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ON A LINEAR HOMOGENEOUS CONGRUENCE

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Abstract. The number of solutions of the congruence $a_1x_1 + \dots + a_kx_k \equiv 0 \pmod n$ in the box $0 \le x_i \le b_i$ is estimated from below in the best possible way, provided for all i, j either $(a_i, n) \mid (a_j, n)$ or $(a_j, n) \mid (a_i, n)$ or $n \mid [a_i, a_j]$.

1. Introduction. We shall consider the following conjecture proposed in [1]:

CONJECTURE. *Let k, n and b_i $(1 \le i \le k)$ be positive integers, and let a_i $(1 \le i \le k)$ be any integers. The number $N(n; a_1, b_1, \dots, a_k, b_k)$ of solutions of the congruence*

$$\sum_{i=1}^{k} a_ix_i \equiv 0 \pmod n \quad \textit{in the box } 0 \le x_i \le b_i \tag{1}$$

satisfies the inequality

$$N(n; a_1, b_1, \dots, a_k, b_k) \ge 2^{1-n} \prod_{i=1}^{k} (b_i + 1). \tag{2}$$

Since for $k = n - 1$,

$$N(n; 1, 1, \dots, 1) = 2^{1-n} \prod_{i=1}^{k} (1 + 1),$$

if the above conjecture is true, then 2^{1-n} is the best possible coefficient independent of a_i, b_i, and dependent only on n, with which the inequality (2) holds. The first named author proved in [1] that (2) holds if $(n, a_i) = 1$ for all $i \le k$. The aim of this paper is to prove

THEOREM. *The inequality* (2) *holds if for all $i, j \le k$ we have either $(n, a_i) \mid (n, a_j)$ or $(n, a_j) \mid (n, a_i)$, or $n \mid [a_i,\, a_j]$.*

COROLLARY 1. *The inequality* (2) *holds for $n = p^\alpha$ and for $n = pq$ (p, q primes).*

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2. Lemmas. We shall use the following lemmas taken from [1]:

LEMMA A. *Inequality* (2) *holds for* $n = 4$, a_1 *and* a_2 *odd*, $b_1 = b_2 = 2$.

LEMMA B. *Let* B *be a set of residues mod* m, *and let* $a, b \in \mathbb{N}$ *with* $(a, m) = 1$. *If* x *runs through the integers of the interval* $[0, b]$ *and* y *through the elements of* B, *then* $ax+y$ *gives at least* $\min\{m, |B|+b\}$ *residues mod* m.

LEMMA C. *For positive integers* a *and* $x \le a$ *we have*

$$\left(1 + \frac{a}{x}\right)^{x+1} \le 2^{a+1},$$

except for $a = 2$ *and* $x = 1$.

From Lemma B we deduce

LEMMA 1. *Let* A *be a set of residues mod* m, *and let* $a, b \in \mathbb{N}$ *with* $(a, m) = 1$ *and* $b \ge m - |A|$. *For every* r *the number of solutions of the congruence* $ax + y \equiv r \pmod{m}$ *such that* $0 \le x \le b$, $y \in A$ *is at least*

$$s = \left[\frac{b+1}{m+1-|A|}\right] \ge \max\left\{1, \frac{b+1}{2(m+1-|A|)-1}\right\}.$$

Proof. Put $m - |A| = c$ and consider the intervals (reduced to a point for $c = 0$)

$$I_i = [ci + i,\, c(i+1) + i], \quad 0 \le i \le s - 1.$$

Each interval I_i contains $c + 1$ consecutive integers, hence by Lemma B, $ax + y$ with $y \in A$ gives $c + |A| = m$ residues mod m, thus in particular r. Since the s intervals I_i are disjoint we obtain the first part of the lemma. The second part (the inequality) follows from the inequality

$$u \ge \frac{uv + v - 1}{2v - 1}$$

valid for $u, v \ge 1$, in which we take $u = \left[\frac{b+1}{m+1-|A|}\right]$, $v = m + 1 - |A|$.

We have further

LEMMA 2. *If* $a > (\log 2)^{-1}$, *then the function* $(a - x)2^x$ *is unimodal in the interval* $[0, a]$ *with the maximum at* $x = a - (\log 2)^{-1}$.

Proof. By differentiation.

For the proof of further lemmas we need the following definitions and corollaries.

DEFINITION 1. $d_i = (n, a_i)$, $n_i = n/d_i$ $(1 \le i \le k)$.

COROLLARY 2. *Under the assumption of the Theorem we have for all* $i, j \le k$ *either* $n_i \mid n_j$ *or* $n_j \mid n_i$ *or* $(n_i, n_j) = 1$.

DEFINITION 2. We write $i \prec j$ if $n_i \mid n_j$ and either $n_i < n_j$ or $i < j$.

COROLLARY 3. $\prec$ *is a partial ordering of the set* $\{1, \ldots, k\}$.

DEFINITION 3. N_l is the number of residues mod n given by the numbers $\sum_{i\preceq l} a_i x_i$, where $0 \le x_i \le b_i$.

DEFINITION 4. $c(i)$ is the number of $j \le i$ such that $n_j = n_i$.

Now we can formulate

LEMMA 3. *If for $0 \le x_i \le b_i$ the sum $\sum_{i\prec l} a_i x_i$ gives at least $1+\sum_{i\prec l} b_i$ residues mod n, then*

$$N_l \ge \min\Big\{n_l, 1+\sum_{i\preceq l} b_i\Big\}.$$

Proof. Since $(n, a_l) = d_l$ divides a_i for $i \prec l$,

$$\sum_{i\prec l} \frac{a_i}{d_l}\, x_i \text{ gives at least } 1+\sum_{i\prec l} b_i \text{ residues mod } n_l.$$

We apply Lemma B with B being the set of these residues and with $m = n_l$, $a = a_l/d_l$. The assumptions are satisfied, since

$$\left(\frac{a_l}{d_l},\, n_l\right) = \frac{(a_l,\, n)}{d_l} = 1.$$

Therefore, the number of residues mod n_l of $\sum_{i\preceq l}(a_i/d_l)x_i$ is at least

$$\min\Big\{n_l, 1+\sum_{i\preceq l} b_i\Big\}$$

and hence

$$\sum_{i\preceq l} a_i x_i \text{ gives at least } \min\Big\{n_l, 1+\sum_{i\preceq l} b_i\Big\} \text{ residues mod } n.$$

LEMMA 4. *If for an $l \le k$ and all $g \prec l$ we have*

$$\sum_{i\preceq g} b_i \le n_g - 1, \tag{3}$$

then

$$N_l \ge \min\Big\{n_l,\, 1+\sum_{i\preceq l} b_i\Big\}.$$

Proof. Let I_h be the set of $i \le k$ for which there exists a sequence $i_1, \ldots, i_h$ such that $i_1 = i$, $n_{i_\nu} \,|\, n_{i_{\nu+1}}$, $n_{i_\nu} < n_{i_{\nu+1}}$ $(1 \le \nu < h)$ and there exists no longer sequence with this property. Clearly, for a certain s,

$$\{1, \ldots, k\} = \bigcup_{h=1}^{s} I_h$$

and $I_g \cap I_h = \emptyset$ for $g \ne h$. Moreover, by Corollary 2,

$$\text{if } i, j \in I_h \text{ and } n_i \ne n_j, \text{ then } (n_i, n_j) = 1. \tag{4}$$

If $l \in I_h$ we shall write $h(l) = h$. We shall prove the lemma by a double induction, with respect to $s - h(l)$ and with respect to $c(l)$. If $s - h(l) = 0$ and $c(l) = 1$, then $i \preceq l$ implies $i = l$. We have two possibilities.

If $b_l + 1 \geq n_l$, then $(a_l/d_l)x_l$ $(0 \leq x_l \leq b_l)$ gives all residues mod n_l, hence $a_l x_l$ gives n_l residues mod n, thus $N_l = n_l$.

If $b_l + 1 < n_l$, then $(a_l/d_l)x_l$ $(0 \leq x_l \leq b_l)$ gives $b_l + 1$ residues mod n_l, hence $a_l x_l$ gives $b_l + 1$ residues mod n, thus $N_l \geq b_l + 1$. Assume now that the assertion is true for $s - h(l') = 0$ and $c(l') = c - 1$ $(c \geq 2)$, and that $s - h(l) = 0$, $c(l) = c$. Then $i \prec l$ if and only if $i \preceq l'$, where $n_{l'} = n_l$ and $c(l') = c - 1$. Clearly $s - h(l') = 0$ and by the inductive assumption and by (3) with $g = l'$,

$$N_{l'} \geq \min\Big\{n_{l'},\ 1 + \sum_{i \preceq l'} b_i\Big\} \geq 1 + \sum_{i \preceq l'} b_i.$$

Hence, by Lemma 3,

$$N_l \geq \min\Big\{n_l, 1 + \sum_{i \prec l} b_i\Big\}.$$

Assume now that the assumption is true for $s - h(l') = s - h - 1$ and that $h(l) = h$, $c(l) = 1$. Put

$$\Lambda_l = \{i \prec l : i \in I_{h+1} \wedge i = \max\{q : n_q = n_i\}\} = \{i_1, \ldots, i_t\}.$$

If $t = 0$, then $i \preceq l$ implies $i = l$ and the proof proceeds as above for $s - h(l) = 0$, $c(l) = 1$. Therefore we assume that $t > 0$ and infer from (4) that

$$(n_{i_\mu}, n_{i_\nu}) = 1 \quad \text{for } \mu \neq \nu. \tag{5}$$

Since $c(l) = 1$, $i \prec l$ implies $i \preceq i_u$ for some $u \leq t$. By the inductive assumption the assertion is true for every $l' = i_u \in \Lambda_l \subset I_{h+1}$, hence by (3) for all $u \leq t$,

$$N_{i_u} \geq 1 + \sum_{i \preceq i_u} b_i. \tag{6}$$

For $i \preceq i_u$ we have

$$n_i \mid n_{i_u}, \quad d_{i_u} \mid d_i \mid a_i,$$

hence for each $u \leq t$,

$$\sum_{i \preceq i_u} \frac{a_i}{d_{i_u}} x_i \quad (0 \leq x_i \leq b_i) \quad \text{gives} \quad N_{i_u} \geq 1 + \sum_{i \preceq i_u} b_i \text{ residues mod } n_{i_u}.$$

Now, by (5) for all integers $z_1, \ldots, z_t, r_1, \ldots, r_t$ we have

$$\sum_{u=1}^{t} \frac{n}{n_{i_u}} z_u \equiv \sum_{u=1}^{t} \frac{n}{n_{i_u}} r_u \pmod n$$

if and only if $z_u \equiv r_u \pmod{n_{i_u}}$ for all $u \leq t$. It follows that the number of residues mod n given by

$$\sum_{u=1}^{t} \frac{n}{n_{i_u}} \sum_{i \preceq i_u} \frac{a_i}{\frac{n}{n_{i_u}}} x_i = \sum_{u=1}^{t} d_{i_u} \sum_{i \preceq i_u} \frac{a_i}{d_{i_u}} x_i = \sum_{u=1}^{t} \sum_{i \preceq i_u} a_i x_i = \sum_{i \prec l} a_i x_i$$

for $0 \leq x_i \leq b_i$ is equal to $\prod_{u=1}^{t} N_{i_u}$, hence by (6) it is at least

$$\prod_{u=1}^{t} \Bigl(1 + \sum_{i \preceq i_u} b_i\Bigr) \geq 1 + \sum_{u=1}^{t} \sum_{i \preceq i_u} b_i = 1 + \sum_{i \prec l} b_i.$$

Using Lemma 3 we obtain

$$N_l \geq \min \Bigl\{ n_l, 1 + \sum_{i \preceq l} b_i \Bigr\},$$

which proves the assertion for $s - h(l) = s - h$, $c(l) = 1$.

Assume now that the assertion is true for $s-h(l') = s-h$ and $c(l') = c-1$ $(c \geq 2)$ and that $s - h(l) = s - h$, $c(l) = c$. Then $i \prec l$ if and only if $i \preceq l'$, where $n_{l'} = n_l$ and $c(l') = c - 1$. Clearly $s - h(l') = s - h$, thus by the inductive assumption and by (3),

$$N_{l'} \geq \min \Bigl\{ n_{l'}, 1 + \sum_{i \preceq l'} b_i \Bigr\} \geq 1 + \sum_{i \preceq l'} b_i.$$

Hence, by Lemma 3,

$$N_l \geq \min \Bigl\{ n_l, 1 + \sum_{i \preceq l} b_i \Bigr\}.$$

Definition 5. $M = \bigcup_{i=1}^{k} \{n_i\}$.

Lemma 5. *Let us order a_i in such a way that $i \leq j$ implies $n_i \leq n_j$. Under the assumption of the Theorem, for every $l \preceq k$ either*

(7) *there exists $m' \mid n_l,\ m' \in M \setminus \{n_l\}$*

such that

$$\sum_{n_i \mid m'} b_i \geq m',$$

or

(8) $\sum_{n_i \mid n_l,\, i \leq l} a_i x_i$ $(0 \leq x_i \leq b_i)$ *gives at least*

$$\min \Bigl\{ n_l,\ 1 + \sum_{n_i \mid n_l,\, i \leq l} b_i \Bigr\} \text{ residues mod } n.$$

Proof. We apply Lemma 4. If there exists g not satisfying (3) such that $n_g \mid n_l$, $n_g < n_l$ then (7) holds with $m' = n_g$. If there exist g not satisfying (3) with $n_g \mid n_l$, but for all of them $n_g = n_l$, then taking the least such g, by Lemma 4 we obtain

$$N_l \geq N_g \geq \min\Big\{n_g, \sum_{n_i \mid n_g,\, i\leq g} b_i\Big\} = n_g = \min\Big\{n_l, \sum_{n_i \mid n_l,\, i\leq l} b_i\Big\},$$

thus (8) holds.

Finally, if (3) is satisfied by all g with $n_g \mid n_l$, $g < l$, then (8) holds by Lemma 4.

LEMMA 6. *Let $t, x_1, \ldots, x_t$ be integers greater than* 1. *Then*

$$\sum_{u=1}^{t}(x_u - 2) \leq \frac{1}{2}\prod_{u=1}^{t} x_u - 2.$$

Proof. Since $x + y \leq xy$ for $x, y \geq 2$, we have

$$\sum_{u=1}^{t}(x_u - 2) = \sum_{u=1}^{t} x_u - 2t \leq \prod_{u=1}^{t-1} x_u + x_t - 2t.$$

Since $x + y - 2 \leq xy/2$ for $x, y \geq 2$, we have

$$\prod_{u=1}^{t-1} x_u + x_t - 2t \leq \frac{1}{2}\prod_{u=1}^{t} x_u - 2(t-1) \leq \frac{1}{2}\prod_{u=1}^{t} x_u - 2.$$

Combining both inequalities we obtain the lemma.

3. Proof of the Theorem. We may assume without loss of generality that if $i \leq j$ then either $n_i < n_j$, or $n_i = n_j$ and $b_i \geq b_j$. By Corollary 2, for all $i, j \leq k$ we have

$$\text{either} \quad n_i \mid n_j \quad \text{or} \quad n_j \mid n_i \quad \text{or} \quad (n_i, n_j) = 1. \tag{9}$$

We proceed by induction on k. For $k = 1$, (2) is trivially true. Assume it is true for all $k' < k$. If $n_1 = 1$ then

$$N(n; a_1, b_1, \ldots, a_k, b_k) = (b_1 + 1)N(n; a_2, b_2, \ldots, a_k, b_k),$$

hence (2) follows from the inductive assumption.

Therefore assume that $n_i \geq 2$, and $n \geq 4$ by the result of [1]. Suppose that there exist $\overline{m} \in M$ such that $\sum_{n_i \mid \overline{m}} b_i \geq \overline{m} - 1$ and let m be the least number with this property. Hence for all $m' < m$, $m' \in M$ we have

$$\sum_{n_i \mid m'} b_i \leq m' - 2. \tag{10}$$

Let m_u $(1 \le u \le t)$ be all maximal elements with respect to divisibility in the set $\{\mu \in M \setminus \{m\} : \mu \mid m\}$. We have $m_u \in M \setminus \{m\}$, $m_u \mid m$, and the m_u are not divisible by one another, hence by (9), $(m_u, m_v) = 1$ for $u \ne v$. It follows that

$$\prod_{u=1}^{t} m_u \mid m. \tag{11}$$

We take the least j such that

$$\sum_{n_i \mid m,\, i \le j} b_i \ge m - 1. \tag{12}$$

By (10) we have

$$\sum_{n_i \mid m_u} b_i \le m_u - 2,$$

hence by Lemma 6,

$$\sum_{u=1}^{t} \sum_{n_i \mid m_u} b_i \le \sum_{u=1}^{t} (m_u - 2) \le \frac{1}{2} \prod_{u=1}^{t} m_u - 2 \le \frac{1}{2} m - 2, \tag{13}$$

unless $t \le 1$. However, for $t \le 1$ the inequality

$$\sum_{u=1}^{t} (m_u - 2) \le \frac{1}{2} m - 2$$

is also true, thus (12) and (13) imply $n_j \nmid m_u$ $(1 \le u \le t)$, hence $n_j = m$.

Also, by Lemma 5, inequality (10) and the choice of j the number of residues mod n given by $\sum_{n_i \mid m,\, i<j} a_i x_i$ $(0 \le x_i \le b_i)$ is at least $1 + \sum_{n_i \mid m,\, i<j} b_i$. For every choice of x_i $(n_i \nmid m$ or $i > j)$ such that

$$\sum_{n_i \nmid m \text{ or } i>j} a_i x_i \equiv 0 \left(\operatorname{mod} \frac{n}{m}\right) \tag{14}$$

there exist, by Lemma 1, at least

$$\max\left\{1, \frac{b_j + 1}{2(m - \sum_{n_i \mid m,\, i<j} b_i) - 1}\right\}$$

solutions of the congruence

$$\frac{m}{n} \sum_{n_i \mid m,\, i<j} a_i x_i + \frac{m a_j}{n} x_j + \frac{m}{n} \sum_{n_i \nmid m \text{ or } i>j} a_i x_i \equiv 0 \pmod{m},$$

satisfying $0 \le x_i \le b_i$ $(n_i \mid m$ and $i \le j)$. However, the number of summands in (14) is less than k, hence, by the inductive assumption, the number of solutions of (14) with $0 \le x_i \le b_i$ is at least

$$2^{1-n/m} \prod_{n_i \nmid m \text{ or } i>j} (b_i + 1).$$

Thus we obtain

$$N(n;a_1,b_1,\dots,a_k,b_k) \geq 2^{1-n/m} \prod_{n_i\nmid m \text{ or } i>j} (b_i+1)\max\left\{1, \frac{b_j+1}{2(m-\sum_{n_i\mid m,\, i<j} b_i)-1}\right\}. \tag{15}$$

We consider three cases:

$$m<n, \tag{16}$$

$$m=n \quad \text{and} \quad \text{either } j=1 \text{ or } n_{j-1}<n, \tag{17}$$

$$m=n, \quad j\geq 2 \quad \text{and} \quad n_{j-1}=n. \tag{18}$$

In the case (16) we have, by (15) and Bernoulli's inequality,

$$\begin{aligned} N(n;\, a_1,b_1,\dots,a_k,b_k)^{-1}\prod_{i=1}^{k}(b_i+1) &\leq 2^{n/m}\Big(m-\sum_{n_i\mid m,\, i<j} b_i-\frac12\Big)\prod_{n_i\mid m,\, i<j}(b_i+1) \\ &\leq 2^{n/m}\Big(m-b-\frac12\Big)2^b, \end{aligned} \tag{19}$$

where $b=\sum_{n_i\mid m,\, i<j} b_i$. By the choice of j we have

$$b\leq m-2<m-\frac12-\frac{1}{\log 2},$$

hence by Lemma 2,

$$\Big(m-b-\frac12\Big)2^b\leq 3\cdot 2^{m-3}<2^{m-1},$$

and by (19),

$$N(n;a_1,b_1,\dots,a_k,b_k)^{-1}\prod_{i=1}^{k}(b_i+1)<2^{n/m+m-1}\leq 2^{n-1},$$

because $n-n/m-m=(n/m-1)(m-1)-1\geq 0$.

In the case (17) we have again (19), but now, by (13),

$$b\leq \frac12\, n-2\leq n-\frac12-\frac{1}{\log 2},$$

hence by Lemma 2 and Bernoulli's inequality

$$\Big(m-b-\frac12\Big)2^b\leq \Big(\frac{n}{2}+\frac32\Big)2^{n/2-2}\leq 2^{n-3/2}.$$

In the case (18) we have, by (15),

$$N(n;a_1,b_1,\ldots,a_k,b_k)^{-1}\prod_{i=1}^{k}(b_i+1)\le\prod_{n_i|m,\,i\le j}(b_i+1)\tag{20}$$
$$\le\prod_{n_i|m,\,n_i<m}(b_i+1)\cdot\prod_{n_i=m,\,i\le j}(b_i+1)$$
$$\le 2^{\sum_{n_i|m,\,n_i<m}b_i}\prod_{n_i=m,\,i\le j}(b_i+1).$$

Now, by the choice of j,

$$\sum_{n_i|m,\,i<j}b_i\le n-2,\qquad \sum_{n_i=m,\,i<j}b_i\le n-2-\sum_{n_i|m,\,n_i<m}b_i,$$

thus $b_j\le b_{j-1}\le a/x$, where

$$a=n-2-\sum_{n_i|m,\,n_i<m}b_i,\qquad x=\sum_{n_i=m,\,i<j}1.$$

By the inequality for the arithmetic and geometric mean and by Lemma C,

$$\prod_{n_i=m,\,i\le j}(b_i+1)\le\left(1+\frac{a}{x}\right)^{x+1}\le 2^{a+1},$$

unless $a=2$, $x=1$. Leaving this case for a further consideration we obtain from (20),

$$N(n;a_1,b_1,\ldots,a_k,b_k)^{-1}\prod_{i=1}^{k}(b_i+1)$$
$$\le 2^{\sum_{n_i|m,\,n_i<m}b_i}\cdot 2^{n-1-\sum_{n_i|m,\,n_i<m}b_i}=2^{n-1}.$$

If $a=2$, $x=1$ we obtain, because of (13),

$$n-4=\sum_{n_i|m,\,n_i<m}b_i\le\frac{1}{2}n-2,$$

hence $n\le 4$, that is, $n=4$; moreover, $j=2$, $n_1=n_2=4$, $b_2\le b_1\le 2$ and

$$N(n;a_1,b_1,\ldots,a_k,b_k)^{-1}\prod_{i=1}^{k}(b_i+1)\le(b_1+1)(b_2+1)\le 2^{n-1}$$

unless $b_1=b_2=2$. However, the last case is covered by Lemma A.

Assume now that for every $m\in M$ we have

$$\sum_{n_i|m}b_i\le m-2.\tag{21}$$

If $n \in M$ it follows that

$$\sum_{i=1}^{k} b_i \le n-2. \tag{22}$$

If $n \notin M$, then for every n_i there exists the greatest $m \in M$ such that $n_i \mid m$. Put $m = f(n_i)$ and $M_0 = f(M)$. It follows from (9) that the elements of M_0 are coprime. Hence

$$\prod_{m\in M_0} m \mid n$$

and, by Lemma 6,

$$\sum_{n\in M_0} (m-2) \le \frac{1}{2}\prod_{m\in M_0} m - 2 \le \frac{1}{2}n - 2$$

unless M_0 has just one element m_0.

However, $m_0 \le \frac{1}{2}n$, thus in each case, by (21),

$$\sum_{i=1}^{k} b_i \le \sum_{m\in M_0}\sum_{n_i\mid m} b_i \le \sum_{m\in M_0}(m-2) \le \frac{1}{2}n-2$$

and (22) holds generally. It follows by Bernoulli's inequality that

$$N(n; a_1, b_1, \ldots, a_k, b_k)^{-1}\prod_{i=1}^{k}(b_i+1) \le 2^{\sum_{i=1}^{k} b_i} \le 2^{n-2}.$$

Added in proof. As proved in [2], the inequality (2) holds if $n = \prod_{j=1}^{l} q_j^{\alpha_j}$, where q_j are primes and $\sum_{j=1}^{l} 1/q_j < 1$.

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