COLLOQUIUM MATHEMATICUM

VOL. 138 2015 NO. 2

FUGLEDE–PUTNAM THEOREM FOR CLASS A OPERATORS

BY

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Abstract. Let $A \in B(H)$ and $B \in B(K)$. We say that A and B satisfy the Fuglede–Putnam theorem if $AX = XB$ for some $X \in B(K, H)$ implies $A^*X = XB^*$. Patel et al. (2006) showed that the Fuglede–Putnam theorem holds for class $A(s,t)$ operators with $s+t < 1$ and they mentioned that the case $s = t = 1$ is still an open problem. In the present article we give a partial positive answer to this problem. We show that if $A \in B(H)$ is a class A operator with reducing kernel and $B^* \in B(K)$ is a class $\mathcal{Y}$ operator, and $AX = XB$ for some $X \in B(K, H)$, then $A^*X = XB^*$.

1. Introduction. Let H, K be infinite-dimensional separable complex Hilbert spaces and $B(H), B(K)$ the algebras of all bounded linear operators on H and K, respectively. An operator $T \in B(H)$ is said to be *p-hyponormal*, for $p \in (0,1]$, if $(T^*T)^p \geq (TT^*)^p$ [3]. A 1-hyponormal operator is *hyponormal* and a $\frac{1}{2}$-hyponormal one is said to be *semi-hyponormal*. An invertible operator T is said to be *log-hyponormal* if $\log|T| \geq \log|T^*|$ [21]. An operator T is said to be *paranormal* if $\|T^2x\| \geq \|Tx\|^2$. It is known [13] that p-hyponormal and log-hyponormal operators are paranormal.

An operator T belongs to the *class* $A(k)$ for $k > 0$ if

$$(T^*|T|^{2k}T)^{1/(k+1)} \geq |T|^2.$$

When $k = 1$ we say that T belongs to the *class* A. Furuta et al. [10] showed that every class A operator is paranormal.

As a further generalization of $A(k)$, Fujii et al. [9] introduced the class $A(s,t)$: an operator T belongs to the class $A(s,t)$ for $s, t > 0$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{1/(t+s)} \geq |T^*|^{2t}.$$

The class $AI(s,t)$ is the class of all invertible class $A(s,t)$ operators for $s, t > 0$. Fujii et al. [9] showed several properties of $A(s,t)$ and $AI(s,t)$ as extensions of the properties of $A(k)$ shown in [9]. They also showed that T is log-hyponormal if and only if T belongs to $AI(s,t)$ for all $s, t > 0$. It is known [24] that the class $A(k,1)$ equals $A(k)$.

2010 *Mathematics Subject Classification*: Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words and phrases: class A operator, Fuglede–Putnam's theorem.

DOI: 10.4064/cm138-2-3

Let T be an operator with polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$. For $s, t > 0$, the *generalized Aluthge transformation* $\tilde{T}_{s,t}$ of T is

$$\tilde{T}_{s,t} = |T|^s U |T|^t.$$

If $s = t = 1/2$, then $\tilde{T}_{s,t}$ is called the *Aluthge transformation* of T, denoted by $\tilde{T}$ [3]. The following equalities for $s + t = 1$ are relations between T and its transform $\tilde{T}_{s,t}$:

$$\tilde{T}_{s,t}|T|^s = |T|^s U|T|^t|T|^s = |T|^s T,$$
$$U|T|^t \tilde{T}_{s,t} = U|T|^t|T|^s U|T|^t = TU|T|^t.$$

Aluthge and Wang [4] introduced ω-hyponormal operators defined as follows: An operator T is said to be *ω-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. Recall that the class $A(1/2, 1/2)$ coincides with the class of ω-hyponormal operators, and $A(1, 1)$ coincides with the class A. An operator $T \in B(H)$ is said to be a *class $\mathcal{Y}_\alpha$ operator* for $\alpha \geq 1$ (or $T \in \mathcal{Y}_\alpha$) if there exists a positive number k_α such that

$$|TT^* - T^*T|^\alpha \leq k_\alpha^2 (T - \lambda I)^*(T - \lambda I) \quad \text{for all } \lambda \in \mathbb{C}.$$

It is known that $\mathcal{Y}_\alpha \subset \mathcal{Y}_\beta$ if $1 \leq \alpha \leq \beta$. Let $\mathcal{Y} = \bigcup_{1 \leq \alpha} \mathcal{Y}_\alpha$. We remark that a class $\mathcal{Y}_1$ operator T is M-hyponormal, i.e., there exists a positive number M such that

$$(T - \lambda I)(T - \lambda I)^* \leq M^2 (T - \lambda I)^*(T - \lambda I) \quad \text{for all } \lambda \in \mathbb{C},$$

and M-hyponormal operators are class $\mathcal{Y}_2$ operators (see [23]). T is said to *dominant* if for any $\lambda \in \mathbb{C}$ there exists a positive number M_λ such that

$$(T - \lambda I)(T - \lambda I)^* \leq M_\lambda^2 (T - \lambda I)^*(T - \lambda I).$$

It is obvious that dominant operators are M-hyponormal. But it is known that there exists a dominant operator which is not a class $\mathcal{Y}_1$ operator, and also there exists a class $\mathcal{Y}_2$ operator which is not dominant [23].

In the following results we recall Fuglede–Putnam's theorem.

THEOREM 1.1 (Fuglede). *Let X, A be bounded linear operators on a complex Hilbert space and assume that A is normal. If $AX = XA$, then $A^*X = XA^*$.*

Colloquially, the theorem claims that commutativity between operators is transitive under the given assumptions. The claim does not hold in general if N is not normal. A simple counterexample is provided by letting N be the unilateral shift and $X = N$. Also, when X is self-adjoint, the claim is trivial regardless of whether N is normal: $XN^* = (NX)^* = (XN)^* = N^*X$. In the following theorem Putnam obtained Fuglede's result as a special case.

THEOREM 1.2 (Putnam). *Let A, B, X be bounded linear operators on a complex Hilbert space and assume that A, B are normal operators. If $AX = XB$, then $A^*X = XB^*$.*

This theorem was originally proved in [8] under the assumption that $A = B$. As stated, the theorem was proved in [19]. In [1] Berberian observed that Putnam's version can be derived from Fuglede's original theorem by the following matrix trick. Let

$$L = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}.$$

Then L is normal on $H \oplus H$ and $LX = XL$. Hence $L^*X = XL^*$, and this gives Putnam's version.

In the past several years, many authors have extended this theorem to several classes of nonnormal operators. In [16, 15], it was shown that Fuglede–Putnam's theorem holds for A p-hyponormal and B^* class A, and for A log-hyponormal and B^* class $\mathcal{Y}$. Recently, Bachir [5] extended the Fuglede–Putnam theorem to ω-hyponormal operators. In [18], Patel et al. extended the Fuglede–Putnam theorem to the class $A(s, t)$ with $s + t < 1$ and they mentioned that the case $s = t = 1$ was an open problem.

Here we give a partial positive answer to this problem. We will extend the Fuglede–Putnam theorem to class A operators. In [17] the authors extended the Fuglede–Putnam theorem to class A operators in the case where A and B^* are class A operators and X is a Hilbert–Schmidt operator. Here we extend that result to all $X \in B(H)$.

2. Preliminaries. Recall that every operator $A \in B(H)$ has a direct sum decomposition $A = A_1 \oplus A_2$, where A_1 and A_2 are the normal and the pure parts, respectively. Of course in the sum decomposition, either A_1 or A_2 may be absent. We begin with the following well known lemmas which will be used in what follows.

LEMMA 2.1 (Hansen's inequality [12]). *If $A, B \in B(H)$ satisfy $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\delta \geq B^*A^\delta B \quad \text{for all } \delta \in (0, 1].$$

LEMMA 2.2 ([20]). *Let $A, B \in B(H)$. Then the following assertions are equivalent:*

(i) *The pair (A, B) satisfies the Fuglede–Putnam theorem.*
(ii) *If $AX = XB$, then $\overline{\operatorname{ran}}\, X$ reduces A, $(\ker X)^\perp$ reduces B, and $A|_{\overline{\operatorname{ran}}\, X}$, $B|_{(\ker X)^\perp}$ are unitarily equivalent normal operators.*

LEMMA 2.3 ([11]). *Let $A \in B(H)$ be a class A operator. If M is an invariant subspace for A, then the restriction $A|_M$ is also a class A operator.*

LEMMA 2.4 ([13, Remark 3.3]). *Let $A \in B(H)$ be a class A operator. Then the generalized Aluthge transform $\tilde{A}_{1,1} = |A|U|A|$ is $\frac{1}{2}$-hyponormal, that is, $\tilde{A}_{1,1}$ is semi-hyponormal.*

LEMMA 2.5 ([6, Theorem 2.1]). *Let $A = U|A| \in B(H)$. If A is of class A, then $\hat{A}_{1,1} = WU|A^2|^{1/2}$ is hyponormal.*

3. Main results. Recall that an operator T is of *class A* if $|T|^2 \leq |T^2|$. We begin by proving a basic property of class A operators.

LEMMA 3.1. *Let $A \in B(H)$ be a class A operator. If M is an invariant subspace of A, and $A|_M$ is an injective normal operator, then M reduces A.*

Proof. Decompose

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \quad \text{on } H = M + M^{\perp}$$

and suppose $A_1 = A|_M$ is an injective normal operator. Let P be the orthogonal projection of H onto M. Since $\ker A_1 = \ker A_1^* = \{0\}$, we have $M = \overline{\operatorname{ran}}\, A_1 \subset \overline{\operatorname{ran}}\, A$. Then

$$\begin{pmatrix} |A_1|^2 & 0 \\ 0 & 0 \end{pmatrix} = P|A|^2P \leq P|A^2|P \leq (P|A^2|^2P)^{1/2}$$
$$\leq \begin{pmatrix} |A_1^2| & 0 \\ 0 & 0 \end{pmatrix}$$

by Lemma 2.1. Since A_1 is normal, we can write

$$|A^2| = \begin{pmatrix} |A_1^2| & C \\ C^* & D \end{pmatrix}.$$

Then

$$\begin{pmatrix} |A_1|^4 & 0 \\ 0 & 0 \end{pmatrix} = PA^*A^*AAP = P|A^2|\,|A^2|P = \begin{pmatrix} |A_1|^4 + CC^* & 0 \\ 0 & 0 \end{pmatrix}$$

and $C = 0$. Hence

$$\begin{pmatrix} |A_1|^4 & 0 \\ 0 & D^2 \end{pmatrix} = |A^2|^2 = A^*A^*AA$$

$$= \begin{pmatrix} A_1^*A_1^*A_1A_1 & A_1^*A_1^*(A_1A_2 + A_2A_3) \\ (A_2^*A_1^* + A_3^*A_2^*)A_1A_1 & (A_2^*A_1^* + A_3^*A_2^*)(A_1A_2 + A_2A_3) + A_3^*A_3^*A_3A_3 \end{pmatrix}.$$

Since A_1 is an injective normal operator, $A_1A_2 + A_2A_3 = 0$ and $D = |A_3^2|$. Since A is a class A operator, we have

$$0 \leq |A^2| - |A|^2 = \begin{pmatrix} 0 & -A_1^*A_2 \\ -A_2A_1 & |A_3^2| - |A_3|^2 - |A_2|^2 \end{pmatrix}.$$

Thus $A_2 = 0$. ∎

THEOREM 3.2. *Let A be a class A operator with reducing kernel and suppose the adjoint B^* of B is a class $\mathcal{Y}$ operator. If there exists an operator C such that $AC = CB$, then $A^*C = CB^*$.*

Proof. Let $A = A_1 \oplus A_2$ and $B_1 \oplus B_2$ where A_1, B_1 and A_2, B_2 are the normal and the pure parts respectively. Let

$$C = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Then

1. $A_1X = XB_1$,
2. $A_1Y = YB_2$,
3. $A_2Z = ZB_1$, and
4. $A_2W = WB_2$.

Since A_1 is normal and B_2^* is a class $\mathcal{Y}$ operator, the relation $A_1Y = YB_2$ implies $A_1^*Y = YB_2^*$ [16, Theorem 7]. Moreover $\overline{\operatorname{ran}}\, Y^*$ reduces B_2 and $B_2|\overline{\operatorname{ran}}\, Y^*$ is normal. This contradicts the fact that B_2 is pure. Therefore $Y = 0$. Next in the equation $A_2Z = ZB_1$, A_2 is of class $A(1,1)$ and B_1 is normal, we have $A_2^*Z = ZB_1^*$ and $\overline{\operatorname{ran}}\, Z$ reduces A_2 and $A_2|\overline{\operatorname{ran}}\, Z$ is normal [18], which is a contradiction.

Finally, we show that the equation $A_2W = WB_2$ implies $W = 0$. Let $A = U|A|$ be the polar decomposition of A and define its generalized Aluthge transform by $\tilde{A} = |A|U|A|$. Then $\tilde{A}$ is semi-hyponormal by Lemma 2.4. Now arguing as in [16] and using Lemmas 2.4 and 2.5, we get

$$|A_2|W(B_2B_2^* - B_2^*B_2) = 0.$$

Now the condition $\ker A \subseteq \ker A^*$ and A_2 is pure implies A_2 must be injective. Hence

$$(*) \qquad W(B_2B_2^* - B_2^*B_2) = 0.$$

Since $A_2W = WB_2$, $\overline{\operatorname{ran}}\, W$ and $(\ker W)^\perp$ are invariant subspaces of A_2 and B_2^* respectively. Therefore

$$A_2 = \begin{pmatrix} A_{11} & T \\ 0 & A_{22} \end{pmatrix} \quad \text{on } H = \overline{\operatorname{ran}}\, W \oplus (\overline{\operatorname{ran}}\, W)^\perp,$$

$$B_2 = \begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix} \quad \text{on } H = (\ker W)^\perp \oplus \ker W$$

and

$$W = \begin{pmatrix} W_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{on } (\ker W)^\perp \oplus \ker W \mapsto \overline{\operatorname{ran}}\, W \oplus (\operatorname{ran} W)^\perp.$$

From $(*)$, it will follow that B_{11}^* is hyponormal. Since $A_2W = WB_2$, we find $A_{11}W_1 = W_1B_{11}$. Again by [18, Theorem 4.8], $\overline{\operatorname{ran}}\, W_1$ reduces A_{11} and $A_{11}|\overline{\operatorname{ran}}\, W_1$ is normal, a contradiction. Therefore W_1 and hence W is zero. Now the equation $A_1X = XB_1$ implies $A_1^*X = XB_1^*$, and hence the result follows. ∎

Let C_2 denote the Hilbert–Schmidt class. Let $T \in C_2$ and assume that $\{e_n\}$ is an orthonormal basis for H. We define the Hilbert–Schmidt norm to be

$$\|T\|_2 = \Big(\sum_{n=1}^{\infty} \|Te_n\|^2\Big)^{1/2}.$$

This is independent of the choice of basis [7]. If $\|T\|_2 < \infty$, then T is said to be a *Hilbert–Schmidt operator*.

Let $A, B \in B(H)$. The operator Γ defined on C_2 by $\Gamma X = AXB$ has been studied in [2]. It is easy to see that $\|\Gamma\| \le \|A\|\,\|B\|$ and the adjoint of Γ is given by $\Gamma^* X = A^*XB^*$. Indeed,

$$\begin{aligned}\langle \Gamma^* X, Y\rangle &= \langle X, \Gamma Y\rangle \\ &= \langle X, AYB\rangle = \operatorname{tr}((AYB)^*X) = \operatorname{tr}(XB^*Y^*A^*) \\ &= \operatorname{tr}(A^*XB^*Y^*) = \langle A^*XB^*, Y\rangle.\end{aligned}$$

If $A, B \ge 0$, then $\Gamma \ge 0$ and

$$\Gamma^{1/2}X = A^{1/2}XB^{1/2}.$$

Indeed,

$$\begin{aligned}\langle AXB, X\rangle &= \operatorname{tr}(AXBX^*) = \operatorname{tr}(A^{1/2}XBX^*A^{1/2}) \\ &= \operatorname{tr}(A^{1/2}XB^{1/2}(A^{1/2}XB^{1/2})^*) \ge 0.\end{aligned}$$

In order to generalize the class of ω-hyponormal operators, Ito [13] introduced the class $\omega A(s,t)$. An operator T belongs to the class $\omega A(s,t)$ for $s, t > 0$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{1/(s+t)} \ge |T^*|^{2t}, \tag{3.1}$$

$$|T|^{2s} \ge (|T|^s|T^*|^{2t}|T|^s)^{s/(s+t)}. \tag{3.2}$$

Ito [13] showed that $\omega A(s,t)$ can be expressed via the generalized Aluthge transformation as follows:

An operator T belongs to the class $\omega A(s,t)$ for $s,t>0$ if and only if

$$|\tilde{T}_{s,t}|^{2t/(s+t)} \geq |T|^{2t} \quad \text{and} \quad |T|^{2s} \geq |\tilde{T}^*|^{2s/(s+t)}. \tag{3.3}$$

An operator T is said to be of *class* $\omega F(s,t,q)$ for $s,t>0$ and $q\geq 1$ if

$$(|T^*|^t|T|^{2s}|T^*|^t)^{1/q} \geq |T^*|^{2(s+t)/q},$$
$$|T|^{2(s+t)(1-1/q)} \geq (|T|^s|T^*|^{2t}|T|^s)^{1-1/q}.$$

Because q and $(1-q^{-1})^{-1}$ with $q>1$ are a couple of conjugate exponents, it is clear that the class $\omega A(s,t)$ equals $\omega F(s,t,(s+t)/t)$.

In the following lemma we exhibit some properties of the operator Γ.

Lemma 3.3. *Let $p,r>0$ and $q\geq 1$. Let A and B^* be of class $\omega F(p,r.q)$. If X is a Hilbert–Schmidt operator, then the operator $\Gamma : C_2 \to C_2$ defined by $\Gamma X = AXB$ is also of class $\omega F(p,r,q)$.*

Proof. We have

$$(|\Gamma^*|^r|\Gamma|^{2p}|\Gamma^*|^r)X = |A^*|^r|A|^{2p}|A^*|^rX|B|^r|B^*|^{2p}|B|^r,$$
$$(|\Gamma^*|^p|\Gamma|^{2r}|\Gamma^*|^p)X = |A^*|^p|A|^{2r}|A^*|^pX|B|^p|B^*|^{2r}|B|^p$$

and

$$|\Gamma^*|^{2(p+r)/q)}X = |A^*|^{2(p+r)/q}X|B|^{2(p+r)/q},$$
$$|\Gamma|^{2(p+r)(1-1/q)}X = |A|^{2(p+r)(1-1/q)}X|B^*|^{2(p+r)(1-1/q)},$$

for any $X\in C_2$. Then

$$(|\Gamma^*|^r|\Gamma|^{2p}|\Gamma^*|^r)^{1/q} = (|A^*|^r|A|^{2p}|A^*|^r)^{1/q}X(|B|^r|B^*|^{2p}|B|^r)^{1/q}$$

and

$$(|\Gamma|^p|\Gamma^*|^{2r}|\Gamma|^p)^{1-1/q}X = (|A|^p|A^*|^{2r}|A|^p)^{1-1/q}X(|B^*|^p|B|^{2r}|B^*|^p)^{1-1/q}.$$

Since A,B^* are of class $\omega F(p,r,q)$, we get

$$\begin{aligned}((|\Gamma^*|^r|\Gamma|^{2p}|\Gamma^*|^r)^{1-1/q} - |\Gamma^*|^{2(p+r)/q})X \\ = ((|A^*|^r|A|^{2p}|A^*|^r)^{1/q} - |A^*|^{2(p+r)/q})X(|B|^r|B^*|^{2p}|B|^r)^{1/q} \\ + |A^*|^{2(p+r)/q}X((|B|^r|B^*|^{2p}|B|^r)^{1/q} - |B|^{2(p+r)/q}) \geq 0\end{aligned}$$

and

$$\begin{aligned}(|\Gamma|^{2(p+r)(1-1/q)}) - (|\Gamma|^p|\Gamma^*|^{2r}|\Gamma|^p)^{1-1/q}X \\ = |A|^{2(p+r)(1-1/q)} - (|A|^p|A^*|^{2r}|A|^p)^{1-1/q}X|B^*|^{2(p+r)(1-1/q)} \\ + (|A|^p|A^*|^{2r}|A|^p)^{1-1/q}X(|B^*|^p)^{2(p+r)(1-1/q)} - (|B^*|^p|B|^{2r}|B^*|^p)^{1-1/q}) \geq 0.\end{aligned}$$

Thus

$$(|\Gamma^*|^r|\Gamma|^{2p}|\Gamma^*|^r)^{1/q} \geq |\Gamma^*|^{2(p+r)/q}$$

and

$$|\Gamma|^{2(p+r)(1-1/q)} \geq (|\Gamma|^p|\Gamma^*|^{2r}|\Gamma|^p)^{1-1/q}. \blacksquare$$

LEMMA 3.4 ([25]). *Let T be a class $\omega F(p,r,q)$ operator for $0 < p+r \leq 1$ and $q \geq 1$. If $Tx = \lambda x$ with $\lambda \neq 0$, then $T^*x = \overline{\lambda}x$, i.e., the non-zero eigenvalues of T are normal eigenvalues.*

LEMMA 3.5 ([25]). *If T is an invertible class $\omega F(p,r,q)$ operator, where $p, r > 0$ and $q > 1$, then T^{-1} is $\omega F(p,r,q')$, where $1/q + 1/q' = 1$.*

THEOREM 3.6. *Let T be a class $\omega F(p,r,q)$ operator, where $0 < p + r \leq 1$ and $q \geq 1$, and S^* an invertible class $\omega F(p,r,q')$ operator, where $1/p + 1/q' = 1$. If $TX = XS$ for some Hilbert–Schmidt operator X, then $T^*X = XS^*$.*

Proof. Let Γ be the Hilbert–Schmidt operator defined by $\Gamma X = TXS^{-1}$ for all $X \in C_2$. Since $(S^*)^{-1} = (S^{-1})^*$ is of class $\omega F(q,r,q')$, where $1/q + 1/q' = 1$, Lemma 2.3 implies that Γ is of class $\omega F(q,r,q')$. The rest follows as in [14]. $\blacksquare$

Acknowledgements. I would like to express my cordial gratitude to the referee for his valuable advice and suggestions. This work was supported by Taibah University research center 1435.

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Received 10 December 2013;
revised 9 September 2014 (6101)