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CB-DEGENERATIONS AND RIGID DEGENERATIONS OF ALGEBRAS

BY

ADAM HAJDUK (Toruń)

Abstract. The main aim of this note is to prove that if k is an algebraically closed field and a k-algebra A_0 is a CB-degeneration of a finite-dimensional k-algebra A_1, then there exists a factor algebra $\overline{A}_0$ of A_0 of the same dimension as A_1 such that $\overline{A}_0$ is a CB-degeneration of A_1. As a consequence, $\overline{A}_0$ is a rigid degeneration of A_1, provided A_0 is basic.

Introduction. There are at least three different concepts of geometric degenerations for k-algebras: degenerations in the classical sense referring to the geometry of orbits in a variety of algebras (the idea goes back to nineteenth century algebraists, see [8]), the so-called rigid degenerations using the notion of degeneration of (ordered) locally bounded categories (see [5]), and the CB-degenerations introduced by Crawley-Boevey in [2] (see [3] for the precise definitions). All these three concepts are useful for deciding in some specific situations whether a fixed algebra is tame. This method is based on three "degeneration theorems" (see [5, 6, 2]), each of which states that, if a finite-dimensional tame k-algebra A_0 is a degeneration of a fixed algebra A_1, then A_1 is also tame. For classical and rigid degenerations this was proved by Geiss, who uses ordered locally bounded categories, avoiding the so-called Gabriel lemma whose proof is rather involved and requires at least the use of projective geometry (see [7] and also [4, 10]). The result of Crawley-Boevey appeared a little later and is mainly applied in the study of biserial algebras.

In the last fifteen years the degeneration technique has found many interesting applications; in particular, it was successfully used in solving several important classification problems for tame algebras. Also certain natural theoretical questions concerning degenerations have been considered. In [3] some interrelations between the three notions of degeneration are studied. It is shown there that a basic algebra A_0 is a CB-degeneration of a (basic) algebra A_1 of the same dimension as A_0 over a field k if and only if A_0 is a rigid degeneration of A_1 ([3, Theorem 5.1]). Moreover, a reduction of CB-

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degeneration problems for nonbasic algebras to those for their basic representatives in Morita equivalence classes is discussed. Finally, it is proved that for every CB-degeneration of an algebra A_1 to A_0, obtained along an affine line, there exists a factor algebra $\overline{A}_0$ of A_0 such that $\dim_k \overline{A}_0 = \dim_k A_1$ and $\overline{A}_0$ is also a CB-degeneration of A_1. As a consequence, $\overline{A}_0$ is also a rigid degeneration of A_1, provided A_0 is basic ([3, Theorem 6.1]).

The aim of this note is to prove a generalization of this result to the case of all CB-degenerations, without any restriction on the variety involved (see Theorem of Section 2). Consequently, the theoretical scope of [2, Theorem B] is exactly the same as that of the original version of the Geiss theorem from [5].

1. Preliminaries. Throughout the paper, we use the well known definitions (see [2, 6]) and notation introduced in [3]. We now briefly recall the most important of them.

Throughout the paper k denotes an algebraically closed field. By an *algebra* we mean a finite-dimensional k-algebra.

For any $m, n \in \mathbb{N}$, we denote by $\mathrm{M}_{m\times n}(k)$ the set of all $m \times n$-matrices with coefficients in k, by $\mathrm{M}_n(k)$ the algebra $\mathrm{M}_{n\times n}(k)$ of square $n \times n$-matrices and by $\mathrm{Gl}_n(k)$ the group of invertible matrices in $\mathrm{M}_n(k)$. For a fixed dimension vector $d \in \mathbb{N}^{n^2}$, we set

$$H_d(k) = \prod_{i,j=1,\ldots,n} \mathrm{Gl}_{d_{i,j}}(k).$$

Following [2], we introduce a useful definition (see [2, Theorem B]).

Definition. Given two algebras A_0 and A_1, the algebra A_0 is a *CB-degeneration* of A_1 if there exists a finite-dimensional algebra A, an irreducible variety X and regular maps $f_1, \ldots, f_r : X \to A$ such that $A_1 \cong A_x$ for all x in some nonempty open subset U of X, and $A_0 \cong A_{x_0}$ for some $x_0 \in X$, where $A_y = A/(f_1(y), \ldots, f_r(y))$ for any $y \in X$.

In the situation as above, the data sequence $\mathcal{D} = (A, X, \mathcal{F}, U, x_0)$ is called a degenerating collection defining a CB-degeneration of A_1 to A_0 along X by use of A, where $\mathcal{F} = \{f_1, \ldots, f_r\}$.

The concept of rigid degenerations is based on the notion of degeneration for finite locally bounded categories R with a fixed linearly ordered set $(x_1, \ldots, x_n)$ of objects and with the dimension vector $d \in \mathbb{N}^{n^2}$, $n \geq 1$, where $d_{i,j} = \dim_k \mathcal{J}_R(x_i, x_j)$ for all i, j and $\mathcal{J}_R$ is the Jacobson radical of R.

Given d as above, we consider a group action

$$\cdot : H_d(k) \times \mathrm{lbc}_d(k) \to \mathrm{lbc}_d(k),$$

where $\mathrm{lbc}_d(k)$ is the affine variety of constant structures for locally bounded k-categories with a fixed object set $\{1, \ldots, n\}$ and dimension vector d (see [3]).

Suppose we are now given two basic k-algebras A_0 and A_1 of the same dimension. We say that A_0 is a *rigid degeneration* of A_1 if there exist complete sequences $e^{(0)} = (e_1^0, \ldots, e_n^0)$ and $e^{(1)} = (e_1^1, \ldots, e_n^1)$ of primitive pairwise orthogonal idempotents in A_0 and A_1, respectively, such that

$$\dim_k(e_i^0 A_0 e_j^0) = \dim_k(e_i^1 A_1 e_j^1)$$

for all $i, j = 1, \ldots, n$, and that for any constant structures $c^{(0)}, c^{(1)} \in \mathrm{lbc}_d(k)$ of finite locally bounded k-categories $R_0 = R(A_0, e^{(0)})$ and $R_1 = R(A_1, e^{(1)})$ respectively, we have the inclusion

$$H_d \cdot c^{(0)} \subset \overline{H_d \cdot c^{(1)}}.$$

Here we treat R_0 and R_1 via the correspondence $e_i^0 \leftrightarrow i \leftrightarrow e_i^1$, as categories with the object set $\{1, \ldots, n\}$; see [3] for more details.

2. Main theorem. Now we are able to formulate the main result of the paper, generalizing [3, Theorem 6.1].

THEOREM. *Let A_0 and A_1 be finite-dimensional k-algebras. Assume that A_0 is a CB-degeneration of A_1 (with respect to a finite-dimensional algebra A). Then A_1 admits a CB-degeneration (with respect to A) to some factor algebra $\overline{A}_0$ of A_0 such that $\dim_k \overline{A}_0 = \dim_k A_1$. In particular, if A_0 is basic, then A_1 admits a rigid degeneration to the same $\overline{A}_0$.*

For the proof we need some auxiliary facts.

LEMMA. *Let X be an irreducible affine k-variety, $X' \subseteq X$ a nonempty open subset, and $x_0 \in X \setminus X'$. Then there exists an irreducible closed curve $\Gamma \subseteq X$ such that $x_0 \in \Gamma$ and $X' \cap \Gamma \neq \emptyset$.*

Proof. We proceed by induction on $\dim X$. If $\dim X = 1$, then obviously $\Gamma = X$. Suppose that $\dim X > 1$ and the lemma is proved for all varieties of dimension less than $\dim X$. We can assume that $X \subseteq \mathring{A}^n(k)$ is a closed set (in the Zariski topology). Let $X \setminus X' = X_1 \cup \cdots \cup X_s$ be a decomposition of $X \setminus X'$ into irreducible components, and $x_1 \in X_1, \ldots, x_s \in X_s$ a fixed selection of elements. Choose a polynomial $F \in k[T_1, \ldots, T_n]$ such that $F(x_0) = 0$ and $F(x_i) \neq 0$ for $i = 1, \ldots, s$. Then the set $V = X \cap V(F)$ contains no X_i for $i = 1, \ldots, s$. Let Z be an irreducible component of V passing through x_0. Then Z contains no X_i since $Z \subseteq V$. By [9, Theorem 3.3] we have $\dim X_i \leq \dim(X \setminus X') \leq \dim X - 1 = \dim Z$. Thus no X_i contains Z, since otherwise $\dim X_i = \dim Z$ and by [9, Proposition 3.2] we get $X_i = Z$, a contradiction. Therefore the open subset $Z' = Z \cap X'$ of Z is nonempty, and by definition of Z the point x_0 belongs to Z. By inductive assumption

($\dim Z = \dim X - 1$) there exists an irreducible affine curve $\varGamma \subseteq Z$ such that $x_0 \in \varGamma$ and $\varGamma \cap Z' \neq \emptyset$. Notice that $\varGamma \subseteq X$ is closed and $\varGamma \cap X' \neq \emptyset$, hence $\varGamma$ is the required curve. ∎

COROLLARY. *Every CB-degeneration A_0 of an algebra A_1 can be obtained along a nonsingular irreducible affine curve.*

Proof. Let A_0, A_1 be fixed finite-dimensional algebras and $\mathcal{D} = (A, X, \mathcal{F}, U, x_0)$ a collection defining a CB-degeneration from A_1 to A_0, where $\mathcal{F} = \{f_1, \ldots, f_s\}$ are regular maps from X to A. Changing X to a suitable principal open set containing x_0, we can assume that X is an irreducible affine variety. By the Lemma there exists an irreducible curve $\varGamma \subseteq X$ such that $x_0 \in \varGamma$ and $\varGamma \cap U \neq \emptyset$. Then replacing $\mathcal{F}$ by $\mathcal{F}_{|\varGamma} = \{f_{1|\varGamma}, \ldots, f_{s|\varGamma}\}$ and U by $U_{|\varGamma} = U \cap \varGamma$ we can assume that X is an irreducible affine curve.

Let $p : \widetilde{X} \to X$ be a normalization of X (see [11]). It is known that $\widetilde{X}$ is a nonsingular curve, since $\dim Y - \dim \operatorname{Sing} Y \geq 2$ for any normal variety Y, where $\operatorname{Sing} Y$ denotes the set of singular points of Y. We now define a collection $\widetilde{\mathcal{D}} = (A, \widetilde{X}, \widetilde{\mathcal{F}}, \widetilde{U}, \widetilde{x}_0)$, where $\widetilde{\mathcal{F}} = \{f_1 \circ p, \ldots, f_s \circ p\}$, $\widetilde{U} = p^{-1}(U)$, $\widetilde{x}_0$ is a fixed point in $p^{-1}(x_0)$. It is easily seen that $\widetilde{\mathcal{D}}$ defines a CB-degeneration from A_1 to A_0. ∎

Now we can prove the main result of this note.

Proof of Theorem. We carry out the proof by induction on $n = \dim_k A_0 - \dim_k A_1$. If $n = 0$ then we simply get $\overline{A}_0 = A_0$. Assume that $n > 0$ and let $\mathcal{D} = (A, X, \mathcal{F}, U, x_0)$ be a collection defining a CB-degeneration from A_1 to A_0, where as usual $\mathcal{F} = \{f_1, \ldots, f_s\}$. Denote by $v_1, \ldots, v_m \in A$ a basis of A, where $m = \dim_k A$. By the Corollary we can assume that X is an irreducible nonsingular curve. Without loss of generality we can also assume that $I_x = \langle f_1(x), \ldots, f_s(x) \rangle$ for all $x \in X$. For any $i = 1, \ldots, s$ we denote by $\{f_i^j\}_{j=1,\ldots,m}$ the family of regular functions on X such that $f_i(x) = \sum_{j=1}^m f_i^j(x) v_j$ for $x \in X$. We set

$$f(x) = [f_i^j(x)]$$

for any $x \in X$ ($[f_i^j(x)] \in \mathrm{M}_{s\times m}(k)$) and $r = \mathrm{r}(f(x_0))$. Note that $r < \mathrm{r}(f(x))$ for all $x \in U$, since $r = \dim_k I_{x_0}$ and $\mathrm{r}(f(x)) = \dim_k I_x$. By the definition of r there exists a nonzero $r \times r$-minor of the matrix $f(x_0)$. We can assume that it is the determinant of the upper-left $r \times r$-submatrix of $f(x_0)$. Let $h : X \to k$ be the regular function defined by $x \mapsto \det(f(x)_r)$ for $x \in X$, where $f(x)_r = [f_i^j(x)]_{i,j=1,\ldots,r} \in \mathrm{M}_r(k)$. Clearly $h(x_0) \neq 0$. Now we use the identification

$$\mathrm{M}_{s\times m}(k) = \begin{bmatrix} \mathrm{M}_{r\times r}(k) & \mathrm{M}_{r\times(m-r)}(k) \\ \mathrm{M}_{(s-r)\times r}(k) & \mathrm{M}_{(s-r)\times(m-r)}(k) \end{bmatrix}$$

($m, s > r$, since $\mathrm{r}(f(x)) > r$ for any $x \in U$). By applying two-step Gaussianrow elimination, we transform $f(x)$ to a matrix $\overline{f}(x) = [\overline{f}_i^j(x)] \in \mathrm{M}_{s\times m}(k)$, for $x \in X$ such that $h(x) \neq 0$, as follows:

$$f(x) \rightsquigarrow \begin{bmatrix} \mathrm{id}_r & * \\ * & * \end{bmatrix} \rightsquigarrow \begin{bmatrix} \mathrm{id}_r & * \\ 0 & * \end{bmatrix} (= \overline{f}(x)),$$

where the first transformation corresponds to multiplication of $f(x)$ from the left by the block diagonal matrix $\left[\begin{smallmatrix} (f(x)_r)^{-1} & 0 \\ 0 & \mathrm{id}_{s-r} \end{smallmatrix}\right]$. In this way all functions $\overline{f}_i^j(x)$ are rational and belong to the local ring $\mathcal{O}_{x_0}(X)$, since $h(x_0) \neq 0$ and therefore $1/h \in \mathcal{O}_{x_0}(X)$. We regard here $\mathcal{O}_{x_0}(X)$ as a subring of $k(X)$ (X is irreducible). Moreover, observe that $\overline{f}_i^j = 0$ for all $r < i \leq s$, $1 \leq j \leq r$, and $\overline{f}_i^j(x_0) = 0$ for all $r < i \leq s$, $r < j \leq m$.

Now, since $\dim_k I_x > \dim_k I_{x_0}$ for $x \in U$, and $I_x = \langle \overline{f}_1(x), \ldots, \overline{f}_s(x)\rangle$ for $x \in X$ such that $h(x) \neq 0$, we infer that, for all $x \in U$ such that $h(x) \neq 0$, there exists a pair (i, j) with $r < i \leq s$, $r < j \leq m$ such that $\overline{f}_i^j(x) \neq 0$. Consequently, all functions $\overline{f}_i^j$, $r < i \leq s$, $r < j \leq m$, belong to the maximal ideal $\mathrm{m}_{x_0}(X) \subseteq \mathcal{O}_{x_0}(X)$ and not all of them are zero. By the Auslander–Buchsbaum theorem (see [1]), $\mathcal{O}_{x_0}(X)$ is a unique factorization domain, hence $\mathrm{m}_{x_0}(X)$ is a principal ideal generated by some $g \in \mathrm{m}_{x_0}(X)$, since $\mathrm{Krull.dim}\, \mathcal{O}_{x_0}(X) = 1$.

Note that $p = \max\{k \in \mathbb{N};\ g^k \mid \overline{f}_i^j, r < i \leq s,\ r < j \leq m\}$ is finite. We get now equations $\overline{f}_i^j = g^p \cdot \widetilde{f}_i^j$, $r < i \leq s$, $r < j \leq m$, in $\mathcal{O}_{x_0}(X)$, for some rational functions $\widetilde{f}_i^j \in \mathcal{O}_{x_0}(X)$. By definition of p, not all $\widetilde{f}_i^j$ belong to $\mathrm{m}_{x_0}(X)$. We can assume there exists $r < j \leq m$ such that $\widetilde{f}_{r+1}^j \notin \mathrm{m}_{x_0}(X)$. We now define a regular map $\widetilde{f}_{s+1} : X' \to A$ by $\widetilde{f}_{s+1}(x) = \sum_{j=r+1}^m \widetilde{f}_{r+1}^j(x) v_j$ for $x \in X'$, where X' is an open set (a neighbourhood of x_0) obtained as the intersection of the domains of all rational functions $\widetilde{f}_{r+1}^j$, $r < j \leq m$. Observe that $\widetilde{f}_{s+1}$ is a regular function on X' and that $\widetilde{f}_{s+1}(x_0) \notin I_{x_0}$, since $\widetilde{f}_{r+1}^j \notin \mathrm{m}_{x_0}(X)$ for some $r < j \leq m$, and $\widetilde{f}_{s+1}(x) \in \mathrm{Span}\{v_{r+1}, \ldots, v_m\}$ for $x \in X'$.

We set $\widetilde{f}_i = f_{i|X'}$ for $i = 1, \ldots, s$, thus obtaining a collection $\mathcal{D}' = (A, X', \mathcal{F}', U', x_0)$ defining a CB-degeneration from A_1 to some factor algebra $\overline{A}_0$ of A_0 such that $\dim_k \overline{A}_0 < \dim_k A_0$, where $\mathcal{F}' = \{\widetilde{f}_1, \ldots, \widetilde{f}_s, \widetilde{f}_{s+1}\}$ and $U' = U \cap X'$. By the inductive assumption ($\dim_k \overline{A}_0 - \dim_k A_1 < n$), A_1 admits a CB-degeneration to a factor algebra $\overline{\overline{A}}_0$ of $\overline{A}_0$ such that $\dim_k \overline{\overline{A}}_0 = \dim_k A_1$. But $\overline{\overline{A}}_0$ is also a factor algebra of A_0. This completes the proof of the first assertion.

The second assertion follows immediately from [3, Theorem 5.1], since $\overline{A}_0$ is a basic algebra, and consequently so is A_1 (see [3, Corollary 4.1]), provided A_0 is basic. ∎

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Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: ahajduk@mat.uni.torun.pl

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