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KEMPISTY'S THEOREM FOR THE INTEGRAL PRODUCT QUASICONTINUITY

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Abstract. A function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the condition $Q_i(x)$ (resp. $Q_s(x)$, $Q_o(x)$) at a point x if for each real $r > 0$ and for each set $U \ni x$ open in the Euclidean topology of $\mathbb{R}^n$ (resp. strong density topology, ordinary density topology) there is an open set I such that $I \cap U \neq \emptyset$ and $|(1/\mu(U \cap I)) \int_{U \cap I} f(t)\,dt - f(x)| < r$. Kempisty's theorem concerning the product quasicontinuity is investigated for the above notions.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and positive reals $r_1, \ldots, r_n$ put

$$I_i = (x_i - r_i, x_i + r_i) \quad \text{for } i = 1, \ldots, n,$$
$$P(x; r_1, \ldots, r_n) = I_1 \times \cdots \times I_n, \quad Q(x, r) = P(x; r, \ldots, r).$$

Denote by μ the Lebesgue measure and by μ_e the outer Lebesgue measure in $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define the *upper* (resp. *lower*) *outer strong density* $D_u(A, x)$ (resp. $D_l(A, x)$) of A at x as

$$\limsup_{h_1, \ldots, h_n \to 0^+} \frac{\mu_e(A \cap P(x; h_1, \ldots, h_n))}{\mu(P(x; h_1, \ldots, h_n))}$$

and

$$\liminf_{h_1, \ldots, h_n \to 0^+} \frac{\mu_e(A \cap P(x; h_1, \ldots, h_n))}{\mu(P(x; h_1, \ldots, h_n))}$$

respectively. Similarly for $A \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$ we define the *upper* (resp. lower) *outer ordinary density* $d_u(A, x)$ (resp. $d_l(A, x)$) of A at x as

$$\limsup_{h \to 0^+} \frac{\mu_e(A \cap Q(x, h))}{\mu(Q(x, h))} \quad \text{and} \quad \liminf_{h \to 0^+} \frac{\mu_e(A \cap Q(x, h))}{\mu(Q(x, h))}$$

respectively. A point x is said to be an *outer strong density point* (resp. a *strong density point*) of A if $D_l(A, x) = 1$ (resp. if there is a Lebesgue measurable set $B \subset A$ such that $D_l(B, x) = 1$).

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Similarly we define the notions of an outer ordinary density point and of an ordinary density point.

The family $T_{s,d}$ (resp. $T_{o,d}$) of all sets all of whose points are strong (resp. ordinary) density points is a topology called the *strong* (resp. *ordinary*) *density topology* ([1, 2, 9, 7]). If $n = 1$ then $T_{s,d} = T_{o,d}$ is called the *density topology*.

If T_e denotes the Euclidean topology in $\mathbb{R}^n$ then evidently $T_e \subset T_{s,d} \subset T_{o,d}$ and all sets in $T_{o,d}$ are Lebesgue measurable ([1, 2, 9]).

The continuity of mappings f from $(\mathbb{R}^n, T_{s,d})$ (resp. from $(\mathbb{R}^n, T_{o,d})$) to $(\mathbb{R}, T_e)$ is called the *strong* (resp. *ordinary*) *approximate continuity* ([1, 2, 9]).

In [5, 6] the following notion is investigated.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *quasicontinuous at a point* x (written $f \in Q(x)$) if for each $r > 0$ and each $U \in T_e$ containing x there is a nonempty set $I \in T_e$ such that $I \subset U$ and $|f(t) - f(x)| < r$ for all $t \in I$.

A function f is *quasicontinuous* if $f \in Q(x)$ for every $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *integrally quasicontinuous at a point* x ($f \in Q_i(x)$, [4]) if for each $r > 0$ and each $U \in T_e$ containing x there is a nonempty set $I \in T_e$ such that $I \subset U$ and

$$\left| \frac{\int_I f(t)\, dt}{\mu(I)} - f(x) \right| < r.$$

A function f is *integrally quasicontinuous* ($f \in Q_i$) if $f \in Q_i(x)$ for all $x \in \mathbb{R}^n$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Q_s(x)$ (resp. $f \in Q_o(x)$, [4]) if for each $\eta > 0$ and each $U \in T_{s,d}$ (resp. $U \in T_{o,d}$) containing x there is a nonempty set $I \in T_e$ such that $I \cap U \neq \emptyset$, f is Lebesgue integrable on $I \cap U$ and

$$\left| \frac{1}{\mu(I \cap U)} \int\limits_{I \cap U} f(t)\, dt - f(x) \right| < \eta.$$

If $f \in Q_s(x)$ (resp. $f \in Q_o(x)$) for all $x \in \mathbb{R}^n$ then we write $f \in Q_s$ (resp. $f \in Q_o$).

The inclusions $Q_o \subset Q_s \subset Q_i$ are true and each measurable quasicontinuous function $f : \mathbb{R}^n \to \mathbb{R}$ is integrally quasicontinuous ([4]). If $n = 1$ then $Q_o = Q_s$.

Now let n_1, n_2 be two positive integers with $n_1 + n_2 = n$ and let $\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For $x = (x_1, \ldots, x_{n_1}) \in \mathbb{R}^{n_1}$ and $y = (x_{n_1+1}, \ldots, x_n) \in \mathbb{R}^{n_2}$ we write $(x, y) = (x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_n) \in \mathbb{R}^n$.

For a function $f : \mathbb{R}^n \to \mathbb{R}$ and for points $t \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ we define the sections $f_t : \mathbb{R}^{n^2} \to \mathbb{R}$ and $f^y : \mathbb{R}^{n_1} \to \mathbb{R}$ by

$$f_t(y) = f(t, y) \quad \text{and} \quad f^y(t) = f(t, y).$$

If $n = n_1 + n_2$ with $n_1, n_2 > 0$ then we refer to different types of quasicontinuity of functions $f : \mathbb{R}^n \to \mathbb{R}$ as *product quasicontinuties*.

The following theorem of Kempisty is well known ([5, 6]).

THEOREM 1. *If all sections f_t and f^y of a function $f : \mathbb{R}^n \to \mathbb{R}$ are quasicontinuous then f is also quasicontinuous.*

To prove that analogues of Kempisty's theorem for integral quasicontinuities are not true, we start from the following lemma.

LEMMA 1. *Let $A, B \subset \mathbb{R}$ be disjoint countable nonempty sets. There are disjoint measurable sets E, G such that*

$$E \supset A, \qquad G \supset B, \qquad E \cup G = \mathbb{R},$$
$$D_u(E, x) > 0 \quad \text{for each } x \in E,$$
$$D_u(G, y) > 0 \quad \text{for each } y \in G.$$

Proof. This is an immediate consequence of Lemma 3 from [3].

REMARK 1. *Assume the Continuum Hypothesis CH. There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that all sections f_t and f^y, $t, y \in \mathbb{R}$, belong to $Q_s = Q_o$ and the restriction $f|A$ is not measurable for any measurable set $A \subset \mathbb{R}^2$ with $\mu(A) > 0$.*

Proof. Let

$$a_0, a_1, \ldots, a_\alpha, \ldots, \qquad \alpha < \omega_1,$$

be a transfinite sequence of all reals such that $a_\alpha \neq a_\beta$ for $\alpha < \beta < \omega_1$, where ω_1 denotes the first uncountable ordinal.

Let $S \subset \mathbb{R}^2$ be such that the inner Lebesgue measures $\mu_i(S)$ and $\mu_i(\mathbb{R}^2 \setminus S)$ are 0 and $\text{card}(p \cap S) \leq 2$ for each straight line p ([8]).

For $\alpha < \omega_1$ we will define by transfinite induction two functions $g_\alpha, h_\alpha : \mathbb{R} \to \{0, 1\}$.

If the vertical straight line p_0 defined by the equation $t = a_0$ is such that $p_0 \cap S = \emptyset$ then we put $h_0(y) = 0$ for $y \in \mathbb{R}$. Analogously if the horizontal straight line q_0 defined by $y = a_0$ is such that $q_0 \cap S = \emptyset$ then we put $g_0(t) = 0$ for $t \in \mathbb{R}$.

If $p_0 \cap S \neq \emptyset$ then we put $h_0(y) = 1$ for $y \in \mathbb{R}$; if $q_0 \cap S \neq \emptyset$ then we put $g_0(t) = 1$ for $t \in \mathbb{R}$.

Fix a countable ordinal number $\alpha > 0$ and assume that we have defined $g_\beta, h_\beta : \mathbb{R} \to \{0, 1\}$ for $\beta < \alpha$.

Let p_α be defined by $t = a_\alpha$ and let q_α be defined by $y = a_\alpha$. Set

$$A_{1,\alpha} = \{a_\beta;\ \beta < \alpha \text{ and } h_\beta(a_\alpha) = 1\} \cup \{(t \in \mathbb{R};\ (t, a_\alpha) \in q_\alpha \cap S\}.$$

Moreover let $A_{2,\alpha} \subset \mathbb{R} \setminus A_{1,\alpha}$ be a countable dense set. By Lemma 1, there are disjoint measurable sets $E_{1,\alpha} \supset A_{1,\alpha}$ and $E_{2,\alpha} \supset A_{2,\alpha}$ such that $\mathbb{R} = E_{1,\alpha} \cup E_{2,\alpha}$, $D_u(E_{1,\alpha}, t) > 0$ for each $t \in E_{1,\alpha}$ and $D_u(E_{2,\alpha}, t) > 0$ for each $t \in E_{2,\alpha}$. Put

$$g_\alpha(t) = \begin{cases} 1 & \text{for } t \in E_{1,\alpha}, \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Set

$$B_{1,\alpha} = \{a_\beta;\ \beta \le \alpha \text{ and } g_\beta(a_\alpha) = 1\} \cup \{y \in \mathbb{R};\ (a_\alpha, y) \in p_\alpha \cap S\}$$

and let $B_{2,\alpha} \subset \mathbb{R} \setminus B_{1,\alpha}$ be a countable dense set. By Lemma 1, there are disjoint measurable sets $G_{1,\alpha} \supset B_{1,\alpha}$, and $G_{2,\alpha} \supset B_{2,\alpha}$ such that $\mathbb{R} = G_{1,\alpha} \cup G_{2,\alpha}$, $D_u(G_{1,\alpha}, t) > 0$ for each $t \in G_{1,\alpha}$ and $D_u(G_{2,\alpha}, t) > 0$ for each $t \in G_{2,\alpha}$. Let

$$h_\alpha(y) = \begin{cases} 1 & \text{for } t \in G_{1,\alpha} \\ 0 & \text{otherwise on } \mathbb{R}. \end{cases}$$

Now for $x \in \mathbb{R}$ we find an ordinal α such that $x = a_\alpha$ and put

$$f(x, v) = h_\alpha(v) \quad \text{for } v \in \mathbb{R},$$
$$f(u, x) = g_\alpha(u) \quad \text{for } u \in \mathbb{R}.$$

Let

$$\Pr(S) = \{t;\ \exists_y (t, y) \in S\}.$$

Since $f(t, y) = 1$ for $(t, y) \in S$ and $\mu_i(f_t^{-1}(0)) > 0$ for $t \in \Pr(S)$, the restriction $f|A$ is not measurable for any measurable $A \subset \mathbb{R}^2$ with $\mu(A) > 0$. If $t = a_\alpha$ then $f^t = g_\alpha \in Q_s$ and $f_t = h_\alpha \in Q_s$ (see [4, Th. 2]). This finishes the proof.

COROLLARY 1. *The function f constructed in the proof of Remark* 1 *is not in Q_i, so analogues of Kempisty's theorem for the integral quasicontinuities are not true.*

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *strongly* (resp. *ordinarily*) *approximately quasicontinuous* at a point $x \in \mathbb{R}^n$ if for each $\eta > 0$ and each $U \in T_{s,d}$ (resp. $U \in T_{o,d}$) containing x there is a nonempty set $V \subset U$ belonging to $T_{s,d}$ (resp. to $T_{o,d}$) for which $f(V) \subset (f(x) - \eta, f(x) + \eta)$ (cf. [3]). If $n = 1$ then the notions of strong and ordinary approximate quasicontinuity are equivalent and in this case we say that f is *approximately quasicontinuous*.

Observe that all sections f_t and f^y, $t, y \in \mathbb{R}$, of the function $f : \mathbb{R}^2 \to \mathbb{R}$ constructed in the proof of Remark 1 are approximately quasicontinuous at each point.

By the Lebesgue density theorem, functions strongly (and ordinarily) approximately quasicontinuous at all points are measurable (cf. [3]).

For $A \subset \mathbb{R}^n$, $t \in \mathbb{R}^{n_1}$ and $y \in \mathbb{R}^{n_2}$ define the sections

$$A_t = \{v \in \mathbb{R}^{n_2};\, (t,v) \in A\} \quad \text{and} \quad A^y = \{u \in \mathbb{R}^{n_1};\, (u,y) \in A\}.$$

Let

$$T^+_{s,d} = \{A \subset \mathbb{R}^n;\, A \text{ is measurable and } A_u, A^v \in T_{s,d} \text{ for all } u \in \mathbb{R}^{n_1} \text{ and } v \in \mathbb{R}^{n_2}\},$$

$$T^+_{o,d} = \{A \subset \mathbb{R}^n;\, A \text{ is measurable and } A_u, A^v \in T_{o,d} \text{ for all } u \in \mathbb{R}^{n_1} \text{ and } v \in \mathbb{R}^{n_2}\}.$$

In connection with Remark 1 we have the following.

THEOREM 2. *If all sections f_u and f^v, $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are strongly (resp. ordinarily) approximately quasicontinuous at all points then for each $(t,y) \in \mathbb{R}^n$, each $\eta > 0$ and each $A \in T^+_{s,d}$ (resp. $A \in T^+_{o,d}$) containing (t,y) there is a mesurable subset $B \subset A$ such that $\mu(B) > 0$ and $f(B) \subset (f(t,y) - \eta, f(t,y) + \eta)$.*

Proof. Fix $(t,y) \in \mathbb{R}^n$, $A \in T^+_{s,d}$ containing (t,y), and $\eta > 0$. Since f^y is strongly approximately quasicontinuous at t, there is a measurable set $U \subset A^y$ such that $\mu(U) > 0$ and $f^y(U) \subset (f(t,y) - \eta/3, f(t,y) + \eta/3)$. Since all sections f_u, $u \in U$, are strongly approximately quasicontinuous at y, for each $u \in U$ there is a measurable set $V(u) \subset A_u$ of positive measure such that $f_u(V(u)) \subset (f(u,y) - \eta/3, f(u,y) + \eta/3)$. Let

$$E = \{(u,v);\, u \in U \text{ and } v \in V(u)\}$$

and let $H \subset A$ be a measurable cover of E, i.e. $H \supset E$ is a measurable set and each measurable subset of $B \setminus E$ is of measure zero. Evidently the set

$$B = H \cap \{(u,v) \in \mathbb{R}^n;\, |f(u,v) - f(t,y)| < \eta\}$$

is as required. The proof for the case of ordinary approximate quasicontinuity is the same.

Theorem 2 implies the following:

THEOREM 3. *If all sections f_u and f^v, $u \in \mathbb{R}^{n_1}$, $v \in \mathbb{R}^{n_2}$, of a bounded measurable function $f : \mathbb{R}^n \to \mathbb{R}$ are strongly (resp. ordinarily) approximately quasicontinuous at all points then for each $(t,y) \in \mathbb{R}^n$, each $\eta > 0$ and each $A \in T^+_{s,d}$ (resp. $A \in T^+_{o,d}$) containing (t,y) there is a bounded set $E \in T_e$ such that $E \cap A \neq \emptyset$ and*

$$\left| \frac{\int_{A \cap E} f}{\mu(A \cap E)} - f(t,y) \right| < \eta.$$

Proof. Fix $(t,y) \in \mathbb{R}^n$, $A \in T^+_{s,d}$ containing (t,y), and $\eta > 0$. By Theorem 2 there is a measurable set $B \subset A$ such that $\mu(B) > 0$ and

$f(B) \subset (f(t,y) - \eta/2, f(t,y) + \eta/2)$. So,

$$\left| \frac{\int_B f}{\mu(B)} - f(t,y) \right| \le \frac{\eta}{2}.$$

From the absolute continuity of the Lebesgue integral it follows that there is a nonempty set $E \subset \mathbb{R}^n$ belonging to T_e such that $E \supset B$ and

$$\left| \frac{\int_{A\cap E} f}{\mu(A \cap E)} - f(t,y) \right| < \eta.$$

This completes the proof.

Let Z be a nonempty set of indices. We will say that functions $h_\alpha : \mathbb{R}^{n_2} \to \mathbb{R},\ \alpha \in Z$, are *strongly* (resp. *ordinarily*) *integrally equiquasicontinuous at a point* $v \in \mathbb{R}^{n_2}$ if for each set $V \subset \mathbb{R}^{n_2}$ containing v and belonging to $T_{s,d}$ (resp. to $T_{o,d}$) and for each $\eta > 0$ there is a set $G \subset \mathbb{R}^{n_2}$ belonging to T_e and such that $\emptyset \ne V \cap G$ and

$$\left| \frac{\int_{V\cap G} f_\alpha}{\mu(G \cap V)} - f_\alpha(v) \right| < \eta \quad \text{for } \alpha \in Z.$$

THEOREM 4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded measurable function such that*

(i) *for each $(u,v) \in \mathbb{R}^n$ there is a set $A(u,v) \subset \mathbb{R}^{n_1}$ belonging to $T_{s,d}$ and containing u for which the sections f_t, $t \in A(u,v)$, are strongly integrally equiquasicontinuous at v.*

If $f^y \in Q_s$ for all $y \in \mathbb{R}^{n_2}$, then f satisfies the following condition:

(a) *for each $(t,y) \in \mathbb{R}^n$, each $\eta > 0$ and all $U \in T_{s,d}$ with $t \in U$ and $V \in T_{s,d}$ with $y \in V$ there are $Z \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$ belonging to T_e and such that $\emptyset \ne U \cap Z$, $\emptyset \ne V \cap Y$ and*

$$\left| \frac{\int_{(U\times V)\cap(Z\times Y)} f}{\mu((U \times V) \cap (Z \times Y))} - f(t,y) \right| < \eta.$$

Proof. Fix $(t,y) \in \mathbb{R}^n$, $\eta > 0$ and $U, V \in T_{s,d}$ such that $(t,y) \in U \times V$. Since $f^y \in Q_s$ and $t \in U \cap A(t,y) \in T_{s,d}$, there is a bounded set $W \in T_e$ such that $K = W \cap U \cap A(t,y) \ne \emptyset$ and

$$\left| \frac{\int_K f^y}{\mu(K)} - f(t,y) \right| < \frac{\eta}{2}.$$

By our hypothesis (i) there is a set $Y \subset \mathbb{R}^{n_2}$ belonging to T_e and such that $V \cap Y \ne \emptyset$ and

$$\left| \frac{\int_{V\cap Y} f_u}{\mu(Y \cap V)} - f(u,y) \right| < \frac{\eta}{2} \quad \text{for } u \in K.$$

Let $H = K \times (Y \cap V)$. Observe that

$$\begin{aligned}
&\left|\frac{\int_H f(u,v)\,du\,dv}{\mu(H)} - f(t,y)\right| \\
&\qquad \le \left|\frac{\int_K(\int_{Y\cap V} f(u,v)\,dv)\,du}{\mu(H)} - \frac{\int_K f(u,y)\mu(V\cap Y)\,du}{\mu(H)}\right| \\
&\qquad\quad + \left|\frac{\int_K f(u,y)\mu(V\cap Y)\,du}{\mu(H)} - f(t,y)\right| \\
&\qquad \le \frac{\int_K \left|\frac{\int_{Y\cap V} f(u,v)\,dv}{\mu(Y\cap V)} - f(u,y)\right|du}{\mu(K)} + \left|\frac{\int_K f(u,y)\,du}{\mu(K)} - f(t,y)\right| \\
&\qquad < \frac{\eta}{2} + \frac{\eta}{2} = \eta.
\end{aligned}$$

Since f is locally bounded and measurable, from the absolute continuity of the Lebesgue integral it follows that there is a bounded set $X \subset \mathbb{R}^{n_1}$ containing K, belonging to T_e and such that for $M = X \times (Y \cap V)$ we have

$$\left|\frac{\int_M f(u,y)\,du\,dv}{\mu(M)} - f(t,y)\right| < \eta.$$

So the proof is finished.

In the same way we can prove the following theorem.

THEOREM 5. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded measurable function such that*

(ii) *for each $(u,v) \in \mathbb{R}^n$ there is a set $A(u,v) \subset \mathbb{R}^{n_1}$ belonging to $T_{o,d}$ and containing u for which the sections f_t, $t \in A(u,v)$, are ordinarily integrally equiquasicontinuous at v.*

If $f^y \in Q_o$ for all $y \in \mathbb{R}^{n_2}$, then f satisfies the following condition:

(b) *for each $(t,y) \in \mathbb{R}^n$, each $\eta > 0$ and all $U \in T_{o,d}$ with $t \in U$ and $V \in T_{o,d}$ with $y \in V$ there are $Z \subset \mathbb{R}^{n_1}$ and $Y \subset \mathbb{R}^{n_2}$ belonging to T_e and such that $\emptyset \neq U \cap Z$, $\emptyset \neq V \cap Y$ and*

$$\left|\frac{\int_{(U\times V)\cap(Z\times Y)} f}{\mu((U\times V)\cap(Z\times Y))} - f(t,y)\right| < \eta.$$

PROBLEM. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally bounded function satisfying condition (i) of Theorem 4 (resp. condition (ii) of Theorem 5) and having measurable sections f^y for all $y \in \mathbb{R}^{n_2}$. Is f measurable?

Let Z be a nonempty set of indices. We will say that functions $h_\alpha : \mathbb{R}^{n_2} \to \mathbb{R}$, $\alpha \in Z$, are *integrally equiquasicontinuous at a point $y \in \mathbb{R}^{n_2}$* if for each set $U \subset \mathbb{R}^{n_2}$ containing y and belonging to T_e and for each $\eta > 0$

there is a nonempty set $V \subset U$ belonging to T_e such that

$$\left|\frac{\int_V f_\alpha}{\mu(V)} - f_\alpha(y)\right| < \eta \quad \text{for } \alpha \in Z.$$

THEOREM 6. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a bounded measurable function such that* $f^v \in Q_i$ *for all* $v \in \mathbb{R}^{n_2}$, *and for each* $(t, y) \in \mathbb{R}^n$ *there is a set* $A(t, y) \subset \mathbb{R}^{n_1}$ *containing* t *and belonging to* T_e *such that the sections* f_u, $u \in A(t, y)$, *are integrally equiquasicontinuous at* y. *Then* $f \in Q_i$.

The proof of Theorem 6 is completely similar to the proof of Theorem 4.

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