4] that $c_p = 1$. In general, we are not even close to this constant. Our proof will yield $c_p \approx p^3$.
(d) The theorem fails for $p = 1$. It suffices to consider $T_j(d_j f) = \pm d_j f$. Then the operator T in (4) changes the signs of the terms in the martingale difference series of f. It is well known that these operators are not uniformly bounded for all N.

5. Proof of the Main Theorem. The proof combines a classical result of Rosenthal and a theorem from a relatively new area in probability, known as tangent sequences, due to Kwapień and Woyczyński [6]. To our knowledge, Theorem 1 is the first application of tangent sequences to harmonic analysis.

We start with the definition of tangent sequences. We take the concrete construction of [7]. Given f in $L^p(\mathbb{T}^N)$ and its martingale difference series (1), the *tangent sequence* to f is another martingale difference series g defined on $\mathbb{T}^N \times \mathbb{T}^N$ by

$$g(t_1, \ldots, t_N, s_1, \ldots, s_N) = d_0(f) + \sum_{j=1}^{N} d_j(f)(t_1, \ldots, t_{j-1}, s_j).$$

(Note that the tangent sequence g does not depend on t_N.) Kwapień and Woyczyński [6] showed the following result for tangent sequences.

THEOREM 2. *Given $1 < p < \infty$, there is a constant $C_p > 0$, depending only on p, such that*

$$C_p^{-1}\|f\|_{L^p(\mathbb{T}^N)} \le \|g\|_{L^p(\mathbb{T}^N \times \mathbb{T}^N)} \le C_p\|f\|_{L^p(\mathbb{T}^N)}.$$

The next result of Rosenthal concerns sums of independent mean-zero random variables over a probability space. Given a sequence of random variables $(X_n)_{n=1}^N$, for $p \in [2, \infty)$, define the quantity

$$|||(X_n)|||_{(p)} = \max\Big\{ \Big(\sum_{n=1}^{N} \|X_n\|_2^2\Big)^{1/2}, \Big(\sum_{n=1}^{N} \|X_n\|_p^p\Big)^{1/p}\Big\}.$$

Rosenthal's result is the following (see [8]).

THEOREM 3. *Given $2 \le p < \infty$, there is a constant $K_p > 0$, depending only on p, such that for any sequence of mean-zero, independent random* *variables, we have*

$$K_p^{-1}|||(X_n)|||_{(p)} \le \Big\|\sum_{n=1}^{N} X_n\Big\|_p \le K_p|||(X_n)|||_{(p)}.$$

Proof od Main Theorem. Since T is a multiplier operator, it is enough to prove that it is bounded from $L^p(\mathbb{T}^N)$ into $L^p(\mathbb{T}^N)$ for $2 \le p < \infty$. The case $1 < p < 2$ follows by a well known duality argument. Let $f \in L^p(\mathbb{T}^N)$, where $2 \le p < \infty$. We may suppose that f has mean zero; otherwise consider $f - \int_{\mathbb{T}^N} f(x)\,dx$. Also, we may suppose that f is a trigonometric polynomial, so all the series that we will consider in this proof are finite sums, and there is no ambiguity in interpreting them. By comparing Fourier transforms, it is clear that

$$T_j(d_j f) = d_j(T_j(d_j f)). \tag{6}$$

Also, using (5) and the bound of T_j on $L^p(\mathbb{T})$, we find

$$\begin{aligned}\int_{\mathbb{T}} |T_j(d_j f)|^p\,dt_j &= \int_{\mathbb{T}} \Big|\sum_k m_j(k) f_{j,k}(t_1,\dots,t_{j-1})e^{ikt_j}\Big|^p\,dt_j \\ &\le \|T_j\|_p^p \int_{\mathbb{T}} \Big|\sum_k f_{j,k}(t_1,\,\dots,t_{j-1})e^{ikt_j}\Big|^p\,dt_j \\ &= \|T_j\|_p^p \int_{\mathbb{T}} |d_j(f)|^p\,dt_j,\end{aligned} \tag{7}$$

where $\|T_j\|_p$ is the norm of the multiplier operator on $L^p(\mathbb{T})$.

For $f \in L^p(\mathbb{T}^N)$, form the tangent sequence of f and apply Theorem 2. Write the inequalities in Theorem 2 in the following convenient notation:

$$\|f\|^p_{L^p(\mathbb{T}^N)} \approx \Big\|\sum_{j=1}^{N} d_j f(t_1,\dots,t_{j-1},s_j)\Big\|^p_{L^p(\mathbb{T}^{2N})}.$$

Note that

$$\Big\|\sum_{j=1}^{N} d_j f(t_1,\dots,t_{j-1},s_j)\Big\|^p_{L^p(\mathbb{T}^{2N})} = \int_{\mathbb{T}^N} \Big\|\sum_{j=1}^{N} d_j f(t_1,\dots,t_{j-1},s_j)\Big\|^p_{L^p(\mathbb{T}^N,ds)} dt.$$

For all (fixed) $t=(t_1,\dots,t_N) \in \mathbb{T}^N$, the functions $(d_j f(t_1,\dots,t_{j-1},s_j))_{j=1}^N$ are independent functions on $\mathbb{T}^N$ in the variable $s = (s_1,\dots,s_N)$. To simplify notation, for each fixed $t = (t_1,\dots,t_N) \in \mathbb{T}^N$ and $j = 1,\dots,N$, write $d_j f(t_1,\dots,t_{j-1},s_j)$ as $(d_j f)_t(s_j)$. Then the sequence of independent random variables on $L^p(\mathbb{T}^N)$ becomes $((d_j f)_t(s_j))_{j=1}^N$. Applying Theorem 3, we find that, for each $t = (t_1, \ldots, t_N) \in \mathbb{T}^N$,

$$\Big\|\sum_{j=1}^{N}(d_j f)_t(s_j)\Big\|^p_{L^p(\mathbb{T}^N,ds)} \approx \max\Big\{\sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)}, \sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^2_{L^2(\mathbb{T}^N,ds)}\Big\}.$$

Thus

$$\|f\|^p_{L^p(\mathbb{T}^N)} \approx \int_{\mathbb{T}^N} \max\Big\{\sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)}, \sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^2_{L^2(\mathbb{T}^N,ds)}\Big\}dt. \tag{8}$$

Applying (8) to the function Tf in place of f and using (6), we find that

$$\|Tf\|^p_{L^p(\mathbb{T}^N)} \approx \int_{\mathbb{T}^N} \max\Big\{\sum_{j=1}^{N} \|(T_j d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)}, \sum_{j=1}^{N} \|(T_j d_j f)_t(s_j)\|^2_{L^2(\mathbb{T}^N,ds)}\Big\}dt. \tag{9}$$

Using (7), we obtain

$$\begin{aligned}\|(T_j d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)} &= \int_{\mathbb{T}^{N-1}} \int_{\mathbb{T}} |(T_j d_j f)(t_1, \ldots, t_{j-1}, s_j)|^p \, ds_j ds_1 \ldots ds_N \\ &\le \|T_j\|_p^p \int_{\mathbb{T}^{N-1}} \int_{\mathbb{T}} |d_j(f)(t_1, \ldots, t_{j-1}, s_j)|^p \, ds_j ds_1 \ldots ds_N \\ &= \|T_j\|_p^p \|(d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)}.\end{aligned}$$

Similarly for the case $p = 2$. Putting this inequality into (9), we get

$$\|Tf\|^p_{L^p(\mathbb{T}^N)} \le a_p A \int_{\mathbb{T}^N} \max\Big\{\sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^p_{L^p(\mathbb{T}^N,ds)}, \sum_{j=1}^{N} \|(d_j f)_t(s_j)\|^2_{L^2(\mathbb{T}^N,ds)}\Big\}\, dt,$$

where $A = \max_{1\le j\le N}\{\|T_j\|_p^p, \|m_j\|_\infty^2\}$ and a_p depends only on p. Applying (8) once more, we find that

$$\|Tf\|^p_{L^p(\mathbb{T}^N)} \le c_p^p A \|f\|^p_{L^p(\mathbb{T}^N)},$$

where c_p depends on p only. This completes the proof of the Main Theorem. ∎

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