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REDUCED SPHERICAL POLYGONS

BY

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Abstract. For every hemisphere K supporting a spherically convex body C of the d-dimensional sphere S^d we consider the width of C determined by K. By the thickness $\Delta(C)$ of C we mean the minimum of the widths of C over all supporting hemispheres K of C. A spherically convex body $R \subset S^d$ is said to be reduced provided $\Delta(Z) < \Delta(R)$ for every spherically convex body $Z \subset R$ different from R. We characterize reduced spherical polygons on S^2. We show that every reduced spherical polygon is of thickness at most $\pi/2$. We also estimate the diameter of reduced spherical polygons in terms of their thickness. Moreover, a few other properties of reduced spherical polygons are given.

1. Introduction. Let S^d be the unit sphere in the $(d+1)$-dimensional Euclidean space E^{d+1}, where $d \geq 2$. The common part of S^d with any hyper-subspace of E^{d+1} is called a $(d-1)$-*dimensional great sphere* of S^d. In particular, for S^2 the 1-dimensional great spheres are called *great circles*. By a pair of *antipodes* of S^d we mean any pair of points of intersection of S^d with a one-dimensional subspace of E^{d+1}. Observe that if two different points a, b are not antipodes, there is exactly one great circle containing them. By the *arc* ab connecting a and b we mean the shorter part of the great circle containing a and b. The *spherical distance* $|ab|$, or briefly *distance*, of these points is the length of the arc connecting them. By the *spherical ball* of radius $r \leq \pi/2$ and center c we mean the set $B = \{p : |pc| \leq r\}$. For every point $c \in S^d$, by the *hemisphere* with *center* c we mean the spherical ball of radius $\pi/2$. An *open hemisphere* is the interior of the hemisphere. Two hemispheres whose centers are antipodes are called *opposite*.

We say that a set $C \subset S^d$ is *spherically convex*, or briefly *convex*, if it does not contain any pair of antipodes and if together with any two points it contains the whole arc connecting them. By a *spherically convex body* on S^d we mean a closed spherically convex set with non-empty interior. For a short survey of other definitions of convexity on S^d we refer to [2, Section 9.1]. The literature concerning this subject is very large; for instance see [3]–[5], [10], [13] and [14].

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Clearly, the intersection of every family of spherically convex sets is also a spherically convex set. Thus for every set $D \subset S^d$ which is a subset of an open hemisphere there exists a unique smallest convex set containing D. It is called the *convex hull of* D.

If a $(d-1)$-dimensional great sphere F of S^d has a common point p with a convex body $C \subset S^d$ and if its intersection with the interior of C is empty, we say that F is a *supporting* $(d-1)$*-dimensional great sphere of* C *passing through* p. We also say that F *supports* C at p. If p is the only point of support, we say that F *strictly supports* C at p. In particular, if $C \subset S^2$, we call F a *supporting great circle of* C. If H is the hemisphere bounded by F and containing C, we say that H *supports* C at p.

By a *spherically convex polytope* we mean the convex hull of a finite number of points of S^d which is a spherically convex body. For $d = 2$, we get the notion of a *spherically convex polygon.* The convex hull V of $k \geq 3$ points on S^2 such that none of them belongs to the convex hull of the remaining points is called a *spherically convex* k*-gon.* These points are called *vertices* of V. We write $V = v_1 \dots v_k$ when $v_1, \dots, v_k$ are the successive vertices of V, when we go around the boundary of V in the positive direction. When we take $k \geq 3$ successive points on a spherical circle of radius less than $\pi/2$ on S^2 with equal distances of any two successive points, we obtain a *regular spherical* k*-gon.*

The set of points of a great circle of S^2 which are at distance at most $\pi/2$ from a fixed point p of this great circle is called a *semicircle.* We say that p is the *center* of this semicircle. We say that a semicircle *supports* (respectively, *strictly supports*) a spherical convex body $C \subset S^2$ at a point if this point belongs to this semicircle and if the great circle containing this semicircle supports (respectively, strictly supports) C at this point.

If hemispheres G and H of S^d are different and not opposite, then $L = G \cap H$ is called a *lune* of S^d. This notion is considered in many books and papers (see for instance [12, p. 18]). Later we consider only lunes on S^2. The semicircles bounding L and contained in the great circles bounding G and H, respectively, are denoted by G/H and H/G. The distance of the centers of G/H and H/G is called the *thickness* of L. The midpoint of these two centers is called the *center of* L.

Let a be a point in a hemisphere different from its center and let F be the great circle bounding the hemisphere. By the *projection of* a *on* F we mean the point $p \in F$ such that the distance $|ap|$ is the smallest of all $|ac|$, where $c \in F$.

By *rotation* of a set $K \subset S^2$ *around a point* $p \in S^2$ we mean rotation of K around the straight line through the origin of E^3 and p.

Let K be a supporting hemisphere of a convex body $C \subset S^d$. Let us recall the notion of *width of* C *determined by* K, presented in [8]. We are just looking for hemispheres K^* supporting C such that the lunes $K \cap K^*$ are of minimum thickness. By compactness arguments we immediately see that at least one such hemisphere K^* exists, and thus at least one corresponding lune $K \cap K^*$ exists. Theorem 1 of [8] explains how to find K^* and thus the lune or lunes $K \cap K^*$. Denote by $\mathrm{width}_K(C)$ the thickness of the lune (or lunes) $K \cap K^*$; we call it the *width of C determined by K*. This notion of width of $C \subset S^d$ is an analogue of the notion of width of a convex body of E^d. The *thickness* $\Delta(C)$ of C is the minimum of the widths of C over all supporting hemispheres K of C.

Following [8] we say that a spherically convex body $R \subset S^d$ is *reduced* if $\Delta(Z) < \Delta(R)$ for every convex body $Z \subset R$ different from R. This is an analogue of the notion of a reduced body in Euclidean space, given in [6]. See also the survey article [9]. Examples of reduced spherical bodies are bodies of constant width on S^d (for the definition see e.g. [8]), and in particular *spherical balls* of radius smaller than $\pi/2$. Each of the 2^d parts of a spherical ball $B \subset S^d$ dissected by d great pairwise orthogonal $(d-1)$-dimensional spheres through the center of B, called a *$1/2^d$-part of the ball*, is also a reduced spherical body. In particular, for $d = 2$, it is called a *quarter of a spherical disk*. Every regular spherical odd-gon of thickness at most $\pi/2$ is a reduced spherical body.

In this paper we continue the research on width and thickness of spherically convex bodies started in [8]. This time we limit the considerations to the spherically convex polygons on S^2, and especially reduced polygons. The main Theorem 3.2 presents a characterization of them. It is similar to Theorem 8 from [7] giving a characterization of reduced polygons in E^2. But this time we cannot apply a theorem analogous to Theorem 3 from [7] and its consequences on the shape of reduced bodies in E^2. The proof of [7, Theorem 3] applied the notion of parallelism, which does not exist on the sphere. The author has not been able to repeat the proof of an analogous theorem on the sphere, so the present approach is partially different. From Theorem 3.2 a number of corollaries on reduced polygons follow. Moreover, we show that every reduced spherically convex polygon is of thickness at most $\pi/2$. We also estimate the diameter of reduced spherical polygons in terms of their thickness.

2. Width and thickness of spherical polygons

PROPOSITION 2.1. *Let $V \subset S^2$ be a spherically convex polygon and let L be a lune of thickness $\Delta(V)$ containing V. Then at least one of the two semicircles bounding L contains a side of V. If $\Delta(V) < \pi/2$, then the center of this semicircle belongs to the relative interior of that side.*

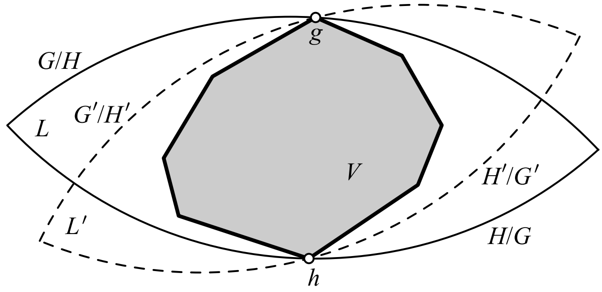

Fig. 1. Rotation of G around g, and of H around h

Proof. Our lune L has the form $G \cap H$, where G and H are different non-opposite hemispheres.

Let us prove the first statement. Assume the contrary, i.e., each of the two semicircles bounding L contains exactly one vertex of V. Denote by g the vertex contained by G/H, and by h the vertex contained by H/G. Claim 2 from [8] says that if L is a lune of minimum thickness containing a convex body $C \subset S^d$, then both the centers of the $(d-1)$-dimensional hemispheres bounding L belong to C. We infer that g is the center of G/H and h is the center of H/G. Now rotate G around g, and H around h, both in the same direction and by the same angle. Since G/H and H/G strictly support V, after any sufficiently small rotation of both in the same direction, we obtain a new lune $L' = G' \cap H'$, where G' and H' are the images of G and H under the corresponding rotations (see Figure 1). It still contains V and has only g and h in common with the boundary of V. But since now g and h are not the centers of the semicircles G'/H', H'/G' which bound L' and they are symmetric with respect to the center of L', we get $\Delta(L') < \Delta(L)$. This contradicts the assumption that $\Delta(L) = \Delta(V)$. We conclude that the first part of our proposition holds true.

For the second part, assume the contrary: the center of the semicircle bounding L does not belong to the relative interior of the side S contained in that semicircle. Corollary 2 of [8], which says that the center belongs to S, now implies that it must be an end-point of S, and thus a vertex of V. Apply $\Delta(V) < \pi/2$. After sufficiently small rotations of G around g, and H around h, both in the same direction, we get a lune narrower than L still containing V, a contradiction. ∎

If $\Delta(V) = \pi/2$, then the center of the semicircle in the formulation of Proposition 2.1 may be an end-point of a side of V. This happens, for instance, for the regular triangle of thickness $\pi/2$.

Proposition 2.2. *Let $V \subset S^2$ be a spherically convex polygon of thickness $> \pi/2$ and let L be a lune of thickness $\Delta(V)$ containing V. Then each of* *the two semicircles bounding L contains a side of V. Moreover, the centers of these semicircles belong to the relative interiors of the sides of V contained in the semicircles.*

Proof. Our argument is partially similar to that in the proof of Proposition 2.1. Assume the opposite to the first statement: at least one the semicircles $G/H, H/G$, say the former, contains exactly one vertex of V, say g. By a similar argument to one in the proof of Proposition 2.1, g is the center of G/H.

After a sufficiently small rotation of G around g, we obtain a new lune L' as the intersection of the new hemisphere G' with H, which still contains V. This and $\Delta(L) > \pi/2$ imply $\Delta(L') < \Delta(L)$, which contradicts the assumption of our proposition that L is of thickness $\Delta(V)$. Hence each of the two semicircles bounding L contains a side of V.

To show the second statement, assume the contrary: the center of a semicircle bounding L, say g, does not belong to the relative interior of the side S of V which is in this semicircle. Then g does not belong to S, or is an end-point of S. After a sufficiently small rotation of G around g, we obtain a new lune L' as the intersection of the new hemisphere G' with H, which still contains V. Since $\Delta(L') < \Delta(L)$, we get a contradiction. ∎

3. Reduced spherical polygons

THEOREM 3.1. *Every reduced spherical polygon is of thickness at most $\pi/2$.*

Proof. Assume that there exists a spherically convex polygon V of thickness larger than $\pi/2$. Let $L \supset V$ be a lune of thickness $\Delta(V)$. By Proposition 2.2 both the semicircles bounding L have with V some sides of V in common, and both their centers belong to the relative interiors of those sides. Thus after we cut off a piece of V by a great circle passing sufficiently close to an end-point of one of these two sides, we obtain a convex polygon $Z \subset V$ different from V such that $\Delta(Z) = \Delta(V)$. Hence V is not reduced, and the proof is finished. ∎

For a convex odd-gon $V = v_1 \dots v_n$, the *opposite side* to the vertex v_i is the side $v_{i+(n-1)/2}v_{i+(n+1)/2}$. The indices are taken modulo n.

THEOREM 3.2. *Every reduced spherical polygon is an odd-gon of thickness at most $\pi/2$. A spherically convex odd-gon V with $\Delta(V) < \pi/2$ is reduced if and only if the projection of each of its vertices on the great circle containing the opposite side belongs to the relative interior of that side and the distance of this vertex from that side is $\Delta(V)$.*

Proof. Let $V = v_1 \dots v_n$ be a reduced spherical polygon. By Theorem 3.1 it is of thickness at most $\pi/2$. By [8, Theorem 4] for every vertex v_i there exists a lune L_i of thickness $\Delta(V)$ containing V one of whose bounding semicircles, say S_i, strictly supports V at v_i. Moreover, v_i is the center of this semicircle. By Proposition 2.1 the opposite semicircle S_i' bounding L_i contains a side of V, which we denote by T_i. Moreover, the center t_i of S_i' belongs to the relative interior of T_i. Of course, t_i is the projection of v_i on S_i'.

Since the segments $v_1t_1, \dots, v_nt_n$ pairwise intersect, the sides $T_1, \dots, T_n$ are different and consecutive. Thus n is odd and $T_i = v_{i+(n-1)/2}v_{i+(n+1)/2}$, so T_i is the opposite side to v_i. Of course the distance from v_i to T_i is $\Delta(V)$.

On the other hand, consider a convex spherical odd-gon $V = v_1 \dots v_n$ fulfilling the "if" part assumptions in the second sentence of the theorem. For every $i \in \{1, \dots, n\}$ the projection of v_i on the great circle containing the opposite side $v_{i+(n-1)/2}v_{i+(n+1)/2}$ belongs to the relative interior of that side. Moreover, the projection of $v_{i+(n-1)/2}$ on the great circle containing the opposite side $v_{i-1}v_i$ belongs to its relative interior, and the projection of $v_{i+(n+1)/2}$ on the great circle containing the opposite side v_iv_{i+1} belongs to its relative interior. Thus the lune whose centers of bounding semicircles are v_i and t_i strictly supports V at v_i. This holds true for $i = 1, \dots, n$. Hence by the assumption, the distance of every vertex of V to the opposite side is $\Delta(V)$. As a result, for every convex body $Z \subset V$ different from V we have $\Delta(Z) < \Delta(V)$. Consequently, V is a reduced spherical polygon. ∎

In Figure 2 we see a spherically convex pentagon. By Theorem 3.2, it is reduced.

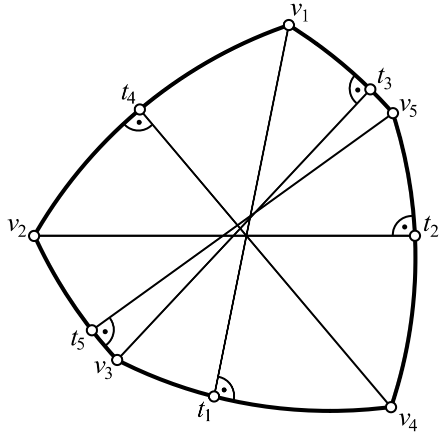

Fig. 2. A reduced spherical pentagon

Corollary 3.3. *Every spherical regular odd-gon of thickness at most $\pi/2$ is reduced.*

Corollary 3.4. *The only reduced spherical triangles are the regular triangles of thickness at most $\pi/2$.*

Proof. By Corollary 3.3 regular triangles of thickness at most $\pi/2$ are reduced. Assume that there exists a reduced spherical triangle W which is not regular. Hence some two of its three heights are different, which implies that the corresponding orthogonal widths are different. The smallest of the three heights equals $\Delta(W)$. We may cut off a sufficiently small piece of W at a vertex of W which is not an end-point of the shortest height, so that the resulting smaller spherically convex body still has thickness $\Delta(W)$. Since W is assumed to be reduced, we get a contradiction. ∎

COROLLARY 3.5. *If K is a supporting hemisphere of a reduced spherical polygon V whose bounding circle contains a side of V, then* $\mathrm{width}_K(V) = \Delta(V)$.

We say that two sets on S^2 are *symmetric with respect to a great circle* if they are symmetric with respect to the plane of E^3 containing that circle.

COROLLARY 3.6. *For every reduced odd-gon $V = v_1 \dots v_n$ with $\Delta(V) < \pi/2$ we have $|v_i t_{i+(n+1)/2}| = |t_i v_{i+(n+1)/2}|$ for $i = 1, \dots, n$, where t_i denotes the projection of v_i on the opposite side.*

Proof. We apply Theorem 3.2. Let $i \in \{1, \dots, n\}$. Take the lune L_i of thickness $\Delta(V)$ such that $V \subset L_i$, one of whose bounding semicircles contains the side $v_{i+(n-1)/2}v_{i+(n+1)/2}$ and in particular the spherical segment $t_i v_{i+(n+1)/2}$ (see Figure 3). Denote the bounding semicircles of L_i by S_i and

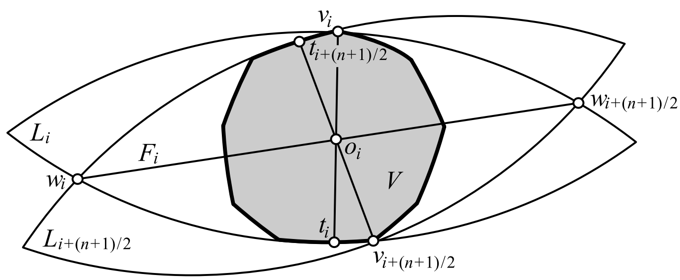

Fig. 3. The lunes L_i and $L_{i+(n+1)/2}$

S_i' so that the first contains $v_{i+(n-1)/2}v_{i+(n+1)/2}$ (and so $t_i v_{i+(n+1)/2}$), and the second supports V at v_i. Also take the lune $L_{i+(n+1)/2}$ of thickness $\Delta(V)$ such that $V \subset L_{i+(n+1)/2}$, and one of whose bounding semicircles contains the side $v_i v_{i+1}$ and in particular the spherical segment $v_i t_{i+(n+1)/2}$. Denote the bounding semicircles of $L_{i+(n+1)/2}$ by $S_{i+(n+1)/2}$ and $S'_{i+(n+1)/2}$ so that the first contains $v_i v_{i+1}$ and the second supports V at $v_{i+(n+1)/2}$.

Let w_i be the point of intersection of S_i and $S_{i+(n+1)/2}$, and let $w_{i+(n+1)/2}$ be the point of intersection of S_i' and $S'_{i+(n+1)/2}$. The common part of L_i and

$L_{i+(n+1)/2}$ is the spherical quadrangle $v_i w_i v_{i+(n+1)/2} w_{i+(n+1)/2}$. Denote by o_i the intersection of $v_i t_i$ and $v_{i+(n+1)/2} t_{i+(n+1)/2}$. Clearly, o_i is the midpoint of $w_i w_{i+(n+1)/2}$. Consider the great circle F_i containing $w_i w_{i+(n+1)/2}$, and thus also o_i. Observe that L_i and $L_{i+(n+1)/2}$ are symmetric with respect to F_i. In particular, v_i and $v_{i+(n+1)/2}$ are symmetric, i.e., their projections on F_i coincide and are at equal distances from v_i and $v_{i+(n+1)/2}$. Also $t_{i+(n+1)/2}$ and t_i are symmetric. Hence the triangles $v_i o_i t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} o_i t_i$ are symmetric with respect to F_i. Thus $|v_i t_{i+(n+1)/2}| = |t_i v_{i+(n+1)/2}|$. ∎

Theorem 3.2 together with Corollary 3.6 allows one to construct reduced spherical polygons of thickness below $\pi/2$. For instance, see the reduced spherical pentagon in Figure 2. The author does not know if there are non-regular reduced polygons of thickness $\pi/2$.

In the proof of Corollary 3.6 it is shown that the triangles $v_i o_i t_{i+(n+1)/2}$ and $v_{i+(n+1)/2} o_i t_i$ are symmetric. This implies the following corollary.

COROLLARY 3.7. *In every reduced odd-gon $V = v_1 \dots v_n$ with $\Delta(V) < \pi/2$, for every $i \in \{1, \dots, n\}$ we have $\angle t_{i+(n+1)/2} v_i t_i = \angle t_i v_{i+(n+1)/2} t_{i+(n+1)/2}$.*

By Corollary 3.6 applied n times, the sum of the lengths of the boundary spherical segments of V from v_i to t_i (with positive orientation) is equal to the sum of the boundary spherical segments of V from t_i to v_i (with positive orientation). Let us formulate this statement as the following corollary.

COROLLARY 3.8. *Let V be a reduced spherical odd-gon and let $i \in \{1, \dots, n\}$. The spherical segment $v_i t_i$ halves the perimeter of V.*

In a spherically convex odd-gon $V = v_1 \dots v_n$, for every $i \in \{1, \dots, n\}$ we set $\alpha_i = \angle v_{i+1} v_i t_i$ and $\beta_i = \angle t_i v_i v_{i+(n+1)/2}$.

COROLLARY 3.9. *If $V = v_1 \dots v_n$ is a reduced spherical polygon with $\Delta(V) < \pi/2$, then $\beta_i \le \alpha_i$ for every $i \in \{1, \dots, n\}$.*

Proof. Denote by u_i the intersection of the great circles containing $v_i v_{i+1}$ and $v_{i+(n-1)/2} v_{i+(n+1)/2}$ such that $v_{i+1} \in v_i u_i$. By Corollaries 3.6 and 3.8 the lengths of the fragments of the boundary of V from $t_{i+(n+1)/2}$ to t_i and from $v_{i+(n+1)/2}$ to v_i are equal. Moreover, the boundary of V from $t_{i+(n+1)/2}$ to t_i is in the triangle $t_{i+(n+1)/2} u_i t_i$, which implies that its length is at most $|t_{i+(n+1)/2} u_i| + |u_i t_i|$. Similarly, the length of the boundary from $v_{i+(n+1)/2}$ to v_i is at least $|v_{i+(n+1)/2} v_i|$. Consequently, $|v_{i+(n+1)/2} v_i| \le |t_{i+(n+1)/2} u_i| + |u_i t_i|$. By Corollary 3.6 we get $|v_i v_{i+(n+1)/2}| < |v_i u_i|$. Since the triangles $v_i v_{i+(n+1)/2} t_i$ and $v_i u_i t_i$ have the common side $v_i t_i$ and right angles at t_i, this implies $\beta_i \le \alpha_i$. ∎

COROLLARY 3.10. *For every reduced polygon and every i we have $\alpha_i > \pi/6 + E$ and $\beta_i < \pi/6 + E$, where E denotes the excess of the triangle $v_i t_i v_{i+(n+1)/2}$.*

The proof of the corollary is left to the reader. The inequality for α_i is stronger, but the one for β_i is weaker than those in [7, Theorem 8]. For instance, in the regular spherical triangle of sides of length $\pi/2$ we have $\alpha_i = \beta_i = \pi/4$.

Recall the problem from [7], repeated in [9], of whether there are reduced polytopes in E^d for $d \geq 3$. For example, simplices are not reduced bodies (see [11]), and in [1] larger classes of Euclidean polytopes containing no reduced polytopes are characterized. For S^d the analogous question has a positive answer. The spherical simplex which is the $1/2^d$-part of the ball of radius $\pi/2$ is a reduced spherical polytope. This follows from the fact that it is also a spherical body of constant width. The problem of whether there are other spherical polytopes on S^d, where $d \geq 3$, remains open.

4. Diameter of reduced spherical polygons

PROPOSITION 4.1. *The diameter of any reduced spherical n-gon is realized only for some pairs of vertices whose indices (modulo n) differ by $(n-1)/2$ or $(n+1)/2$.*

Proof. We apply Theorem 3.2. We see that n is odd. Let $V = v_1 \dots v_n$ be our reduced odd-gon. Of course, the diameter of V equals the maximum distance between its vertices. It is sufficient to show that this maximum distance is of the form $|v_i v_{i+(n-1)/2}|$ or $|v_i v_{i+(n+1)/2}|$, where $i \in \{1, \dots, n\}$.

Recall that Theorem 3 of [8] says that the diameter of a spherically convex body equals its maximum width. So in order to prove the proposition it is sufficient to show that for every vertex v_i of V and for every hemisphere H supporting V at v_i the only vertices of V that may be on the semicircle H^*/H are $v_{i+(n-1)/2}$ and $v_{i+(n+1)/2}$.

When changing the supporting hemisphere H of V from the position where the bounding great circle contains the side $T_{i+(n-1)/2}$ to the position where the bounding great circle contains $T_{i+(n+1)/2}$ (by rotating H around v_i), we see that the hemisphere H first supports V only at the points of the spherical segment $T_{i+(n-1)/2}$, then it supports V only at the point v_i, and finally it supports V at the points of $T_{i+(n+1)/2}$. Thus by Theorem 3.2 and Corollary 3.3, and also by Theorems 1 and 2 of [8] and their proofs, the corresponding hemisphere H^* first supports V only at $v_{i+(n-1)/2}$, then it supports V once at the side T_i, and at the end only at $v_{i+(n+1)/2}$ for some time. Hence during the above described change of position of H, each H^*/H supports V only at $v_{i+(n-1)/2}$ or $v_{i+(n+1)/2}$. ∎

Theorem 4.2. *For every reduced spherical polygon V on the sphere we have*

$$\mathrm{diam}(V) \leq \arccos\left(\sqrt{1 - \frac{\sqrt{2}}{2}\sin\varDelta(V)} \cdot \cos\varDelta(V)\right),$$

with equality for V being a regular spherical triangle.

Proof. We apply Proposition 4.1. Assume that $\mathrm{diam}(V) = |v_i v_{i+(n+1)/2}|$ for an $i \in \{1, \dots, n\}$ (the case when $\mathrm{diam}(V) = |v_i v_{i+(n-1)/2}|$ is analogous).

Set $p_i = |t_i v_{i+(n+1)/2}|$, $s_i = |v_i v_{i+(n+1)/2}|$ and $\gamma_i = \angle t_i v_{i+(n+1)/2} v_i$.

By Corollary 3.7 we have $\gamma_i = \alpha_i + \beta_i$. Corollary 3.9 now implies $\gamma_i \geq 2\beta_i$. From $|v_i t_i| = \varDelta(V)$ and the sine theorem on the sphere we have

$$\frac{\sin p_i}{\sin\beta_i} = \frac{\sin\varDelta(V)}{\sin\gamma_i}.$$

Thus from $\gamma_i \geq 2\beta_i$ we get

$$\frac{\sin p_i}{\sin\beta_i} \leq \frac{\sin\varDelta(V)}{\sin 2\beta_i}, \quad \text{so} \quad \sin p_i \leq \frac{\sin\varDelta(V)}{2\cos\beta_i}.$$

Moreover we have $\beta_i \leq \pi/4$, since otherwise from $\beta_i \leq \alpha_i$ (Corollary 3.9) and $\gamma_i = \alpha_i + \beta_i$ it follows that $\gamma_i > 2\beta_i > \pi/2$, which is impossible because by Theorem 3.2 we have $\varDelta(V) \leq \pi/2$.

Thus $\sin p_i \leq \frac{\sqrt{2}}{2}\sin\varDelta(V)$. Hence $\sin^2 p_i \leq \frac{1}{2}\sin^2\varDelta(V)$. So

$$\sqrt{1 - \frac{1}{2}\sin^2\varDelta(V)} \leq \cos p_i.$$

Moreover, by the Pythagorean theorem on the sphere we have $\cos s_i = \cos p_i \cos\varDelta(V)$. Consequently,

$$\cos s_i \geq \sqrt{1 - \frac{\sqrt{2}}{2}\sin\varDelta(V)} \cdot \cos\varDelta(V).$$

Proposition 4.1 now gives the desired inequality.

If V is a regular spherical triangle, then $v_i v_{i+(n+1)/2}$, where $i \in \{1, 2, 3\}$, are the sides of V. The distance of the end-points of each of them is just $\arccos\big(\sqrt{1 - \frac{\sqrt{2}}{2}\sin\varDelta(V)} \cdot \cos\varDelta(V)\big)$, by the Pythagorean theorem on the sphere. ∎

By Theorem 4.2 every reduced spherical polygon of thickness $\pi/4$ is of diameter at most $\pi/3$. Clearly, the regular triangle of thickness $\pi/4$ has diameter $\pi/3$.

We conjecture that equality in Theorem 4.2 holds only for regular triangles. We also conjecture that the diameter of every reduced convex body $C \subset S^2$, where $\varDelta(C) \leq \pi/2$, is at most $\arccos(\cos^2\varDelta(C))$, which would imply that $\mathrm{diam}(C) \leq \sqrt{2}\,\varDelta(C)$. The only extreme case seems to be the quarter Q of a disk. Its diameter is just $\arccos(\cos^2 \varDelta(Q))$. By L'Hospital's rule we find that $\arccos(\cos^2 \varDelta(Q))/\varDelta(Q)$ tends to $\sqrt{2}$ as $\varDelta(Q)$ tends to 0. So the limit factor is as in the planar case (see [7, Theorem 9]).

We conjecture that the perimeter of every reduced spherical polygon V is not larger than for the spherical regular triangle of the same thickness, so it is at most

$$6 \arccos \frac{\cos \varDelta(V) + \sqrt{8 + \cos^2 \varDelta(V)}}{4},$$

and that it is attained only for this regular triangle. We expect that of all reduced spherical polygons of fixed thickness and with at most n vertices, only the regular spherical n-gon has the minimal perimeter. We also conjecture that the area of every reduced spherical polygon V is less than $2\big(1 - \cos \frac{\varDelta(V)}{2}\big)\pi$ and that this estimate cannot be improved in general. This is the limit value for the area of the regular spherical odd-gons whose number of vertices tends to infinity. We also expect that every reduced spherical non-regular n-gon of fixed thickness has area smaller than the regular spherical n-gon of that thickness.

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